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LUMINOSITY VERSUS ROTATION IN A SUPERMASSIVE STAR

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ABSTRACT

We determine the effect of rotation on the luminosity of supermassive stars. We apply the Roche model to calculate analytically the emitted radiation from a uniformly rotating, radiation-dominated supermassive configuration. We find that the luminosity at maximum rotation, when mass at the equator orbits at the Kepler period, is reduced by $\sim 36\%$ below the usual Eddington luminosity from a corresponding nonrotating star. A supermassive star is believed to evolve in a quasi-stationary manner along such a maximally rotating "mass-shedding" sequence before reaching the point of dynamical instability; hence this reduced luminosity determines the evolutionary timescale. Our result therefore implies that the lifetime of a supermassive star prior to dynamical collapse is $\sim 36\%$ longer than the value typically estimated by employing the usual Eddington luminosity.

Subject headings: stars: formation — stars: interiors — stars: rotation

1. INTRODUCTION

Recent observations provide strong evidence that supermassive black holes (SMBHs) exist and are the sources that power active galactic nuclei and quasars (for a review and references, see, e.g., Rees 1998). However, the scenario by which SMBHs form is still very uncertain (for an overview, see, e.g., Rees 1984). One promising route is the collapse of a supermassive star (SMS). Once they form out of primordial gas, sufficiently massive stars will evolve in a quasi-stationary manner via radiative cooling, slowly contracting until reaching the point of onset of relativistic radial instability. At this point, such stars undergo catastrophic collapse on a dynamical timescale, possibly leading to the formation of an SMBH (Bisnovatyi-Kogan, Zeldovich, & Novikov 1967; Zeldovich & Novikov 1971; Shapiro & Teukolsky 1983).

Because most objects formed in nature have some angular momentum, rotation is likely to play a significant role in the quasi-stationary evolution as well as the final collapse of an SMS. The slow contraction of even a slowly rotating SMS will likely spin it up to the mass-shedding limit, because such stars are so centrally condensed. At the mass-shedding limit, matter on the equator moves in a Keplerian orbit about the star, supported against gravity by centrifugal force and not by an outward pressure gradient. The SMS evolves in a quasi-stationary manner along the mass-shedding curve, simultaneously emitting radiation, matter, and angular momentum until reaching the onset of radial instability.

In this paper, we derive the luminosity of a uniformly rotating SMS as a function of its spin rate, up to the mass-shedding limit. The magnitude of the luminosity is crucial because it determines the evolutionary timescale of the star as it evolves. Elsewhere we use the result to follow the slow contraction of a cooling, rotating SMS to the onset of dynamical instability (Baumgarte & Shapiro 1999, hereafter Paper II). Here, however, we focus on the emitted flux and total integrated luminosity from a stationary SMS as a function of its rate of rotation (for an overview of previous work on SMSs and references, see, e.g., Zeldovich & Novikov 1971; Shapiro & Teukolsky 1983; Paper II).

Even though our calculation, which is analytic up to quadrature, is rather simple and straightforward, we have not been able to find a similar argument in the literature. Previous analytical arguments have dealt with more general rotation laws, but they adopt the slow rotation approximation and emphasize gas-pressure atmospheres (see, e.g., Kippenhahn 1977). While detailed numerical calculations of luminosities of rotating stars have been carried out for select main-sequence and massive stars (see, e.g., Tassoul 1978, Table 12.1, and references therein; Langer & Heger 1997), we cannot find a calculation for an SMS. Hence, independent of its relevance to the evolution of SMSs prior to catastrophic collapse, our result may be of interest to stellar modeling of rapidly rotating stars in the limit of very high mass, where our calculation is applicable.

Our paper is organized as follows: in § 2 we enumerate and justify our assumptions. In § 3 we briefly review the Roche approximation, which we use to describe the outer layers of a rotating SMS. In § 4 we derive the flux and luminosity from the star. In § 5 we briefly summarize our results and compare them with previous calculations of rotating main-sequence stars.

2. BASIC ASSUMPTIONS

Our analysis relies on several explicit assumptions, all of which we expect to hold to high accuracy in SMSs. In particular, we assume that the star is

1. dominated by thermal radiation pressure,
2. fully convective,
3. uniformly rotating,
4. characterized by a Rosseland mean opacity that is independent of density,
5. governed by Newtonian gravitation, and
6. described by the Roche model in the outer envelope.

For large masses, the ratio between radiation pressure, P_r, and gas pressure, P_g, satisfies

$$\beta \equiv \frac{P_g}{P_r} = 8.49\left(\frac{M}{M_\odot}\right)^{-1/2} \tag{1}$$

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(see, e.g., Shapiro & Teukolsky 1983, eqs. [17.2.8] and [17.3.5]); here the coefficient has been evaluated for a composition of pure ionized hydrogen. For stars with $M \gtrsim 10^4$ $M_\odot$, we can therefore neglect the pressure contributions of the plasma in determining the equilibrium profile, even though the plasma may be important for determining the stability of the star (Zeldovich & Novikov 1971; Shapiro & Teukolsky 1983). A simple proof that SMSs are convective in this limit is given in Loeb & Rasio (1994). This result implies that the photon entropy per baryon,

$$s_r = \frac{4}{3}\frac{aT^3}{n_B}\,, \tag{2}$$

is constant throughout the star and so is $\beta \approx 8(s_r/k)^{-1}$. Here a is the radiation density constant, n_B is the baryon density, and k is Boltzmann's constant. As a consequence, the equation of state of an SMS is that of an $n = 3$ polytrope:

$$P = K\rho^{4/3}\,, \quad K = \left[\left(\frac{k}{\bar{\mu}m}\right)^4 \frac{3}{a}\frac{(1+\beta)^3}{\beta^4}\right]^{1/3} = \text{const}\,, \tag{3}$$

where P is the pressure, ρ the mass density, m the atomic mass unit, and $\bar{\mu}$ the mean molecular weight [cf. Clayton 1983, eq. (2-289); note that Clayton adopts a different definition of β, which is related to ours by $\beta_{\rm Clayton} = \beta/(1+\beta)$].

The third assumption, that the star is uniformly rotating, is probably the most uncertain of our assumptions. Nevertheless, it has been argued that convection and magnetic fields provide an effective turbulent viscosity, which dampens differential rotation and brings the star into uniform rotation (Bisnovatyi-Kogan et al. 1967; Wagoner 1969).

In the high-temperature, low-density, strongly ionized plasma of an SMS, Thomson scattering off free electrons is the dominant source of opacity. This opacity is independent of density and justifies our fourth assumption.

We assume that gravitational fields are sufficiently weak that we can apply Newtonian gravity. SMSs of interest here have $R/M \gtrsim 400$ (see Paper II), so this assumption certainly holds. Relativistic corrections are important for the stability of SMSs but can be neglected in the analysis of the equilibrium state.

Finally, the Roche approximation provides a very accurate description of the envelope of a rotating stellar model with a soft equation of state, as in the case of an $n = 3$ polytrope (for numerical demonstrations, see, e.g., Papaloizou & Whelan 1973 and Paper II). Since our analysis is based on this approximation, we will briefly review it together with some of its predictions in the following section.

In applications to SMSs our analysis neglects electron-positron pairs and Klein-Nishina corrections to the electron-scattering opacity, which is valid for $M \gtrsim 10^5$ $M_\odot$ (see, e.g., Fuller, Woosley, & Weaver 1986).

3. REVIEW OF THE ROCHE MODEL

Stars with soft equations of state are extremely centrally condensed: they have an extended, low-density envelope, while the bulk of the mass is concentrated in the core. For an $n = 3$ polytrope, for example, the ratio of central density to average density is $\rho_c/\bar{\rho} = 54.2$. The gravitational force in the envelope is therefore dominated by the massive core, and it is thus legitimate to neglect the self-gravity of the envelope. In the equation of hydrostatic equilibrium,

$$\frac{\nabla P}{\rho} = -\nabla(\Phi + \Phi_c)\,, \tag{4}$$

this neglect amounts to approximating the Newtonian potential Φ by

$$\Phi = -\frac{M}{r} \tag{5}$$

(here we adopt gravitational units by setting $G \equiv 1$). In equation (4) we introduce the centrifugal potential Φ_c, which, for constant angular velocity Ω about the z-axis, can be written

$$\Phi_c = -\tfrac{1}{2}\Omega^2(x^2 + y^2) = -\tfrac{1}{2}\Omega^2 r^2 \sin^2\theta\,. \tag{6}$$

Integrating equation (4) yields the Bernoulli integral

$$h + \Phi + \Phi_c = H\,, \tag{7}$$

where H is a constant of integration and

$$h = \int \frac{dP}{\rho} = (n+1)\frac{P}{\rho} \tag{8}$$

is the enthalpy per unit mass. Evaluating equation (7) at the pole yields

$$H = -\frac{M}{R_p}\,, \tag{9}$$

since $h = 0$ on the surface of the star and $\Phi_c = 0$ along the axis of rotation. In the following we assume that the polar radius R_p of a rotating star is always the same as in the nonrotating case. This assumption has been shown numerically to be very accurate (e.g., Papaloizou & Whelan 1973).

A rotating star reaches mass shedding when the equator orbits with the Kepler frequency. Using equations (6) and (7), it is easy to show that at this point the ratio between equatorial and polar radius is

$$\left(\frac{R_e}{R_p}\right)_{\rm shedd} = \frac{3}{2}\,. \tag{10}$$

The corresponding maximum orbital velocity is

$$\Omega_{\rm shedd} = \left(\frac{2}{3}\right)^{3/2}\left(\frac{M}{R_p^3}\right)^{1/2} \tag{11}$$

(Zeldovich & Novikov 1971; Shapiro & Teukolsky 1983).

4. LUMINOSITY OF ROTATING STARS

According to our assumptions, the pressure in supermassive stars is dominated by radiation pressure

$$P \approx P_r = \tfrac{1}{3}aT^4\,. \tag{12}$$

In the diffusion approximation, the radiation flux is everywhere given by

$$\boldsymbol{F} = -\frac{1}{3\kappa\rho}\nabla U\,. \tag{13}$$

Here U is the energy density of the radiation,

$$U = aT^4 = 3P\,, \tag{14}$$

and κ is the opacity (which we assume to be dominated by electron scattering, $\kappa = \kappa_{\rm es}$). Inserting equations (13) and

(14) into the equation of hydrostatic equilibrium (4) yields

$$\kappa \boldsymbol{F} = \nabla(\Phi + \Phi_c) . \quad (15)$$

In polar coordinates in an orthonormal basis, the magnitude of the flux is

$$F = (F_{\hat{r}}^2 + F_{\hat{\theta}}^2)^{1/2} . \quad (16)$$

Evaluating the gradients of Φ and Φ_c in the envelope yields

$$F = \frac{M}{\kappa r^2}\left[1 - 2\,\frac{\Omega^2 \sin^2\theta}{M/r^3} + \left(\frac{\Omega^2 \sin\theta}{M/r^3}\right)^2\right]^{1/2} . \quad (17)$$

Introducing the dimensionless spin and radius parameters

$$\alpha \equiv \frac{\Omega}{\Omega_{\rm shedd}} \quad (18)$$

and

$$z \equiv \frac{r}{R_p} , \quad (19)$$

and denoting

$$F_{\rm Edd} = \frac{M}{\kappa r^2} , \quad (20)$$

the usual Eddington flux from a spherical star, we can rewrite equation (17) as

$$\frac{F}{F_{\rm Edd}} = [1 - 2(\tfrac{2}{3})^3\alpha^2 z^3 \sin^2\theta + (\tfrac{2}{3})^6\alpha^4 z^6 \sin^2\theta]^{1/2} . \quad (21)$$

Note that, from equation (7), the surface of the star is defined by

$$\Phi + \Phi_c - H = 0 , \quad (22)$$

or equivalently[2]

$$\tfrac{4}{27}\alpha^2 z^3 \sin^2\theta - z + 1 = 0 . \quad (23)$$

Given α and θ, the value of z on the surface can be found by solving this cubic equation. Equations (15) and (22) immediately imply that the flux is normal to the surface of the star.

Evaluating equation (21) at the surface, we plot the emergent flux F as a function of θ for different values of α in Figure 1. Note that at the mass-shedding limit, when $\alpha = 1$, the flux vanishes at the equator (where $z = 3/2$ and $\sin\theta = 1$). This, of course, is an immediate consequence of hydrostatic equilibrium: at mass shedding, the centrifugal force exactly balances the gravitational force at the equator, so that the pressure gradient vanishes (eq. [4]). For radiation-dominated stars, equation (15) then implies that the flux has to vanish.

The total luminosity can be found by integrating

$$L = \int_{\mathscr{A}} \boldsymbol{F} \cdot d\mathscr{A} = \int_{\mathscr{A}} F\, d\mathscr{A} \quad (24)$$

over the surface $\mathscr{A}$ of the star. The surface element $d\mathscr{A}$ can be written

$$d\mathscr{A} = 2\pi r \sin\theta\, ds = 2\pi r \sin\theta (dr^2 + r^2\, d\theta^2)^{1/2}$$
$$= 2\pi r^2 \sin\theta\, d\theta\left(1 + \frac{1}{r^2}\frac{dr}{d\theta}\right)^{1/2} \quad (25)$$

or

$$d\mathscr{A} = 2\pi r^2\, d\mu\left[1 + (1 - \mu^2)\,\frac{z'}{z}\right]^{1/2} . \quad (26)$$

Here we have introduced

$$\mu \equiv \cos\theta \quad (27)$$

and

$$z' \equiv \frac{dz}{d\mu} . \quad (28)$$

Differentiating equation (23), the latter can be expressed as

$$\frac{z'}{z} = \frac{8}{27}\,\frac{z^3\alpha^2\mu}{2z - 3} . \quad (29)$$

Putting the pieces together, we find that the luminosity is given by

$$L = 2\int_0^1 2\pi r^2\, d\mu\left[1 + (1 - \mu^2)\left(\frac{z'}{z}\right)^2\right]^{1/2} F \quad (30)$$

or

$$\frac{L}{L_{\rm Edd}} = \int_0^1 d\mu\left[1 + (1 - \mu^2)\left(\frac{z'}{z}\right)^2\right]^{1/2}$$
$$\times\left[1 - 2\left(\frac{2}{3}\right)^3\alpha^2 z^3(1 - \mu^2) + \left(\frac{2}{3}\right)^6\alpha^4 z^6(1 - \mu^2)\right]^{1/2} , \quad (31)$$

where $L_{\rm Edd}$ is the usual Eddington luminosity

$$L_{\rm Edd} = \frac{4\pi M}{\kappa} . \quad (32)$$

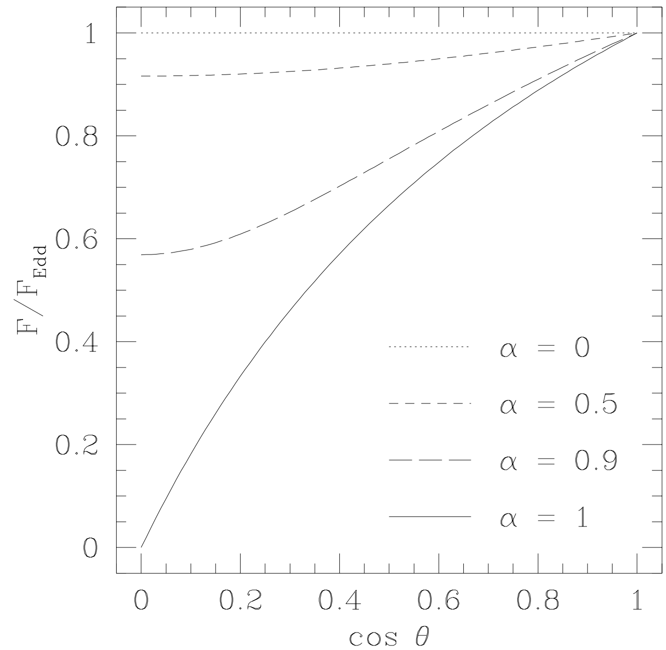

FIG. 1.—The flux F as a function of polar angle for different values of $\alpha = \Omega/\Omega_{\rm shedd}$. Note that, at the mass-shedding limit $\alpha = 1$, the flux vanishes on the equator.

[2] Expanding eqs. (17) and (23) to lowest order in α^2 shows that they are in perfect agreement with eqs. (39) and (30) in Kippenhahn (1977) for uniform rotation. Kippenhahn's treatment allows for nonuniform rotation but is restricted to slow rotation.

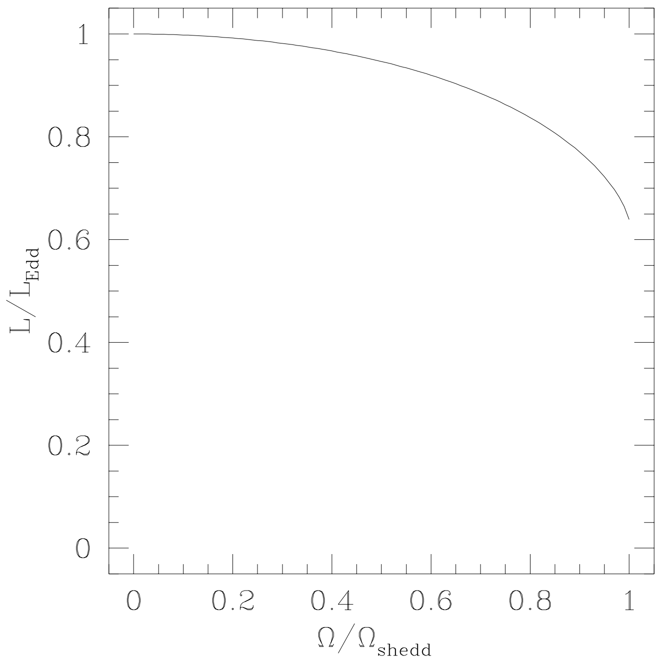

FIG. 2.—Luminosity L as a function of the orbital velocity Ω. At the mass-shedding limit, the luminosity is reduced by 36%.

It proves most convenient to evaluate equation (31) numerically. In Figure 2, we plot the resulting luminosity L as a function of the spin parameter α. Obviously, for nonrotating stars with $\alpha = 0$, we recover $L = L_{\rm Edd}$. For maximally rotating stars, however, the luminosity is reduced by about 36%:

$$L_{\rm shedd} = 0.639 L_{\rm Edd} . \tag{33}$$

Accordingly, adopting the Eddington luminosity for a supermassive star that evolves along the mass-shedding limit would underestimate its lifetime by about 36%.

5. DISCUSSION

We find that the luminosity of an SMS rotating at breakup velocity is reduced by about 36% compared with the luminosity of a nonrotating SMS of the same mass.

It is difficult to compare this result with previous numerical calculations of massive, rotating stars, which are summarized in Table 12.1 in Tassoul (1978; compare the discussion in Kippenhahn 1977). No calculations seem to have been performed for stellar masses greater than 62.7 $M_\odot$. The luminosities of these 62.7 $M_\odot$ models at breakup velocity are indeed reduced below the nonrotating, spherical luminosities, but only by 7%, much less than what we find. However, the physical conditions in 62.7 $M_\odot$ stars are very different from those in SMSs and do not satisfy our assumptions (see § 2). For example, at these moderate masses, the stars are not dominated by radiation pressure; according to equation (1), $\beta \approx 1$ for these stars and β varies with both the location in the star and the orbital velocity.[3] Also, the total opacity, at least close to the surface of the moderate-mass stars considered previously, is no longer dominated by electron scattering and contains non-negligible contributions from bound-bound and bound-free absorption. These contributions introduce a dependence on density, so our assumption 4 no longer holds. More specifically, spinning up the star may decrease the density in the envelope and therefore decrease the opacity and hence increase the luminosity. This would partly compensate for the decrease in the luminosity due to rotation and would decrease the effect that we find for true SMSs. Finally, we note that moderate-mass stars are not fully convective.

Our result is important because the luminosity determines the timescale of the evolution and hence the lifetime of SMSs, which are believed to evolve along a mass-shedding sequence. This lifetime has been used in several calculations of SMS evolution. We adopt our new result in Paper II, where we analyze the secular evolution of SMSs up to the onset of radial instability.

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[3] Note that, if β were strictly independent of position and spin rate, the right-hand sides of eqs. (20) and (32) would be reduced by a constant factor of $(\beta + 1)^{-1}$, and the luminosity of a rotating star (eq. [31]) would still be reduced by the same amount below the luminosity of its nonrotating counterpart.

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