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COSMIC MICROWAVE BACKGROUND ANISOTROPIES IN THE COLD DARK MATTER MODEL: A COVARIANT AND GAUGE-INVARIANT APPROACH

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ABSTRACT

We present a fully covariant and gauge-invariant calculation of the evolution of anisotropies in the cosmic microwave background radiation. We use the physically appealing covariant approach to cosmological perturbations, which ensures that all variables are gauge-invariant and have a clear physical interpretation. We derive the complete set of frame-independent linearized equations describing the (Boltzmann) evolution of anisotropy and inhomogeneity in an almost Friedmann-Robertson-Walker cold dark matter (CDM) universe. These equations include the contributions of scalar, vector, and tensor modes in a unified manner. Frame-independent equations for scalar and tensor perturbations, which are valid for any value of the background curvature, are obtained straightforwardly from the complete set of equations. We discuss the scalar equations in detail, including the integral solution and relation with the line-of-sight approach, analytic solutions in the early radiation-dominated era, and the numerical solution in the standard CDM model. Our results confirm those obtained by other groups, who have worked carefully with noncovariant methods in specific gauges, but ours are derived here in a completely transparent fashion.

Subject headings: cosmic microwave background — cosmology: theory — gravitation — large-scale structure of universe

1. INTRODUCTION

The cosmic microwave background radiation (CMB) occupies a central role in modern cosmology. It provides us with a unique record of conditions along our past light cone back to the epoch of decoupling (last scattering), when the optical depth to Thomson scattering rises suddenly because of hydrogen recombination. Accurate observations of the CMB anisotropy should allow us to distinguish between models of structure formation and, in the case of nonseeded models, to infer the spectrum of initial perturbations in the early universe. Essential to this program is the accurate and reliable calculation of the anisotropy predicted in viable cosmological models.

Such calculations have a long history, beginning with Sachs & Wolfe (1967), who investigated the anisotropy on large scales ($\gtrsim 1°$) by calculating the redshift back to last scattering along null geodesics in a perturbed universe. On smaller angular scales one must address the detailed local processes occurring in the electron/baryon plasma prior to recombination and the effects of noninstantaneous last scattering. These processes, which give rise to a wealth of structure in the CMB power spectrum on intermediate scales and damping on small scales (see, for example, Silk 1967, 1968), are best addressed by following the photon distribution function directly from an early epoch in the history of the universe to the current point of observation. This requires a numerical integration of the Boltzmann equation, and it has been carried out by many groups, of which Peebles & Yu (1970), Bond & Efstathiou (1984, 1987), Hu & Sugiyama (1995), Ma & Bertschinger (1995), Seljak & Zaldarriaga (1996) is a representative sample.

The calculation of CMB anisotropies is simple in principle but is plagued with subtle gauge issues in reality (Stoeger, Ellis, & Schmidt 1991; Stoeger et al. 1995; Challinor & Lasenby 1998). These problems arise because of the gauge freedom in specifying a map Φ between the real universe (denoted by S) and the unperturbed background model (denoted by $\bar{S}$; Ellis & Bruni 1989), which is usually taken to be a Friedmann-Robertson-Walker (FRW) universe. The map Φ identifies points in the real universe with points in the background model, thus defining the perturbation in any quantity of interest. For example, for the density ρ as measured by some physically defined observer, the perturbation at $x \in S$ is defined to be $\delta\rho(x) \equiv \rho(x) - \bar{\rho}(\bar{x})$, where $\bar{\rho}$ is the equivalent density in the background model and x maps to $\bar{x}$ under Φ. The map Φ is usually (partially) specified by imposing coordinate conditions in S and $\bar{S}$. Any residual freedom in the map Φ after the imposition of the coordinate conditions (gauge-fixing) gives rise to the following gauge problems: (1) the map cannot be reconstructed from observations in S alone, so that quantities such as the density perturbation, which depend on the specific map Φ, are necessarily not observable; and (2) if the residual gauge freedom allows points in $\bar{S}$ to be mapped to physically inequivalent points in $\bar{S}$ in the limit that $S = \bar{S}$, then unphysical gauge mode solutions to the linearized perturbation equations will exist.

There are several ways to deal with the gauge problems described above. In the earliest approach (Lifshitz 1946), one retains the residual gauge freedom (in the synchronous gauge) but keeps track of it so that gauge mode solutions can be eliminated. Furthermore, the final results of such a calculation must be expressed in terms of the physically relevant, gauge-invariant quantities. Although there is nothing fundamentally wrong with this approach if carried out correctly, it suffers from a long history littered with confusion and errors. The need to express results in terms of gauge-invariant variables suggests that it might be beneficial to employ such variables all along as the dynamical degrees of freedom in the calculation. A further advantage of such an approach is that gauge modes are automatically eliminated from the perturbation equations when expressed in terms of gauge-invariant variables. This is the approach adopted by Bardeen (1980), who showed how to construct gauge-invariant variables for scalar, vector, and tensor modes in linearized perturbation theory by taking suitable linear combinations of the gauge-dependent perturbations in the metric and matter variables. This approach has been used in several calculations of the CMB anisotropy (see, for example, Abbott & Schaefer 1986; Panek 1986). However, the Bardeen variables are not entirely satisfactory. The approach is inherently linear, so that the variables are only defined for small departures from FRW symmetry. Furthermore, the approach assumes a nonlocal decomposition of the perturbations into scalar, vector, and tensor modes at the outset, each of which is then treated independently. As a result, the Bardeen variables are only gauge-invariant for the restricted class of gauge transformations that respect the scalar, vector, and tensor splitting. Finally, although the Bardeen variables are gauge-invariant, they are not physically transparent in that, in a general gauge, they do not characterize the perturbations in a manner that is amenable to simple physical interpretation.

An alternative scheme for the gauge-invariant treatment of cosmological perturbations was given by Ellis & Bruni (1989; see also Ellis, Hwang, & Bruni 1989), who built upon earlier work by Hawking (1966). In this approach, which is derived from the covariant approach to cosmology/hydrodynamics of Ehlers and Ellis (Ehlers 1993; Ellis 1971), the perturbations are described by gauge-invariant variables that are covariantly defined in the real universe. This ensures that the variables have simple physical interpretations in terms of the inhomogeneity and anisotropy of the universe. Since the definition of the covariant variables does not assume any linearization, exact equations can be found for their evolution, which can then be linearized around the chosen background model. Furthermore, the covariant approach does not employ the nonlocal decomposition into scalar, vector, or tensor modes at a fundamental level. If required, the decomposition can be performed at a late stage in the calculation to aid solving the equations. Even if one denies that working with gauge-invariant variables is a significant advantage, the key advantage of the covariant approach, however, is that one is able to work exclusively with physically relevant quantities, satisfying equations that make manifest their physical consequences.

The covariant and gauge-invariant approach has already been applied to the line-of-sight calculation of CMB anisotropies under the instantaneous recombination approximation (Dunsby 1997; Challinor & Lasenby 1998), and it has been used to obtain model-independent limits on the inhomogeneity and anisotropy from measurements of the CMB anisotropy on large scales (Maartens, Ellis, & Stoeger 1995). In this paper, we extend the methodology developed in these earlier papers to give a full kinetic theory calculation of CMB anisotropies valid on all angular scales. Our motivation for reconsidering this problem is twofold. First, it is our belief that the covariant and gauge-invariant description of cosmological perturbations provides a powerful set of tools for the formulation of the basic perturbation equations and their subsequent interpretation that are superior to the techniques usually employed in such calculations for the reasons discussed above. In particular, by applying covariant methods for the problem of the generation of CMB anisotropies, we can expect the same advantages of physical clarity and unification that have already been demonstrated in other areas, (Ellis et al. 1989; Bruni, Ellis, & Dunsby 1992b; Dunsby, Bruni, & Ellis 1992; Dunsby, Bassett, & Ellis 1996; Tsagas & Barrow 1997). The approach described here brings the underlying physics to the fore and can only help to consolidate our rapidly growing understanding of the physics of CMB anisotropies. Furthermore, although we only consider the linearized calculation here, the extension of these methods to the full nonlinear case is quite straightforward (Maartens, Gebbie, & Ellis 1998). Our second motivation is to perform an independent verification of the results of other groups (for example, Ma & Bertschinger 1995) with a methodology that is free from any of the gauge ambiguities that have caused problems and confusion in the past. Given the potential impact on cosmology of the next generation of CMB data, we believe that the above comments provide ample justification for reconsidering this problem.

For definiteness we consider the cold dark matter (CDM) model, although the methods we describe are straightforward to extend to other models. We have endeavored to make this paper reasonably self-contained, so we begin with a brief overview of the covariant approach to cosmology and define the key variables we use to characterize the perturbations in § 2. In § 3 we go on to present a complete set of frame-independent equations describing the evolution of the matter components and radiation in an almost FRW universe (with arbitrary spatial curvature). These equations, which employ only covariantly defined, gauge-invariant variables, are independent of any harmonic analysis; they describe scalar, vector, and tensor perturbations in a unified manner. Many of the equations have simple Newtonian analogues, and their physical consequences are far more transparent than the equations that underlie the metric-based approaches. Equations pertinent to scalar (see § 5) and tensor modes (see § 7) can be obtained from the full set of equations with very little effort and are useful at this late stage in the calculation as an aid to solving the linearized equations. A significant feature of this approach is that a covariant angular decomposition of the distribution functions is made early on in the calculation before any splitting into scalar, vector, and tensor modes. This allows scalar, vector, and tensor modes to be treated in a more unified manner. In particular, the azimuthal dependence of the moments of the distribution functions does not have to be put in by hand (after inspection of the azimuthal dependence of the other terms in the Boltzmann equation), as happens in most metric-based calculations. This is particularly significant for tensor modes where the required azimuthal dependence is nontrivial and is different for the two polarizations of gravitational waves. We consider the equations for scalar modes in considerable detail. We present the integral solution of the Boltzmann multipole equations in a $K = 0$ almost FRW universe, and we discuss the relation between line-of-sight methods (which employ lightlike integrations along the light cone) and the Boltzmann multipole approach (where a timelike integration is performed). We derive analytic solutions for scalar modes in the early radiation-dominated universe, which are used as initial conditions for the numerical solution of the scalar equations, the results of which we describe in § 6. In § 7 we give a brief discussion of the tensor equations in the covariant approach. The covariant angular decomposition naturally gives rise to a set of variables that describe the temperature anisotropy in a more direct manner than in the conventional metric-based approaches. This is particularly apparent for tensor perturbations, where the CMB power spectrum at a given multipole l is determined by the $l-2$, l, and $(l+2)$th moments of the conventional decomposition of the photon distribution function, which obscures the physical interpretation of these moments. Finally, we end with our conclusions in § 8. Ultimately, our results confirm those of other groups (for example, Ma & Bertschinger 1995) who have performed similar calculations by working carefully in specific gauges, but ours are obtained with a unified methodology that is more physically transparent and less prone to lead to confusion over subtle gauge effects.

We employ standard general relativity and use a $(+---)$ metric signature. Our conventions for the Riemann and Ricci tensors are fixed by $[\nabla_a, \nabla_b]u^c = -\mathscr{R}_{abc}{}^c u^d$ and $\mathscr{R}_{ab} \equiv \mathscr{R}_{abc}{}^c$. Parentheses around indices denote symmetrization on the indices enclosed, and square brackets denote antisymmetrization. We use units with $c = G = 1$ throughout and a unit of distance of Mpc for numerical work.

2. THE COVARIANT APPROACH TO COSMOLOGY

In this section, we summarize the covariant approach to cosmology (Ehlers 1993; Ellis 1971; Hawking 1966) and the gauge-invariant perturbation theory of Ellis & Bruni (1989) that is derived from it. We begin by choosing a velocity field u^a, which satisfies the following criterion: *the velocity must be physically defined in such a way that it reduces to the four-velocity of the fundamental observers in the FRW limit.* This restriction on u^a is essential to ensure gauge-invariance of the Ellis & Bruni (1989) perturbation theory. Note that, in a general perturbed spacetime, there is no unique choice for u^a. Acceptable choices for u^a include the four-velocity of a given matter component and the timelike eigenvector of the stress-energy tensor. In the covariant approach to perturbations in cosmology, covariant variables are introduced that describe the inhomogeneity and anisotropy of the universe. These variables employ the velocity field u^a in their definition, and so, in a given spacetime, their values depend on how we choose u^a (the exact transformation laws are given in Maartens et al. 1998). For a given choice of u^a, the covariant variables, defined below, describe the results of observations made by observers comoving with the velocity u^a, and their frame-dependence reflects the fact that the observations depend on the velocity of the observer. It might be thought that the freedom in the choice of velocity would introduce similar ambiguities as the choice of map Φ does in conventional approaches. However, this is not the case because of the restriction on u^a that we emphasized above. It is certainly true that, with a suitable choice of u^a, we can eliminate some aspect of the inhomogeneity and isotropy observed. For example, we can always choose u^a so that the CMB dipole vanishes. However, in a given spacetime, the covariant variables cannot be forced to take arbitrary values through some particular choice of u^a. In particular, if, for some timelike velocity field (not necessarily restricted to satisfy the criterion emphasized above), *all* of the gauge-invariant variables defined below vanish identically, then the universe is necessarily FRW. This gives a covariant condition that characterizes the FRW limit, but note that, if u^a is unrestricted, we could have the situation where the universe is FRW; however, we are not viewing it from the perspective of the fundamental observers, and so some of the variables would not vanish. (This is similar to the presence of gauge mode solutions in the metric-based approach.) However, if we ensure that u^a is defined physically, so that in the FRW limit it necessarily reduces to the velocity of the fundamental observers, this situation cannot arise, and the variables used to characterize the anisotropy and inhomogeneity are genuinely gauge-invariant.

We refer to the choice of velocity as a frame choice. In this paper, we defer making a frame choice until we have derived all the relevant equations, so that we have available a set of equations valid for any choice of u^a. However, to actually solve the equations, we must make a definite choice for u^a (the system of equations is underdetermined until such a choice is made). Here, it will be convenient to choose u^a to coincide with the velocity of the CDM component, since u^a is then geodesic.

The velocity u^a defines a projection tensor h_{ab}, which projects into the space perpendicular to u^a (the instantaneous rest space of observers moving with velocity u^a):

$$h_{ab} \equiv g_{ab} - u_a u_b \,, \tag{1}$$

where g_{ab} is the spacetime metric. Since h_{ab} is a projection tensor, it satisfies

$$h_{ab} = h_{(ab)}, \quad h_a^c h_{cb} = h_{ab}, \quad h_a^a = 3, \quad u^a h_{ab} = 0 \,. \tag{2}$$

We employ the projection tensor to define a spatial covariant derivative ${}^{(3)}\nabla^a$, which acting on a tensor $T^{b\ldots c}{}_{d\ldots e}$ returns a tensor that is orthogonal to u^a on every index:

$${}^{(3)}\nabla^a T^{b\ldots c}{}_{d\ldots e} \equiv h^a_p h^b_r \ldots h^c_s h^t_d \ldots h^u_e \nabla^p T^{r\ldots s}{}_{t\ldots u} \,, \tag{3}$$

where ∇^a denotes the usual covariant derivative. If the velocity field u^a has vanishing vorticity (see later) ${}^{(3)}\nabla^a$ reduces to the covariant derivative in the hypersurfaces orthogonal to u^a.

The covariant derivative of the velocity decomposes as

$$\nabla_a u_b = \varpi_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} + u_a w_b \,, \tag{4}$$

where $w_a \equiv u^b \nabla_b u_a$ is the acceleration (which satisfies $u^a w_a = 0$), the scalar $\theta \equiv \nabla^a u_a = 3H$ is the volume expansion rate (H is the local Hubble parameter), $\varpi_{ab} \equiv \nabla_{[a} u_{b]} + w_{[a} u_{b]}$ is the vorticity tensor (which satisfies $\varpi_{ab} = \varpi_{[ab]}$ and $u^a \varpi_{ab} = 0$), and $\sigma_{ab} \equiv {}^{(3)}\nabla_{(a} u_{b)} - \theta h_{ab}/3$ is the shear tensor (which satisfies $\sigma_{ab} = \sigma_{(ab)}$, $\sigma^a_a = 0$, and $u^a \sigma_{ab} = 0$). The nontrivial integrability condition

$${}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}\phi = -\varpi_{ab}\dot{\phi} \tag{5}$$

for any scalar field ϕ—where an overdot denotes the action of the operator $u^a\nabla_a$—follows from the Ricci identity. Note in particular that in an evolving universe ($\dot{\phi} \neq 0$), spatial gradients are necessarily nonvanishing in the presence of vorticity. This behavior, which is a consequence of there being no global hypersurfaces that are everywhere orthogonal to u^a if the vorticity does not vanish, is central to the discussion of vector perturbations. For vanishing vorticity, the 3 Ricci scalar (or intrinsic-curvature scalar) ${}^{(3)}\mathscr{R}$ in the hypersurfaces orthogonal to u^a evaluates to

$${}^{(3)}\mathscr{R} = 2\kappa\rho - \frac{2}{3}\theta^2 + \sigma_{ab}\sigma^{ab} \,, \tag{6}$$

where ρ is the total energy density in the u^a frame.

In an exact FRW universe, the vorticity, shear, and acceleration vanish identically. In an almost FRW universe, these variables, when suitably normalized to make them dimensionless, are regarded as first order in a smallness parameter ϵ (Maartens et al. 1995). We use the convenient notation, $O(n)$, to denote that a variable is $O(\epsilon^n)$. We assume that products of first-order variables can be neglected from any expression in the linearized calculation considered here.

Other first-order variables may be obtained by taking the spatial gradient of scalar quantities. Such quantities are gauge-invariant by construction, since they vanish identically in an exact FRW universe. We shall make use of the comoving fractional spatial gradient of the density $\rho^{(i)}$ of a species i,

$$\mathscr{X}_a^{(i)} \equiv \frac{S}{\rho^{(i)}} \, {}^{(3)}\nabla_a \rho^{(i)} , \tag{7}$$

and the comoving spatial gradient of the expansion

$$\mathscr{Z}_a \equiv S^{(3)}\nabla_a \theta . \tag{8}$$

The scalar S is a local scale factor satisfying

$$\dot{S} \equiv u^a \nabla_a S = HS, \qquad {}^{(3)}\nabla^a S = O(1) , \tag{9}$$

which removes the effects of the expansion from the spatial gradients defined above. The vector $\mathscr{X}_a^{(i)}$ is a manifestly covariant and gauge-invariant characterization of the density inhomogeneity.

The matter stress-energy tensor $\mathscr{T}_{ab}$ decomposes with respect to u^a as

$$\mathscr{T}_{ab} \equiv \rho u_a u_b + 2u_{(a} q_{b)} - p h_{ab} + \pi_{ab} , \tag{10}$$

where $\rho \equiv \mathscr{T}_{ab} u^a u^b$ is the density of matter (measured by a comoving observer), $q_a \equiv h_a^b \mathscr{T}_{bc} u^c$ is the energy (or heat) flux and is orthogonal to u^a, $p \equiv -h_{ab} \mathscr{T}^{ab}/3$ is the isotropic pressure, and the symmetric traceless tensor $\pi_{ab} \equiv h_a^c h_b^d \mathscr{T}_{cd} + p h_{ab}$ is the anisotropic stress, which is also orthogonal to u^a. In an exact FRW universe, isotropy restricts $\mathscr{T}_{ab}$ to perfect-fluid form, so that in an almost FRW universe the heat flux and isotropic stress may be treated as first-order variables. The final first-order gauge-invariant variables we require derive from the Weyl tensor $\mathscr{W}_{abcd}$, which vanishes in an exact FRW universe because of isotropy. The electric and magnetic parts of the Weyl tensor, denoted by $\mathscr{E}_{ab}$ and $\mathscr{B}_{ab}$, respectively, are symmetric traceless tensors, orthogonal to u^a, which we define by

$$\mathscr{E}_{ab} \equiv u^c u^d \mathscr{W}_{acbd} , \tag{11}$$

$$\mathscr{B}_{ab} \equiv -\frac{1}{2} u^c u^d \eta_{ac}{}^{ef} \mathscr{W}_{efbd} , \tag{12}$$

where η_{abcd} is the covariant permutation tensor with $\eta_{0123} = -(-g)^{1/2}$.

2.1. *Linearized Perturbation Equations for the Total Matter Variables*

Exact equations describing the propagation of the total matter variables (such as the total density ρ), the kinematic variables, and the electric and magnetic parts of the Weyl tensor and the constraints between them follow from the Ricci identity and the Bianchi identity. The Riemann tensor is expressed in terms of $\mathscr{E}_{ab}$, $\mathscr{B}_{ab}$ and the Ricci tensor, $\mathscr{R}_{ab}$, and the Einstein equation is used to substitute for the Ricci tensor in terms of the matter stress-energy tensor. On linearizing the equations that result from this procedure (Bruni, Dunsby, & Ellis 1992a), one obtains five constraint equations,

$$\mathscr{B}_{ab} + ({}^{(3)}\nabla^c \varpi_{d(a} + {}^{(3)}\nabla^c \sigma_{d(a}) \eta_{b)ce}{}^d u^e = 0 , \tag{13}$$

$${}^{(3)}\nabla^b \mathscr{B}_{ab} - \tfrac{1}{2}\kappa[(\rho + p)\eta_{ab}{}^{cd} u^b \varpi_{cd} + \eta_{abcd} u^{b(3)}\nabla^c q^d] = 0 , \tag{14}$$

$${}^{(3)}\nabla^b \mathscr{E}_{ab} - \frac{1}{6}\kappa(2^{(3)}\nabla_a \rho + 2\theta q_a + 3^{(3)}\nabla^b \pi_{ab}) = 0 , \tag{15}$$

$${}^{(3)}\nabla^b \varpi_{ab} + {}^{(3)}\nabla^b \sigma_{ab} - \frac{2}{3}\,{}^{(3)}\nabla_a \theta - \kappa q_a = 0 , \tag{16}$$

$${}^{(3)}\nabla^c(\eta_{abcd} u^d \varpi^{ab}) = 0 , \tag{17}$$

and seven propagation equations,

$$\dot{\mathscr{E}}_{ab} + \theta \mathscr{E}_{ab} + {}^{(3)}\nabla^c \mathscr{B}_{d(a} \eta_{b)ce}{}^d u^e + \frac{1}{6}\kappa\Big[3(\rho + p)\sigma_{ab} + 3\Big({}^{(3)}\nabla_{(a} q_{b)} - \frac{1}{3} h_{ab}\,{}^{(3)}\nabla^c q_c\Big) - 3\dot{\pi}_{ab} - \theta \pi_{ab}\Big] = 0 , \tag{18}$$

$$\dot{\mathscr{B}}_{ab} + \theta \mathscr{B}_{ab} - ({}^{(3)}\nabla^c \mathscr{E}_{d(a} + \tfrac{1}{2}\kappa^{(3)}\nabla^c \pi_{d(a}) \eta_{b)ce}{}^d u^e = 0 , \tag{19}$$

$$\dot{\sigma}_{ab} + \frac{2}{3}\theta \sigma_{ab} - \Big({}^{(3)}\nabla_{(a} w_{b)} - \frac{1}{3} h_{ab}\,{}^{(3)}\nabla^c w_c\Big) + \mathscr{E}_{ab} + \tfrac{1}{2}\kappa \pi_{ab} = 0 , \tag{20}$$

$$\dot{\varpi}_{ab} - {}^{(3)}\nabla_{[a} w_{b]} + \frac{2}{3}\theta \varpi_{ab} = 0 , \tag{21}$$

$$\dot{q}_a + \frac{4}{3}\theta q_a + (\rho + p) w_a + {}^{(3)}\nabla^b \pi_{ab} - {}^{(3)}\nabla_a p = 0 , \tag{22}$$

$$\dot{\theta} + \frac{1}{3}\theta^2 - {}^{(3)}\nabla^a w_a + \tfrac{1}{2}\kappa(\rho + 3p) = 0 , \tag{23}$$

$$\dot{\rho} + \theta(\rho + p) + {}^{(3)}\nabla^a q_a = 0 , \tag{24}$$

where $\dot{T}_{ab\ldots c} \equiv u^d \nabla_d T_{ab\ldots c}$. The constraint equations do not involve time derivatives, and so they serve to constrain initial data for the problem. The propagation equations are consistent with the constraint equations in the sense that the constraints are preserved in time by the propagation equations if they are satisfied initially. The consistency of the exact equations follows from their derivation from the exact field equations and is preserved by the linearization procedure. Including a cosmological constant Λ in the above equations is straightforward; one adds a contribution Λ/κ to the total density ρ and subtracts the same term from the total pressure p.

Many of the equations given above have simple Newtonian analogs (Ellis 1971) and thus are simple to interpret physically. The analogs arise because many of the covariantly defined variables have counterparts in (self-gravitating) Newtonian hydrodynamics. An important exception is that there is no Newtonian analog of the magnetic part of the Weyl tensor (the electric part is analogous to the tidal part of the Newtonian gravitational potential) and no equation analogous to the $\dot{\mathscr{E}}_{ab}$ propagation equation (Ellis & Dunsby 1997a). These exceptions arise because of the instantaneous interaction in Newtonian gravity, which excludes the possibility of gravitational wave solutions to the Newtonian equations. However, there is a close analogy between the $\dot{\mathscr{E}}_{ab}$ and $\dot{\mathscr{B}}_{ab}$ propagation equations and the constraint equations (14) and (15) and Maxwell's equations split with respect to an arbitrary timelike velocity field (see, for example, Maartens & Bassett 1998).

There is some redundancy in the full set of linear equations. (See Maartens 1997 for a discussion of the redundancy in the exact nonlinear equations for an irrotational dust universe.) For example, equation (13), which determines $\mathscr{B}_{ab}$ in terms of the vorticity and the shear, along with equation (16) and the integrability condition given as equation (5) imply equation (14). Similarly, equation (19) follows from equation (13), and the propagation equations for the shear (eq. [20]) and the vorticity (eq. [21]). It follows that $\mathscr{B}_{ab}$ may be eliminated from the equations in favor of the vorticity and the shear by making use of equation (13). This elimination is useful when discussing the propagation of vector and tensor modes (see § 7).

The usual Friedmann equations describing homogeneous and isotropic cosmological models are readily obtained from the full set of covariant equations, since in an exact FRW universe the only nontrivial propagation equations are the Raychaudhuri equation (eq. [23]) and the energy conservation equation (eq. [24]), which reduce to the Friedmann equation

$$\dot{H} + H^2 = -\frac{1}{6}\kappa(\rho + 3p) , \tag{25}$$

and the usual equation for the density evolution

$$\dot{\rho} = -3H(\rho + p) . \tag{26}$$

The second Friedmann equation is obtained as a first integral of these two equations:

$$H^2 + \frac{K}{S^2} = \frac{1}{3}\kappa\rho , \tag{27}$$

where $6K/S^2$ is the intrinsic curvature scalar of the surfaces of constant cosmic time.

The fractional comoving spatial gradient of the density, $\mathscr{X}_a$, and the comoving spatial gradient of the expansion rate, $\mathscr{Z}_a$, are the key variables in the covariant discussion of the growth of inhomogeneity in the universe (Ellis & Bruni 1989; Ellis et al. 1989). It is useful to have available the propagation equations for these variables. For $\mathscr{X}_a$, we take the spatial gradient of the density evolution equation (eq. [24]) and commute the space and time derivatives to obtain

$$\rho\dot{\mathscr{X}}_a + (\rho + p)(\mathscr{Z}_a - S\theta w_a) + S^{(3)}\nabla_a{}^{(3)}\nabla^b q_b + S\theta^{(3)}\nabla_a p - \theta p \mathscr{X}_a = 0 . \tag{28}$$

For $\mathscr{Z}_a$, we take the spatial gradient of the Raychaudhuri equation (eq. [23]), which gives

$$\dot{\mathscr{Z}}_a + \frac{2}{3}\theta\mathscr{Z}_a - S\left[\frac{1}{3}\theta^2 + \frac{1}{2}\kappa(\rho + 3p)\right]w_a + \frac{1}{2}\kappa S({}^{(3)}\nabla_a\rho + 3{}^{(3)}\nabla_a p) - S^{(3)}\nabla_a{}^{(3)}\nabla^b w_b = 0 , \tag{29}$$

For an ideal fluid ($q_a = \pi_{ab} = 0$ when we choose u^a to be the fluid velocity) with a barotropic equation of state $p = p(\rho)$, the propagation equations for $\mathscr{X}_a$ and $\mathscr{Z}_a$ combine with the momentum conservation equation (eq. [22]) and the integrability condition (given as eq. [5]) to give an inhomogeneous second-order equation for $\mathscr{X}_a$ (Ellis, Bruni, & Hwang 1990). For a simple equation of state $p = (\gamma - 1)\rho$, where γ is a constant, the second-order equation is

$$\ddot{\mathscr{X}}_a + \left(\frac{5}{3} - \gamma\right)\theta\dot{\mathscr{X}}_a + \frac{1}{2}(\gamma - 2)(3\gamma - 2)\left(\frac{1}{3}\theta^2 + \frac{3K}{S^2}\right)\mathscr{X}_a + (\gamma - 1)\left({}^{(3)}\nabla^2\mathscr{X}_a + \frac{2K}{S^2}\mathscr{X}_a\right) + 2\gamma(\gamma - 1)S\theta^{(3)}\nabla^b\varpi_{ab} = 0 . \tag{30}$$

From this equation, it is straightforward to recover the usual results for the growth of inhomogeneities in an almost FRW universe (Ellis & Bruni 1989). The inhomogeneous term describes the coupling between the vorticity and the spatial gradient of the density, which arises from the lack of global hypersurfaces orthogonal to u^a in the presence of nonvanishing vorticity. In reality, the universe cannot be described by a barotropic perfect fluid. A more careful analysis of the individual matter components is required, which we present in the next section.

3. EQUATIONS FOR INDIVIDUAL MATTER COMPONENTS

In this paper we concentrate on CDM models, so the matter components that we must consider are the photons and neutrinos, which are the only relativistic species, and the tightly coupled baryon/electron system and the CDM, which are both nonrelativistic over the epoch of interest. We consider the description of each of these components separately in this section.

3.1. *Photons*

In relativistic kinetic theory (see, for example, Misner, Thorne, & Wheeler 1973), the photons are described by a scalar-valued distribution function $f^{(\gamma)}(x, p)$. An observer sees $f^{(\gamma)}(x, p)\, d^3x\, d^3p$ photons at the spacetime point x in a proper volume d^3x, with covariant momentum p^a in a proper volume d^3p of momentum space. The photon momentum p^a decomposes with respect to the velocity u^a as

$$p^a = E(u^a + e^a) , \tag{31}$$

where $E = p^a u_a$ is the energy of the photon, as measured by an observer moving with velocity u^a and e_a is a unit spacelike vector that is orthogonal to u^a:

$$e^a e_a = -1 , \qquad e^a u_a = 0 , \tag{32}$$

which describes the propagation direction of the photon in the instantaneous rest space of the observer. With this decomposition of the momentum, we may write the photon distribution function in the form $f^{(\gamma)}(E, e)$ when convenient, where the dependence on spacetime position x has been left implicit. The stress-energy tensor $\mathscr{T}^{(\gamma)}_{ab}$ for the photons may then be written as

$$\mathscr{T}^{(\gamma)}_{ab} = \int dE\, d\Omega\, E f^{(\gamma)}(E, e) p_a p_b , \tag{33}$$

where the measure $d\Omega$ denotes an integral over solid angles. The photon energy density $\rho^{(\gamma)}$, the heat flux $q^{(\gamma)}_a$, and the anisotropic stress $\pi^{(\gamma)}_{ab}$ are given by integrals of the three lowest moments of the photon distribution function:

$$\rho^{(\gamma)} = \int dE\, d\Omega\, E^3 f^{(\gamma)}(E, e) , \tag{34}$$

$$q^{(\gamma)}_a = \int dE\, d\Omega\, E^3 f^{(\gamma)}(E, e) e_a , \tag{35}$$

$$\pi^{(\gamma)}_{ab} = \int dE\, d\Omega\, E^3 f^{(\gamma)}(E, e) e_a e_b + \frac{1}{3}\rho^{(\gamma)} h_{ab} . \tag{36}$$

In the absence of scattering, the photon distribution is conserved in phase space. Denoting the photon position by $x^a(\lambda)$ and the momentum by $p^a(\lambda)$, the path in phase space is described by the equations

$$\frac{dx^a}{d\lambda} = p^a , \tag{37}$$

$$p^a \nabla_a p^b = 0 , \tag{38}$$

where λ is an affine parameter along the null geodesic $x^a(\lambda)$. Denoting the Liouville operator by $\mathscr{L}$, we have

$$\mathscr{L} f^{(\gamma)}(x, p) = \frac{d}{d\lambda} f^{(\gamma)}[x^a(\lambda), p^a(\lambda)] = 0 , \tag{39}$$

in the absence of collisions. Over the epoch of interest here, the photons are not collisionless but instead are interacting with a thermal distribution of electrons and baryons. The dominant contribution to the scattering comes from Compton scattering off free electrons, which have number density n_e in the baryon/electron rest frame. Since the average energy of a CMB photon is small compared to the electron mass well after electron-positron annihilation, we may approximate the Compton scattering by Thomson scattering. Furthermore, since the kinetic temperature of the electrons (which equals the radiation temperature prior to recombination) is small compared to the electron mass, the electrons are nonrelativistic, and we may ignore the effects of thermal motion of the electrons (in the average rest frame of the baryon/electron system) on the scattering. Our final assumption is to ignore polarization of the radiation. Thomson scattering of an unpolarized but anisotropic distribution of radiation leads to the generation of polarization, which then affects the temperature anisotropy because of the polarization dependence of the Thomson cross section σ_{T}. In this manner, polarization of the CMB is generated through recombination and its neglect leads to errors of a few percent (Hu et al. 1995) in the predicted temperature anisotropy. We hope to develop a covariant version of the radiative-transfer equations, including polarization, in the near future, which should simplify their physical interpretation.

In the presence of scattering, the photon distribution function evolves according to the collisional Boltzmann equation,

$$\mathscr{L} f^{(\gamma)}(x, p) = \mathscr{C} , \tag{40}$$

where the collision operator for Thomson scattering is

$$\mathscr{C} = n_e \sigma_{\mathrm{T}} p^a u_a^{(b)} [f_+^{(\gamma)}(x, p) - f^{(\gamma)}(x, p)] , \tag{41}$$

where $u_a^{(b)}$ is the covariant velocity of the baryon/electron system and $f_+^{(\gamma)}(x, p)$ describes scattering into the phase space element under consideration:

$$f_+^{(\gamma)}(x, p) = \frac{3}{16\pi} \int f^{(\gamma)}(x, p')[1 + (g^{ab} e_a^{(b)} e_b'^{(b)})^2] \, d\Omega_{e'(b)} , \tag{42}$$

where $e_a^{(b)}$ is the photon direction relative to $u_a^{(b)}$,

$$p_a = E^{(b)}(u_a^{(b)} + e_a^{(b)}), \qquad E^{(b)} = p^a u_a^{(b)} , \tag{43}$$

and $e_a'^{(b)}$ is the initial direction (relative to $u_a^{(b)}$) of the photon whose initial momentum is p_a' and final momentum is p_a. We write the baryon velocity in the form

$$u_a^{(b)} = \gamma^{(b)}(u_a + v_a^{(b)}) , \tag{44}$$

where $v_a^{(b)}$ is the relative velocity of the baryons, which satisfies $u^a v_a^{(b)} = 0$, and $\gamma^{(b)} \equiv (1 + g^{ab} v_a^{(b)} v_b^{(b)})^{-1/2}$. Note that to first order we have $u_a^{(b)} = u_a + v_a^{(b)}$, since the relative velocities of the individual matter components are first-order in an almost FRW universe. Multiplying the Boltzmann equation by E^2 and integrating over energies, we find

$$\begin{aligned} \int dE\, E^2 \mathscr{L} f^{(\gamma)}(E, e) &= n_e \sigma_{\mathrm{T}} [\gamma^{(b)}(1 + e^a v_a^{(b)})]^{-3} \\ &\quad \times \int dE^{(b)}\, E^{(b)3} f_+^{(\gamma)}(x, p) - n_e \sigma_{\mathrm{T}} \gamma^{(b)}(1 + e^a v_a^{(b)}) \\ &\quad \times \int dE\, E^3 f^{(\gamma)}(E, e) , \end{aligned} \tag{45}$$

where we have used

$$E^{(b)} = \gamma^{(b)} E(1 + e^a v_a^{(b)}) \tag{46}$$

to replace the integral over E by an integral over $E^{(b)}$ in the first term on the right. This term can be rewritten as an integral over $E'^{(b)} \equiv p'^a u_a^{(b)}$ using the fact that there is no energy transfer in Thomson scattering in the rest frame of the scattering electron, so that

$$\begin{aligned} \int dE^{(b)}\, E^{(b)3} f_+^{(\gamma)}(x, p) &= \frac{3}{16\pi} \\ &\quad \times \int dE'^{(b)} \, d\Omega_{e'(b)} \, E'^{(b)3} [1 + (g^{ab} e_a^{(b)} e_b'^{(b)})^2] f^{(\gamma)}(x, p) . \end{aligned} \tag{47}$$

Using the definition of the radiation stress-energy tensor (eq. 33) in the right-hand side, we have

$$\int dE^{(b)}\, E^{(b)3} f_+^{(\gamma)}(x, p) = \frac{3}{16\pi} g^{ab} g^{cd} \mathscr{T}_{bd}^{(\gamma)} (u_a^{(b)} u_c^{(b)} + e_a^{(b)} e_c^{(b)}) , \tag{48}$$

where $e_a^{(b)}$ can be expressed as

$$e_a^{(b)} = [\gamma^{(b)}(1 + e^c v_c^{(b)})]^{-1}(u_a + e_a) - \gamma^{(b)}(u_a + v_a^{(b)}) . \tag{49}$$

It follows that the energy-integrated Boltzmann equation reduces to

$$\begin{aligned} \int dE\, E^2 \mathscr{L} f^{(\gamma)}(E, e) &= \frac{3}{16\pi} n_e \sigma_{\mathrm{T}} [\gamma^{(b)}(1 + e^f v_f^{(b)})]^{-3} \\ &\quad \times g^{ab} g^{cd} \mathscr{T}_{bd}^{(\gamma)} (u_a^{(b)} u_c^{(b)} + e_a^{(b)} e_c^{(b)}) \\ &\quad - n_e \sigma_{\mathrm{T}} \gamma^{(b)}(1 + e^a v_a^{(b)}) \int dE\, E^3 f^{(\gamma)}(E, e) . \end{aligned} \tag{50}$$

This equation is exact under the assumption of Thomson scattering and the neglect of polarization. Here we shall only require the linearized version of equation (50); for a covariant discussion of the second-order effects in this equation, see Maartens et al. (1998). On linearizing equation (50) around an almost FRW universe, we find

$$\begin{aligned} \int dE\, E^2 \mathscr{L} f^{(\gamma)}(E, e) &= \frac{3}{16\pi} n_e \sigma_{\mathrm{T}} \left[\frac{4}{3} (1 - 4 e^a v_a^{(b)}) \rho^{(\gamma)} + \pi_{ab}^{(\gamma)} e^a e^b \right] \\ &\quad - n_e \sigma_{\mathrm{T}} \int dE\, E^3 f^{(\gamma)}(E, e) . \end{aligned} \tag{51}$$

This covariant form of the Boltzmann equation was used in Challinor & Lasenby (1998) to discuss CMB anisotropies from scalar perturbations on angular scales above the damping scale. Note that the equation is fully covariant with all variables observable in the real universe, is valid for arbitrary type of perturbation (scalar, vector, and tensor), employs no harmonic decomposition, and is valid for any background FRW model.

The numerical solution of the Boltzmann equation (eq. 51) is greatly facilitated by decomposing the equation into covariantly defined angular moments. The majority of recent calculations (for example, Seljak & Zaldarriaga 1996) perform an angular decomposition of the Boltzmann equation after specifying the perturbation type and performing the appropriate harmonic expansions. The procedure is straightforward for scalar perturbations in a $K = 0$ universe, where the Fourier mode of the perturbation in the distribution function may be assumed to be axisymmetric about the wavevector $\boldsymbol{k}$ (this assumption is consistent with the evolution implied by the Boltzmann equation), allowing an angular expansion in Legendre polynomials alone. However, for tensor perturbations the situation is not so straightforward (see, for example, Kosowsky 1996), since the Boltzmann equation does not then support axisymmetric modes. Instead, the necessary azimuthal dependence of the Fourier components of the perturbation in the distribution function, which is different for the two polarizations of the tensor modes, must be put in by hand prior to a Legendre expansion in the polar angle. This procedure may be eliminated by performing a covariant angular expansion of $f^{(\gamma)}(x, p)$ prior to specifying the perturbation type or background FRW model. The covariant (tensor) moment equations that result may then be solved for any type of perturbation (and any background curvature K) by expanding in covariant tensors derived from the appropriate harmonic functions (see § 5 for the case of scalar perturbations and § 7 for tensor perturbations). This procedure automatically takes care of the required angular dependencies of the harmonic components of the distribution function, allowing a streamlined and unified treatment of all perturbation types in background FRW models with arbitrary spatial curvature.

The covariant angular expansion of the photon distribution function takes the form

$$f^{(\gamma)}(E, e) = \sum_{l=0}^{\infty} F^{(l)}_{a_1 \ldots a_l} e^{a_1} e^{a_2} \ldots e^{a_l} \tag{52}$$

(Ellis, Matravers, & Treciokas 1983; Thorne 1981), where the tensors $F^{(l)}_{a_1 \ldots a_l}$ have an implicit dependence on spacetime position x and energy E and are totally symmetric, traceless, and orthogonal to u^a:

$$F^{(l)}_{a_1 \ldots a_l} = F^{(l)}_{(a_1 \ldots a_l)}, \quad g^{a_1 a_2} F^{(l)}_{a_1 a_2 \ldots a_l} = 0, \quad u^{a_1} F^{(l)}_{a_1 \ldots a_l} = 0 . \tag{53}$$

Employing the expansion given in equation (52), the action of the Liouville operator on $f^{(\gamma)}(E, e)$ reduces to

$$\mathscr{L} f^{(\gamma)}(E, e) = \sum_{l=0}^{\infty} (\partial_E F^{(l)}_{a_1 \ldots a_l} e^{a_1} \partial_\lambda E \; . \\ + p^b \nabla_b F^{(l)}_{a_1 \ldots a_l} e^{a_1} + l F^{(l)}_{a_1 \ldots a_l} p^b \nabla_b e^{a_1}) e^{a_2} \ldots e^{a_l} \tag{54}$$

Using the geodesic equation, we find that

$$h_{ab} p^c \nabla_c e^b = -E(\sigma_{bc} e^b e^c e_a + \sigma_{ab} e^b + w^b e_a e_b + w_a - \varpi_{ab} e^b) , \tag{55}$$

which is first order. In an exact FRW universe, isotropy restricts $F^{(l)}_{a_1 \ldots a_l} = 0$ for $l > 0$, so that in an almost FRW universe $F^{(l)}_{a_1 \ldots a_l} = O(1)$ for l not equal to zero. It follows that the last term in equation (54) makes only a second-order contribution and may be dropped in the linear calculation considered here.

Inserting the expansion given in equation (52) into the Boltzmann equation (eq. [51]) and performing a covariant angular expansion of the resulting equation gives a set of moment equations that are equivalent to the original Boltzmann equation. The linearized calculation is straightforward, although a little care is needed for the first three moments, since $F^{(0)}$ is a zero-order quantity. (The exact expansion of the left-hand side of the Boltzmann equation, eq. [54], is given in Ellis et al. 1983 and Thorne 1981.) For $l = 0$, 1, and 2, we find

$$\dot{\rho}^{(\gamma)} + \frac{4}{3} \theta \rho^{(\gamma)} + {}^{(3)}\nabla^a q^{(\gamma)}_a = 0 , \tag{56}$$

$$\dot{q}^{(\gamma)}_a + \frac{4}{3} \theta q^{(\gamma)}_a + {}^{(3)}\nabla^b \pi^{(\gamma)}_{ab} + \frac{4}{3} \rho^{(\gamma)} w_a - \frac{1}{3} {}^{(3)}\nabla_a \rho^{(\gamma)} \\ = n_e \sigma_{\rm T} \left(\frac{4}{3} \rho^{(\gamma)} v^{(b)}_a - q^{(\gamma)}_a \right) , \tag{57}$$

$$\dot{\pi}^{(\gamma)}_{ab} + \frac{4}{3} \theta \pi^{(\gamma)}_{ab} + {}^{(3)}\nabla^c J^{(3)}_{abc} - \frac{2}{5} \\ \times \left({}^{(3)}\nabla_{(a} q^{(\gamma)}_{b)} - \frac{1}{3} h_{ab} {}^{(3)}\nabla^c q^{(\gamma)}_c \right) - \frac{8}{15} \rho^{(\gamma)} \sigma_{ab} \\ = -\frac{9}{10} n_e \sigma_{\rm T} \pi^{(\gamma)}_{ab} , \tag{58}$$

and, for $l \geq 3$,

$$\dot{J}^{(l)}_{a_1 \ldots a_l} + \frac{4}{3} \theta J^{(l)}_{a_1 \ldots a_l} + {}^{(3)}\nabla^b J^{(l+1)}_{b a_1 \ldots a_l} - \frac{l}{(2l+1)} \\ \times \left[{}^{(3)}\nabla_{(a_1} J^{(l-1)}_{a_2 \ldots a_l)} - \frac{(l-1)}{(2l-1)} {}^{(3)}\nabla^b J^{(l-1)}_{b(a_1 \ldots a_{l-2}} h_{a_{l-1} a_l)} \right] \\ = -n_e \sigma_{\rm T} J^{(l)}_{a_1 \ldots a_l} . \tag{59}$$

The tensors $J^{(l)}_{a_1 \ldots a_l}$, which are traceless, totally symmetric, and orthogonal to u^a, are derived from the $F^{(l)}_{a_1 \ldots a_l}$ by integrating over energy:

$$J^{(l)}_{a_1 \ldots a_l} \equiv \frac{4\pi(-2)^l (l!)^2}{(2l+1)(2l)!} \int_0^\infty dE \, E^3 F^{(l)}_{a_1 \ldots a_l} . \tag{60}$$

The constant factor is chosen to simplify algebraic factors in the moment equations. Using equations (34)–(36), the lowest three moments relate simply to the energy density, heat flux, and anisotropic stress:

$$\rho^{(\gamma)} = J^{(0)}, \quad q^{(\gamma)}_a = J^{(1)}_a, \quad \pi^{(\gamma)}_{ab} = J^{(2)}_{ab} . \tag{61}$$

It is straightforward to show that the tensor

$${}^{(3)}\nabla_{(a_1} J^{(l-1)}_{a_2 \ldots a_l)} - \frac{(l-1)}{(2l-1)} {}^{(3)}\nabla^b J^{(l-1)}_{b(a_1 \ldots a_{l-2}} h_{a_{l-1} a_l)} , \tag{62}$$

which appears in equation (59), is traceless, symmetric, and orthogonal to u^a, as required.

It will be observed that for $l \geq 3$, the moment equations link the $l-1$, l, and $l+1$ angular moments of the (integrated) distribution function, while the $l = 2$ equation also involves the density $\rho^{(\gamma)}$, which is the $l = 0$ moment. The exact moment equations that arise from expanding the Liouville equation in covariant harmonics also couple the $l+2$ and $l-2$ moments to $J^{(l)}_{a_1 \ldots a_l}$ (Ellis et al. 1983), but these terms are second-order for $l \geq 3$ and so do not appear in the linearized equations presented here. In the exact expansion of the Liouville equation, the coefficient of the $l+2$ angular moment in the exact propagation equation for $J^{(l)}_{a_1 \ldots a_l}$ is the shear σ_{ab}, which leads to the result that the angular expansion of the distribution function for noninteracting radiation can only truncate ($J^{(l)}_{a_1 \ldots a_l} = 0$ for all l greater than some L) if the shear vanishes (Ellis 1996). This exact result, which is lost in linearized theory that permits truncated distribution functions with nonvanishing shear, is an example of a linearization instability (see Ellis & Dunsby 1997b for more examples). However, this is not problematic for the linearized calculation of CMB anisotropies, since it is never claimed that the higher order moments of the photon distribution vanish exactly. Instead, the series is truncated (with suitable care to avoid reflection of power back down the series) for numerical convenience. The truncation is performed with L large enough so that there is no significant effect on the $J^{(l)}_{a_1 \ldots a_l}$ for the range of l of interest.

Finally, by taking the spatial gradient of equation (56) and commuting the space and time derivatives, we find the propagation equation for the comoving fractional spatial gradient of the photon density, $\mathscr{X}^{(\gamma)}_a$,

$$\dot{\mathscr{X}}^{(\gamma)}_a + \frac{4}{3} \mathscr{Z}_a + \frac{S}{\rho^{(\gamma)}} {}^{(3)}\nabla_a {}^{(3)}\nabla^b q^{(\gamma)}_b - \frac{4}{3} S\theta w_a = 0 , \quad (63)$$

where $\mathscr{Z}_a$ is the comoving spatial gradient of the volume expansion.

3.2. *Neutrinos*

We consider only massless neutrinos, and these are noninteracting over the epoch of interest. It follows that their distribution function $f^{(\nu)}(x, p)$ satisfies the Liouville equation $\mathscr{L} f^{(\nu)}(x, p) = 0$. Expanding the neutrino distribution function in covariant angular harmonics, we arrive at the moment equations for the tensors $G^{(l)}_{a_1 \ldots a_l}$, which are defined in the same manner as the $J^{(l)}_{a_1 \ldots a_l}$ but with the photon distribution function replaced by the neutrino distribution. These moment equations are the same as the photon equations but with the scattering terms omitted:

$$\dot{\rho}^{(\nu)} + \frac{4}{3} \theta \rho^{(\nu)} + {}^{(3)}\nabla^a q^{(\nu)}_a = 0 , \quad (64)$$

$$\dot{q}^{(\nu)}_a + \frac{4}{3} \theta q^{(\nu)}_a + {}^{(3)}\nabla^b \pi^{(\nu)}_{ab} + \frac{4}{3} \rho^{(\nu) w_a} - \frac{1}{3} {}^{(3)}\nabla_a \rho^{(\nu)} = 0 , \quad (65)$$

$$\dot{\pi}^{(\nu)}_{ab} + \frac{4}{3} \theta \pi^{(\nu)}_{ab} + {}^{(3)}\nabla^c G^{(3)}_{abc} - \frac{2}{5} \times \left({}^{(3)}\nabla_{(a} q^{(\nu)}_{b)} - \frac{1}{3} h_{ab} {}^{(3)}\nabla^c q^{(\nu)}_c \right) - \frac{8}{15} \rho^{(\nu)} \sigma_{ab} = 0 , \quad (66)$$

and for $l \geq 3$,

$$\dot{G}^{(l)}_{a_1 \ldots a_l} + \frac{4}{3} \theta G^{(l)}_{a_1 \ldots a_l} + {}^{(3)}\nabla^b G^{(l+1)}_{b a_1 \ldots a_l} - \frac{l}{(2l+1)} \times \left[{}^{(3)}\nabla_{(a_1} G^{(l-1)}_{a_2 \ldots a_l)} - \frac{(l-1)}{(2l-1)} {}^{(3)}\nabla^b G^{(l-1)}_{b(a_1 \ldots a_{l-2}} h_{a_{l-1} a_l)} \right] = 0 . \quad (67)$$

The propagation equation for the comoving fractional spatial gradient of the neutrino density, $\mathscr{X}^{(\nu)}_a$, follows from equation (64):

$$\dot{\mathscr{X}}^{(\nu)}_a + \frac{4}{3} \mathscr{Z}_a + \frac{S}{\rho^{(\nu)}} {}^{(3)}\nabla_a {}^{(3)}\nabla^b q^{(\nu)}_b - \frac{4}{3} S\theta w_a = 0 . \quad (68)$$

3.3. *Baryons*

Over the epoch of interest here, the electrons and baryons are nonrelativistic and may be approximated by a tightly coupled ideal fluid (the coupling arising from Coulomb scattering). The energy density of the fluid is $\rho^{(b)}$, which includes contributions from both the baryonic species and the electrons, the fluid pressure is $p^{(b)}$, and the velocity of the fluid is $u^{(b)}_a = u_a + v^{(b)}_a$ to first order, where the $O(1)$ relative velocity $v^{(b)}_a$ satisfies $u^a v^{(b)}_a = O(2)$.

The linearized baryon stress-energy tensor evaluates to

$$\mathscr{T}^{(b)}_{ab} = \rho^{(b)} u_a u_b - p^{(b)} h_{ab} + 2(\rho^{(b)} + p^{(b)}) u_{(a} v^{(b)}_{b)} , \quad (69)$$

which shows that there is a heat flux $(\rho^{(b)} + p^{(b)}) v^{(b)}_a$ due to the baryon motion relative to the u^a frame. The equations of motion for $\rho^{(b)}$ and $v^{(b)}_a$ follow from the conservation of baryon plus photon stress-energy (the baryons and photons interact through nongravitational effects only with themselves):

$$\nabla^a \mathscr{T}^{(b)}_{ab} + \nabla^a \mathscr{T}^{(\gamma)}_{ab} = 0 . \quad (70)$$

Using the $l = 0$ and $l = 1$ moment equations for the photon distribution, we find the propagation equation for the baryon energy density,

$$\dot{\rho}^{(b)} + (\rho^{(b)} + p^{(b)})\theta + (\rho^{(b)} + p^{(b)}) {}^{(3)}\nabla^a v^{(b)}_a = 0 , \quad (71)$$

and a propagation equation for $v^{(b)}_a$,

$$(\rho^{(b)} + p^{(b)})(\dot{v}^{(b)}_a + w_a) + \frac{1}{3} (\rho^{(b)} + p^{(b)}) \theta v^{(b)}_a + \dot{p}^{(b)} v^{(b)}_a - {}^{(3)}\nabla_a p^{(b)} + n_e \sigma_{\mathrm{T}} \left(\frac{4}{3} \rho^{(\gamma)} v^{(b)}_a - q^{(\gamma)}_a \right) = 0 , \quad (72)$$

which must be supplemented by an equation of state linking $p^{(b)}$ and $\rho^{(b)}$. The final term in equation (72) describes the exchange of momentum between the radiation and the baryon/electron fluid as a result of Thomson scattering. There is no such term in equation (71), since both the radiation drag force and the baryon velocity relative to the u^a frame are first-order, which give only a second-order rate of energy transfer in the u^a frame. Energy transfer due to thermal motion of the electrons in the baryon rest frame has a negligible effect on $\rho^{(b)}$, since the electrons are nonrelativistic; $k_B T^{(b)} \ll m_e$, where $T^{(b)}$ is the baryon kinetic temperature (assumed equal to the electron kinetic temperature) and m_e is the electron mass.

Taking the spatial gradient of equation (71) gives the propagation equation for $\mathscr{X}^{(b)}_a$, the fractional comoving spatial gradient of the baryon energy density:

$$\rho^{(b)}\dot{\mathscr{X}}_a^{(b)} + (\rho^{(b)} + p^{(b)})(\mathscr{Z}_a + S^{(3)}\nabla_a{}^{(3)}\nabla^b v_b^{(b)} - S\theta w_a) + S\theta^{(3)}\nabla_a p^{(b)} - \theta p^{(b)}\mathscr{X}_a^{(b)} = 0 \,. \quad (73)$$

We have retained all terms involving the baryon pressure $p^{(b)}$ in the equations of this section. In practice, over epochs where the baryons are nonrelativistic ($p^{(b)} \ll \rho^{(b)}$), the only pressure term that need be retained is the term ${}^{(3)}\nabla^a p^{(b)}$, which appears in equation (72). This term appears as a small correction to the total sound speed in the tightly coupled baryon/photon plasma and is potentially significant during the acoustic oscillations in the plasma.

3.4. *Cold Dark Matter*

We will only consider CDM here, which may be described as a pressureless ideal fluid. Hot dark matter (HDM), which requires a phase space description, is considered in Ma & Bertschinger (1995), where the calculations for scalar perturbations are performed in the synchronous and conformal Newtonian gauges. The CDM has energy density $\rho^{(c)}$ in its rest frame, which has velocity $u_a^{(c)} = u_a + v_a^{(c)}$, with the first-order relative velocity $v_a^{(c)}$ satisfying $u^a v_a^{(c)} = O(2)$. The linearized CDM stress-energy tensor evaluates to

$$\mathscr{T}_{ab}^{(c)} = \rho^{(c)} u_a u_b + 2\rho^{(c)} u_{(a} v_{b)}^{(c)} \,, \quad (74)$$

which is conserved, since the CDM interacts with other species only through gravity. The conservation of $\mathscr{T}_{ab}^{(c)}$ gives the propagation equations for $\rho^{(c)}$ and $v_a^{(c)}$:

$$\dot{\rho}^{(c)} + \theta\rho^{(c)} + \rho^{(c)(3)}\nabla^a v_a^{(c)} = 0 \,, \quad (75)$$

$$\dot{v}_a^{(c)} + \frac{1}{3}\theta v_a^{(c)} + w_a = 0 \,. \quad (76)$$

Since the CDM moves on geodesics, the velocity $u_a^{(c)}$ provides a convenient frame choice. With this choice, the acceleration w_a vanishes. We use the CDM frame to define the fundamental velocity u^a in § 5, where we discuss scalar perturbations in the CDM model. For the moment, however, we continue to leave the choice of frame unspecified for generality. The final equation that we require is the propagation equation for the fractional comoving spatial gradient of the density, $\mathscr{X}_a^{(c)}$. This follows from equation (75) on taking the spatial gradient:

$$\dot{\mathscr{X}}_a^{(c)} + \mathscr{Z}_a + S^{(3)}\nabla_a{}^{(3)}\nabla^b v_b^{(c)} - S\theta w_a = 0 \,. \quad (77)$$

The equations for the matter components that we have described in this section combine with the covariant equations of § 2 to give a complete description of the evolution of inhomogeneity and anisotropy in a fully covariant and gauge-invariant manner. The equations given in § 2 make use of the total energy density and pressure, heat flux, and anisotropic stress. These quantities are related to the individual matter components in the CDM model by

$$\rho = \rho^{(\gamma)} + \rho^{(\nu)} + \rho^{(b)} + \rho^{(c)} \,, \quad (78)$$

$$p = \frac{1}{3}\rho^{(\gamma)} + \frac{1}{3}\rho^{(\nu)} + p^{(b)} \,, \quad (79)$$

$$q_a = q_a^{(\gamma)} + q_a^{(\nu)} + (\rho^{(b)} + p^{(b)})v_a^{(b)} + \rho^{(c)}v_a^{(c)} \,, \quad (80)$$

$$\pi_{ab} = \pi_{ab}^{(\gamma)} + \pi_{ab}^{(\nu)} \,. \quad (81)$$

The equations given here are both covariant and gauge-invariant. Employing gauge-invariant variables ensures that the problem of gauge-mode solutions does not arise and that all quantities are independent of the choice of map between the real universe and a background FRW model. We have only considered the linearized equations here, but the linearization procedure is not fundamental to the covariant and gauge-invariant approach. It is straightforward to extend the treatment to include nonlinear effects (Maartens et al. 1998), which should provide a systematic footing for the discussion of second-order effects on the CMB. Indeed, the simplicity with which exact, nonlinear equations can be written down and manipulated is a significant virtue of the covariant approach. Unlike in Bardeen's gauge-invariant approach (Bardeen 1980), the definition of the variables employed here does not require that the perturbations be in the linear regime, and, furthermore, the variables do not depend on the nonlocal decomposition of the perturbations into scalar, vector, and tensor type and the associated harmonic analysis. The covariant approach describes scalar, vector, and tensor modes in a unified manner, although decomposing the linear perturbations is useful to aid solution of the linearized equations later on in the calculation. A further advantage of the covariant and gauge-invariant approach over that introduced by Bardeen is that only covariantly defined variables are employed, which are simple to interpret physically. In contrast, the Bardeen variables are constructed by taking linear combinations of (gauge-dependent) metric and matter perturbations in such a way that the resulting variable is gauge-invariant (for small gauge transformations that preserve the scalar, vector or tensor structure of the metric perturbation). These variables have simple physical interpretations only for certain specific gauge choices. Finally, note that we have not yet had to specify whether the background FRW model is open, flat, or closed. However, we have made the implicit assumption that the universe is almost FRW when specifying the zero and first-order variables in the linearization procedure.

4. THE CMB TEMPERATURE ANISOTROPY

The energy-integrated moments $J_{a_1\ldots a_l}^{(l)}$ of the photon distribution function provide a fully covariant description of the CMB temperature anisotropy. In the u^a frame, we denote the average bolometric temperature on the sky at the point x by $T_0(x)$, so that

$$T_0^4 \propto \frac{1}{4\pi}\int dE\, d\Omega\, E^3 f^{(\gamma)}(x, p) = \frac{1}{4\pi} J^{(0)} \,, \quad (82)$$

which is simply the Stefan-Boltzmann law. We use the fractional temperature variation $\delta_{\mathrm{T}}(e)$ from the full-sky average T_0 to characterize the temperature perturbation along the spatial direction e^a in a gauge-invariant and covariant manner (Maartens et al. 1995; Dunsby 1997). It follows that

$$[1 + \delta_{\mathrm{T}}(e)]^4 = \frac{4\pi}{J^{(0)}}\int dE\, E^3 f^{(\gamma)}(x, p) \,, \quad (83)$$

so that to first-order

$$\delta_{\mathrm{T}}(e) = \frac{1}{4\rho^{(\gamma)}}\sum_{l=1}^{\infty}\frac{(2l+1)(2l)!}{(-2)^l(l!)^2} J_{a_1\ldots a_l}^{(l)} e^{a_1}\ldots e^{a_l} \,. \quad (84)$$

The right-hand side of equation (84) is the covariant angular expansion of the temperature anisotropy. The tensors $J_{a_1\ldots a_l}^{(l)}$ thus provide a natural covariant description of the CMB anisotropy. They may be related to the more familiar a_{lm} components in the spherical harmonic expansion of $\delta_T(e)$ by introducing an orthogonal triad in the instantaneous rest space at x, so that $e^a = (\sin\theta\cos\phi, \sin\theta \sin\phi, \cos\theta)$. Then the two expansions are related by

$$4\rho^{(\gamma)} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) = \frac{(2l+1)(2l)!}{(-2)^l (l!)^2} J^{(l)}_{a_1 \ldots a_l} e^{a_1} \ldots e^{a_l} \,. \tag{85}$$

Squaring this expression and integrating over solid angles, we find the important rotational invariant

$$\frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 = \frac{4\pi}{(4\rho^{(\gamma)})^2} \frac{(2l)!}{(-2)^l (l!)^2} h^{a_1 b_1} \ldots h^{a_l b_l} J^{(l)}_{a_1 \ldots a_l} J^{(l)}_{b_1 \ldots a_l} \,. \tag{86}$$

The quantity on the left is a quadratic estimator for the CMB power spectrum (the C_l), which we see is related to the covariant tensors $J^{(l)}_{a_1 \ldots a_l}$ in a very simple manner. Further properties of the covariant and gauge-invariant description of CMB temperature anisotropies are given by Gebbie & Ellis (1998).

5. SCALAR PERTURBATIONS

Up to this point, we have treated the scalar, vector, and tensor modes of linear theory in a unified manner. However, to obtain solutions to the covariant equations, it proves useful to consider scalar, vector, and tensor modes separately. In this section we consider scalar modes; tensor modes are discussed briefly in § 7. (Vector modes decay in an expanding universe in the absence of defects and so are not likely to have a significant effect on the CMB in inflationary models.) In the covariant approach to perturbations in cosmology, we characterize scalar perturbations by demanding that the magnetic part of the Weyl tensor and the vorticity be at most second-order. Setting $\mathscr{B}_{ab} = O(2)$ ensures that gravitational waves are excluded to first-order, and demanding that $\varpi_{ab} = O(2)$ ensures that the density gradients seen by an observer in the u^a frame arise from clumping of the density, ${}^{(3)}\nabla^2 \rho = O(1)$, and not from kinematic effects due to vorticity (the absence of flow-orthogonal hypersurfaces), which give ${}^{(3)}\nabla^2 \rho = O(2)$ in an almost FRW universe. Note that we do not classify scalar perturbations as having $\mathscr{B}_{ab} = \varpi_{ab} = 0$ (to all orders), which is only a highly restricted subset of the full set of scalar solutions. For example, in an (exactly) irrotational dust-filled universe (a "silent" universe), it can be shown from the exact nonlinear equations that demanding $\mathscr{B}_{ab} = 0$ forces the solution into a very small class, which probably all have high symmetry (Ellis 1996) and so cannot represent a very general perturbation. This arises because requiring that $\mathscr{B}_{ab} = 0$ be preserved along the flow lines introduces a series of complex constraints that greatly reduce the size of the solution set. However, requiring only that $\mathscr{B}_{ab}$ and ϖ_{ab} be at most second order gives a much larger class of solutions, because only two new constraints are introduced and these are necessarily preserved by the propagation equations. The solutions with $\mathscr{B}_{ab}$ and ϖ_{ab} vanishing exactly comprise a very small subset of the larger class of exact solutions, which we classify as scalar perturbations.

On setting $\mathscr{B}_{ab} = 0$ and $\varpi_{ab} = 0$ (equality to zero in the linearized theory should be taken to imply that the quantity is at most second order), we see from equation (13) that

$${}^{(3)}\nabla^c \sigma_{d(a} \eta_{b)ce}{}^d u^e = 0 \quad \Rightarrow \quad {}^{(3)}\nabla_{[a} {}^{(3)}\nabla^c \sigma_{b]c} = 0 \,, \tag{87}$$

where the antisymmetrization is on the indices a and b in the right-hand equation. This is a necessary condition for σ_{ab} to be constructed from a scalar potential. It follows from equations (16) and (21) that

$${}^{(3)}\nabla_{[a} q_{b]} = 0, \qquad {}^{(3)}\nabla_{[a} w_{b]} = 0 \,, \tag{88}$$

so that the heat flux and acceleration may be written as spatial gradients of scalar fields (making use of the integrability condition given as eq. [5]). Consistency of ${}^{(3)}\nabla_{[a} q_{b]} = 0$ with equation (22) for q_a then requires that

$${}^{(3)}\nabla_{[a} {}^{(3)}\nabla^c \pi_{b]c} = 0 \quad \Rightarrow \quad {}^{(3)}\nabla_{[a} {}^{(3)}\nabla^c \mathscr{E}_{b]c} = 0 \,, \tag{89}$$

with the implication following from equation (15). It follows that all vector variables, such as q_a and ${}^{(3)}\nabla^b \mathscr{E}_{ab}$, may be derived from scalar potentials. The new constraint, given as equation (87), is only consistent with the propagation equations if $\mathscr{E}_{ab}$ and π_{ab} satisfy

$${}^{(3)}\nabla^c \mathscr{E}_{d(a} \eta_{b)ce}{}^d u^e = -\frac{1}{2} \kappa {}^{(3)}\nabla^c \pi_{d(a} \eta_{b)ce}{}^d u^e \,. \tag{90}$$

In the absence of anisotropic stress, we see that the left-hand side of equation (90) is constrained to be zero, which is consistent with the propagation equation for $\mathscr{E}_{ab}$, given as equation (18), with $\pi_{ab} = 0$. If the anisotropic stress does not vanish, we include the constraint

$${}^{(3)}\nabla^c \mathscr{E}_{d(a} \eta_{b)ce}{}^d u^e = 0 \quad \Rightarrow \quad {}^{(3)}\nabla^c \pi_{d(a} \eta_{b)ce}{}^d u^e = 0 \tag{91}$$

in the definition of a scalar mode, which is easily shown to be consistent with the propagation equation for $\mathscr{E}_{ab}$. Requiring consistency of equation (91) with the propagation equation for π_{ab} implied by the photon and neutrino Boltzmann hierarchy yields a series of constraints on the moments $J^{(l)}_{a_1 \ldots a_l}$ and $G^{(l)}_{a_1 \ldots a_l}$, which are necessary conditions for them to be derived from scalar potentials.

The new constraint equations that we have introduced, by restricting the solution to be a scalar mode, may be satisfied by constructing the covariant and gauge-invariant variables from tensors derived from scalar potentials by taking appropriate spatial covariant derivatives of the scalar functions. It proves convenient to separate the temporal and spatial aspects of the problem by expanding the scalar potentials in the eigenfunctions $Q^{(k)}$ of the generalized Helmholtz equation

$${}^{(3)}\nabla^2 Q^{(k)} \equiv {}^{(3)}\nabla^{a(3)}\nabla_a Q^{(k)} = \frac{k^2}{S^2} Q^{(k)} \tag{92}$$

(Hawking 1966; Ellis et al. 1989), which are constructed to satisfy

$$\dot{Q}^{(k)} = 0 \,. \tag{93}$$

These equations are correct to zero order only; in such equations, equality should be understood to this order only. In general, we cannot impose equation (92) and $\dot{Q}^{(k)} = 0$ exactly, since the constraint equation is inconsistent with the evolution implied by $\dot{Q}^{(k)} = 0$ at first order. The allowed values of the eigenvalues k^2/S^2 are determined by the scalar curvature of the background model (since $Q^{(k)}$ are only needed to zero order). In a flat model, $K = 0$, k is a comoving continuous wavenumber ≥ 0. In closed models, $K > 0$, k takes only discrete values with $k^2 = \gamma(\gamma + 2)K$, where γ is a nonzero, positive integer. In open models, $K < 0$, k again takes continuous values but with the restriction $k^2 \geq |K|$. More details may be found in Harrison (1967). The eigenfunctions $Q^{(k)}$ are labeled by the lumped index k. This index, which determines the eigenvalue k^2/S^2, should be understood to distinguish implicitly the distinct degenerate eigenfunctions, which all have the same eigenvalue k^2/S^2. This multiple use of the symbol k should not cause any confusion, since the lumped index will always appear as a superscript or subscript. A function of the eigenvalue k will be denoted with the eigenvalue as an argument, for example, $A(k)$, to distinguish it from the quantity A_k, which depends on the mode label k and not just the eigenvalue. From the $Q^{(k)}$ we form a vector $Q_a^{(k)}$,

$$Q_a^{(k)} \equiv \frac{S}{k} {}^{(3)}\nabla_a Q^{(k)} , \tag{94}$$

which is orthogonal to u^a and is parallel transported at zero order along the flow lines:

$$u^a Q_a^{(k)} = 0, \qquad \dot{Q}_a^{(k)} = 0 . \tag{95}$$

We define totally symmetric tensors of rank l, $Q_{a_1 \ldots a_l}^{(k)}$, by the recursion formula (for $l > 1$; see also Gebbie & Ellis 1998)

$$Q_{a_1 \ldots a_l}^{(k)} = \frac{S}{k} \left({}^{(3)}\nabla_{(a_1} Q_{a_2 \ldots a_l)}^{(k)} - \frac{l-1}{2l-1} {}^{(3)}\nabla^b Q_{b(a_1 \ldots a_{l-2}}^{(k)} h_{a_{l-1} a_l)} \right) . \tag{96}$$

These tensors satisfy the zero-order properties

$$u^{a_1} Q_{a_1 a_2 \ldots a_l}^{(k)} = 0, \qquad h^{a_1 a_2} Q_{a_1 a_2 \ldots a_l}^{(k)} = 0, \qquad \dot{Q}_{a_1 a_2 \ldots a_l}^{(k)} = 0 , \tag{97}$$

which are readily proved by induction.

The scalar functions $Q^{(k)}$ are the covariant generalizations of the scalar eigenfunctions of the Laplace-Beltrami operator on the homogeneous spatial sections of the background FRW model, which are usually employed in the harmonic decomposition of perturbed quantities (see, for example, Bardeen 1980). In the covariant approach, attention is focused on a velocity field u^a rather than a spatial slicing of spacetime, so it is natural to employ harmonic functions defined by equation (92). Some of the differential properties of the derived tensors $Q_{a_1 \ldots a_l}^{(k)}$ (for $l \leq 2$) are given in the appendix to Bruni et al. (1992a). We add two more results to this list that will be useful later:

$${}^{(3)}\nabla^{a_1} Q_{a_1 a_2 \ldots a_l}^{(k)} = \frac{l}{2l-1} \frac{k}{S} \left[1 - (l^2 - 1) \frac{K}{k^2} \right] Q_{a_2 \ldots a_l}^{(k)} , \tag{98}$$

$${}^{(3)}\nabla^2 Q_{a_1 \ldots a_l}^{(k)} = \frac{k^2}{S^2} \left[1 - l(l+1) \frac{K}{k^2} \right] Q_{a_1 \ldots a_l}^{(k)} , \tag{99}$$

which may be derived from the recursion relation given as equation (96) and the definition in equation (92). Further properties of the scalar harmonics are given by Gebbie & Ellis (1998).

To relate the timelike integration employed in the Boltzmann multipole approach, adopted here, to the lightlike integration employed in line-of-sight methods (Seljak & Zaldarriaga 1996), which we consider in § 5.2, it is convenient to note the following zero-order results for the variation of the $Q_{a_1 \ldots a_l}^{(k)}$ along the line of sight through some point R: Let $x^a(\lambda)$ be a null geodesic with tangent vector parallel to $u_a + e_a$ and λ satisfying $(u_a + e_a)\nabla^a \lambda = 1$. Define a positive parameter $y(\lambda)$ along the past null geodesic by $dy/d\lambda = -k/S$ with $y = 0$ at the point of observation R. Then the evolution of the quantities $Q_{a_1 \ldots a_l}^{(k)} e^{a_1} \ldots e^{a_l}$ is given to zero order by the hierarchy

$$\frac{d}{dy} (Q_{a_1 \ldots a_l}^{(k)} e^{a_1} \ldots e^{a_l}) = -Q_{a_1 \ldots a_{l+1}}^{(k)} e^{a_1} \ldots e^{a_{l+1}} + \frac{l^2}{(4l^2 - 1)} \left[1 - \frac{K}{k^2} (l^2 - 1) \right] Q_{a_1 \ldots a_{l-1}}^{(k)} e^{a_1} \ldots e^{a_{l-1}} , \tag{100}$$

which follows from the recursion relation given as equation (96). We shall only consider the solution to this hierarchy in a $K = 0$ universe, in which case we find the following variation of $Q^{(k)}$ along the line of sight:

$$(Q^{(k)})_y = \sum_{l=0}^{\infty} \frac{(2l)!(2l+1)}{(-2)^l (l!)^2} j_l(y) (Q_{a_1 \ldots a_l}^{(k)} e^{a_1} \ldots e^{a_l})_{y=0} , \tag{101}$$

where the $j_l(y)$ are spherical Bessel functions. Equation (101) expresses $Q^{(k)}$, a parameter distance y down the line of sight in terms of the $Q_{a_1 \ldots a_l}^{(k)} e^{a_1} \ldots e^{a_l}$ at the point of observation, R. If required, the value of $Q_{a_1 \ldots a_l}^{(k)} e^{a_1} \ldots e^{a_l}$ down the light cone can be found from the solution for $Q^{(k)}$ (eq. [101]) and the hierarchy (eq. [100]).

The additional constraints introduced by the conditions for a scalar mode are satisfied identically if we construct the gauge-invariant variables in the following manner:

$$\mathscr{X}_a^{(i)} = \sum_k k \mathscr{X}_k^{(i)} Q_a^{(k)} , \tag{102}$$

$$q_a^{(i)} = \rho^{(i)} \sum_k q_k^{(i)} Q_a^{(k)} , \tag{103}$$

$$v_a^{(i)} = \sum_k v_k^{(i)} Q_a^{(k)} , \tag{104}$$

$$\pi_{ab}^{(i)} = \rho^{(i)} \sum_k \pi_k^{(i)} Q_{ab}^{(k)} , \tag{105}$$

$$\mathscr{Z}_a = \sum_k \frac{k^2}{S} \mathscr{Z}_k Q_a^{(k)} , \tag{106}$$

$$\mathscr{E}_{ab} = \sum_k \frac{k^2}{S^2} \Phi_k Q_{ab}^{(k)} , \tag{107}$$

$$\sigma_{ab} = \sum_k \frac{k}{S} \sigma_k Q_{ab}^{(k)} , \tag{108}$$

$$w_a = \sum_k \frac{k}{S} w_k Q_a^{(k)} , \tag{109}$$

where i labels the particle species (and we omit the label when referring to total fluid variables). The symbolic summation in these expressions is a sum over eigenfunctions of equation (92). For closed background models the sum is discrete, but in the flat and open cases the summation should be understood as an integral over the continuous label k, which distinguishes distinct eigenfunctions. The scalar expansion coefficients, such as $\mathscr{X}_k^{(i)}$, are themselves first-order gauge-invariant variables, which satisfy

$${}^{(3)}\nabla^a \mathscr{X}_k^{(i)} = O(2) . \tag{110}$$

They are labeled by the lumped index k. Finally, we assume that the higher order angular moments of the photon and neutrino distribution functions may also be expanded in the $Q^{(k)}_{a_1\ldots a_l}$ harmonics. By considering the zero-order form of the scalar harmonics $Q^{(k)}$ and derived tensors, it is straightforward to show that this condition is equivalent to the usual assumption that the Fourier components of the distribution functions are axisymmetric about the wavevector $\boldsymbol{k}$ (see, for example, Seljak & Zaldarriaga 1996). With this condition, we have

$$J^{(l)}_{a_1\ldots a_l} = \rho^{(\gamma)}\sum_k J^{(l)}_k Q^{(k)}_{a_1\ldots a_l}\,, \qquad G^{(l)}_{a_1\ldots a_l} = \rho^{(\nu)}\sum_k G^{(l)}_k Q^{(k)}_{a_1\ldots a_l}\,, \tag{111}$$

for photons and neutrinos, respectively.

5.1. *The Scalar Equations*

It is now a simple matter to substitute the harmonic expansions of the covariant variables into the constraint and propagation equations given in §§ 2 and 3 in order to obtain equations for the scalar expansion coefficients that describe scalar perturbations in a covariant and gauge-invariant manner. To simplify matters, we assume that the variations in baryon pressure $p^{(b)}$ due to entropy variations are negligible compared to those arising from variations in $\rho^{(b)}$, so that we may write

$$\nabla^a p^{(b)} = c_s^2\ \nabla^a \rho^{(b)}\,, \tag{112}$$

where c_s is the adiabatic sound speed in the baryon/electron fluid (this is different from the total sound speed in the tightly coupled baryon/photon fluid).

With this assumption, we obtain the following equations for scalar perturbations: for the spatial gradients of the densities, we find

$$\dot{\mathscr{X}}^{(\gamma)}_k = -\frac{k}{S}\left(\frac{4}{3}\mathscr{Z}_k + q^{(\gamma)}_k\right) + \frac{4}{3}\theta w_k\,, \tag{113}$$

$$\dot{\mathscr{X}}^{(\nu)}_k = -\frac{k}{S}\left(\frac{4}{3}\mathscr{Z}_k + q^{(\nu)}_k\right) + \frac{4}{3}\theta w_k\,, \tag{114}$$

$$\dot{\mathscr{X}}^{(c)}_k = -\frac{k}{S}\left(\mathscr{Z}_k + v^{(c)}_k\right) + \theta w_k\,, \tag{115}$$

$$\dot{\mathscr{X}}^{(b)}_k = \left(1 + \frac{p^{(b)}}{\rho^{(b)}}\right)\left[-\frac{k}{S}\left(\mathscr{Z}_k + v^{(b)}_k\right) + \theta w_k\right] + \left(\frac{p^{(b)}}{\rho^{(b)}} - c_s^2\right)\theta\mathscr{X}^{(b)}_k\,; \tag{116}$$

for the spatial gradient of the expansion, we find

$$\dot{\mathscr{Z}}_k = -\frac{1}{3}\theta\mathscr{Z}_k - \frac{1}{2}\frac{S}{k}\kappa\left[2\left(\rho^{(\gamma)}\mathscr{X}^{(\gamma)}_k + \rho^{(\nu)}\mathscr{X}^{(\nu)}_k\right) + \rho^{(c)}\mathscr{X}^{(c)}_k + (1+3c_s^2)\rho^{(b)}\mathscr{X}^{(b)}_k\right] + \frac{S}{k}w_k\left(\frac{3}{2}\kappa\rho\gamma - \frac{3K}{S^2}\right), \tag{117}$$

where γ is defined in terms of the total pressure p and density ρ by $p = (\gamma - 1)\rho$ (note that we do not assume that γ is constant). The heat fluxes satisfy

$$\dot{q}^{(\gamma)}_k + \frac{1}{3}\frac{k}{S}\left(2\pi^{(\gamma)(1-3K/k^2)}_k - \mathscr{X}^{(\gamma)}_k + 4w_k\right) = n_e\sigma_{\mathrm{T}}\left(\frac{4}{3}v^{(b)}_k - q^{(\gamma)}_k\right), \tag{118}$$

$$\dot{q}^{(\nu)}_k + \frac{1}{3}\frac{k}{S}\left(2\pi^{(\nu)(1-3K/k^2)}_k - \mathscr{X}^{(\nu)}_k + 4w_k\right) = 0\,, \tag{119}$$

and, for the baryon and CDM peculiar velocities,

$$\dot{v}^{(c)}_k + \frac{1}{3}\theta v^{(c)}_k + \frac{k}{S}w_k = 0\,, \tag{120}$$

$$\left(1 + \frac{p^{(b)}}{\rho^{(b)}}\right)\left[\dot{v}^{(b)}_k + \frac{1}{3}(1-3c_s^2)\theta v^{(b)}_k + \frac{k}{S}w_k\right] - \frac{k}{S}c_s^2\mathscr{X}^{(b)}_k = -n_e\sigma_{\mathrm{T}}\frac{\rho^{(\gamma)}}{\rho^{(b)}}\left(\frac{4}{3}v^{(b)}_k - q^{(\gamma)}_k\right). \tag{121}$$

The propagation equations for the anisotropic stresses are

$$\dot{\pi}^{(\gamma)}_k + \frac{3}{5}\frac{k}{S}\left(1 - \frac{8K}{k^2}\right)J^{(3)}_k - \frac{2}{5}\frac{k}{S}q^{(\gamma)}_k - \frac{8}{15}\frac{k}{S}\sigma_k = -\frac{9}{10}n_e\sigma_{\mathrm{T}}\pi^{(\gamma)}_k\,, \tag{122}$$

$$\dot{\pi}^{(\nu)}_k + \frac{3}{5}\frac{k}{S}\left(1 - \frac{8K}{k^2}\right)G^{(3)}_k - \frac{2}{5}\frac{k}{S}q^{(\nu)}_k - \frac{8}{15}\frac{k}{S}\sigma_k = 0\,, \tag{123}$$

and the remaining moment equations, for $l \geq 3$, are

$$\dot{J}^{(l)}_k + \frac{k}{S}\left\{\frac{l+1}{2l+1}\left[1 - l(l+2)\frac{K}{k^2}\right]J^{(l+1)}_k - \frac{l}{2l+1}J^{(l-1)}_k\right\} = -n_e\sigma_{\mathrm{T}}J^{(l)}_k\,, \tag{124}$$

$$\dot{G}^{(l)}_k + \frac{k}{S}\left\{\frac{l+1}{2l+1}\left[1 - l(l+2)\frac{K}{k^2}\right]G^{(l+1)}_k - \frac{l}{2l+1}G^{(l-1)}_k\right\} = 0\,. \tag{125}$$

The propagation equations for $\mathscr{E}_{ab}$ and σ_{ab} become

$$\left(\frac{k}{S}\right)^2\left(\dot{\Phi}_k + \frac{1}{3}\theta\Phi_k\right) + \frac{1}{2}\frac{k}{S}\kappa\rho(\gamma\sigma_k + q_k) + \frac{1}{6}\kappa\rho\theta(3\gamma - 1)\pi_k - \frac{1}{2}\kappa\rho\dot{\pi}_k = 0\,, \tag{126}$$

$$\left(\frac{k}{S}\right)\left(\dot{\sigma}_k + \frac{1}{3}\theta\sigma_k\right) + (k/S)^2(\Phi_k - w_k) + \frac{1}{2}\kappa\rho\pi_k = 0\,. \tag{127}$$

Finally, the remaining constraint equations become

$$2\left(\frac{k}{S}\right)^3\left(1 - \frac{3K}{k^2}\right)\Phi_k - \frac{k}{S}\kappa\rho\left[\mathscr{X}_k + \left(1 - \frac{3K}{k^2}\right)\pi_k\right] - \kappa\rho\theta q_k = 0\,, \tag{128}$$

$$\frac{2}{3}\left(\frac{k}{S}\right)^2\left[\mathscr{Z}_k - \left(1 - \frac{3K}{k^2}\right)\sigma_k\right] + \kappa\rho q_k = 0\,. \tag{129}$$

The variables $\mathscr{X}_k$, q_k and π_k refer to the total matter and are given in terms of the component variables by

$$\rho\mathscr{X}_k = \rho^{(\gamma)}\mathscr{X}^{(\gamma)}_k + \rho^{(\nu)}\mathscr{X}^{(\nu)}_k + \rho^{(b)}\mathscr{X}^{(b)}_k + \rho^{(c)}\mathscr{X}^{(c)}_k\,, \tag{130}$$

$$\rho q_k = \rho^{(\gamma)}q^{(\gamma)}_k + \rho^{(\nu)}q^{(\nu)}_k + (\rho^{(b)} + p^{(b)})v^{(b)}_k + \rho^{(c)}v^{(c)}_k\,, \tag{131}$$

$$\rho\pi_k = \rho^{(\gamma)}\pi^{(\gamma)}_k + \rho^{(\nu)}\pi^{(\nu)}_k\,. \tag{132}$$

These equations give a complete description of the evolution of inhomogeneity and anisotropy from scalar perturbations in an almost FRW universe with any spatial curvature. The system closes up once a choice for the velocity u^a is made, and it is straightforward to check that the constraint equations are consistent with the propagation equations. The equations for $J_k^{(l)}$ and $G_k^{(l)}$ for $l \geq 3$ are equivalent to those usually found in the literature (see, for example, Ma & Bertschinger 1995 and set $K = 0$). This is because the moments of the perturbed distribution function used in such gauge-dependent calculations are gauge-invariant for $l \geq 1$. (The $l = 1$ moment does depend on the choice of coordinates in the real universe, but it is independent of the mapping onto the background model, since the background distribution function has no angular dependence.) Gauge-invariant versions of the usual synchronous-gauge equations (Ma & Bertschinger 1995) are obtained by taking u^a to coincide with the CDM velocity, so that w_a and $v_a^{(c)}$ vanish.

5.2. *The Integral Solution*

The integral solution to the Boltzmann multipole equations is central to the line-of-sight integration method for the calculation CMB anisotropies. This method, which has been implemented in the CMBFAST code of Seljak & Zaldarriaga (1996), provides a very fast route to calculating the CMB power spectrum. Although we do not make use of this method for the numerical calculations presented in this paper, it may be useful to some readers to have available the integral solution to the covariant and gauge-invariant Boltzmann hierarchy, not least because it provides the link between the lightlike integrations along the observational null cone, employed in line-of-sight methods (which includes the original calculation by Sachs & Wolfe 1967), and the timelike integrations along the flow lines of the velocity field, u^a, employed in the Boltzmann multipole approach. For simplicity, we restrict attention to $K = 0$, almost FRW universes.

The (linearized) integral solution to the hierarchy of Boltzmann multipole equations given in the previous subsection is, for $l \geq 1$,

$$
\begin{aligned}
(J_k^{(l)})_R = 4\int_0^{t_R} e^{-\tilde{\tau}}\bigg\{&\left(\frac{k}{S}\sigma_k + \frac{3}{16}n_e\sigma_T\pi_k^{(\gamma)}\right)\\
&\times\left[\frac{1}{3}j_l(\tilde{y}) + \frac{d^2}{d\tilde{y}^2}j_l(\tilde{y})\right] - \left(\frac{k}{S}w_k - n_e\sigma_T v_k^{(b)}\right)\frac{d}{d\tilde{y}}j_l(\tilde{y})\\
&-\left[\frac{1}{3}\left(\frac{k}{S}\mathscr{Z}_k - \theta w_k\right) - \frac{1}{4}n_e\sigma_T\mathscr{X}_k^{(\gamma)}\right]j_l(\tilde{y})\bigg\}dt' , \quad (133)
\end{aligned}
$$

where $(-)_R$ denotes the quantity evaluated at R and the integral is taken along the flow line of u^a through the point R. Here, t is proper time along the flow line (with $t = t_R$ at R), $\tilde{y} = \tilde{y}(t_R, t')$ is k times the conformal time difference along the flow line between t' and t_R [$\tilde{y} \equiv \int_{t'}^{t_R} k/S\, dt$], and $\tilde{\tau} = \tilde{\tau}(t_R, t')$ is an "optical depth" along the flow line [$\tilde{\tau} \equiv \int_{t'}^{t_R} n_e\sigma_T\, dt$]. In deriving equation (133) we have neglected any contribution from initial conditions, since these are exponentially suppressed by a factor $\exp[-\tilde{\tau}(t_R, 0)]$. It is straightforward to verify, by differentiating with respect to t_R, that equation (133) is the solution to the Boltzmann hierarchy for scalar perturbations in a $K = 0$, almost FRW universe. Verification for the $l = 1$ moment requires the following formal solution for $\mathscr{X}_k^{(\gamma)}$:

$$
\begin{aligned}
(\mathscr{X}_k^{(\gamma)})_R = 4\int_0^{t_R} e^{-\tilde{\tau}}\bigg\{&\left(\frac{k}{S}\sigma_k + \frac{3}{16}n_e\sigma_T\pi_k^{(\gamma)}\right)\\
&\times\left[\frac{1}{3}j_0(\tilde{y}) + \frac{d^2}{d\tilde{y}^2}j_0(\tilde{y})\right] - \left(\frac{k}{S}w_k - n_e\sigma_T v_k^{(b)}\right)\frac{d}{d\tilde{y}}j_0(\tilde{y})\\
&-\left[\frac{1}{3}\left(\frac{k}{S}\mathscr{Z}_k - \theta w_k\right) - \frac{1}{4}n_e\sigma_T\mathscr{X}_k^{(\gamma)}\right]j_0(\tilde{y})\bigg\}dt' , \quad (134)
\end{aligned}
$$

where again we have neglected the exponentially suppressed contribution from the initial conditions. In numerical applications, it is convenient to manipulate equation (133) further by integrating by parts (Seljak & Zaldarriaga 1996).

The integral in equation (133) is taken along the flow line of u^a through R. However, in the linearized calculation considered here, the integral can be performed along (any) null geodesic through R also. To see this, regard $\tilde{y}$ and $\tilde{\tau}$ as the restrictions to the flow line of (zero-order) fields in the past light cone of R, with ${}^{(3)}\nabla_a\tilde{y} = O(1)$ and similarly for $\tilde{\tau}$. Replacing the measure dt' by $u_a\, dx^a$ in the integral in equation (133) and noting that $\nabla_{[a}u_{b]} = O(1)$ and, for example, ${}^{(3)}\nabla_a\sigma_k = O(2)$, it follows that the line integral of u^a times the integrand in equation (133) is path-independent. The flow line and null geodesic through R can be joined at early times by a spacelike curve with $u_a\, dx^a = O(1)$, which therefore makes only a second-order contribution to the integral around the closed loop. To zero order, the restriction of the fields $\tilde{y}$ and $\tilde{\tau}$ to the null geodesic through R define quantities $y = y(\lambda_R, \lambda')$ and $\tau = \tau(\lambda_R, \lambda')$ on the null curve, where $y \equiv \int_{\lambda'}^{\lambda_R} k/S\, d\lambda$ and $\tau \equiv \int_{\lambda'}^{\lambda_R} n_e\sigma_T\, d\lambda$, so that τ is the optical depth along the line of sight. (The parameter λ along the null geodesic was defined in § 5.) Using the integral along the line of sight, we can now reassemble the gauge-invariant temperature perturbations from the mean, $\delta_T(e)$, at R using equations (84) and (111). Recalling equation (101) for the variation of the quantities $Q_{a_1\ldots a_l}^{(k)}e^{a_1}\ldots e^{a_l}$ down the line of sight, the temperature anisotropy from scalar perturbations in an almost FRW universe reduces to

$$
\begin{aligned}
[\delta_T(e)]_R = &-\frac{1}{4}\sum_k(\mathscr{X}_k^{(\gamma)}Q^{(k)})_R + \sum_k\int^{\lambda_R}e^{-\tau}\\
&\times\left[\frac{k}{S}\sigma_k Q_{ab}^{(k)}e^a e^b + \frac{k}{S}w_k Q_a^{(k)}e^a - \frac{1}{3}\left(\frac{k}{S}\mathscr{Z}_k - \theta w_k\right)Q^{(k)}\right]d\lambda'\\
&+\sum_k\int^{\lambda_R}n_e\sigma_T e^{-\tau}\\
&\times\left(\frac{3}{16}\pi_k^{(\gamma)}Q_{ab}^{(k)}e^a e^b - v_k^{(b)}Q_a^{(k)}e^a + \frac{1}{4}\mathscr{X}_k^{(\gamma)}Q^{(k)}\right)d\lambda' . \quad (135)
\end{aligned}
$$

Equation (135), first given in this covariant form in Challinor & Lasenby (1998), is valid for any value of the spatial curvature, even though the derivation given here considers the $K = 0$ case only. Equation (135) is most easily obtained by a direct integration of the covariant Boltzmann equation for the temperature anisotropy $\delta_T(e)$ along the line of sight (Challinor & Lasenby 1998). However, the route followed here makes clear the link between the lightlike and timelike integrations, employed in the line of sight and Boltzmann multipole methods, respectively.

5.3. *Initial Conditions on Super-Horizon Scales*

In this subsection, we analytically extract the solution of the scalar perturbation equations in the radiation-dominated era. We shall only consider modes with $|K|/k^2 \ll 1$ so that we may ignore terms involving K in the scalar equations. Associated with each mode there is a characteristic length scale, S/k. The condition $|K|/k^2 \ll 1$ is equivalent to requiring that this length scale be small compared to the curvature radius of the universe. For such modes, k is effectively a comoving wavenumber. We shall also require that the mode be well outside the horizon scale $1/H$, so that we consider only those modes satisfying

$$1 \ll \mathscr{H}_k^2 \ll \frac{H^2 S^2}{|K|}, \tag{136}$$

where $\mathscr{H}_k \equiv SH/k$ is the ratio of the characteristic length scale to the horizon scale and $H^2S^2/|K|$ is the (squared) ratio of the curvature radius to the horizon scale. If the universe may be approximated by a $K = 0$ universe to zero-order, equation (136) reduces to $\mathscr{H}_k \gg 1$. The approximate analytic solution may be used to provide initial conditions for a numerical integration of the scalar equations (see § 6).

At this point, it is convenient to make a choice of frame. In the CDM model, the rest frame of the CDM defines a geodesic frame, which provides a convenient choice for u^a, since the acceleration then vanishes identically. We assume that this frame choice has been made in the rest of this paper.

Well before decoupling, the baryons and photons are tightly coupled because of the high opacity to Thomson scattering. This scattering damps the photon moments for $l \geq 2$, but a dipole ($l = 1$) moment can survive if the baryon velocity does not coincide with the CDM velocity. To a good approximation, we may ignore the $J_k^{(l)}$ for $l \geq 2$ and set $v_k^{(b)} = 3q_k^{(\gamma)}/4$ so that the radiation is isotropic in the rest frame of the baryons. This is the lowest order term in the tight-coupling approximation (see § 6.2). Similarly, we expect that the higher order neutrino moments will also be small in the early universe, since the neutrinos were in thermal equilibrium prior to their decoupling. Furthermore, the baryon and CDM densities, $\rho^{(b)}$ and $\rho^{(c)}$, are negligible compared to the radiation and neutrino densities, $\rho^{(\gamma)}$ and $\rho^{(\nu)}$, in the radiation-dominated era.

A useful first approximation to the full set of scalar equations is obtained by setting the neutrino moments, $G_k^{(l)}$, to zero for $l \geq 2$. It is convenient to take the dependent variable to be $x \equiv \mathscr{H}_k^{-1}$ instead of the proper time t along the flow lines, so that the scalar propagation equations of the previous section reduce to the following set:

$$x^2\mathscr{Z}_k' + x\mathscr{Z}_k + 3[(1-R)\mathscr{X}_k^{(\gamma)} + R\mathscr{X}_k^{(\nu)}] = 0, \tag{137}$$

$$x^2\Phi_k' + x\Phi_k + 2\sigma_k + \frac{3}{2}[(1-R)q_k^{(\gamma)} + Rq_k^{(\nu)}] = 0, \tag{138}$$

$$x\sigma_k' + \sigma_k + x\Phi_k = 0, \tag{139}$$

$$\mathscr{X}_k^{(\gamma)\prime} + \frac{4}{3}\mathscr{Z}_k + q_k^{(\gamma)} = 0, \tag{140}$$

$$\mathscr{X}_k^{(\nu)\prime} + \frac{4}{3}\mathscr{Z}_k + q_k^{(\nu)} = 0, \tag{141}$$

$$q_k^{(\gamma)\prime} - \frac{1}{3}\mathscr{X}_k^{(\gamma)} = 0, \tag{142}$$

$$q_k^{(\nu)\prime} - \frac{1}{3}\mathscr{X}_k^{(\nu)} = 0, \tag{143}$$

where a prime denotes differentiation with respect to x and we have used the zero-order Friedmann equation in the form $H^2 = \kappa\rho/3$, since the curvature term may be neglected by equation (136). We have followed Ma & Bertschinger (1995) by introducing the dimensionless quantity R defined by

$$R \equiv \frac{\rho^{(\nu)}}{\rho^{(\nu)} + \rho^{(\gamma)}}. \tag{144}$$

After neutrino decoupling, R is a constant that depends only on the number of neutrino species. The remaining equations that we require are the two scalar constraints, which reduce to

$$2x^3\Phi_k - 3x[(1-R)\mathscr{X}_k^{(\gamma)} + R\mathscr{X}_k^{(\nu)}] - 9[(1-R)q_k^{(\gamma)} + Rq_k^{(\nu)}] = 0, \tag{145}$$

$$2x^2(\mathscr{Z}_k - \sigma_k) + 9[(1-R)q_k^{(\gamma)} + Rq_k^{(\nu)}] = 0. \tag{146}$$

This set of equations gives a closed equation for Φ_k:

$$3x\Phi_k'' + 12\Phi_k' + x\Phi_k = 0. \tag{147}$$

This equation should be compared to the fourth-order equation for the metric perturbation variable in the synchronous gauge (see, for example, Ma & Bertschinger 1995). The fourth-order equation admits four linearly independent solutions, but two of the solutions are gauge modes that arise from mapping an exact FRW universe to itself. The gauge-invariant approach adopted here ensures that such gauge modes do not arise. This is evident from equation (147), which is only a second-order equation. The two linearly independent solutions of this equation both describe physical perturbations in the Weyl tensor, which vanishes for an exact FRW universe. It is now straightforward to find the general solution of equations (137)–(146). There are two solutions with nonvanishing Weyl tensor ($\Phi_k \neq 0$), which we write as

$$\Phi_k = -3y^{-3}[(Cy - D)\cos y - (C + Dy)\sin y], \tag{148}$$

$$\mathscr{Z}_k = 3\sqrt{3}y^{-3}[2(C + Dy)\cos y + 2(Cy - D) \times \sin y - C(2 + y^2)], \tag{149}$$

$$\sigma_k = 3\sqrt{3}y^{-2}[D\cos y + C\sin y - Cy], \tag{150}$$

$$q_k^{(\gamma)} = -4\sqrt{3}y^{-1}[C\cos y - D\sin y - C], \tag{151}$$

$$q_k^{(\nu)} = -\frac{2\sqrt{3}}{R}y^{-1}[(2RC + Dy)\cos y + (Cy - 2RD)\sin y - 2RC], \tag{152}$$

$$\mathscr{X}_k^{(\gamma)} = 12y^{-2}[(C + Dy)\cos y + (Cy - D)\sin y - C], \tag{153}$$

$$\mathscr{X}_k^{(\nu)} = \frac{6}{R}y^{-2}[(2RC - Cy^2 + 2RDy)\cos y + (2RCy - 2RD + Dy^2)\sin y - 2RC], \tag{154}$$

where $y \equiv x/(3)^{1/2}$ and C and D are constants. There are also three solutions with vanishing Weyl tensor ($\Phi_k = 0$), which we write as

$$\mathscr{Z}_k = \frac{\sqrt{3}}{4} A_3 y^{-3}(2+y^2), \tag{155}$$

$$\sigma_k = \frac{\sqrt{3}}{4} A_3 y^{-1}, \tag{156}$$

$$q_k^{(\gamma)} = -\frac{1}{\sqrt{3}}(A_1 \cos y - A_2 \sin y + A_3 y^{-1}), \tag{157}$$

$$q_k^{(\nu)} = \frac{R-1}{\sqrt{3}R}(-A_1 \cos y + A_2 \sin y) - \frac{1}{\sqrt{3}} A_3 y^{-1}, \tag{158}$$

$$\mathscr{X}_k^{(\gamma)} = A_1 \sin y + A_2 \cos y + A_3 y^{-2}, \tag{159}$$

$$\mathscr{X}_k^{(\nu)} = \frac{R-1}{R}(A_1 \sin y + A_2 \cos y) + A_3 y^{-2}, \tag{160}$$

where A_1, A_2, and A_3 are further constants. The solution with only A_3 nonzero describes a radiation-dominated universe, which is exactly FRW except that the CDM has a peculiar velocity (relative to the velocity of the fundamental observers) $v_a^{(c)} = v_a^{(0)}/S$, where $v_a^{(0)}$ is a first-order vector, orthogonal to the fundamental velocity, which is parallel transported along the fundamental flow lines: $\dot{v}_a^{(0)} = O(2)$. This can be seen most clearly by adopting the energy frame, defined by the condition $q_a = 0$. This is arguably a better choice to make in the early universe, since u^a is then defined in terms of the dominant matter components rather than a minority component, such as the CDM, which has little effect on the gravitational dynamics. Choosing the energy frame and ignoring anisotropic stresses (which are frame-independent in linear theory), the CDM relative velocity evolves according to

$$\dot{v}_a^{(c)} + \frac{1}{3}\theta v_a^{(c)} + \frac{1}{4S}[(1-R)\mathscr{X}_a^{(\gamma)} + R\mathscr{X}_a^{(\nu)}] = 0 \tag{161}$$

in the radiation-dominated era. Since the CDM interacts with the other matter components through gravity alone and since the gravitational influence of the CDM on the dominant matter components may be ignored during radiation domination, equation (161) is the only equation governing the evolution of the perturbations that makes reference to the CDM. It follows that any solution of equation (161) defines a valid solution to the linearized perturbation equations. The solution, which has $v_a^{(c)} = v_a^{(0)}/S$ in an otherwise FRW universe corresponds to the solution labeled by A_3 in equations (155)–(160). Note that this solution decays in an expanding universe. Following standard practice, we assume that this mode may be ignored ($A_3 = 0$), since it would require highly asymmetric initial conditions at the end of the inflationary epoch if the decaying mode was significant during the epoch of interest here. Similar comments apply to the mode labeled by D in equations (148)–(154).

An important subclass of these solutions describe adiabatic modes. We assume that the appropriate covariant and gauge-invariant definition of adiabaticity is that

$$\frac{{}^{(3)}\nabla_a \rho^{(i)}}{\rho^{(i)} + p^{(i)}} = \frac{{}^{(3)}\nabla_a \rho^{(j)}}{\rho^{(j)} + p^{(j)}}, \tag{162}$$

for all species i and j (Bruni et al. 1992a). This condition, which is frame-independent in linear theory, is the natural covariant generalization of the (gauge-invariant) condition

$$\frac{\delta\rho^{(i)}}{\bar{\rho}^{(i)} + \bar{p}^{(i)}} = \frac{\delta\rho^{(j)}}{\bar{\rho}^{(j)} + \bar{p}^{(j)}}, \tag{163}$$

where $\delta\rho^{(i)} = \rho^{(i)} - \bar{\rho}^{(i)}$ is the usual gauge-dependent density perturbation, and overbars denote the background quantity. Demanding approximate adiabaticity between the photons and the neutrinos leaves only one free constant of integration, which we take to be C. The remaining constants are $A_1 = A_3 = D = 0$ and $A_2 = -6C$. Note that the constants of integration will depend on the mode label k, in general, so we have $C = C_k$.

This adiabatic solution may be developed further by including higher-moments of the neutrino distribution function and finding a series expansion of the (enlarged) system in x. To obtain solutions correct to $O(x^3)$, it is necessary to retain $\pi_k^{(\nu)}$ and $G_k^{(3)}$. The series solution that results is

$$\begin{aligned}
\Phi_k &= C\left[1 - \frac{389R+700}{168(2R+25)}x^2\right],\\
\mathscr{X}_k^{(\gamma)} &= \frac{(4R+15)C}{6(R+5)}x^2,\\
\mathscr{Z}_k &= -\frac{(4R+15)C}{4(R+5)}\left[x - \frac{4R+5}{18(4R+15)}x^3\right],\\
\mathscr{X}_k^{(\nu)} &= \frac{(4R+15)C}{6(R+5)}x^2,\\
\sigma_k &= -\frac{5C}{2(R+5)}\left[x + \frac{112R^2-16R-1050}{2520(2R+25)}x^3\right],\\
q_k^{(\gamma)} &= \frac{(4R+15)C}{54(R+5)}x^3, \quad \pi_k^{(\nu)} = -\frac{2C}{3(R+5)}x^2,\\
q_k^{(\nu)} &= \frac{(4R+23)C}{54(R+5)}x^3, \quad G_k^{(3)} = -\frac{5C}{63(R+5)}x^3,\\
G_k^{(l)} &= O(x^l) \text{ for } l > 3.
\end{aligned} \tag{164}$$

Note that on large scales ($x \ll 1$), the harmonic coefficient Φ_k of $\mathscr{E}_{ab}$ is constant along the flow lines. It follows that, on large scales, we may write $\mathscr{E}_{ab} = \mathscr{E}_{ab}^{(0)}/S^2$, where $\dot{\mathscr{E}}_{ab}^{(0)} = O(2)$. The series solution given in equation (164) is adiabatic between the photons and neutrinos to $O(x^3)$, but the adiabaticity is broken by the higher order terms. This difference in the dynamic behavior of radiation and neutrinos is due to their different kinetic equations; the neutrinos are collisionless, which allows higher order angular moments in the distribution function to grow, but the radiation is tightly coupled to the baryon fluid, which prevents the growth of higher order moments. The baryon relative velocity $v_a^{(b)}$ is determined by the condition that the radiation be nearly isotropic in the rest frame of the baryons,

$$v_k^{(b)} = \frac{3}{4} q_k^{(\gamma)}, \tag{165}$$

and the spatial gradients of the baryon and CDM follow from the adiabaticity condition,

$$\mathscr{X}_k^{(b)} = \mathscr{X}_k^{(c)} = \frac{3}{4}\mathscr{X}_k^{(\gamma)} = \frac{3}{4}\mathscr{X}_k^{(\nu)}, \tag{166}$$

where we have neglected the small effect of baryon pressure. The series solution given as equation (164) was used to provide adiabatic initial conditions for the numerical solution of the perturbation equations, discussed in the next section.

6. ADIABATIC SCALAR PERTURBATIONS IN A $K = 0$ UNIVERSE

In this section, we discuss the calculation of the CMB power spectrum from initially adiabatic scalar perturbations in an almost FRW universe with negligible spatial curvature. The evolution of anisotropy in the CMB and inhomogeneities in the density fields resulting from scalar perturbations may be found by solving numerically the equations presented in § 5, with initial conditions determined from the analytic solutions of the previous section. For adiabatic perturbations, the specification of initial conditions is particularly simple; there is a single function C_k of the mode label k to set. This function gives the (constant) amplitude of the harmonic component of the electric part of the Weyl tensor on super-horizon scales.

6.1. *The CMB Power Spectrum*

The gauge-invariant temperature perturbation from the mean, denoted by $\delta_{\mathrm{T}}(e)$, is given by equation (84). Substituting for the harmonic expansion of the angular moments $J^{(l)}_{a_1 \ldots a_l}$, we find

$$\delta_{\mathrm{T}}(e) = \frac{1}{4} \sum_{l=1}^{\infty} \frac{(2l+1)(2l)!}{(-2)^l (l!)^2} \left[\sum_k C_k \, T^{(l)}(k) Q^{(k)}_{a_1 \ldots a_l} e^{a_1} \ldots e^{a_l} \right], \tag{167}$$

where we have introduced the radiation-transfer function $T^{(l)}(k)$, which is a function of the eigenvalue k only. The transfer function is defined to be the value of $J^{(l)}_k$ for the initial condition $C_k = 1$. Since the dynamics of a scalar mode labeled by the index k depends only on the eigenvalue of the eigenfunction $Q^{(k)}$, the transfer function is a function of the eigenvalue k only. For the linearized theory considered here, we have $J^{(l)}_k = C_k \, T^{(l)}(k)$.

We have come as far as we can without making a specific choice for the scalar harmonic functions $Q^{(k)}$. To proceed, we introduce an almost FRW coordinate system (Ellis 1996) as follows: If the perturbations in the universe are only of scalar type, then the velocity u^a is hypersurface orthogonal, so that we may label the orthogonal hypersurfaces with a time label t. Furthermore, since we have chosen u^a to be the CDM velocity, which is geodesic, the flow-orthogonal hypersurfaces may be labeled unambiguously with proper time along the flow lines, so that $u^a = \nabla^a t$. The orthogonal hypersurfaces depart from being spaces of constant curvature only at first order, so we can introduce comoving spatial coordinates x^i in such a way that our (synchronous) coordinate system is almost FRW in form. (Latin indices, such as i, run from 1 to 3.) It is then straightforward to show that the functions $e^{i\boldsymbol{k} \cdot \boldsymbol{x}}$, where $\boldsymbol{k} \cdot \boldsymbol{x} = k^i x^i$ and k^i are constants, satisfy the defining equations for the scalar harmonic functions (eqs. [92]–[93]), with $k^2 = k^i k^i$, in an almost FRW universe with negligible spatial curvature. It follows that we may take

$$Q^{(k)} = e^{i\boldsymbol{k} \cdot \boldsymbol{x}} . \tag{168}$$

For the open and flat cases, the appropriate generalizations of the $e^{i\boldsymbol{k} \cdot \boldsymbol{x}}$ (Harrison 1967) should be used for the $Q^{(k)}$. Note that the expansion coefficients, such as $J^{(l)}_k$, depend on the detailed choice of the scalar harmonics $Q^{(k)}$, but that covariant tensors such as $\sum_k J^{(l)}_k Q^{(k)}_{a_1 \ldots a_l}$ are independent of this choice. If vector perturbations are also significant, we cannot use the velocity u^a to define a time coordinate in the manner described above. Instead, an almost FRW coordinate system should be constructed using an irrotational and geodesic velocity field $\hat{u}^a$, which is close to our chosen fundamental velocity u^a. (One possibility is to take $\hat{u}^a \propto \nabla^a \rho$.) Using this velocity field, almost FRW coordinates can be constructed by the above procedure (Ellis 1996). The resulting $Q^{(k)}$ will satisfy the defining (zero-order) properties of the scalar harmonics in the u^a frame, since the relative velocity of $\hat{u}^a$ is first-order.

Since $u^a e_a = 0$ and $u^a = \nabla^a t$, we can always choose the x^i so that at our observation point $e^a = (0, S^{-1} e^i)$, with $e^i e^i = 1$ (for example, one can choose the x^i so that $S x^i$ are locally Cartesian coordinates in the constant time hypersurface). Then it follows that, to zero order,

$$Q^{(k)}_{a_1 \ldots a_l} e^{a_1} \ldots e^{a_l} = \frac{(2i)^l (l!)^2}{(2l)!} P_l(\mu) Q^{(k)} , \tag{169}$$

where $\mu \equiv k^i e^i / k$ and $P_l(\mu)$ are the Legendre polynomials. This result demonstrates that expanding the angular moments of the distribution function in the covariant tensors $Q^{(k)}_{a_1 \ldots a_l}$ is equivalent to the usual Legendre expansion of the Fourier modes of the distribution function (which are axisymmetric about the wave vector $\boldsymbol{k}$) in an almost FRW universe, where the spatial curvature may be neglected.

Following standard practice, we make the assumption that we inhabit one realization of a stochastic ensemble of universes, so that the C_k are random variables. (The physical basis on which this assumption rests is that initial fluctuations were generated from causal quantum processes in the early universe, such as during a period of inflation; see for example, Kolb & Turner 1990.) Given our chosen form for the $Q^{(k)}$, statistical isotropy of the ensemble demands that the covariance matrix for the C_k takes the following form:

$$\langle C_k C^*_{k'} \rangle = C^2(k) \delta_{kk'} , \tag{170}$$

where $C^2(k)$ is the primordial power spectrum, which is a function of the eigenvalue k. The $\delta_{kk'}$ appearing in equation (170) is defined by $\sum_k \delta_{kk'} A_k = A_{k'}$, where A_k is an arbitrary function of the mode label k. The CMB power spectrum C_l is defined by $C_l \equiv \langle | a_{lm} |^2 \rangle$, where a_{lm} are the coefficients in the spherical harmonic expansion of the temperature anisotropy (see § 4). Substituting the harmonic expansion of the $J^{(k)}_{a_1 \ldots a_l}$ into equation (86) and using the zero-order result

$$h^{a_1 b_1} \ldots h^{a_l b_l} Q^{(k)}_{a_1 \ldots a_l} Q^{(k)*}_{a_1 \ldots a_l} = \frac{(-2)^l (l!)^2}{(2l)!} , \tag{171}$$

which follows from $Q^{(k)} = e^{i\boldsymbol{k} \cdot \boldsymbol{x}}$, we find the familiar expression for the CMB power spectrum in terms of the transfer functions and the primordial power:

$$C_l = \pi^2 \int d \ln k \, C^2(k) [T^{(l)}(k)]^2 . \tag{172}$$

We make the standard assumption that, on large scales, the primordial power spectrum may be approximated by a power law of the form $C^2(k) \propto k^{n_s - 1}$. Many inflationary models predict that the scalar index n_s will be close to unity (Kolb & Turner 1990). The case $n_s = 1$ describes the scale-invariant spectrum. This term arises from considering the logarithmic power spectrum in Fourier space of the (gauge-dependent) fractional density perturbation δ_ρ evaluated at horizon crossing. An analogous result can be found in the covariant and gauge-invariant approach: we evaluate the logarithmic power spectrum of the dimensionless scalar $\Delta \equiv {}^{(3)}\nabla^2 \rho/(\rho H^2)$ in the energy frame ($q_a = 0$). Making use of equation (15), with the contribution from anisotropic stress neglected and the frame-invariance of ${}^{(3)}\nabla^a \mathscr{E}_{ab}$ in linear theory, we find that

$$\partial_{\ln k} \langle \Delta^2 \rangle = \frac{16}{9} \pi \mathscr{H}_k^{-8} C^2(k) . \tag{173}$$

Note that Δ only receives a contribution at linear order from scalar modes (Ellis et al. 1990). In deriving equation (173), which is valid before the modes labeled by k reenter the Hubble radius, we have assumed that only the fastest growing scalar mode is significant, so that Φ_k is constant before horizon crossing. For given k, the logarithmic power in Δ evolves in time because of the presence of $\mathscr{H}_k$ on the right-hand side of equation (173). However, at horizon crossing, $\mathscr{H}_k$ falls below some critical value of order unity that is independent of k. It follows that, for the scalar index $n_s = 1$, the logarithmic power in Δ at horizon crossing is independent of scale.

6.2. *The Tight-Coupling Approximation*

At early times, when the baryons and photons are tightly coupled, the radiation is nearly isotropic in the frame of the baryons. In this limit, it is convenient to replace the propagation equations for the $J^{(l)}_{a_1 \ldots a_l}$ with $l \geq 1$ and for the baryon relative velocity $v_a^{(b)}$ with approximate equations that may be developed by an expansion in the photon mean free path $1/n_e \sigma_T$. The approximate equations are simpler to solve numerically than the exact equations, since the former do not include the large Thomson scattering terms present in the latter.

For scalar perturbations, it is simplest to work directly with the harmonic expansion coefficients, $J_k^{(l)}$ and $v_k^{(b)}$. The relevant timescales in the problem are the photon mean free time $t_c \equiv (n_e \sigma_T)^{-1}$, the expansion timescale $t_H \equiv H^{-1}$, and the light travel time across the wavelength of the mode under consideration, $t_k \equiv S/k$. In the tight-coupling approximation, we expand in the small dimensionless numbers t_c/t_H and t_c/t_k, so that the procedure is valid for $t_c \ll \min(t_H, t_k)$. While a mode is outside the horizon, $\min(t_H, t_k) = t_H$, whereas $\min(t_H, t_k) = t_k$ during the acoustic oscillations. In the CDM frame ($w_a = 0$), the procedure is similar to that usually employed (Peebles & Yu 1970; Ma & Bertschinger 1995) in the synchronous gauge. We combine the propagation equations given in § 5.1 for the photon moments $J_k^{(l)}$ ($l \geq 1$) and the baryon relative velocity $v_k^{(b)}$ to get the exact (in linear theory) equations:

$$(1+R)\dot{v}_k^{(b)} = -\frac{3}{4} R \dot{\Delta}_k - t_H^{-1} v_k^{(b)} + \frac{1}{4} R t_k^{-1} \mathscr{X}_k^{(\gamma)} + c_s^2 t_k^{-1} \mathscr{X}_k^{(b)} - \frac{1}{2} R t_k^{-1} \pi_k^{(\gamma)} , \tag{174}$$

$$(1+R)\Delta_k = -t_c \left[\dot{\Delta}_k - \frac{4}{3} t_H^{-1} v_k^{(b)} - \frac{1}{3} t_k^{-1} \mathscr{X}_k^{(\gamma)} + \frac{4}{3} c_s^2 t_k^{-1} \mathscr{X}_k^{(b)} + \frac{2}{3} t_k^{-1} \pi_k^{(\gamma)} \right] , \tag{175}$$

$$\pi_k^{(\gamma)} = -\frac{10}{9} t_c \left[\dot{\pi}_k^{(\gamma)} + \frac{3}{5} t_k^{-1} J_k^{(3)} - \frac{2}{5} t_k^{-1} \Delta_k - \frac{8}{15} t_k^{-1} (\sigma_k + v_k^{(b)}) \right] , \tag{176}$$

$$J_k^{(l)} = -t_c \left[\dot{J}_k^{(l)} + t_k^{-1} \left(\frac{l+1}{2l+1} J_k^{(l+1)} - \frac{l}{2l+1} J_k^{(l-1)} \right) \right] \quad \text{for } l \geq 3 , \tag{177}$$

where $\Delta_k \equiv q_k^{(\gamma)} - 4 v_k^{(b)}/3$ and, for this section, $R \equiv 4\rho^{(\gamma)}/3\rho^{(b)}$. Iterating these equations gives the tight-coupling expansions

$$\dot{v}_k^{(b)} = \dot{v}_0^{(b)} + \dot{v}_1^{(b)} + \ldots , \tag{178}$$

$$\Delta_k = \Delta_1 + \Delta_2 + \ldots , \tag{179}$$

$$\pi_k^{(\gamma)} = \pi_1^{(\gamma)} + \pi_2^{(\gamma)} + \ldots , \tag{180}$$

$$J_k^{(l)} = J_{l-1}^{(l)} + J_l^{(l)} + \ldots , \tag{181}$$

where the subscript on the variables on the right-hand side denotes the order in the expansion parameter $\epsilon = \max(t_c/t_H, t_c/t_k)$. To avoid cluttering of indices, we leave the mode label k implicit. We shall only require the results to first order in ϵ:

$$\dot{v}_0^{(b)} = \frac{1}{1+R} \left(\frac{1}{4} \frac{k}{S} R \mathscr{X}_k^{(\gamma)} + \frac{k}{S} c_s^2 \mathscr{X}_k^{(b)} - H v_k^{(b)} \right) , \tag{182}$$

$$\dot{v}_1^{(b)} = -\frac{3R^2}{2(1+R)^2} H \Delta_1 - \frac{R}{2(1+R)} \frac{k}{S} \pi_1^{(\gamma)} - \frac{R}{4(1+R)^2} \times \left[4 \frac{t_c}{t_H} (\dot{H} + 2H^2) H^{-1} v_k^{(b)} + \frac{t_c}{t_k} (2H \mathscr{X}_k^{(\gamma)} + \dot{\mathscr{X}}_k^{(\gamma)} - 4 c_s^2 \dot{\mathscr{X}}_k^{(b)}) \right] , \tag{183}$$

$$\Delta_1 = \frac{1}{3(1+R)} \left[\frac{t_c}{t_k} \left(\mathscr{X}_k^{(\gamma)} - 4 c_s^2 \mathscr{X}_k^{(b)} \right) + 4 \frac{t_c}{t_H} v_k^{(b)} \right] , \tag{184}$$

$$\pi_1^{(\gamma)} = \frac{16}{27} \frac{t_c}{t_k} (\sigma_k + v_k^{(b)}) . \tag{185}$$

The propagation equation for $q_k^{(\gamma)}$ in the tight-coupling approximation may be obtained from the exact equation (174), with $\dot{\Delta}_k$ replaced by $\dot{q}_k^{(\gamma)} - 4\dot{v}_k^{(b)}/3$ and $\dot{v}_k^{(b)}$ and $\pi_k^{(\gamma)}$ replaced by their tight-coupling expansions.

6.3. *Numerical Results*

We are now in a position to evolve an initial set of perturbations from early times to the present in an almost FRW universe with negligible spatial curvature and to calculate the power spectrum of the CMB anisotropies that result. In this section we present the results of a numerical simulation in the standard CDM model with the parameters $H_0 = 50$ km s^{-1} Mpc^{-1}, baryon fraction $\Omega_b = 0.05$, CDM fraction $\Omega_c = 0.95$, helium fraction 0.24, zero cosmological constant, and a scale-invariant spectrum of initially adiabatic conditions ($n_s = 1$).

Our code to solve the covariant and gauge-invariant perturbation equations in the CDM frame, including the Boltzmann hierarchies for the photons and neutrinos, was based on the serial COSMICS code developed by Bertschinger and Bode and described in Ma & Bertschinger (1995).[1] We modified the COSMICS code to solve the covariant equations given in this paper for the matter variables and for the spatial gradient of the expansion, $\mathscr{Z}_a$. The shear, which is required to solve the Boltzmann hierarchies for the photons and neutrinos, was determined from equation (16). The electric part of the Weyl tensor, $\mathscr{E}_{ab}$, could then be determined from equation (15). Our calculations of the zero-order ionization history of the universe, which fully include the effects of helium and hydrogen recombination, followed Ma & Bertschinger (1995), as did our truncation schemes for the photon and neutrino Boltzmann hierarchies. The first-order tight-coupling approximation was used at sufficiently early times that $\max(t_c/t_H, t_c/t_k) \ll 1$.

In Figure 1 we show the variation of the harmonic coefficients $\mathscr{X}_k^{(i)}$ of the comoving fractional spatial gradients in the CDM frame against redshift in the standard CDM model. Similar plots were given by Ma & Bertschinger (1995) for the Fourier components of the (gauge-dependent) density perturbations $\delta_\rho^{(i)} \equiv (\rho^{(i)} - \bar{\rho}^{(i)})/\bar{\rho}^{(i)}$, where $\bar{\rho}^{(i)}$ is the density of the species i in the background model. Our results, given in Figure 1, agree well with the synchronous-gauge results of Ma & Bertschinger (1995). This is because the constant time surfaces in this gauge are orthogonal to the CDM velocity, so that $\mathscr{X}_a^{(i)}$ is a covariant measure of the density inhomogeneity in these surfaces. Although $\delta_\rho^{(i)}$ is gauge-dependent in the synchronous gauge, the gauge conditions restrict this gauge dependence to transformations of the form $\delta_\rho^{(i)} \mapsto \delta_\rho^{(i)} - \alpha\dot{\bar{\rho}}/\bar{\rho}$, where α is a first-order constant. It follows that the Fourier coefficients of $\delta_\rho^{(i)}$ are gauge-invariant away from $\boldsymbol{k} = 0$ in Fourier space.

The qualitative behavior of the comoving density gradients can be seen directly from their propagation equations. For scalar perturbations, it is simplest to work directly with the equations of motion (113)–(116) for the harmonic coefficients $\mathscr{X}_k^{(i)}$ in the CDM frame. Eliminating the spatial gradients of the expansion, $\mathscr{Z}_a$, we find the following second-order equations:

$$\ddot{\mathscr{X}}_k^{(\gamma)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(\gamma)} + \frac{1}{3}\frac{k^2}{S^2}\mathscr{X}_k^{(\gamma)} = \frac{2}{3}\kappa\sum_i(\rho^{(i)} + 3p^{(i)})\mathscr{X}_k^{(i)} - \frac{1}{3}\frac{k}{S}\theta q_k^{(\gamma)} + \frac{2}{3}\frac{k^2}{S^2}\pi_k^{(\gamma)} - n_e\sigma_{\mathrm{T}}\frac{k}{S}\left(\frac{4}{3}v_k^{(b)} - q_k^{(\gamma)}\right), \quad (186)$$

$$\ddot{\mathscr{X}}_k^{(\nu)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(\nu)} + \frac{1}{3}\frac{k^2}{S^2}\mathscr{X}_k^{(\nu)} = \frac{2}{3}\kappa\sum_i(\rho^{(i)} + 3p^{(i)})\mathscr{X}_k^{(i)} - \frac{1}{3}\frac{k}{S}\theta q_k^{(\nu)} + \frac{2}{3}\frac{k^2}{S^2}\pi_k^{(\nu)}, \quad (187)$$

$$\ddot{\mathscr{X}}_k^{(b)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(b)} + c_s^2\frac{k^2}{S^2}\mathscr{X}_k^{(b)} = \frac{1}{2}\kappa\sum_i(\rho^{(i)} + 3p^{(i)})\mathscr{X}_k^{(i)} + n_e\sigma_{\mathrm{T}}\frac{\rho^{(\gamma)}}{\rho^{(b)}}\frac{k}{S}\left(\frac{4}{3}v_k^{(b)} - q_k^{(\gamma)}\right), \quad (188)$$

$$\ddot{\mathscr{X}}_k^{(c)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(c)} = \frac{1}{2}\kappa\sum_i(\rho^{(i)} + 3p^{(i)})\mathscr{X}_k^{(i)}, \quad (189)$$

where we have ignored baryon pressure except in the acoustic term $c_s^2(k^2/S^2)\mathscr{X}_k^{(b)}$, which can be significant on small scales. In the limiting case that the mode is well outside the Hubble radius ($\mathscr{H}_k \gg 1$), the equations of motion for each component reduce to the common form

$$\ddot{\mathscr{X}}_k^{(i)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(i)} = \kappa\frac{1}{2}\left(1 + \frac{p^{(i)}}{\rho^{(i)}}\right)\sum_j(\rho^{(j)} + 3p^{(j)})\mathscr{X}_k^{(j)}. \quad (190)$$

For adiabatic initial conditions, it is clear that the adiabatic condition, given as equation (162), is maintained while the mode is outside the Hubble radius. Solving equation (190) for adiabatic perturbations gives growing modes proportional to t and $t^{2/3}$ during radiation and matter domination, respectively.

If a mode enters the Hubble radius prior to last scattering, the photon/baryon fluid, which is still tightly coupled, undergoes acoustic oscillations. To lowest order in the tight-coupling parameter, $\max(t_c/t_H, t_c/t_k)$, the photon and baryon perturbations remain adiabatic, evolving according to

$$\ddot{\mathscr{X}}_k^{(\gamma)} + \frac{2}{3}\theta\dot{\mathscr{X}}_k^{(\gamma)} + \frac{R + 3c_s^2}{3(1 + R)}\frac{k^2}{S^2}\mathscr{X}_k^{(\gamma)} = \frac{2}{3}\kappa\sum_i(\rho^{(i)} + 3p^{(i)})\mathscr{X}_k^{(i)} - \frac{4R}{9(1 + R)}\frac{k}{S}\theta v_k^{(b)}, \quad (191)$$

where $R \equiv 4\rho^{(\gamma)}/3\rho^{(b)}$. The solution of the homogeneous equation describes acoustic oscillations in a fluid with sound speed squared $(R + 3c_s^2)/3(1 + R)$, which are damped by the expansion of the universe. However, the oscillations are driven gravitationally by the gradient ${}^{(3)}\nabla_a(\rho + 3p)$, which gives an almost constant amplitude oscillation in the radiation-dominated era. The Silk damping that is visible in Figure 1 for $k = 1.0$ Mpc^{-1} at $z \simeq 10^{-3.5}$ arises from photon diffusion (which is not described by the lowest order tight-coupling approximation), and so it is not described by equation (191). The neutrino perturbation also oscillates once inside the Hubble radius in the radiation-dominated region, while the power-law growth of the CDM is impeded by the gravitational attraction of the oscillating dominant component (the inhomogeneous term in eq. [189]). In the matter-dominated era, the CDM becomes the dominant component, so we again see power-law growth of the CDM perturbation on all scales. Before last scattering, the photons and baryons remain tightly coupled, but the character of the ${}^{(3)}\nabla_a(\rho + 3p)$ driving term in equation (191) changes from an oscillation to a power law as the CDM becomes dominant. At last scattering, the photons and baryons decouple. The baryons no longer feel the pressure support provided by the photons; the Jeans length of the baryons is very small, and the acoustic term in equation (188) is negligible. The ${}^{(3)}\nabla_a(\rho + 3p)$ driving term attracts the baryons into the potential wells caused principally by inhomogeneity of the CDM, so that $\mathscr{X}_a^{(b)}$ relaxes to $\mathscr{X}_a^{(c)}$ as a power law. After last scattering, the photons and neutrinos continue to undergo driven oscillations, which decay toward the particular integral $\mathscr{X}_k^{(\gamma)} = \mathscr{X}_k^{(\nu)} = 6\mathscr{H}_k^2\mathscr{X}_k^{(c)}$.

In Figure 2 we show the CMB power spectrum calculated from a simulation in the standard CDM model. On large scales, the plateau arises from the usual potential fluctuations $\sum_k\Phi_k Q^{(k)}/3$ on the last scattering surface (Sachs & Wolfe 1967). The oscillations in the CMB power spectrum on smaller scales (the Doppler peaks) arise from the acoustic oscillations in the baryon/photon fluid. These oscillations give rise to strongly scale-dependent gradients of the

[1] The COSMICS package, including full documentation, is available at http://arcturus.mit.edu/cosmics.

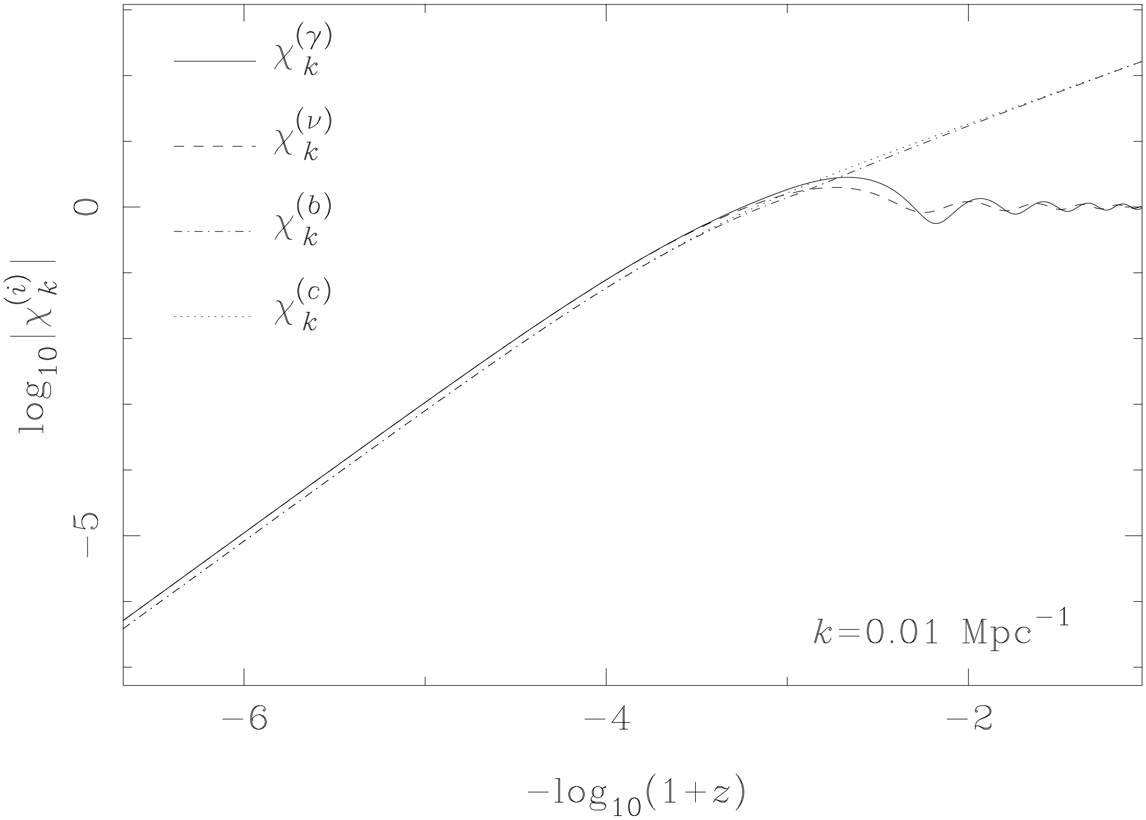

FIG. 1*a*

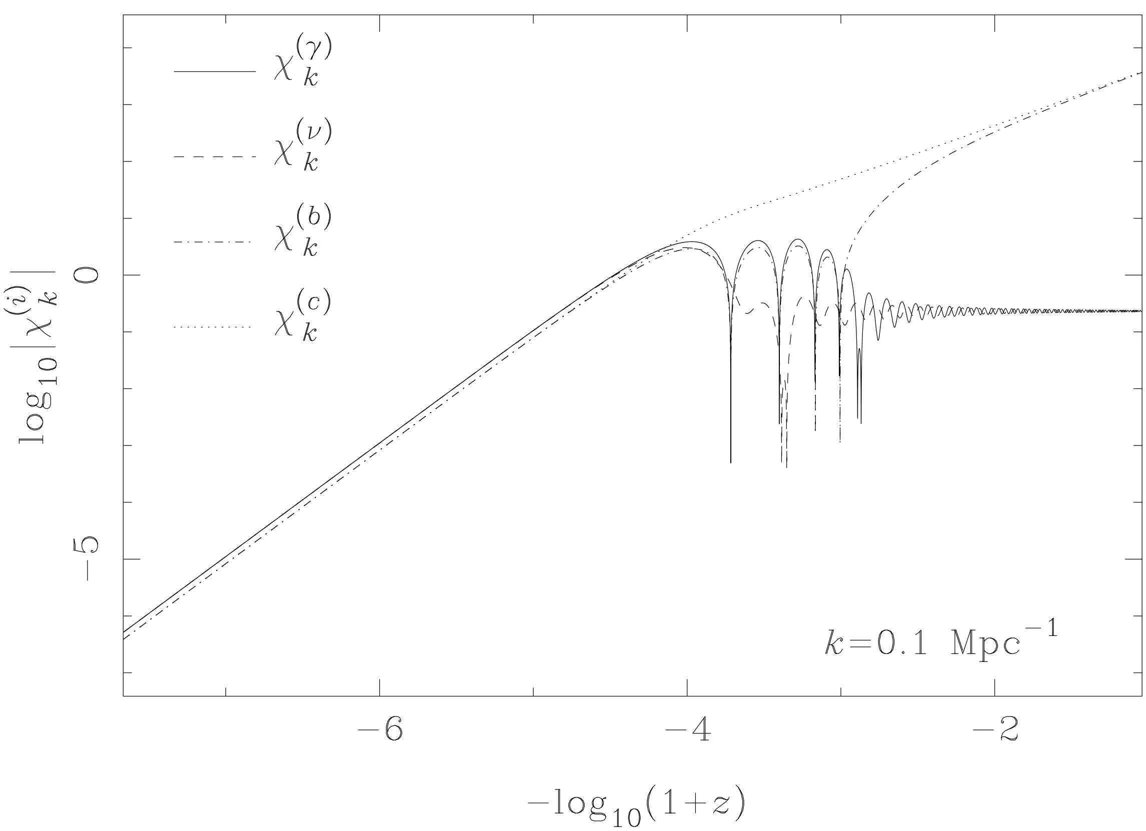

FIG. 1*b*

FIG. 1.—Variation of the harmonic coefficients of the fractional comoving density gradients in the CDM frame with redshift in the standard CDM model: $\Omega_c = 0.95$, $\Omega_b = 0.05$, $\Omega_\Lambda = 0$, $H_0 = 50$ km s^{-1} Mpc^{-1}, with helium fraction 0.24, for $k = 0.01$ (*a*), 0.1 (*b*), and 1.0 (*c*) Mpc^{-1}. The normalization is chosen so that $\Phi_k = 1$ for all k at early times.

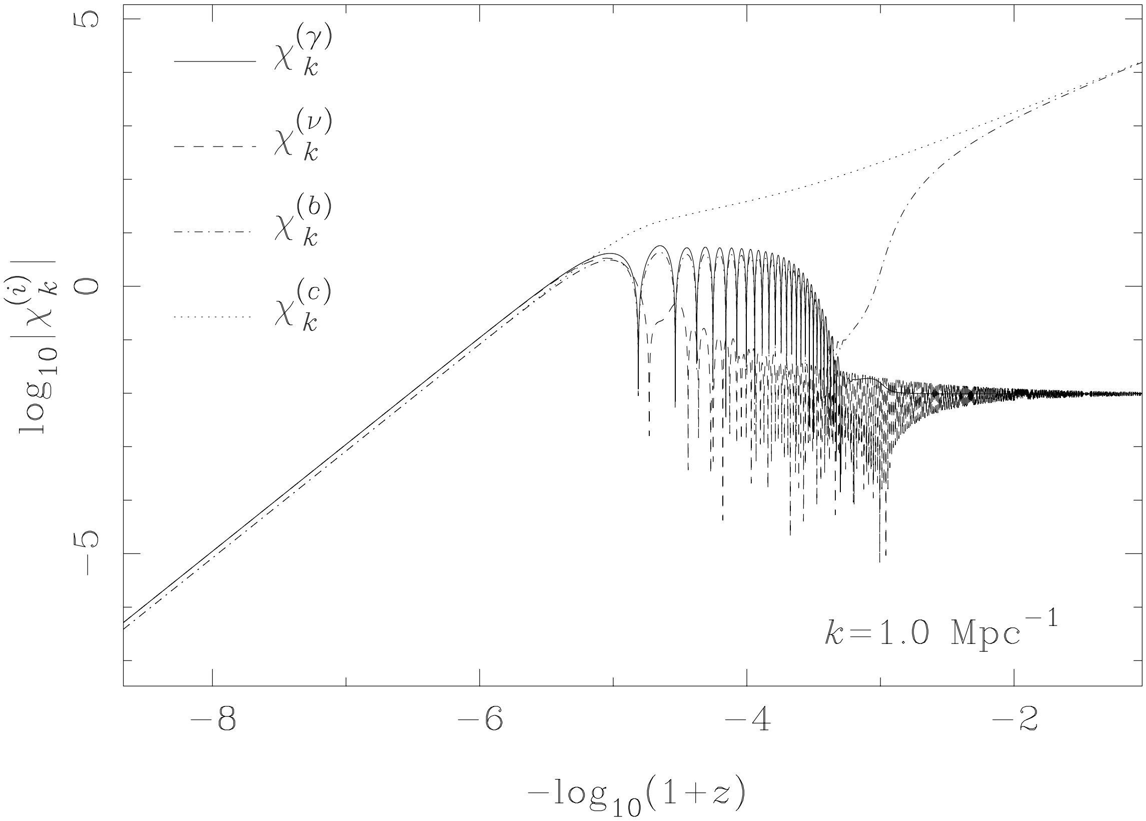

FIG. 1*c*

photon energy density in the energy frame—which in the approximation of instantaneous recombination can be interpreted as temperature variations across the last scattering surface—and a local scale-dependent distortion of the last scattering surface relative to the energy frame. Since the last scattering surface is well approximated by a hypersurface of constant radiation temperature (so that recombination does occur there), it is more correct to interpret the Doppler peaks in terms of the local variations in redshift along null geodesics back to the last scattering surface than in terms of temperature variations on the last scattering surface. (There is another significant contribution to the Doppler peaks, which is of dipole nature on the last scattering surface and tends to fill in the power spectrum near the first Doppler peak; see Hu & Sugiyama 1995 and Challinor & Lasenby 1998 for more details.) On the smallest scales, the power spectrum is damped because of photon diffusion in the photon/baryon plasma prior to recombination.

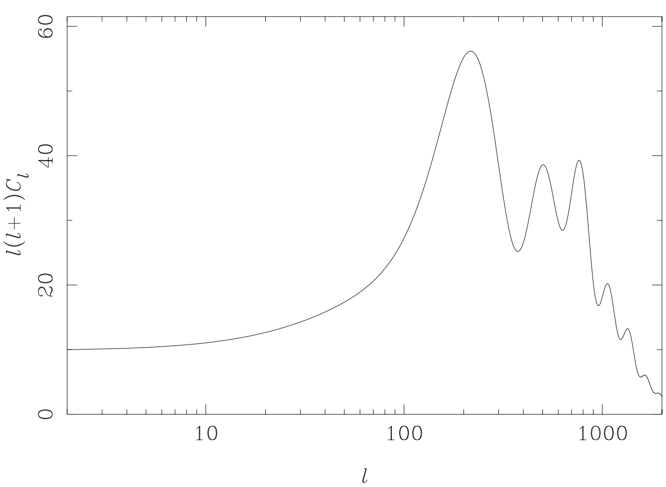

FIG. 2.— Power spectrum of scalar CMB anisotropies in the standard CDM model: $\Omega_c = 0.95$, $\Omega_b = 0.05$, $\Omega_\Lambda = 0$, $H_0 = 50$ km s^{-1} Mpc^{-1}, $n_s = 1$, with helium fraction 0.24. The normalization of the vertical scale is arbitrary.

7. TENSOR PERTURBATIONS

The covariant equations of § 3 are independent of any nonlocal splitting of the perturbations into scalar, vector, and tensor modes. In the linear approximation, we have seen that the equations describing scalar perturbations can be obtained from the full set of equations in a straightforward manner. The same is also true for vector and tensor perturbations. Vector perturbations are not expected to be important today in nonseeded models, since the vorticity decays in an expanding universe (Hawking 1966). However, most inflationary models do predict the generation of a primordial spectrum of gravitational waves (tensor modes) during an epoch of inflation (see Lidsey et al. 1997 for a comprehensive review). The existence of such a background of relic gravitational waves would have a significant effect on the CMB anisotropy power spectrum on large scales (Crittenden et al. 1993). For completeness, we give the tensor multipole equations in this section. We defer a detailed derivation of the equations and a discussion of their relation to the multipole equations usually employed in calculations of the effects of tensor modes (Crittenden et al. 1993) to a future paper (Challinor 1998).

In the covariant approach to cosmology, gravitational waves are characterized by requiring that the vorticity and all gauge-invariant vectors and scalars vanish at first order (Dunsby et al. 1997), so that the spatial gradients of the density and expansion vanish as well as the acceleration and the heat fluxes. The electric and magnetic parts of the Weyl tensor, the shear tensor, and the anisotropic stress are constrained to be transverse:

$$^{(3)}\nabla^a \mathscr{E}_{ab} = 0, \quad ^{(3)}\nabla^a \mathscr{B}_{ab} = 0, \quad ^{(3)}\nabla^a \sigma_{ab} = 0, \quad ^{(3)}\nabla^a \pi_{ab} = 0 . \tag{192}$$

It is straightforward to show that these conditions are consistent with the linearized propagation equations given in § 2 (see also Maartens 1997 for a discussion of consistency of the exact equations for irrotational dust spacetimes). As with scalar perturbations, it is convenient to expand harmonically the first-order covariant variables in tensors derived from solutions of a generalized Helmholtz equation. For tensor perturbations, we employ the tensor-valued solutions $Q^{(k)}_{ab} = Q^{(k)}_{(ab)}$ (Hawking 1966; note that we use the same symbol for the tensor harmonics and the second-rank tensors derived from the scalar harmonic functions to avoid cluttering formulae with additional labels), which satisfy the zero-order relations

$$Q^{(k)}_{ab}\, u^a = 0, \quad ^{(3)}\nabla^a Q^{(k)}_{ab} = 0, \quad \dot{Q}^{(k)}_{ab} = 0 . \tag{193}$$

Expanding $\mathscr{E}_{ab}$, $\mathscr{B}_{ab}$, σ_{ab}, and π_{ab} in these tensors (as in § 5 but with $Q^{(k)}_{ab}$ replaced by the tensor harmonics and the scalar-valued variables Φ_k replaced by $\mathscr{E}_k$), we obtain simple propagation equations for the electric part of the Weyl tensor and the shear:

$$\left(\frac{k}{S}\right)^2\left(\dot{\mathscr{E}}_k + \frac{1}{3}\,\theta\mathscr{E}_k\right) - \frac{k}{S}\left(\frac{k^2}{S^2} + \frac{3K}{S^2} - \frac{1}{2}\,\gamma\kappa\rho\right)\sigma_k = -\frac{1}{6}\,(3\gamma - 1)\kappa\rho\theta\pi_k + \frac{1}{2}\,\kappa\rho\dot{\pi}_k , \tag{194}$$

$$\frac{k}{S}\left(\dot{\sigma}_k + \frac{1}{3}\,\theta\sigma_k\right) + \left(\frac{k}{S}\right)^2 \mathscr{E}_k = -\frac{1}{2}\,\kappa\rho\pi_k . \tag{195}$$

Note that we have used the constraint equation (13) to eliminate the magnetic part of the Weyl tensor from the propagation equation for the electric part. Equations (194) and (195) close up, with the anisotropic stress treated as a known field, to give a second-order equation for the shear:

$$\ddot{\sigma}_k + \theta\dot{\sigma}_k + \left[\frac{k^2}{S^2} + \frac{2K}{S^2} - \frac{1}{3}\,(3\gamma - 2)\kappa\rho\right]\sigma_k = \kappa\rho\,\frac{S}{k}\left[\frac{1}{3}\,(3\gamma - 2)\theta\pi_k - \dot{\pi}_k\right], \tag{196}$$

which generalizes the homogeneous equation derived in Dunsby et al. (1997) to include anisotropic stress.

For the photon and neutrino angular moments, $J^{(l)}_{a_1 \ldots a_l}$ and $G^{(l)}_{a_1 \ldots a_l}$, we expand in tensors $Q^{(k)}_{a_1 \ldots a_l}$ derived from the tensor harmonics using the same recursion relation as for scalar perturbations,

$$Q^{(k)}_{a_1 \ldots a_l} = \frac{S}{k}\left({}^{(3)}\nabla_{(a_1} Q^{(k)}_{a_2 \ldots a_l)} - \frac{l-1}{2l-1}\, {}^{(3)}\nabla^b Q^{(k)}_{b(a_1 \ldots a_{l-2}} h_{a_{l-1} a_l)} \right), \tag{197}$$

for $l \geq 2$. This procedure gives the following covariant Boltzmann multipole equations for tensor perturbations: for the anisotropic stress ($l = 2$),

$$\dot{\pi}^{(\gamma)}_k + \frac{1}{3}\frac{k}{S}\left(1 - \frac{6K}{k^2}\right)J^{(3)}_k - \frac{8}{15}\frac{k}{S}\,\sigma_k = -\frac{9}{10}\,n_e\sigma_T\pi^{(\gamma)}_k , \tag{198}$$

$$\dot{\pi}^{(\nu)}_k + \frac{1}{3}\frac{k}{S}\left(1 - \frac{6K}{k^2}\right)G^{(3)}_k - \frac{8}{15}\frac{k}{S}\,\sigma_k = 0 , \tag{199}$$

and for $l \geq 3$,

$$\dot{J}^{(l)}_k + \frac{k}{S}\left(\frac{(l+3)(l-1)}{(l+1)(2l+1)}\left\{1 - \left[(l+1)^2 - 3\right]\frac{K}{k^2}\right\} \times J^{(l+1)}_k - \frac{l}{2l+1}\,J^{(l-1)}_k\right) = -n_e\sigma_T J^{(l)}_k , \tag{200}$$

$$\dot{G}^{(l)}_k + \frac{k}{S}\left(\frac{(l+3)(l-1)}{(l+1)(2l+1)}\left\{1 - \left[(l+1)^2 - 3\right]\frac{K}{k^2}\right\} \times G^{(l+1)}_k - \frac{l}{2l+1}\,G^{(l-1)}_k\right) = 0 . \tag{201}$$

As we stressed earlier, by performing the covariant angular expansion before harmonically expanding the moments, the necessary angular dependence of the moments appears automatically; whereas in the metric-based approach, this (azimuthal) dependence must be put in by hand and is different for the two polarizations of gravitational waves. It should be noted that the moment equations given here are not the same as those satisfied by the $\tilde{\Delta}^i_{Il}$ variables—where $i = +, \times$ labels the polarization of the gravitational wave and I denotes the intensity Stokes parameter (we follow the notation employed in Kosowsky 1996)—that are usually employed in metric-based calculations of the effects of tensor modes. This is because the covariant angular expansion gives rise to a more natural set of variables, the $J^{(l)}_k$, which are related to the temperature anisotropy in a simpler manner than the $\tilde{\Delta}^i_{Il}$. In particular, the lth multipole C_l of the anisotropy power spectrum depends only on $J^{(l)}_k$, whereas C_l depends on the $(l-2)$, l, and $(l+2)$th moments, $\tilde{\Delta}^i_{Il}$, thus obscuring the physical interpretation of these variables. The relation between the two sets of variables will be discussed further in Challinor (1998).

The first-order propagation equations for the shear and electric part of the Weyl tensor, along with the Boltzmann multipole equations for the photons and neutrinos, give a closed set of equations that can be solved to calculate the temperature anisotropy for given initial conditions. The numerical solution of these equations will be considered elsewhere (Challinor 1998).

8. CONCLUSION

We have shown how the full kinetic-theory calculation of the evolution of CMB anisotropies and density inhomogeneities can be performed in the covariant approach to cosmology (Ehlers 1993; Ellis 1971) using the gauge-invariant perturbation theory of Ellis & Bruni (1989). Adopting covariantly defined, gauge-invariant variables throughout ensured that our discussion avoided the gauge ambiguities that appear in certain gauges and that all variables had a clear, physical interpretation. We presented a unified set of equations describing the evolution of photon and neutrino anisotropies and cosmological perturbations in the CDM model, which were independent of a decomposition into scalar, vector, or tensor modes and the associated harmonic analysis. Although we only considered the case of linear perturbations around an FRW universe here, it is straightforward to extend the approach to include nonlinear effects (Maartens et al. 1998), which should allow a physically transparent discussion of second-order effects on the CMB. Indeed, the ease with which one can write down the exact equations for the physically relevant variables is one of the major strengths of the covariant approach.

The linear equations describing scalar and tensor modes were obtained from the full set of equations in a straightforward and unified manner, highlighting the advantage of having the full equations (independent of the decomposition into scalar, vector, and tensor modes) available. For the scalar case, the Boltzmann multipole equations for the moments of the distribution functions obtained here were equivalent to those usually seen in the literature. However, for tensor modes, the covariant approach led naturally to a set of moment variables that more conveniently describe the temperature anisotropy than those usually employed. For scalar modes, we discussed the solution of the perturbation equations in detail, including the integral solution of the Boltzmann multipole equations and the relation between the timelike integrations performed in the multipole approach to calculating CMB anisotropies and the lightlike integrations of the line-of-sight approach. The numerical solution of the scalar equations in a $K = 0$, almost FRW, CDM universe were also discussed. Our numerical results provide independent confirmation of those of other groups (see, for example, Ma & Bertschinger 1995; Seljak & Zaldarriaga 1996), who have obtained their results by employing noncovariant methods in specific gauges. Typically, these methods require one to keep careful track of all residual gauge freedom, both to enable identification of any gauge-mode solutions and to ensure that the final results quoted are gauge-invariant (and hence observable). Fortunately, the isotropy of the photon distribution function in an exact FRW universe ensures that the CMB power spectrum, as calculated from the gauge-dependent perturbation to the distribution function, is gauge-invariant for $l \geq 1$.

We hope to have shown the ease with which the covariant approach to cosmology can be applied to the problem of calculating CMB anisotropies. The covariant and gauge-invariant method discussed here frees one from the gauge problems that have caused confusion in the past and focuses attention on the physically relevant variables in the problem and the underlying physics. Future work in this area will include the discussion of nonlinear effects (Maartens at al. 1998), the inclusion of polarization, and the effects of HDM, all of which can be expected to bring the same advantages of physical clarity and transparency that we hope to have demonstrated here.

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