

EFFICIENCY OF MAGNETIZED THIN ACCRETION DISKS IN THE KERR METRIC

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ABSTRACT

The efficiency of thin disk accretion onto black holes depends on the inner boundary condition, specifically the torque applied to the disk at the last stable orbit. This is usually assumed to vanish. I estimate the torque on a magnetized disk using a steady magnetohydrodynamic inflow model originally developed by Takahashi et al. I find that the efficiency ϵ can depart significantly from the classical thin disk value. In some cases $\epsilon > 1$, i.e., energy is extracted from the black hole.

Subject headings: accretion, accretion disks — black hole physics

1. INTRODUCTION

The dynamics of accretion close the event horizon of black holes is of considerable interest to astronomers because that is where most of the accretion energy is released and because strong-field gravitational effects may be evident there.

Much of the work on thin disks around black holes assumes that the disk ends at or near $r = r_{\text{mso}}$, the radius of the marginally stable circular orbit. Inside r_{mso} , material is assumed to nearly free-fall onto the black hole (e.g., Bardeen 1970; Thorne 1974; Abramowicz, Jaroszynski, & Sikora 1978). This implies the so-called “no-torque” boundary condition on the disk and leads to a disk whose surface brightness vanishes at the inner edge (Shakura & Sunyaev 1973). The no-torque boundary condition leads to the classical estimate for the accretion efficiency $\epsilon_0 = 1 - \mathcal{E}(r_{\text{mso}})/c^2$, where $\mathcal{E}(r_{\text{mso}}) = -u_t(r_{\text{mso}})$ is the “energy at infinity” per unit mass of a particle at the marginally stable orbit; ϵ_0 ranges between $1 - 2\sqrt{2}/3 \approx 0.057$ for a nonrotating ($a = 0$) black hole to $1 - 1/\sqrt{3} \approx 0.42$ for a prograde disk around a maximally rotating ($a = 1$) black hole.

It was realized early on, however, that magnetic fields might alter the dynamics of the accreting material and hence the accretion efficiency (see, e.g., J. M. Bardeen quoted in Thorne 1974). More recently, it has been argued that the magnetic fields threading the accreting material should be strong enough to be dynamically important, but not so strong that the fields can be regarded as force-free (Krolik 1999).

In this Letter I estimate the magnetic torque on the inner edge of a thin accretion disk in the Kerr metric. I use a model for the inflow that is based on earlier work by Takahashi et al. (1990), following Phinney (1983) and Camenzind (1986a, 1986b, 1989).

2. MODEL

Consider a thin disk in the equatorial plane of the Kerr metric. Because the disk is thin, $c_s^2/c^2 \ll 1$, so the relativistic enthalpy $\eta \approx 1$. The disk is magnetically turbulent (Balbus & Hawley 1991) with magnetic energy density $B^2/(8\pi) \approx 2\alpha\rho c_s^2$ (Hawley, Gammie, & Balbus 1995), where α is the usual dimensionless viscosity of accretion disk theory. I will assume that $\alpha \ll 1$ so that magnetic fields make a negligible contribution to the hydrostatic equilibrium of the disk. Then the disk has an inner surface at $r = r_{\text{in}}$, with $r_{\text{mso}} - r_{\text{in}} \sim r_{\text{mso}}(c_s/c)^2$, so

$r_{\text{in}} \approx r_{\text{mso}}$. At the inner surface of the disk, I imagine that magnetic field lines rise approximately radially through the disk atmosphere and force the atmosphere to corotate with the disk surface. High enough in this corotating atmosphere, the gradient of the effective potential changes sign and the gas begins to stream inward along field lines toward the black hole. As gas leaves the disk, the character of the flow changes, becoming less turbulent and more nearly laminar.

This picture leads one to consider a steady, axisymmetric inflow close to the equatorial plane of the Kerr metric. In what follows, I will work exclusively in Boyer-Lindquist coordinate t, r, θ, ϕ and follow the notational conventions of Misner, Thorne, & Wheeler (1973). I assume that, in the poloidal plane, the fluid velocity and magnetic field are purely radial, i.e., that $u^\theta \equiv 0$ and $F_{r\phi} \equiv 0$, where u^μ is the 4-velocity and $F_{\mu\nu}$ is the electromagnetic field (“Maxwell”) tensor; recall that in the non-relativistic limit (denoted with an arrow), $F_{r\phi}(\theta = \pi/2) \rightarrow rB_\theta$, where \mathbf{B} is the magnetic field 3-vector. I also assume that all flow variables are functions only of r , i.e., that $\partial_\theta = 0$. This one-dimensional flow is similar in spirit to the old Weber & Davis (1967) model for the solar wind, turned inside out. As in the Weber-Davis model, the magnetic field has a monopolar geometry.

I will also assume perfect conductivity, so that the electric field vanishes in a frame comoving with the fluid:

$$u^\mu F_{\mu\nu} = 0. \quad (1)$$

Together with the symmetry conditions, this leaves $F_{\mu\nu}$ with six nonzero components, $F_{t\theta} (\rightarrow -rE_\theta)$, $F_{r\theta} (\rightarrow -rB_\phi)$, and $F_{\theta\phi} (\rightarrow -r^2B_r)$, and gives the relation $F_{r\theta} = -(F_{t\theta}u^t - F_{\theta\phi}u^\phi)/u^r$. Thus the electromagnetic field has only 2 degrees of freedom.

The flow symmetries reduce Maxwell’s equations to

$$\partial_r F_{t\theta} = 0 \quad (2)$$

and

$$\partial_r F_{\theta\phi} = 0. \quad (3)$$

The first equation is the relativistic “isorotation” law, which can be rewritten $F_{t\theta} = \Omega_r F_{\theta\phi}$, where Ω_r is the rotation frequency u^ϕ/u^t at the radius where $u^r = 0$. The second equation is the relativistic equivalent of $\nabla \cdot \mathbf{B} = 0$.

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Conservation of particle number leads to a conserved “rest mass flux,” per unit θ ,

$$F_M = 2\pi r^2 \rho u^r \rightarrow 2\pi r^2 \rho v_r, \quad (4)$$

where $\rho \equiv$ particle number density multiplied by the rest mass per particle and \mathbf{v} is the 3-velocity. The conserved angular momentum flux is

$$\begin{aligned} F_L &= 2\pi r^2 T_\phi^r = 2\pi r^2 \left(\rho u_\phi u^r - \frac{\mathcal{D} F_{\theta\phi} F_{r\theta}}{4\pi r^2} \right) \\ &\rightarrow 2\pi r^3 \left[\rho v_r v_\phi - \frac{B_\phi B_r}{4\pi} \right], \end{aligned} \quad (5)$$

where $T_{\mu\nu}$ is the stress-energy tensor, $\mathcal{D} \equiv 1 - 2r_g/r + a^2 r_g^2/r^2$, $r_g \equiv GM/c^2$. The conserved mass-energy flux is

$$\begin{aligned} F_E &= -2\pi r^2 T_t^r = 2\pi r^2 \left(-\rho u_t u^r - \frac{\mathcal{D} F_{t\theta} F_{r\theta}}{4\pi r^2} \right) \\ &\rightarrow 2\pi r^2 \left\{ \rho v_r \left[c^2 + \frac{1}{2} (v_r^2 + v_\phi^2) - \frac{GM}{r} \right] \right. \\ &\quad \left. + v_r \frac{B_\phi^2}{4\pi} - v_\phi \frac{B_r B_\phi}{4\pi} \right\}, \end{aligned} \quad (6)$$

and, consistent with the thin disk approximation, I have neglected the thermal energy's contribution to F_E . Notice that the accretion efficiency is given by $\epsilon = 1 - F_E/F_M$.

The final equation needed to close the system is the normalization of 4-velocity,

$$u^\mu u_\mu = -c^2. \quad (7)$$

Henceforth I will set $r_g = c = -F_M = 1$.²

Equations (1)–(7) describe a one-dimensional inflow model with a series of algebraic relations. This model is physically identical to that developed by Takahashi et al. (1990), although cast in somewhat different notation. Also, unlike Takahashi et al. (1990), I specialize to the case in which the inflow is anchored in a thin disk at the marginally stable orbit. This sets the inflow in a specific astrophysical context and allows one to relate it to an accretion efficiency. These inflow solutions have also been explored by Camenzind (1994, 1996) and, in unpublished work, by M. Camenzind & P. Englmaier.

3. BOUNDARY CONDITIONS

The inflow model has six dynamical variables (ρ , u^t , u^r , u^ϕ , $F_{t\theta}$, and $F_{\theta\phi}$) and six conserved quantities (F_M , F_L , F_E , $F_{\theta\phi}$, Ω_F , and $u^\mu u_\mu$), so the equations are fully integrated. What sets the conserved quantities?

The mass flux F_M has been normalized to -1 , but more physically it is set by conditions in the disk at large radius. The normalization of 4-velocity gives $u^\mu u_\mu = -1$. $F_{\theta\phi}$ is related to the magnetic flux emerging from the inner edge of the disk. This is presumably determined by the action of magnetohydrodynamic (MHD) turbulence in the disk and the interaction of the disk with the inflow itself. I will treat it as a parameter

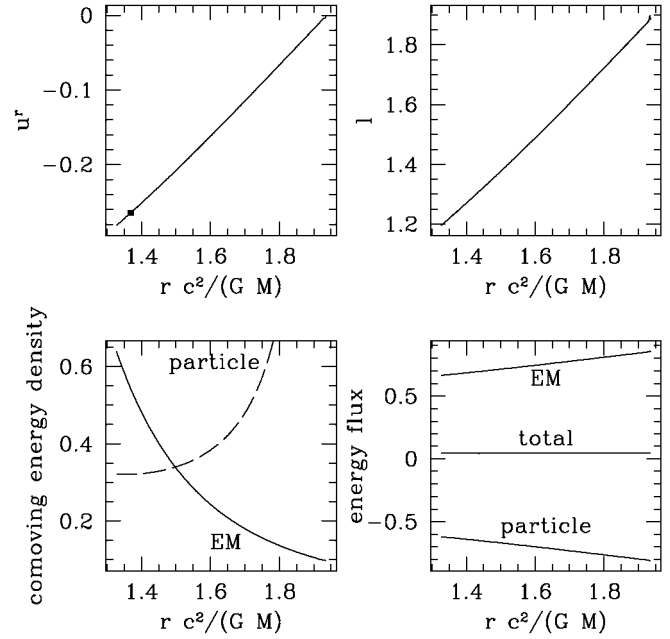


FIG. 1.—Example solution with $a = 0.95$ and $F_{\theta\phi} = 6$. The upper left panel shows $u^r(r)$, the upper right $l = u_\phi(r)$. The dot shows the location of the fast critical point. The lower left panel shows the electromagnetic energy density [$T_{\mu\nu}(\text{EM})u^\mu u^\nu$; solid line] and the rest-mass energy density of the accreting particles (ρ ; dashed line) measured in a frame comoving with the fluid. The lower right panel shows the electromagnetic, particle, and total energy fluxes as a function of radius.

of the problem and estimate reasonable values for it later. Three quantities remain to be specified.

It seems reasonable to require that the energy and angular momentum fluxes be continuous across the boundary between the inflow and the disk. It also seems reasonable to require that the inflow 4-velocity match continuously onto the disk. Happily, these requirements do not overconstrain the problem. If $\Omega_F = \Omega(r_{\text{in}})$ and $u^r(r_{\text{in}}) = 0$, then one can show that $F_E = \Omega F_L + (\mathcal{E} - l\Omega)F_M$ at r_{in} ; here $l \equiv u_\phi$. This eliminates 2 degrees of freedom by expressing F_E in terms of F_L and by fixing Ω_F . The energy and angular momentum flux are then continuous if the viscous (more properly, turbulent) angular momentum flux of the disk just outside r_{in} matches the electromagnetic angular momentum flux of the inflow just inside r_{in} :

$$T_\phi^r(\text{visc}) = T_\phi^r(\text{EM}). \quad (8)$$

This condition will be automatically satisfied once the disk relaxes to a steady state, since the disk will increase its surface density until it can carry off the entire outward electromagnetic angular momentum (and energy) flux from the inflow.

The final degree of freedom (F_L , or equivalently, F_E) is fixed by the condition that the flow pass smoothly through the fast critical point.³ Thus, F_L and F_E emerge as “eigenvalues” of the solution.

4. SOLUTIONS

Once the model parameters are fixed, the resulting set of nonlinear algebraic equations must be solved numerically. I

² Physical units may be recovered as follows: length, r_g ; time, r_g/c ; mass, $-F_M c/r_g$; $F_{\theta\phi}$, $GM(-F_M)^{1/2} c^{-3/2}$, while $F_M \approx -Mr/(2H)$.

³ The Alfvén point does not impose any new condition on the flow, since all trans-Alfvénic solutions pass smoothly through the Alfvén point; see, e.g., Phinney (1983). The slow point is absent because the flow is cold.

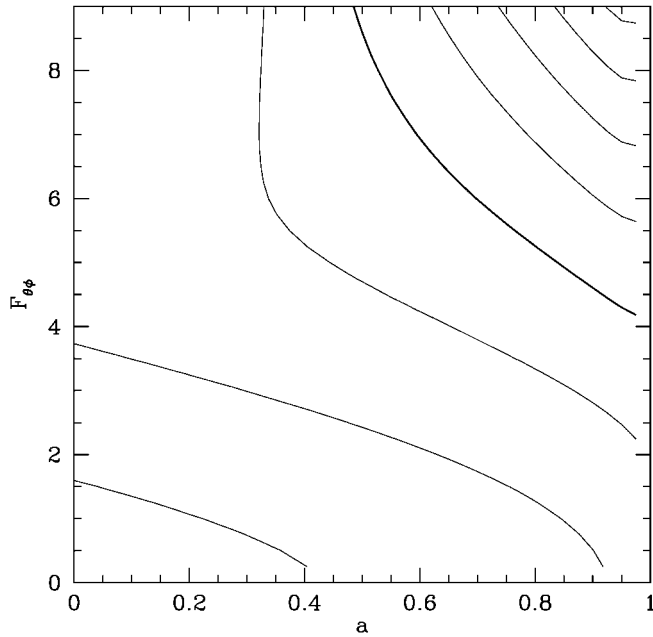


FIG. 2.—Contours of constant angular momentum flux F_L in the a - $F_{\theta\phi}$ plane. The zero angular momentum flux contour is marked as a heavy solid line; above and to the right of this line $F_L > 0$. The light contours are linearly spaced at $\Delta F_L = 1$.

obtain F_L and the location of the fast critical point (r_f, u_f) via simultaneous solution of

$$\partial_{u'} F_E(r_f, u_f, F_L) = 0, \quad (9)$$

$$\partial_r F_E(r_f, u_f, F_L) = 0, \quad (10)$$

$$F_E(r_f, u_f, F_L) - F_E(r_{\text{mso}}, 0, F_L) = 0. \quad (11)$$

Here I have used the fact that the critical point is a saddle point of $F_E(r, u; F_L)$. I use the multidimensional Newton-Raphson method of Press et al. (1992), and for simplicity I evaluate the derivatives numerically.

There is one subtlety involved in the solution. In calculating $F_E(r, u; F_L)$, one must solve a quadratic equation for u' (or equivalently u^ϕ), so one must decide which root is physical. In general this is a nontrivial matter, since the physical solution skirts a region in the r - u' plane where the discriminant of the quadratic vanishes (the solution makes a smooth transition from one branch of the solution to another). Fortunately, it turns out that one branch is appropriate in the neighborhood of r_{mso} and the other in the neighborhood of the fast point, so in practice this subtlety is easily dealt with.

Figure 1 shows $u'(r)$ for a solution with $a = 0.95$ and $F_{\theta\phi} = 6$. The dot marks the fast point at $(r, u') = (1.37, -0.26)$. This solution has $F_E \approx 0.04$ and $F_L \approx 1.23$, so $\epsilon \equiv 1 + F_E \approx 1.04$, i.e., energy is extracted from the black hole. Evidently magnetic fields can significantly alter the efficiency from the classical value.

Figure 2 shows contours of the eigenvalue F_L in a survey over the a - $F_{\theta\phi}$ plane (see also Fig. 3 of Camenzind 1994). For strong field and large black hole spin (up and to the right of the heavy solid line in the figure), $F_L > 0$. Takahashi et al. (1990) have shown that $F_L > 0$ if the field rotation frequency Ω_F is exceeded by the characteristic rotation frequency of the spacetime, $2a/[r^3(1 + a^2/r^2 + 2a^2/r^3)]$, at the Alfvén point.

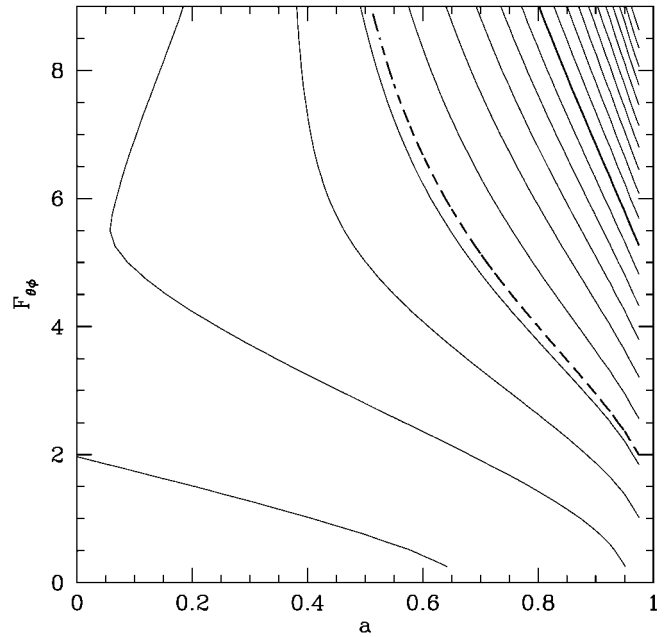


FIG. 3.—Contours of constant efficiency $\epsilon \equiv 1 + F_E$ in the a - $F_{\theta\phi}$ plane. The unit efficiency contour is marked as a heavy solid line; above and to the right of this line $F_E > 0$, i.e., energy is being extracted from the black hole. The light contours are linearly spaced at $\Delta \epsilon = 0.1$. The heavy dashed contours lies at $\epsilon \approx 0.42$, the classical efficiency of a prograde disk around a maximally rotating black hole.

Figure 3 shows contours of the accretion efficiency ϵ evaluated over a portion of the a - $F_{\theta\phi}$ plane (see also Fig. 4 of Camenzind 1994). The contours in Figure 3 are located at intervals of $\Delta \epsilon = 0.05$. For strong field and large black hole spin (up and to the right of the heavy solid line in the figure), $F_E > 0$, that is, energy is extracted from the black hole. As Takahashi et al. (1990) have shown, this can only happen if the Alfvén point lies within the ergosphere ($r < 2$). The heavy dashed line corresponds to the classical efficiency of a prograde disk around a maximally rotating black hole, $\epsilon \approx 0.42$.

5. ASTROPHYSICAL DISCUSSION

What is an astrophysically sensible value for the crucial magnetic field strength parameter $F_{\theta\phi}$? I will make a purely Newtonian estimate for clarity and because the relativistic corrections are likely to be smaller than the other sources of uncertainty. Suppose that the radial field leaving the disk is equal to fB_d , where the subscript d denotes a quantity evaluated in the disk and $f \lesssim 1$. It is a result of numerical models of disk turbulence that $B_d^2/(8\pi) \approx 2\alpha\rho_d c_{s,d}^2$ (e.g., Hawley, Gammie, & Balbus 1995). Using the usual steady state disk equation $3\pi\Sigma\nu \approx \dot{M} \approx -2F_M(H/r)$, I find

$$F_{\theta\phi} \approx r^2 B_r \approx 2.3 f r_d^{3/4}. \quad (12)$$

If the inner edge of the disk is at $r_d = 6$, $F_{\theta\phi} \approx 8.8f$, so the region of parameter space shown in Figures 2 and 3 is likely relevant to disks.

The inflow model thus suggests that the presence of a modest magnetic field and the accompanying torque on the inner edge of the disk can significantly increase the efficiency of thin disk accretion onto black holes. For $a = 0$ and $F_{\theta\phi} \ll 1$, $\epsilon \approx \epsilon_0 + 0.01|F_{\theta\phi}|$, while for $F_{\theta\phi} = 4$, $\epsilon = 0.165$. The added luminosity manifests itself as an increase in the surface brightness of the

disk due to the torque applied at its inner boundary. In the same quasi-Newtonian spirit as the estimate of $F_{\theta\phi}$, the surface brightness profile becomes $\propto 1 - \beta(r_{\text{mso}}/r)^{1/2}$ (see Shapiro & Teukolsky 1983) and $\beta \approx 1$ in the Shakura-Sunyaev model. Here $\beta \equiv 1 - F_L(\text{EM})/l_{\text{mso}} \lesssim 1$.

The inflow model also suggests that because material accretes with a smaller specific angular momentum than it would in the absence of magnetic fields, it is more difficult to manufacture a rapidly spinning hole by disk accretion. An equilibrium spin is reached when $F_L = 2aF_E$ (e.g., eq. [10] of Popham & Gammie 1998). For $F_{\theta\phi} = 4$, this equilibrium value is reached at $a \approx 0.7$.

A major limitation of this study (which should be regarded as an instructive example rather than a source for estimating efficiencies) is that I have ignored the vertical structure of the inflow. Crudely speaking, one might expect that field lines that do not lie in the midplane are more lightly loaded ($|F_{\theta\phi}|$ is larger) so that specific energy and angular momentum fluxes

might increase away from the midplane, until at sufficient latitude one reaches a field line that would rather inflate away than remain tied between the inflow and the disk. At high latitude, then, the outward electromagnetic energy flux might emerge in the form of a wind and be better described by the force-free magnetosphere model of Blandford & Znajek (1977).

Another major limitation is the simplified, monopolar field geometry. This limitation can be overcome by direct numerical integration of the basic equations. Evidently numerical models of inflow inside the marginally stable orbit would be enormously interesting. Fortunately they are now practical, at least within the MHD approximation.

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