

# DISENTANGLING THE COSMIC WEB. I. MORPHOLOGY OF ISODENSITY CONTOURS

JENS SCHMALZING,<sup>1,2</sup> THOMAS BUCHERT,<sup>1,3</sup> ADRIAN L. MELOTT,<sup>4</sup> VARUN SAHNI,<sup>5</sup>

B. S. SATHYAPRAKASH,<sup>6</sup> AND SERGEI F. SHANDARIN<sup>4,7</sup>

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## ABSTRACT

We apply Minkowski functionals and various derived measures to decipher the morphological properties of large-scale structure seen in simulations of gravitational evolution. Minkowski functionals of isodensity contours serve as tools to test global properties of the density field. Furthermore, we identify coherent objects at various threshold levels and calculate their partial Minkowski functionals. We propose a set of two derived dimensionless quantities, planarity and filamentarity, which reduce the morphological information in a simple and intuitive way. Several simulations of the gravitational evolution of initial power-law spectra provide a framework for systematic tests of our method.

*Subject headings:* large-scale structure of universe — methods: numerical — methods: statistical

## 1. INTRODUCTION

The existence of the large-scale clustering of galaxies had already been well established by the early 1970s, mainly due to the pioneering work of Totsuji & Kihara (1969) and Peebles (1974), who showed that the two-point correlation function for galaxies in the Lick and Zwicky catalogs was positive and had the power-law form  $\xi(r) \propto r^{-1.8}$  on scales  $\lesssim 10 h^{-1}$  Mpc. Their result was later extended to three-dimensional galaxy catalogs as well. Although the clustering of galaxies is now a well-known fact, a complete description of clustering that includes its geometrical features has so far eluded researchers. This is perhaps due to the fact that the galaxy density field that we observe appears to be strongly non-Gaussian. A Gaussian random field is uniquely described by its power spectrum,  $P(k)$ , or its two-point correlation function,  $\xi(r)$ , since  $\xi(r)$  and  $P(k)$  form a Fourier transform pair. This is no longer true for a non-Gaussian field, for which  $\xi$  must be complemented by other statistical descriptors that are sensitive to the structure of matter on large scales. In the so-called “standard model” of structure formation, an initially Gaussian density distribution becomes non-Gaussian due to mode coupling and the resulting build-up of phase correlations during the nonlinear regime. These phase correlations give rise to the amazing diversity of form that is characteristic of a highly evolved distribution of matter, and is often referred to as being cellular, filamentary, sheetlike, network-like, sponge-like, a cosmic web, etc. Most of these descriptions are based either on the visual appearance of large-scale structure or on the presence of features that are absent in the reference

Gaussian distribution, which, by definition, is assumed to be featureless. Gravitational instability, for example, may cause CDM-like Gaussian initial perturbations to evolve toward a density field that percolates at a higher density threshold, i.e., at a lower filling factor, than a Gaussian field (Melott et al. 1983). Such distributions display greater connectivity and are sometimes referred to as being “network-like” (Yess & Shandarin 1996).

In order to come to grips with the rich textural possibilities inherent in large-scale structure, a number of geometrical indicators of clustering have been proposed in the past, including minimal spanning trees (Barrow, Sonoda, & Bhavsar 1985), the genus curve (Gott, Melott, & Dickinson 1986), percolation theory (Zeldovich & Shandarin 1982; Shandarin 1983) and shape analysis (Sathyaprakash, Sahni, & Shandarin 1998; Sahni & Coles 1995, and references therein). A major recent advance in our understanding of gravitational clustering has been associated with the application of Minkowski functionals (MFs) to cosmology (Mecke, Buchert, & Wagner 1994). The four MFs  $V_0, \dots, V_3$  provide an excellent description of the geometrical properties of a collection of point objects (galaxies) or, alternatively, of continuous distributions such as density fields in large-scale structure or brightness contours in the cosmic microwave background. The scope and descriptive power of the MFs is enhanced by the fact that both percolation analysis and the genus curve are members of the family. In addition, as demonstrated by Sahni, Sathyaprakash, & Shandarin (1998), ratios of MFs provide us with an excellent “shape statistic” with which one can attempt to quantify the morphology of large-scale structure, including the shapes of individual superclusters and voids. Spurred by the success of MFs in quantifying the geometrical properties of large-scale structure, we apply the MFs to scale-invariant  $N$ -body simulations of gravitational clustering, in an attempt to probe both the global properties and the individual “bits and pieces” that might make up the “cosmic web” (Bond, Kofman, & Pogosyan 1996).

## 2. METHOD

### 2.1. Minkowski Functionals

Minkowski functionals (named after Minkowski 1903) were introduced into cosmology by Mecke, Buchert, & Wagner (1994), employing the generalized Boolean grain

<sup>1</sup> Ludwig-Maximilians-Universität, Theresienstraße 37, 80333 München, Germany.

<sup>2</sup> Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Straße 1, 85740 Garching, Germany; jensen@mpa-garching.mpg.de.

<sup>3</sup> Theory Division, CERN, 1211 Genève 23, Switzerland; buchert@theorie.physik.uni-muenchen.de.

<sup>4</sup> Department of Physics and Astronomy, University of Kansas, Lawrence, Kansas 66045; melott@kusmos.phsx.ukans.edu, sergei@kusmos.phsx.ukans.edu.

<sup>5</sup> Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Pune 411007, India; varun@iucaa.ernet.in.

<sup>6</sup> Department of Physics and Astronomy, University of Wales, College of Cardiff, Cardiff CF2 3YB, UK; spxbss@astro.cf.ac.uk.

<sup>7</sup> Theoretical Astrophysics Center, Juliane Maries Vej 30, 2100 København Ø, Denmark.

model. This model associates a body with a point process (in our case given by the location of galaxies or clusters) by decorating each point with a ball of radius  $r$ . The union set of the covering balls is then studied morphologically, whereby the radius of the ball serves as a diagnostic parameter probing the spatial scale of the body.

In this paper we use the excursion set approach, which is applicable to continuous fields (which can be constructed from point processes). Schmalzing & Buchert (1997) pointed out how to apply Minkowski functionals to isocontours of continuous fields, where the contour level (the threshold) is employed as a diagnostic parameter. The excursion set approach inherits two diagnostic parameters, because we can also vary the smoothing scale used in constructing the continuous field.

In three dimensions there exist four Minkowski functionals,  $V_\mu$ ,  $\mu = 0, 1, 2$ , and  $3$ . They provide a complete and unique description of a pattern's global morphology in the sense of Hadwiger's theorem (Hadwiger 1957). While reducing the information contained in the full hierarchy of correlation functions, this small set of numbers incorporates correlations of arbitrary order, and therefore provides a complementary look at large-scale structure. The geometric interpretations of all Minkowski functionals in three dimensions are summarized in Table 1.

We calculate global Minkowski functionals of the isodensity contours of the density field, as described in Appendix A. Furthermore, we separately calculate the partial Minkowski functionals of each isolated part of the isodensity contour. Since the total isodensity contour is the union of all its parts, the global functionals are given as sums of the partial functionals at the same threshold. This follows the spirit of Mecke et al. (1994), who introduced partial Minkowski functionals to measure local morphology for the generalized Boolean grain model.

Partial Minkowski functionals (PMFs) offer the possibility of probing the morphology of individual objects, or the object's environmental morphological properties. We expect that this concept will be more powerful when applied to continuous fields at high spatial resolution, so that the details of structures are not smoothed out. Their application to point processes also delivers more direct information. PMFs provide a wide range of possibilities for morphological studies, which we will explore in a forthcoming paper.

## 2.2. Shapefinders

One important task of morphological statistics is quantifying strongly non-Gaussian features such as filaments and pancakes. Given the four Minkowski functionals, we aim to reduce their morphological information content to

two measures of planarity and filamentarity, as has been done, for example, with various geometrical quantities (Mo & Buchert 1990), moments of inertia (Babul & Starkman 1992), and cumulants of counts-in-cells (Luo & Vishniac 1995).

Recently, Sahni et al. (1998) proposed a set of shapefinders derived from Minkowski functionals. One starts from the three independent ratios of Minkowski functionals that have a dimension of length. Requiring that they yield the radius  $R$  if applied to a ball, we define

$$\begin{aligned} \text{Thickness, } T &\equiv \frac{V_0}{2V_1} ; & \text{Width, } W &\equiv \frac{2V_1}{\pi V_2} ; \\ \text{Length, } L &\equiv \frac{3V_2}{4V_3} . \end{aligned} \quad (1)$$

By the isoperimetric inequalities (eq. [B6]), we have  $L \geq W \geq T$  for any convex body.

Going one step further, Sahni et al. (1998) also define dimensionless shapefinders by

$$\text{Planarity, } \mathcal{P} \equiv \frac{W - T}{W + T} ; \quad \text{Filamentarity, } \mathcal{F} \equiv \frac{L - W}{L + W} . \quad (2)$$

Some examples are in order.

## 2.3. Simple Examples

Let us consider some simple families of convex bodies in three-dimensional space that can take both filamentary and planar shape.

A spheroid with two axes of length  $r$  and one axis of length  $\lambda r$  has Minkowski functionals

$$\begin{aligned} V_0 &= \frac{4\pi}{3} r^3 \lambda , & V_1 &= \frac{\pi}{3} r^2 \left[ 1 + f\left(\frac{1}{\lambda}\right) \right] , \\ V_2 &= \frac{2}{3} r [\lambda + f(\lambda)] , & V_3 &= 1 , \end{aligned} \quad (3)$$

where<sup>8</sup>

$$f(x) = \frac{\arccos x}{\sqrt{1 - x^2}} . \quad (4)$$

By varying the parameter  $\lambda$  from zero to infinity, we can change the morphology of the spheroid from a filament to a pancake via a spherical cluster.

A different way of deforming a filament into a pancake is via generic triaxial ellipsoids, whereby one of the smaller axes of a strongly prolate spheroid is increased in size until it matches the larger axis and an oblate spheroid has been reached. An integral expression can be found in Sahni et al. (1998).

Yet another transition from prolate to oblate shape is provided by cylinders of radius  $r$  and height  $\lambda r$ . Here the

TABLE 1

MINKOWSKI FUNCTIONALS EXPRESSED IN TERMS OF THE CORRESPONDING GEOMETRIC QUANTITIES

Geometric Quantity	$\mu$	$V_\mu$
$V$ , volume .....	0	$V_0 = V$
$A$ , surface .....	1	$V_1 = A/6$
$H$ , mean curvature .....	2	$V_2 = H/3\pi$
$\chi$ , Euler characteristic .....	3	$V_3 = \chi$

<sup>8</sup> Note that for arguments  $x > 1$ , one can use the relation  $i \arccos x = \ln[x + (x^2 - 1)^{1/2}]$  to recover an explicitly real-valued expression.

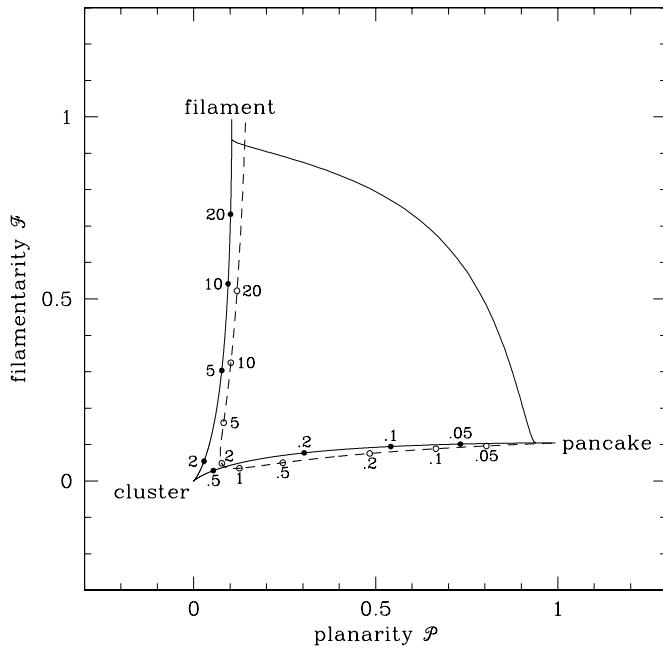


FIG. 1.—Shapefinders of some convex bodies. The solid line shows the transition from a ball to a filament to a pancake and back to a ball via triaxial ellipsoids of various shapes. The dashed line corresponds to cylinders that also undergo the transition from a filament to a pancake by varying their height. The dots on the spheroid and cylinder curves indicate typical values of the  $\lambda$  parameter from eqs. (3) and (5), respectively.

Minkowski functionals are given by

$$\begin{aligned} V_0 &= \pi r^3 \lambda, \quad V_1 = \frac{\pi}{3} r^2 (1 + \lambda), \\ V_2 &= \frac{1}{3} r (\pi + \lambda), \quad V_3 = 1. \end{aligned} \quad (5)$$

Figure 1 shows a Blaschke diagram, which is a plot of planarity,  $\mathcal{P}$ , and filamentarity,  $\mathcal{F}$ , summarizing the morphological properties of these simple convex bodies.

### 3. A SET OF $N$ -BODY SIMULATIONS

#### 3.1. Description

We start from a family of initial power-law spectra  $P(k) \propto k^n$ , where  $n \in \{-2, -1, 0, +1\}$ , set before an Einstein-de Sitter background ( $\Omega = 1, \Lambda = 0$ ). We conduct numerical experiments using a PM code (consult Melott & Shandarin 1993 for details). Four sets of phases were used for each model, making a total of 16 simulation runs. Each run consists of  $128^3$  particles sampled at an epoch well in the nonlinear regime. This epoch is chosen such that the scale of nonlinearity,  $k_{nl}$ , defined in terms of the evolved spectrum

$$\sigma_{k_{nl}}^2 = \int_0^{k_{nl}} d^3 k P(k) = 1, \quad (6)$$

is equal to 8 in units of the fundamental mode of the simulation box. By using the stage  $k_{nl} = 8$ , we make sure that structure is already sufficiently developed on scales much

larger than the simulation's resolution, while it is not yet influenced by boundary effects.

Using a cloud-in-cell kernel, these particles were put onto a  $256^3$  grid, which is the maximum value an ordinary workstation can tackle with acceptable time and memory consumption. Subsequently, the density field was smoothed with a Gaussian kernel  $\propto \exp(-x^2/2\lambda^2)$ , where  $x$  is the distance in mesh units, and the width  $\lambda$  is set to 3. Tests have shown that this value both leads to a reasonably smooth field and preserves at least some detail on smaller scales. Throughout this article, we rescale the density to the density contrast  $\delta$ , ranging from  $-1$  to infinity with zero mean.

#### 3.2. The Global Field

The global Minkowski functionals calculated from the density fields described in the previous section are shown in Figure 2. The four different line styles correspond to the different spectral indices. Figure 3 shows the Minkowski functionals for the same set of models, but instead of the density threshold  $\delta$ , the rescaled threshold  $v$  is used as the  $x$ -axis. The value of  $v$  is calculated from the volume Minkowski functional  $v_0$ , which is the filling factor  $f$ , as described by Gott, Weinberg, & Melott (1987). Essentially, its use forces exact Gaussian behavior of the volume  $v_0$ , by the implicit connection

$$v_0(\delta) = \frac{1}{2} - \frac{1}{2} \Phi\left(\frac{v}{\sqrt{2}}\right). \quad (7)$$

Thus, the deviations from Gaussianity that are due to changes in the one-point probability distribution function are removed, and deviations due to higher-order correlations are emphasized.

Obviously, the global functionals clearly discriminate between the various models. However, in order to make this statement more quantitative, let us take a closer look at the individual coherent objects composing the isodensity contours.

#### 3.3. The Largest Objects

At intermediate thresholds, the excursion sets consist of numerous isolated objects. We identify them by grouping adjacent occupied grid cells into one object, where adjacent means that the cells have a common face. Since the Minkowski functionals of the global field are calculated by integrating over quantities that can be assigned to individual grid cells, the partial Minkowski functionals of each object can be obtained at no extra cost once the cells belonging to each object have been identified.

Several plots in Figures 4, 5, 6, and 7 illustrate the behavior of the Minkowski functionals of these objects.

Obviously, the contribution of smaller objects to the volume is almost negligible compared to the largest one. Note that in all figures, the mean and standard deviation over all four realizations are shown instead of the individual curves. It is worth noting that the variance is largest in the  $n = -2$  and  $-1$  models, which are dominated by structures on large scales and hence show the strongest sample variance.

The models with various initial spectral indices  $n$  show a qualitatively similar behavior. At small filling factors (high density thresholds), the two largest clusters make negligible contributions to each of the global characteristics. Then at the percolation transition the largest cluster quickly

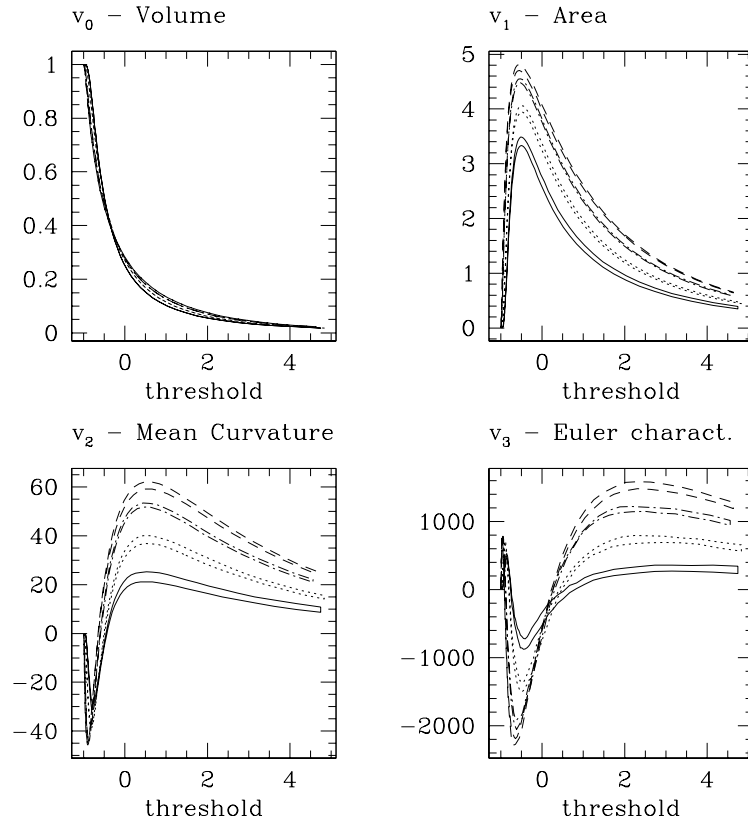


FIG. 2.—Minkowski functionals for the evolutionary stage  $k_{nl} = 8$  of four different models with initial power law spectra  $P(k) \propto k^n$ . The area between two lines gives the  $1\sigma$  deviations of four realizations from their mean. Solid lines,  $n = -2$ ; dotted lines,  $n = -1$ ; dash-dotted lines,  $n = 0$ ; dashed lines,  $n = +1$ .

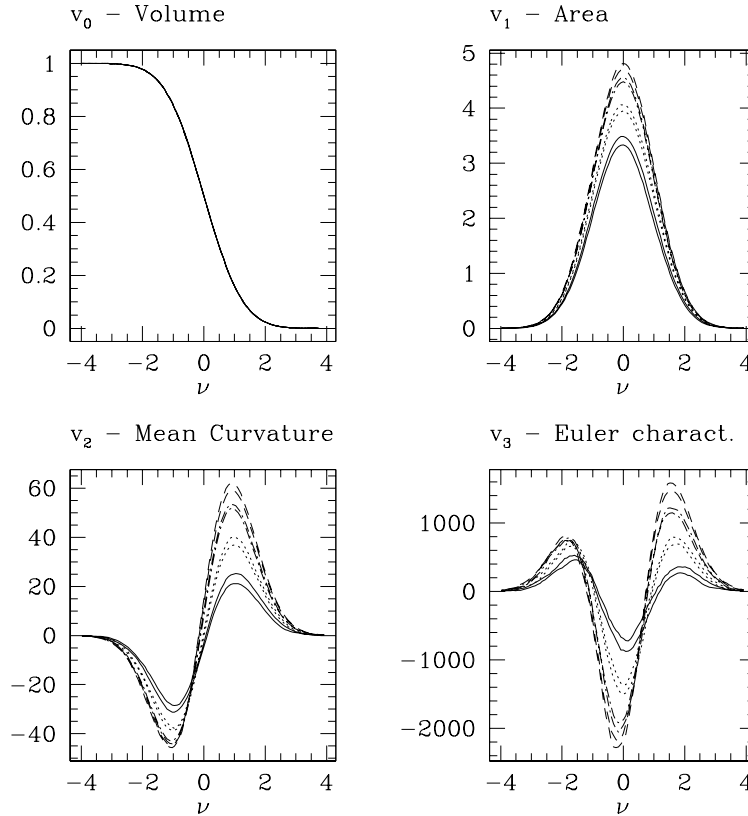


FIG. 3.—Minkowski functionals for the evolutionary stage  $k_{nl} = 8$  of four different models with initial power law spectra. Line styles are as in Fig. 3, but instead of the density threshold  $\delta$ , the rescaled threshold  $\nu$  introduced by Gott et al. (1987) is used.

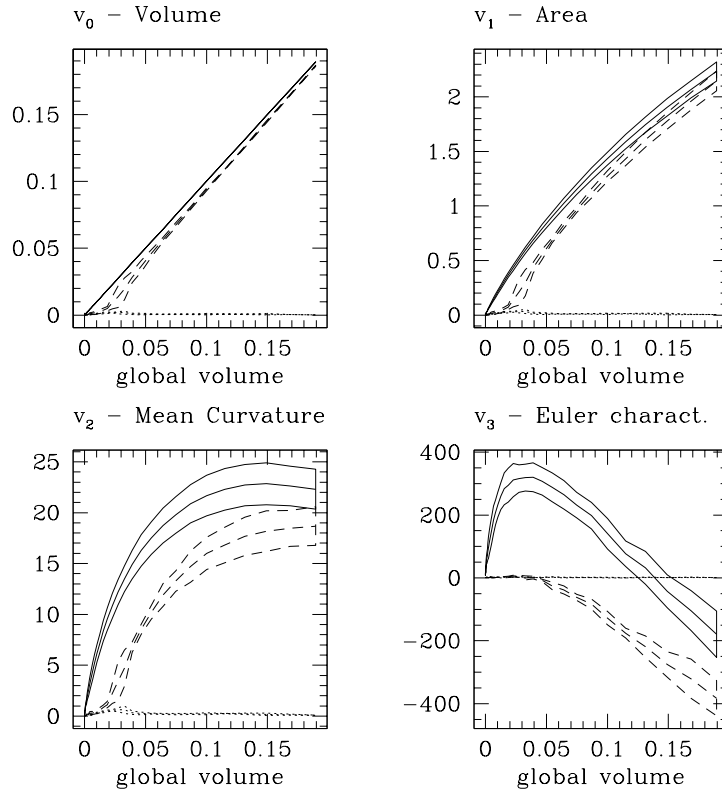


FIG. 4.—Comparison of the global Minkowski functionals of the whole density field (*solid line*) to the Minkowski functionals of the largest (*dashed line*) and second-largest objects (*dotted line*). The areas indicate the mean and standard deviation over the four realizations of each model. In order to emphasize the region around the percolation transition, the Minkowski functionals are plotted against the filling factor, which is equal to the volume enclosed in all isocontours of the global field. In this particular plot, the values for an initial power-law spectrum with an index of  $n = -2$  are shown.

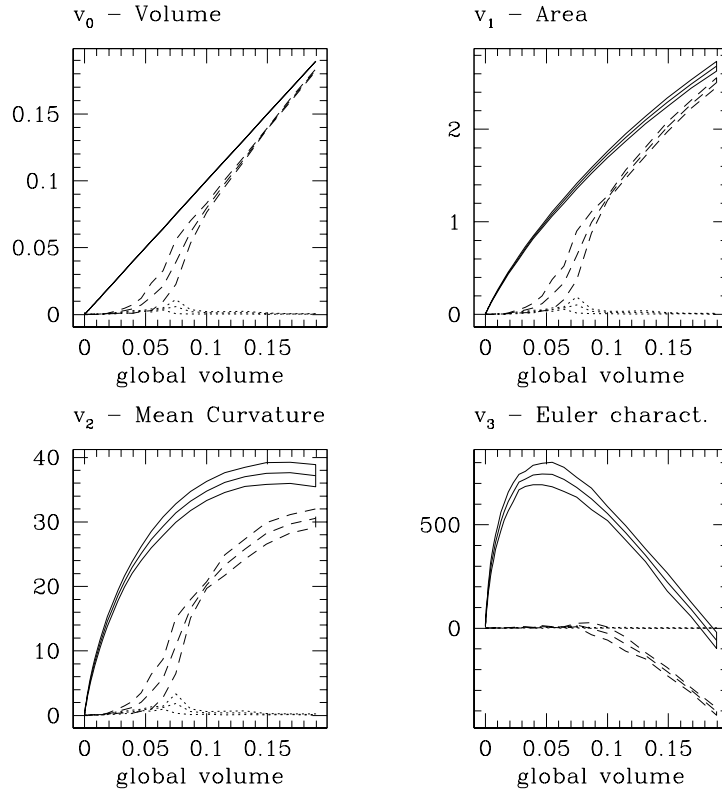


FIG. 5.—Same tendency as in Fig. 4, visible in this case where  $n = -1$ . As before, the dominance of large networked structures is clearly visible; for example, the Euler characteristic stays well below zero even for high thresholds, which indicates a spongy structure with many tunnels in it. However, the networked structures have fewer loops, since they tend to split up at lower thresholds.

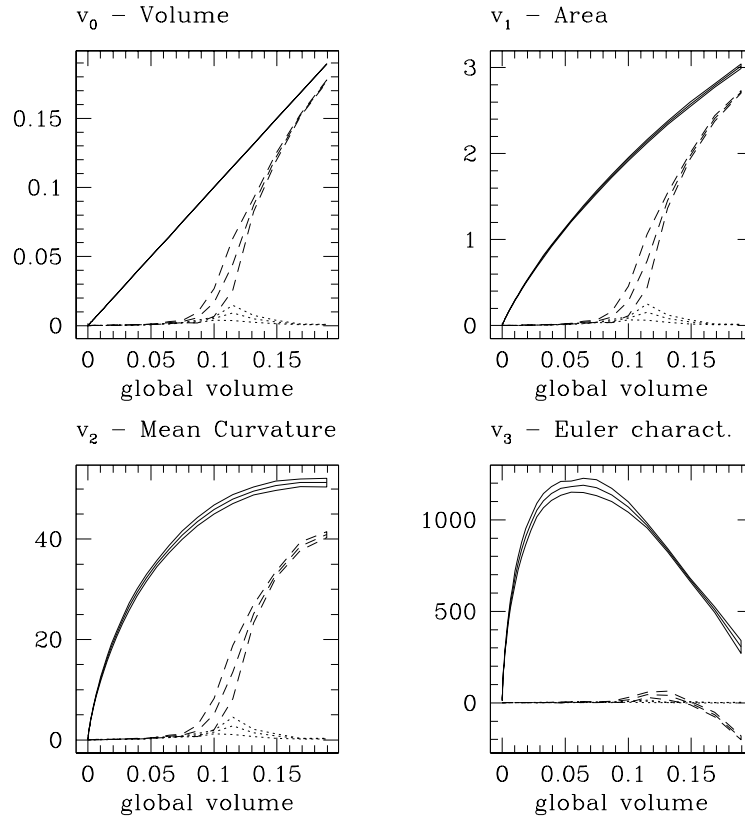


FIG. 6.—The case of  $n = 0$ , showing a different situation. The largest object splits up rapidly, and is already simply connected at fairly low density contrast ( $\delta \approx 0.5$ ). In addition, at thresholds around percolation, the second-largest object becomes comparable to the largest one, so the field is not dominated by a single networked structure, but by several chunks.

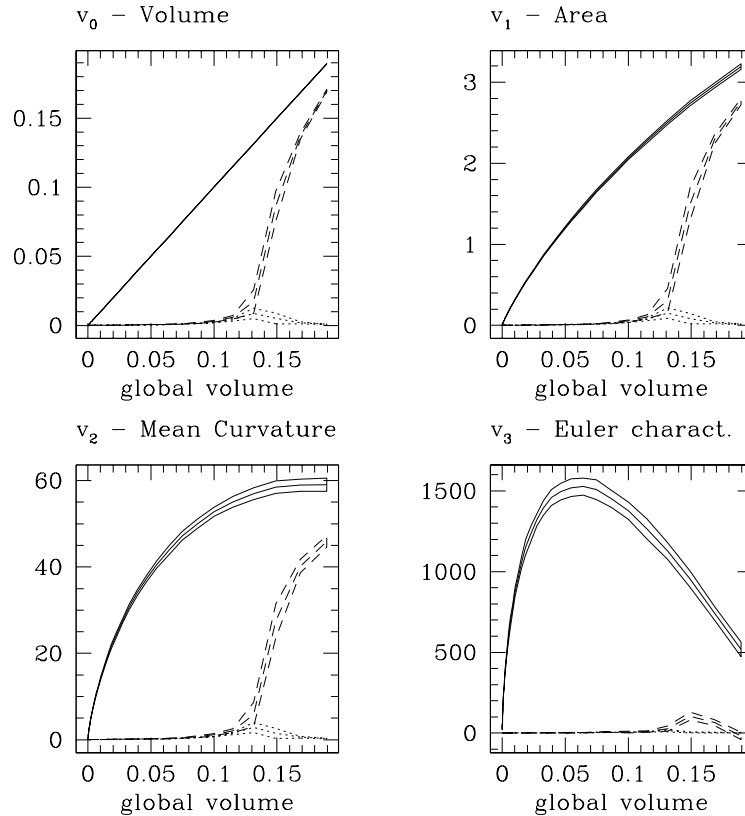


FIG. 7.—Same effect as in Fig. 6, even more pronounced for  $n = +1$ , since this model is dominated by small-scale structures. Contrary to the models with negative index  $n$ , where the Euler characteristic of the largest object approaches the value 1 for a simply connected structure from below, one can even see a small positive peak around the percolation threshold.

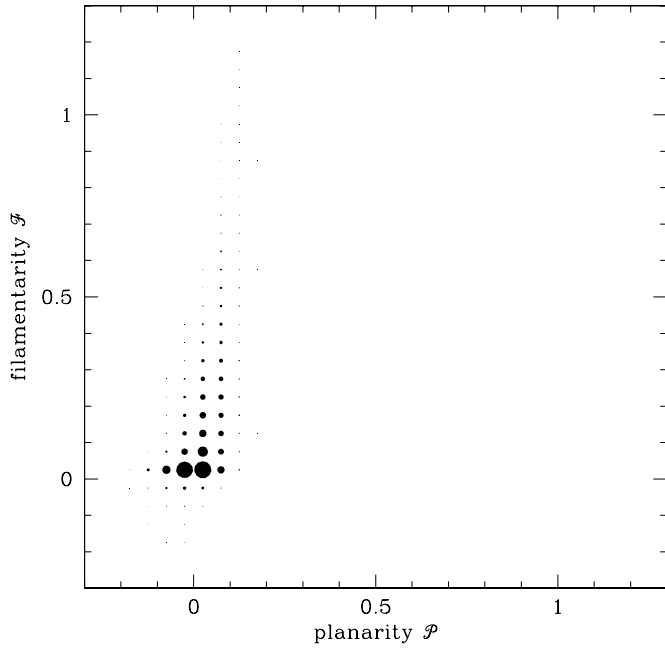


FIG. 8.—Shapefinder or Blaschke diagram of the model with  $n = -2$ . The area of each dot corresponds to the number of coherent objects whose shapefinders  $(\mathcal{P}, \mathcal{F})$  lie around the center of this dot. Note that the shapefinders of all objects at all thresholds are used in the construction of the diagram. Nevertheless, most information comes from thresholds close to percolation, where small objects are abundant.

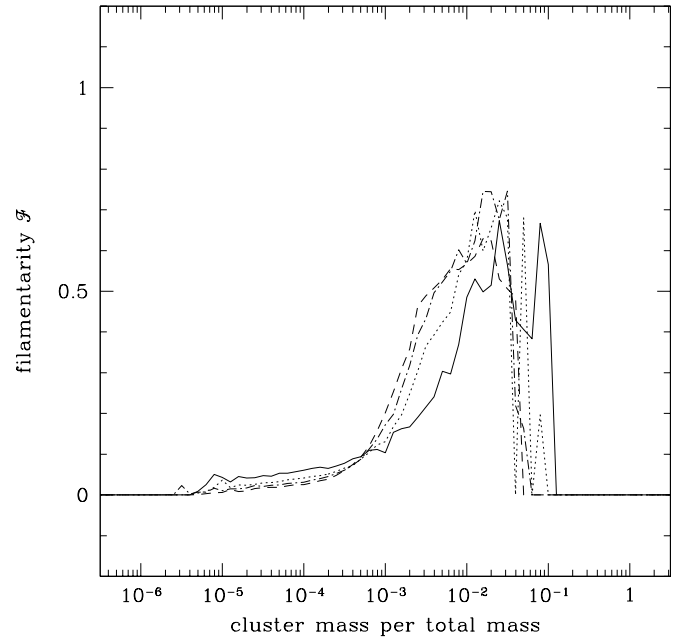


FIG. 10.—Plot of the same type of distribution as in Fig. 9, but here the average filamentarity shapefinder,  $\mathcal{F}$ , is plotted as a function of the cluster mass. Lines are as in Fig. 2.

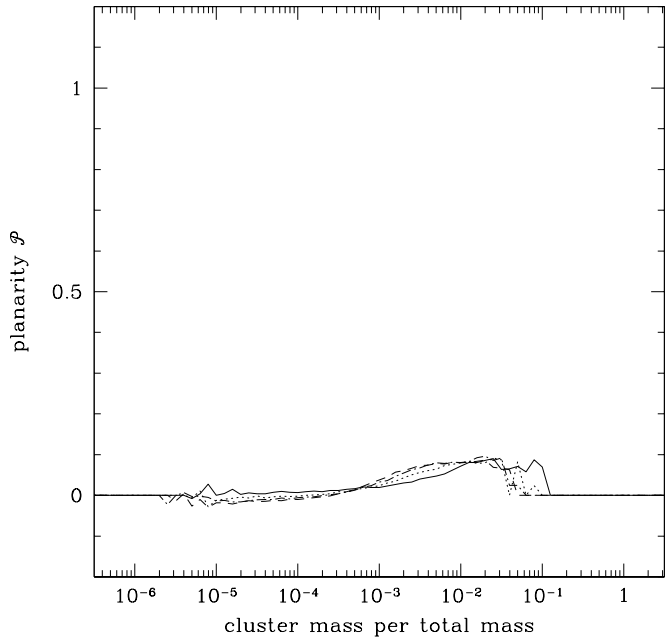


FIG. 9.—Average planarity shapefinder,  $\mathcal{P}$ , as a function of the cluster mass. Lines are as in Fig. 2. As in Fig. 8, all objects at all thresholds are used to improve statistics.

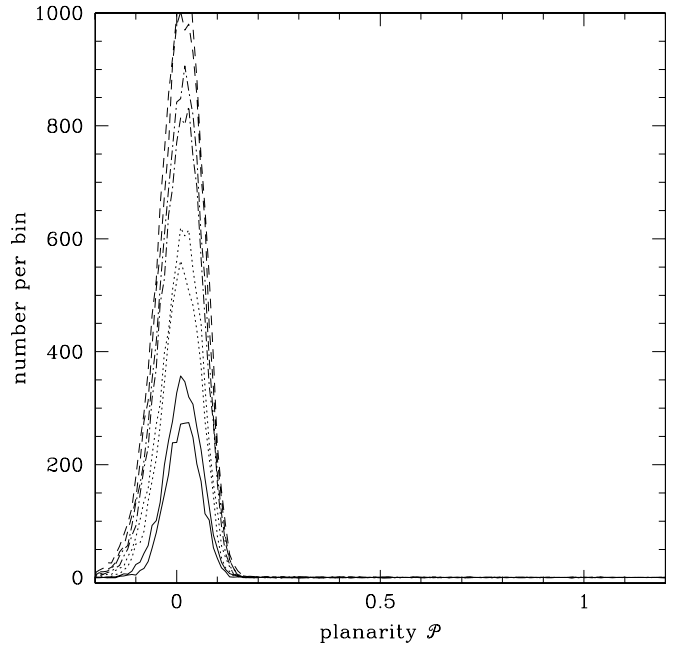


FIG. 11.—Distribution of the planarity shapefinder,  $\mathcal{P}$ . The histogram only includes objects larger than  $10^{-5}$  times the simulation box, a value of the order of the smoothing volume. Note that although theoretically the value should lie between 0 and 1, measurement errors lead to negative values. The curves give total numbers of objects per histogram interval; hence, they differ widely in normalization, but still agree reasonably well in shape. Lines are as in Fig. 2.

becomes the dominant structure in terms of volume, area, and integrated mean curvature. The second-largest cluster also grows at the percolation threshold, but just a little, and then it quickly diminishes. The percolation transition is clearly marked in all three characteristics of the largest cluster by their sudden growth. However, this transition does not happen at a well-defined threshold. Instead, clusters gradually merge into the largest objects as the threshold is decreased (the filling factor grows). This continuous transition has also been observed using percolation analysis, i.e., the zeroth Minkowski functional alone (Shandarin & Zeldovich 1983; Shandarin & Yess 1998; Klypin & Shandarin 1993; Sahni et al. 1997), and is explained by the finite size of the sample.

Nevertheless, the percolation transitions happen within fairly narrow ranges of the filling factor that are clearly distinct for different models in question; the filling factors are approximately  $0.03 \pm 0.01$ ,  $0.07 \pm 0.015$ ,  $0.11 \pm 0.015$ , and  $0.14 \pm 0.015$  in the  $n = -2, -1, 0$ , and  $+1$  models, respectively. It is remarkable that all percolation transitions occur at smaller filling factors than in Gaussian fields (about 0.16), indicating that even in the most hierarchical model ( $n = +1$ ) the structures tend to be more connected than in the “structureless” Gaussian field. Pauls & Melott (1995) showed positive correlation with networks based on the same phases all the way to  $n = +3$ . This confirms the conclusion of Yess & Shandarin (1996) that the universality of the network structures results from the evolution of Gaussian initial conditions through gravitational instability. The Euler characteristic of the largest cluster also marks the percolation threshold, but in a different manner: before percolation it is zero and after percolation it becomes negative in every model; however, in the  $n = 0$  and  $n = +1$  models, it grows to a small positive peak before becoming negative. All global functionals have no particular features at the percolation threshold.

### 3.4. Small Objects

As an example, the Blaschke diagram for the model  $n = -2$  is shown in Figure 8. The distributions for the other models look qualitatively very similar, and the average quantities for other models are presented in Figures 9 and 10. Figure 9 shows that most of the small objects are either spherical or slightly planar (two largest dots in Fig. 8). There is also a considerable number of elongated clusters with filamentarities from 0.1 to 0.5. In some cases, filamentarity reaches large values,  $\sim 1$ . In contrast, planarity is much weaker; it hardly reaches the value of 0.2 (which is partly a consequence of the smoothing). There is a hint of a small correlation between filamentarity and planarity: the objects with the largest filamentarity also tend to have larger planarity.

Figures 9 and 10 display the shapefinders as functions of the cluster mass. The curves give averages over the realizations of each model. Apparently, the signal for filamentarity is much stronger than for planarity, regardless of the model, which is in full agreement with Figure 8. The planarity and filamentarity distributions qualitatively look very similar except in amplitude. Small objects ( $5 \times 10^{-6} \lesssim m \lesssim 5 \times 10^{-4}$ ) display stronger planarity and filamentarity for models with more power on large scales. However, for greater masses ( $m \gtrsim 5 \times 10^{-4}$ ) the situation is reversed; the less large-scale power, the greater the filamentarity and planarity. If the former condition seems to be natural and

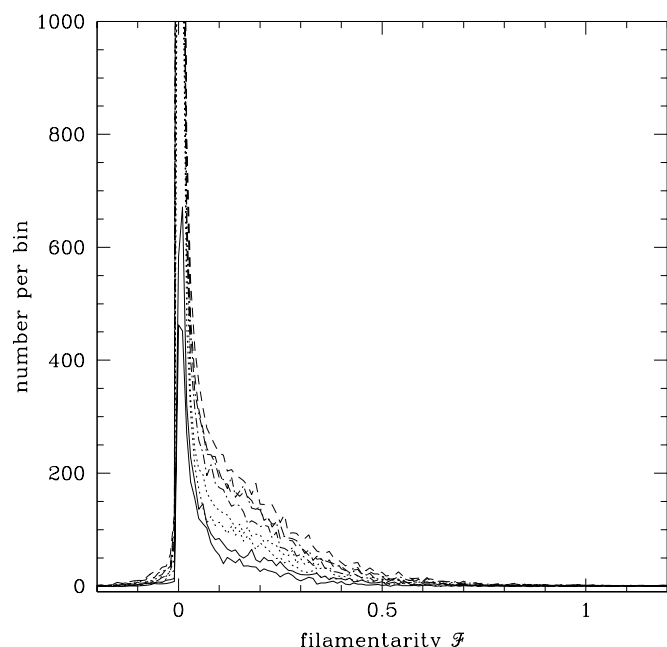


FIG. 12.—Same type of distribution as in Fig. 11, but showing the filamentarity shapefinder,  $\mathcal{F}$ . Lines are as in Fig. 2.

was expected, the latter has been unexpected. Both the planarity and filamentarity monotonically grow and reach their maxima at the largest clusters:  $\mathcal{P}_m \approx 0.1$  and  $\mathcal{F}_m \approx 0.5$  in all models. As expected, the largest objects possess the largest planarities and filamentarities, but the independence of the maxima from the model was again unexpected. Figures 11 and 12 show the histograms for the shapefinders  $\mathcal{P}$  and  $\mathcal{F}$ , respectively. They clearly show the large difference in the total number of structures; the more power on small scales, the greater the abundance of clusters. These figures are in general agreement with Figures 9 and 10.

## 4. CONCLUSION AND OUTLOOK

Global Minkowski functionals do discriminate (Figs. 2 and 3). Only the  $n = -2$  model shows slight drawbacks as far as robustness is concerned, but that is due to the method's sensitivity to large-scale features of the smoothed density field. The total area, mean curvature, and Euler characteristic are sensitive to abundances of the structures and are easy to interpret; the more power on small scales (greater  $n$ ), the more abundant structures and therefore the greater the amplitude of the curve.

Even more valuable information is obtained from looking at the Minkowski functionals of the largest coherent object at each threshold (Figs. 4, 5, 6, and 7). All four Minkowski functionals of the largest cluster clearly consistently detect the percolation transition. Two points are worth stressing: (1) in all models, the percolation transition happens at smaller filling factors than in the “structureless” Gaussian fields; and (2) the more power on large scales (i.e., the smaller  $n$ ), the smaller the filling factor at percolation. Both conclusions confirm the results of Yess & Shandarin (1996) regarding the universality of the network structures in the power-law models with  $n \leq 1$ . The results of Pauls & Melott (1995) present evidence that this should be expected all the way up to  $n = +3$ ; at  $n = +4$ , mode coupling effects



from smaller scales should begin to fully disrupt the network structure.

Small objects, on the other hand, give different results. Their abundance discriminates well, but is already determined by the difference in the Euler characteristic, as well as by the total area and mean curvature of the whole contour (Figs. 2 and 3), which are also sensitive to the abundance of structures. The morphology of small objects as measured by shapefinders shows little differences between models so far (Figs. 9, 10, 11, and 12). Both the maximum average planarity ( $\mathcal{P} \approx 0.1$ ) and the maximum average filamentarity ( $\mathcal{F} \approx 0.5$ ) are reached in the most massive nonpercolating objects. None of the models showed ribbon-like objects characterized by both large planarity and large filamentarity. We may speculate that the smaller objects are ones that formed earlier, are more nonlinear, and therefore more decoupled from initial conditions.

However, all models used Gaussian initial conditions and evolve under the influence of gravity. Hence, the similar morphology of the clumps may point toward universal

behavior. Note that phenomena such as string wakes might produce totally different results.

The grouping and measurement techniques used in this study may be less accurate for small objects than for large clusters. It is worth trying to study the morphology of small objects by applying more accurate methods of measuring partial Minkowski functionals, such as a Boolean grain model (Schmalzing 1996, and a follow-up to this article) or the interpolation method of Novikov, Feldman, & Shandarin (1999), generalized to three dimensions.

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## APPENDIX A

### CALCULATING MINKOWSKI FUNCTIONALS OF ISODENSITY CONTOURS

Using methods from integral geometry and scale-space theory, Schmalzing & Buchert (1997) developed numerical methods to quantify the morphology of isodensity contours of a random field. They present two complementary possibilities for estimating Minkowski functionals.

Given a density field sampled at the grid points of a cubic lattice with lattice constant  $a$ , Crofton’s formula (Crofton 1868) requires counting the number of grid cells per unit volume contained in the excursion set. We obtain

$$\hat{V}_0(v) = n_3, \quad (A1)$$

$$\hat{V}_1(v) = \frac{2}{9a} (n_2 - 3n_3), \quad (A2)$$

$$\hat{V}_2(v) = \frac{2}{9a^2} (n_1 - 2n_2 + 3n_3), \quad (A3)$$

$$\hat{V}_3(v) = \frac{1}{a^3} (n_0 - n_1 + n_2 - n_3), \quad (A4)$$

where the quantity  $n_j$  is the number density of  $j$ -dimensional elementary cells; to be specific,  $n_3$  is the number of grid volumes,  $n_2$  counts the grid faces, and  $n_1$  and  $n_0$  denote the numbers of grid edges and lattice points, respectively.

Alternatively, it is sufficient to estimate the derivatives of the random field at the grid points and perform a spatial average,  $\langle \dots \rangle_D = (1/D) \int_D d^3x$  over Koenderink invariants:<sup>9</sup>

$$\hat{V}_0(v) = \langle \Theta(u - v) \rangle_D, \quad (A5)$$

$$\hat{V}_1(v) = \left\langle \frac{1}{6} \delta(u - v) (u_{,i} u_{,i})^{1/2} \right\rangle_D, \quad (A6)$$

$$\hat{V}_2(v) = \left\langle \frac{1}{3\pi} \delta(u - v) \frac{\epsilon_{ijm} \epsilon_{klm} u_{,i} u_{,jk} u_{,l}}{2u_{,n} u_{,n}} \right\rangle_D, \quad (A7)$$

$$\hat{V}_3(v) = \left\langle \frac{1}{4\pi} \delta(u - v) \frac{\epsilon_{ijk} \epsilon_{lmn} u_{,i} u_{,l} u_{,jm} u_{,kn}}{2(u_{,p} u_{,p})^{3/2}} \right\rangle_D. \quad (A8)$$

<sup>9</sup> The notation is chosen to emphasize the coordinate invariance of the integrand. We use indices following a comma to indicate differentiation with respect to the corresponding coordinate. The terms  $\epsilon_{ijk}$  denotes the components of the totally antisymmetric third-rank tensor normalized to  $\epsilon_{123} = 1$ . Summation over pairwise indices is understood.

Careful analysis reveals that both families of estimators are biased by the finite lattice constant,  $a$ , encountered with any practical realization of a random field. The deviations are of order  $a^2$  and can be evaluated analytically for a Gaussian random field (see Winitzki & Kosowsky 1997 for some results). It turns out that the two estimates using Crofton's formula and Koenderink invariants, respectively, deviate from the true value in opposite directions. Hence, their difference gives at least a rough idea of the errors associated with binning the continuous field onto a finite lattice.

Yet another method of calculating Minkowski functionals of isocontours was suggested by Novikov et al. (1999). However, their interpolation method has so far only been implemented in two dimensions.

## APPENDIX B

### ISOPERIMETRIC INEQUALITIES AND SHAPEFINDERS

In his original article, Minkowski (1903) introduced the mixed volumes  $V(K_1, \dots, K_n)$  of  $n$  convex bodies  $K_1$  to  $K_n$ . In terms of ordinary volumes of Minkowski sums of the bodies, we have

$$V(K_1, \dots, K_n) \equiv \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{i_1 < \dots < i_k} V(K_{i_1} \oplus \dots \oplus K_{i_k}), \quad (\text{B1})$$

where the Minkowski sum is defined as

$$K \oplus L \equiv \{x + y | x \in K, y \in L\}. \quad (\text{B2})$$

In the case of a single body, the mixed volume reduces to its ordinary volume. Furthermore, these quantities inherit many interesting properties from the volume and hence play a central role in the Brunn-Minkowski theory (see Schneider 1993 for an introduction).

A most useful inequality, proved by Alexandrov (1937) and Fenchel (1936), states that

$$V^2(K_1, K_2, \dots, K_n) \geq V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n). \quad (\text{B3})$$

The Minkowski functionals of a body  $K$  in  $d$  dimensions can be related to mixed volumes of two bodies via<sup>10</sup>

$$V_j(K) = \frac{1}{\omega_j} V(\underbrace{K, \dots, K}_{d-j}, \underbrace{B, \dots, B}_j) \quad (\text{B4})$$

where  $B$  denotes the  $d$ -dimensional unit ball. Consult the book by Burago & Zalgaller (1988) for a broad discussion of the Alexandrov-Fenchel inequality and related issues.

In three dimensions, the Alexandrov-Fenchel inequality (eq. [B3]) leads to two independent nontrivial inequalities for the Minkowski functionals, namely,

$$V_1^2 \geq \frac{\pi}{4} V_0 V_2, \quad (\text{B5})$$

$$V_2^2 \geq \frac{8}{3\pi} V_1 V_3. \quad (\text{B6})$$

$$(\text{B7})$$

These two inequalities motivate the introduction of the Blaschke diagram (Hadwiger 1955). A convex body with Minkowski functionals  $V_j$  is mapped to a point  $(x, y)$  with coordinates

$$x \equiv \frac{\pi V_0 V_2}{4 V_1^2}, \quad y \equiv \frac{8 V_1 V_3}{3\pi V_2^2}. \quad (\text{B8})$$

A ball has  $x = y = 1$ . For convex bodies, all Minkowski functionals are nonnegative and satisfy the inequalities given in equation (B6), so these points are confined to the unit square. Nevertheless, the convex bodies do not fill the whole unit square—the limiting isoperimetric inequality has yet to be found (Schneider 1993 and references therein).

The dimensionless shapefinders introduced in equation (2) are related to the isoperimetric ratios via

$$\mathcal{P} = \frac{1-x}{1+x}, \quad \mathcal{F} = \frac{1-y}{1+y}. \quad (\text{B9})$$

Obviously, a scatter plot of shapefinders is almost equivalent to the ordinary Blaschke diagram reflected at the point (0.5, 0.5), so either method of presentation should convey the same morphological information. Throughout this article, we refer to plots of shapefinders ( $\mathcal{P}$ ,  $\mathcal{F}$ ) as Blaschke diagrams.

<sup>10</sup> The volume of the  $j$ -dimensional unit ball is denoted by  $\omega_j$ . Some special values are  $\omega_0 = 1$ ,  $\omega_1 = 2$ ,  $\omega_2 = \pi$ , and  $\omega_3 = 4\pi/3$ . The general formula reads  $\omega_j = \pi^{j/2} / [\Gamma(j/2 + 1)]$ .

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