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DETERMINING THE EQUATION OF STATE OF THE EXPANDING UNIVERSE USING A NEW INDEPENDENT VARIABLE

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ABSTRACT

To determine the equation of state of the universe, we propose to use a new independent variable $R \equiv (H_0/c)[d_L(z)/(1+z)]$, where H_0 and $d_L(z)$ are the present Hubble parameter and the luminosity distance, respectively. For the flat universe suggested by the observation of the anisotropy of cosmic microwave backgrounds, the density and the pressure are expressed as $\rho/\rho_0 = 4(df/dR)^2/f^6$ and $p/\rho_0 = -4/3(d^2f/dR^2)/f^5$, where ρ_0 is the present density and $f(R) = 1/[1+z(R)]^{1/2}$. In the (R,f) plane the sign as well as the strength of the pressure is in proportion to the curvature of the curve $f(R)$. We propose to adopt a Pade-like expression of $f(R) = 1/u^{1/2}$ with $u \equiv 1 + \sum_{n=1}^{N} u_n R^n$. For the flat Λ models, the expansion up to $N = 7$ has at most an error less than 0.2% for $z < 1.7$ and any value of Λ. We also propose a general method to determine the equation of state of the universe that has $N - 1$ free parameters. If the number of parameters are smaller than $N - 1$, there is a consistency check of the equation of state so that we may confirm or refute each model.

Subject headings: cosmology: theory — dark matter — distance scale

1. INTRODUCTION

Recent measurements of the luminosity distance $d_L(z)$ using Type Ia supernovae (Riess et al. 1998; Schmidt et al. 1998; Perlmutter et al. 1999) suggest that accurate $d_L(z)$ may be obtained in the near future. SNAP[1] especially will give us the luminosity distance of $\sim$2000 Type Ia supernovae with an accuracy of a few percent up to $z \sim 1.7$ every year. On the other hand, from the observation of the first Doppler peak of the anisotropy of the cosmic microwave background, it is now suggested that the universe is flat (de Bernardis et al. 2000; Lange et al. 2000), which may be proved in the future by the *Microwave Anistropy Probe* and *Planck* satellite. If the flat universe case is correct, the density $\rho(z)$ and the pressure $p(z)$ in principle can be determined only from $d_L(z)$ (Nakamura & Chiba 1999), so that the equation of the state of the universe is uniquely determined. If not, the determination of the present curvature of the universe and the determination of $\rho(z)$ and $p(z)$ will be coupled in general (Nakamura & Chiba 1999).

Now let us assume that the universe is flat. Even in this case at least two problems exist: (1) How to express the continuous function $d_L(z)$, which is accurate enough from $z = 0$ to $z \sim 1.7$, using several free parameters. (2) How to obtain accurate $\rho(z)$ and $p(z)$ from $d_L(z)$, that is, the equation of state of the universe (Starobinsky 1998; Huterer & Turner 1999; Nakamura & Chiba 1999; Chiba & Nakamura 2000; Saini et al. 2000).

In this paper, we propose to use a new independent variable $R \equiv (H_0/c)[d_L(z)/(1+z)]$ instead of z, where H_0 is the present Hubble parameter. We show that $\rho(R)/\rho_0 = 4(df/dR)^2/f^6$ and $p(R)/\rho_0 = -4/3(d^2f/dR^2)/f^5$, where ρ_0 is the present density and $f(R) = 1/[1+z(R)]^{1/2}$. This means that the pressure is in proportion to the curvature of the curve $f(R)$. For an accurate expression of $f(R)$, we propose a Pade-like form of $f(R) = 1/u^{1/2}$ with $u \equiv 1 + \sum_{n=1}^{N} u_n R^n$. For the flat Λ model, the expansion up to $N = 7$ has at most an error less than 0.2% for $z < 1.7$ and all values of Λ. We also propose a general method to determine the equation of state of the universe that has fewer than $N - 1$ free parameters.

2. NEW VARIABLES

The luminosity distance $d_L(z)$ is given by $d_L(z) = a_0(1+z)f(\chi)$, with $\chi = 1/a_0 \int_0^z dz'/H(z')$, where $H(z)$ and a_0 are the Hubble parameter at z and the present scale factor, respectively, and $f(\chi) = \chi$, $\sinh\chi$, and $\sin\chi$ for a flat, open, and closed universe, respectively. Let us define $R = (H_0/c)[d_L(z)/(1+z)]$, where H_0 is the present Hubble parameter. Then $\rho(z)$ and $p(z)$ are expressed as

$$\frac{\rho(z)}{\rho_0} = \frac{1}{(dR/dz)^2} + \left[(1+z)^2 - \frac{R^2}{(dR/dz)^2}\right] H_0^2 \Omega_{k0} ,$$

$$\frac{3p(z)}{\rho_0} = -\frac{3}{(dR/dz)^2} + (1+z)\frac{d}{dz}\left[\frac{1}{(dR/dz)^2}\right] - \left\{(1+z)^2 - \frac{3R^2}{(dR/dz)^2} + (1+z)\frac{d}{dz}\left[\frac{R^2}{(dR/dz)^2}\right]\right\} H_0^2 \Omega_{k0} , \quad (1)$$

[1] Supernova/Acceleration Probe at http://snap.lbl.gov.

where ρ_0 is the present density and $\Omega_{k0} \equiv k/(a_0^2 H_0^2)$ (Nakamura & Chiba 1999). Since the flat universe is suggested from both observations (de Bernardis et al. 2000; Lange et al. 2000) and theory (inflation paradigm), we consider only the $\Omega_{k0} = 0$ case in this paper.

Now we adopt R as an independent variable instead of z. Then $\rho(R)$ and $p(R)$ are expressed as $\rho(R)/\rho_0 = 4(df/dR)^2 f^{-6}$ and $p(R)/\rho_0 = -4/3(d^2f/dR^2)f^{-5}$, where $f(R) = 1/[1 + z(R)]^{1/2}$. These expressions of $\rho(R)$ and $p(R)$ have quite interesting physical meanings. The density is in proportion to the square of the first derivative of f with respect to the new independent variable R, while the pressure is in proportion to the second derivative of f, that is, the curvature. Therefore, if the pressure is zero, $f(R)$ is the straight line, while the negative pressure corresponds to the positive curvature of the curve $f(R)$ in the (R,f) plane. This is completely in contrast to the $[z, d_L(z)]$ plane, where as far as $\rho > 3p$, the curve $d_L(z)$ has the positive curvature. Therefore, in the $[z, d_L(z)]$ plane it is difficult to distinguish by eye if the pressure is negative or not. However, in the (R,f) plane it is quite easy to distinguish the sign of the curvature of the curve so that the negative pressure can be identified by eye. To demonstrate this we show in Figure 1 the data from the Supernova Cosmology Project (Perlmutter et al. 1999) in the (R,f) plane.[2] Although the error bar is considerably large, a glance shows that the curvature is positive.

The first way to analyze the data quantitatively is to expand f in a power series of R as $f(R) = 1 + \sum_{n=1}^{N} f_n R^n$, where f_n are constants. We did this expansion with $N = 4$ for the data from the Supernova Cosmology Project (Perlmutter et al. 1999) and obtained[3] $f_1 = -0.5$, $f_2 = 0.292^{+0.018}_{-0.015}$, $f_3 = -0.257 \pm 0.034$, and $f_4 = 0.046^{+0.060}_{-0.075}$. Therefore, the apparent positive curvature by eye is confirmed quantitatively since f_2 is positive.[4]

[2] Similar analysis may be possible using data from the High-Z Supernova Search (Schmidt et al. 1998). As an example we use only the data from the Supernova Cosmology Project in this article. This does not mean that other data are not important.

[3] In the likelihood analysis, we used the inverse function $R = R(z)$ by solving the quartic equation.

[4] Note here that for the flat Λ model with $\Omega_M = 0.3$ and $z = 1$, which roughly corresponds to the data of the Supernova Cosmology Project (Perlmutter et al. 1999), the expansion of f with $N = 4$ has an accuracy of 3.6%. Here the accuracy means the relative error of f computed from the expansion (eq. [3]) to the given f in eq. (2).

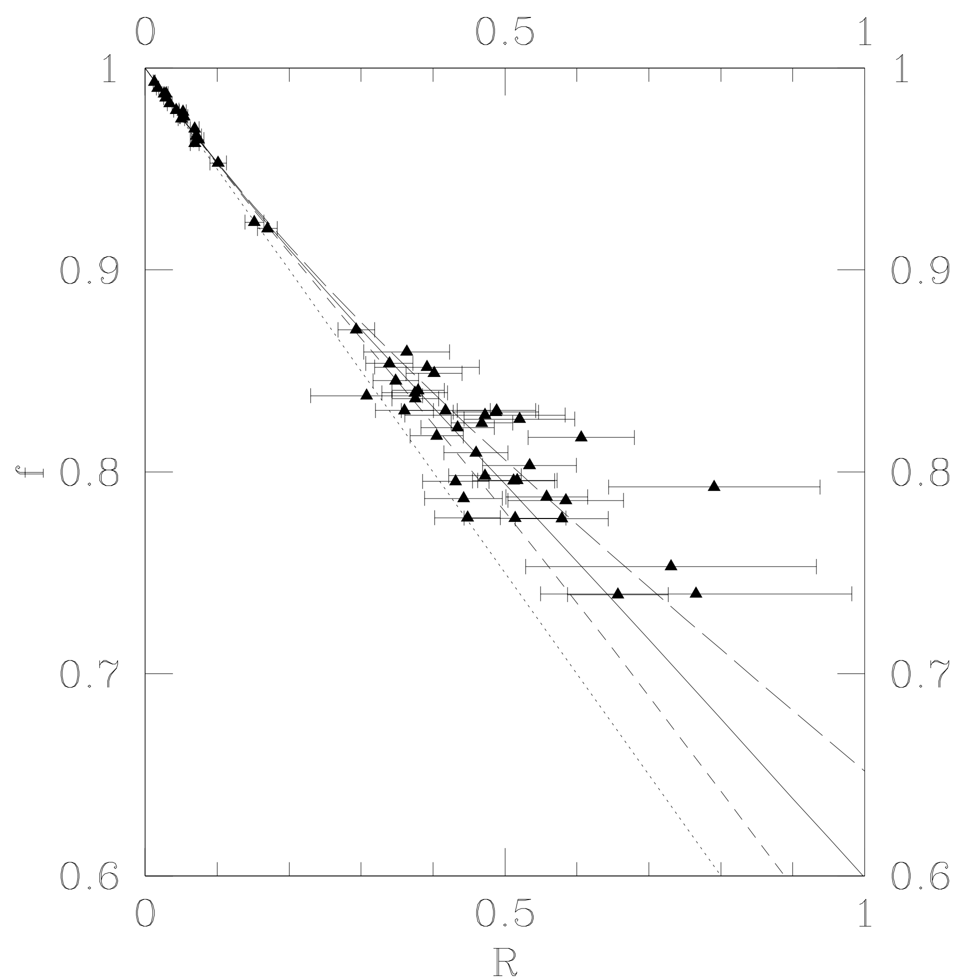

FIG. 1.—54 Type Ia supernovae data sets from the Supernova Cosmology Project (Perlmutter et al. 1999) in the (R,f) plane. The variable $f = 1/(1 + z)^{1/2}$, and R is a new independent variable defined by $R \equiv (H_0/c)[d_L(z)/(1 + z)]$, where H_0 and $d_L(z)$ are the present Hubble parameter and the luminosity distance, respectively. The dotted line shows the flat dust case $f = 1 - 0.5R$. The solid line is given by the likelihood analysis $f = 1/u^{1/2}$ and $u = 1 + R + 0.084R^2 + 0.343R^3 + 0.360R^4$ with $\chi^2 = 47.72$ for $54 - 4 = 50$ dof. The long- and short-dashed lines correspond to $\pm 1\ \sigma$ values of u_2, u_3, and u_4.

To know the convergence property of the expansion of f in a power series of R, let us consider the flat Λ model. In this model, R is expressed as

$$R = 2\int_f^1 df/\sqrt{\Omega_M + (1-\Omega_M)f^6}\ , \tag{2}$$

where Ω_M is the present density parameter of the dark matter. For $N = 7$, the expansion is expressed as

$$\begin{aligned} f = 1 &- \tfrac{1}{2}R + \tfrac{3}{8}(1-\Omega_M)R^2 - \tfrac{5}{16}(1-\Omega_M)R^3 + \tfrac{5}{128}(1-\Omega_M)(7-3\Omega_M)R^4 - \tfrac{3}{256}(1-\Omega_M)(17\Omega_M - 21)R^5 \\ &+ \tfrac{1}{1024}(1-\Omega_M)[8\Omega_M + 168(1-\Omega_M) + 12\Omega_M(1-\Omega_M) + 63(1-\Omega_M)^2]R^6 \\ &- \tfrac{1}{14336}(1-\Omega_M)[8\Omega_M + 1176(1-\Omega_M) + 132\Omega_M(1-\Omega_M) + 1827(1-\Omega_M)^2]R^7\ . \end{aligned} \tag{3}$$

However, in this expansion the sign of f_n changes alternately so that the convergence is extremely slow. The accuracy of the expansion is only 16% for $\Omega_M = 0.3$ and $f = 0.6$, corresponding to $z = 1.77$ and $R = 1.13$, while for low z, the accuracy is better. For example, for $\Omega_M = 0.3$ and $f = 0.7 (R = 0.79, z = 1.04)$, the accuracy is 0.95%.

The above expansion might not be accurate enough to analyze the data from such projects as SNAP, so we need a Pade-like approximation in a power-series expansion of f. We adopt the following Pade-like approximation:

$$f(R) = 1/\sqrt{u}\ , \qquad u = 1 + \sum_{n=1}^{N} u_n R^n\ . \tag{4}$$

For the flat Λ model, R is related to u as

$$R = \int_1^u du/\sqrt{\Omega_M u^3 + (1-\Omega_M)}\ . \tag{5}$$

Up to $N = 7$, u is expanded as

$$u = 1 + R + \tfrac{3}{4}\Omega_M R^2 + \tfrac{1}{2}\Omega_M R^3 + \tfrac{1}{16}\Omega_M(2+3\Omega_M)R^4 + \tfrac{3}{16}\Omega_M^2 R^5 + \tfrac{1}{64}\Omega_M^2(4+3\Omega_M)R^6 + \tfrac{1}{112}\Omega_M^2(1+6\Omega_M)R^7\ . \tag{6}$$

All u_n are positive so that the convergence is very rapid. In reality, for $\Omega_M = 0.3$ and $f = 0.6 (R = 1.13, u = 2.77)$, the relative error of f is 0.059%. In this expansion the error is the largest for $\Omega_M = 1$. However, even in this case, the error is 0.2% for $f = 0.6 (R = 0.8, u = 2.77)$. Therefore, if accurate values of R for various z are obtained observationally, we may determine seven parameters, $u_n (n = 1,2,\ldots,7)$, which may be enough to express $f(R)$ in less than 0.1% accuracy.

3. EQUATION OF STATE

Let us assume that accurate u_n for $n = 1,\ldots, N$ are obtained from the analysis of observational data. In this section we discuss how to determine and confirm the equation of state from u_n. First, $\rho(R)$ and $p(R)$ are expressed as $\rho(R)/\rho_0 = (du/dR)^2$ and $p(R)/\rho_0 = -(du/dR)^2 + 2u/3(d^2u/dR^2)$, respectively. Second, the square of the sound velocity $dp/d\rho$ is expressed as

$$\frac{dp}{d\rho} = -\frac{2}{3} + \frac{u(d^3u/dR^3)}{3[(du/dR)(d^2u/dR^2)]}\ . \tag{7}$$

3.1. *One-Parameter Equation of State*

The flat Λ model has only one parameter, Ω_M. This parameter is equal to either $4u_2/3$ or $2u_3$ so that the identity of $\Omega_M = 4u_2/3 = 2u_3$ will be the consistency check of the Λ model (see also Chiba & Nakamura 1998). From the data of the Supernova Cosmology Project (Perlmutter et al. 1999) we obtained $u_1 = 1$, $u_2 = 0.084^{+0.076}_{-0.063}$, $u_3 = 0.343^{+0.14}_{-0.13}$, and $u_4 = 0.360^{+0.32}_{-0.24}$. Therefore, $4u_2/3 = 0.112^{+0.101}_{-0.084}$, while $2u_3 = 0.686^{+0.28}_{-0.26}$. Note here that for the flat Λ model with $\Omega_M = 0.3$ and $z = 1$, which roughly corresponds to the data of the Supernova Cosmology Project (Perlmutter et al. 1999), the expansion of u with $N = 4$ has an accuracy of 0.19%. There is another null test of the Λ model for all R. From equation (5), we can derive

$$du/dR = \sqrt{\Omega_M u^3 + (1-\Omega_M)}\ , \qquad d^2u/dR^2 = \tfrac{3}{2}\Omega_M u^2\ , \qquad d^3u/dR^3 = 3\Omega_M u\sqrt{\Omega_M u^3 + (1-\Omega_M)}\ . \tag{8}$$

Using these expressions of derivatives, we can easily prove $dp/d\rho = 0$ for all u. Therefore, the null test of $dp/d\rho$ in equation (7) observationally can confirm the Λ model. At present from the above values of u_2 and u_3, we have $dp/d\rho$ at $u = 1$ as $dp/d\rho = 3.41^{+4.7}_{-5.2}$, which means the present data are not accurate enough to confirm the Λ model.

3.2. *Two-Parameter Equation of State*

The flat *w-cosmology* is an example of this class. In w-cosmology the universe contains x-matter with $p_x = w\rho_x$, where w is a constant. R is expressed as $R = \int_1^u du/[\Omega_M u^3 + (1-\Omega_M)u^{3(1+w)}]^{1/2}$. Then derivatives are given by

$$\begin{aligned} \frac{du}{dR} &= \sqrt{\Omega_M u^3 + (1-\Omega_M)u^{3(1+w)}}\ , \\ \frac{d^2u}{dR^2} &= \frac{3}{2}\Omega_M u^2 + \frac{3}{2}(1+w)(1-\Omega_M)u^{3w+2}\ , \\ \frac{d^3u}{dR^3} &= \left[3\Omega_M u + \frac{3}{2}(1+w)(3w+2)(1-\Omega_M)u^{3w+1}\right]\sqrt{\Omega_M u^3 + (1-\Omega_M)u^{3(1+w)}}\ . \end{aligned} \tag{9}$$

The constants u_2 and u_3 are given by

$$u_2 = \frac{3\Omega_M}{4} + \frac{3(1+w)(1-\Omega_M)}{4}\,, \qquad u_3 = \frac{\Omega_M}{2} + \frac{(1+w)(3w+2)(1-\Omega_M)}{4}\,. \tag{10}$$

The constants Ω_M and w are determined from u_2 and u_3 as

$$\Omega_M = 1 - \frac{(4u_2-3)^2}{12u_3 - 6 - 5(4u_2-3)}\,, \qquad w = \frac{4u_3-2}{4u_2-3} - \frac{5}{3}\,. \tag{11}$$

Then $dp/d\rho\ [= w(1+w)(1-\Omega_M)u^{2+3w}/(\Omega_M u^2 + (1+w)(1-\Omega_M)u^{2+3w})]$ should agree with $dp/d\rho$ in equation (7) so that we can confirm or refute the w-cosmology.

3.3. *General Equation of State*

Let us now consider the general case in which w is a function of u in w-cosmology. R is expressed as

$$R = \int_1^u \frac{du}{\sqrt{\Omega_M u^3 + (1-\Omega_M)\exp[q(u)]}}\,, \qquad q(u) = \int_1^u \frac{3[1+w(u)]}{u}\,du\,. \tag{12}$$

Then derivatives are given by

$$\frac{du}{dR} = \sqrt{\Omega_M u^3 + (1-\Omega_M)\exp[q(u)]}\,,$$

$$\frac{d^2u}{dR^2} = \frac{3}{2}\Omega_M u^2 + \frac{3}{2}(1-\Omega_M)\frac{[1+w(u)]}{u}\exp[q(u)]\,,$$

$$\frac{d^3u}{dR^3} = \left(3\Omega_M u + \frac{3}{2}\left\{\frac{[1+w(u)]}{u^2}[3w(u)+2] + \frac{dw}{udu}\right\}(1-\Omega_M)\exp[q(u)]\right)\sqrt{\Omega_M u^3 + (1-\Omega_M)\exp[q(u)]}\,.$$

The constants u_2 and u_3 are given by

$$u_2 = \frac{3\Omega_M}{4} + \frac{3(1+w_0)(1-\Omega_M)}{4}\,, \qquad u_3 = \frac{\Omega_M}{2} + \frac{(1-\Omega_M)[(1+w_0)(3w_0+2)+w_1]}{4}\,, \tag{13}$$

where $w_0 = w(1)$ and $w_1 = dw/du(u=1)$. Now we have only two equations for three unknown constants Ω_M, w_0, and w_1. To resolve this one may use the expression for u_4. However, in the expression of u_4, a new unknown constant $w_2 = d^2w/du^2$ $(u=1)$ appears so that we have to make the closure. One way is to determine Ω_M from other data such as $d_L(z)$ for $z > 3$ from the *Next Generation Space Telescope* (Efstathiou 1999). In this case w_0 and $w_1 \cdots$ are determined as

$$w_0 = \frac{4u_2 - 3\Omega_M}{3(1-\Omega_M)} - 1\,, \qquad w_1 = 4u_3 - 2\Omega_M - (1+w_0)(3w_0+2)\,, \qquad w_2 = \cdots \tag{14}$$

The other case is to assume $w_2 = 0$. Then Ω_M, w_0, and w_1 are determined as a solution to simultaneous nonlinear equations as

$$u_2 = u_2(\Omega_M, w_0, w_1)\,, \qquad u_3 = u_3(\Omega_M, w_0, w_1)\,, \qquad u_4 = u_4(\Omega_M, w_0, w_1)\,. \tag{15}$$

In both cases $dp/d\rho$ is given by

$$\frac{dp}{d\rho} = \frac{w(u)[1+w(u)] + (u/3)w_1(1-\Omega_M)\exp[q(u)]}{\Omega_M u^2 + (1-\Omega_M)[1+w(u)/u]\exp[q(u)]}\,, \tag{16}$$

where $w(u) = w_0 + w_1(u-1)$. As before this $dp/d\rho$ should agree with $dp/d\rho$ in equation (7), which is used to confirm or refute each model. In general, if u_n are determined up to N, the equation of state with $w = w_0 + \sum_{n=1}^{N-1} w_n(u-1)^n$ can be determined in principle.

4. DISCUSSION

Now if x-matter consists of the scalar field ϕ with the potential $V(\phi)$, they are related to ρ_x and p_x as

$$\left(\frac{d\phi}{dt}\right)^2 = \rho_x + p_x\,, \qquad V(\phi) = \tfrac{1}{2}(\rho_x - p_x)\,.$$

Using $\rho(R)$ and $p(R)$, we have

$$\phi - \phi_0 = \frac{1}{\sqrt{8\pi G}}\int_0^R \sqrt{-3\Omega_M u + \frac{2}{u}\frac{d^2u}{dR^2}}\,dR \equiv g(R)\,, \tag{17}$$

$$V(\phi) = \frac{3H_0^2}{\sqrt{16\pi G}}\left[2\left(\frac{du}{dR}\right)^2 - \Omega_M u^3 - \frac{2u}{3}\frac{d^2u}{dR^2}\right] \equiv h(R)\,, \tag{18}$$

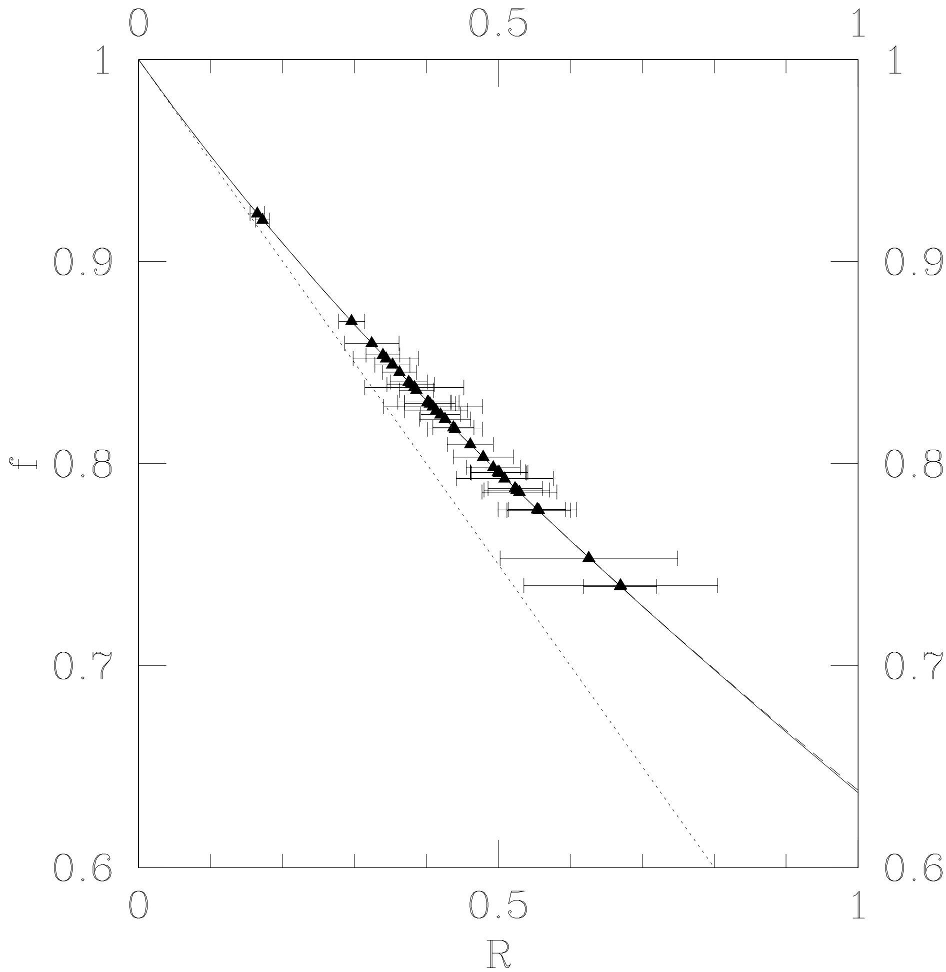

FIG. 2.—38 Simulated Type Ia supernovae data sets in the (R, f) plane. See text for how this data is made. The dotted line shows the flat dust case $f = 1 - 0.5R$. The solid line is the result of likelihood analysis of $f = 1/(1 + R + 0.2240R^2 + 0.1505R^3 + 0.0585R^4 + 0.0187R^5 + 0.0095R^6 + 0.0053R^7)^{1/2}$ with $\chi^2 = 36.16$ for $38 - 7 = 31$ dof. The dashed line is the theoretical curve of $f = 1/(1 + R + 0.225R^2 + 0.15R^3 + 0.054375R^4 + 0.016875R^5 + 0.0068906R^6 + 0.00225R^7)^{1/2}$. Note that error bars are extended by a factor 100.

where ϕ_0 is the present value of the scalar field. From equation (17) we have $R = g^{-1}(\phi - \phi_0)$. Then the potential is expressed as $V(\phi) = h[g^{-1}(\phi - \phi_0)]$.

In § 3 we assumed that accurate u_n for $n = 1, \ldots, N$ are obtained. We here show an example of the determination of u_n for $n = 1, \ldots, 7$. We adopt the redshifts of 38 data sets with $z > 0.17$ from Perlmutter et al. (1999). We also adopt the relative error of R for each data set as X_i. Now let us assume that our universe obeys the Λ model with $\Omega_M = 0.3$. Then we know the theoretical value of R_i^t for each z_i. To simulate the real observation, we set $R_i = R_i^t(1 \pm SX_i)$, where S is a scale factor. Let us assume that an accurate observation gives us 38 luminosity distances with $\sim 0.1\%$ accuracy so that the scale factor S is chosen to make the relative error of R for 38 data sets be $\sim 0.1\%$.[5] We performed the likelihood analysis for this simulated data and obtained $u_2 = 0.2240$, $u_3 = 0.1505$, $u_4 = 0.0585$, $u_5 = 0.0187$, $u_6 = 0.0095$, and $u_7 = 0.0053$ with $\chi^2 = 36.16$ for $38 - 7 = 31$ degrees of freedom (dof), while theoretical values are $u_2 = 0.225$, $u_3 = 0.15$, $u_4 = 0.054375$, $u_5 = 0.016875$, $u_6 = 0.0068906$, and $u_7 = 0.00225$. From this, $\Omega_M = 4/3u_2 = 0.29866$ or $\Omega_M = 2u_3 = 0.3010$ is obtained, while $dp/d\rho = 0.0052$ at $u = 1$. Assuming the more general equation of state with $w_2 = w_3 \ldots = w_7 = 0$, we have $\Omega_M = 0.31095$, $w_0 = -1.01783$, and $w_1 = -0.04768$. This suggests that we may confirm the Λ model if such accurate luminosity distances are available. We show in Figure 2 the simulated data and the results of the likelihood analysis. Note that error bars are extended by a factor of 100. The theoretical curve (*dashed line*) and the observational curve (*solid line*) are almost indistinguishable.

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[5] This might be possible if the statistical error approaches the systematic error in, for example, the SNAP project.

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