

AN INCOHERENT α - Ω DYNAMO IN ACCRETION DISKS

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ABSTRACT

We use the mean-field dynamo equations to show that spatially and temporally incoherent fluctuations in the helicity in mirror-symmetric turbulence in a shearing flow can generate a large-scale, coherent magnetic field. We illustrate this effect with simulations of a few simple systems. For statistically homogeneous turbulence, we find that the dynamo growth rate is roughly $\tau_{\text{eddy}}^{-1/3} \tau_{\text{shear}}^{-2/3} N_{\text{eddy}}^{-1/3} (\lambda_{\text{eddy}}/L_B)^{2/3}$, where τ_{eddy} is the eddy turnover time, τ_{shear}^{-1} is the local shearing rate, N_{eddy} is the number of eddies per magnetic domain, λ_{eddy} is the size of an eddy, and L_B is the extent of a magnetic domain perpendicular to the mean flow direction. Even in the presence of turbulence and shear the dynamo can be stopped by turbulent dissipation if (for example) the eddy scale is close to the magnetic domain scale and $\tau_{\text{shear}} > \tau_{\text{eddy}}$. We also identify a related incoherent dynamo in a system with a stationary distribution of helicity with a high-spatial frequency and an average value of zero. In accretion disks, the incoherent dynamo can lead to axisymmetric magnetic domains the radial and vertical dimensions of which will be comparable to the disk height. This process may be responsible for dynamo activity seen in simulations of dynamo-generated turbulence involving, for example, the Balbus-Hawley instability. However, although it explains the generation of a magnetic field in numerical simulations without significant large-scale average helicity and the occasional field reversals, it also predicts that the dimensionless viscosity will scale as $\sim (h/r)^2$, which is not seen in the simulations. On the other hand, this result is consistent with phenomenological models of accretion disks, although these suggest a slightly shallower dependence on h/r . We discuss some possible resolutions to these contradictions.

Subject headings: accretion, accretion disks — MHD — turbulence

1. INTRODUCTION

The emergence of dynamically important and well-ordered magnetic fields from initial conditions in which the magnetic fields are weak and largely random (the dynamo problem) is probably the single most important topic in astrophysical magnetohydrodynamics. In the usual approach to mean-field dynamo theory, magnetic field growth is driven by an asymmetry in the underlying fluid flow that tends to twist the magnetic field and yields an extra net electromotive force in the toroidal direction (see, for example, Parker 1979; Moffatt 1978). It is usually assumed that in the absence of such an asymmetry the dynamo process will fail. Nevertheless, there is a potential loophole in that statistically mirror-symmetric turbulence will still have a fluctuating local helicity. This point has been studied previously (see Moffatt 1978; Zel'dovich et al. 1988) without the discovery of any situations likely to lead to a successful dynamo.

In this paper, we will show that under some circumstances the presence of a strong local shear will lead to a successful incoherent dynamo. We will also see that this dynamo is particularly vulnerable to strong turbulent dissipation, so that shear and turbulence are by no means sufficient to guarantee a successful dynamo.

One of the more important applications of this process is to the generation of magnetic fields in accretion disks and the consequent transport of angular momentum. The traditional approach (Shakura & Sunyaev 1973) is to assume

that accretion disks are characterized by an effective viscosity arising from an unspecified collective process, given by $\alpha_{\text{SS}} c_s h$, where c_s is the local sound speed, h is the disk half-thickness, and α_{SS} is a constant of order unity. More recently, there has been the realization (Balbus & Hawley 1991) that a previously discovered magnetic field instability in a shearing flow (Velikhov 1959; Chandrasekhar 1961) will act to produce a positive angular momentum flux in an accretion disk. This has given rise to two separate but related claims. The first is the proposal that this is the dominant mechanism of angular momentum transport in ionized accretion disks. The second is the proposal that this instability, by itself, leads to a turbulent dynamo that drives the magnetic field into equipartition with the ambient pressure—i.e., $V_A \sim c_s$, where V_A is the Alfvén speed in the disk. This leads to the conclusion that α_{SS} is a constant of order unity. More precisely, in numerical simulations (e.g., Brandenburg et al. 1995) α_{SS} saturates at a value less than 10^{-2} , both because the magnetic pressure saturates at a fraction of the gas pressure and because the horizontal off-diagonal components of $\langle \mathbf{B} \cdot \mathbf{B} \rangle$ are a fraction of $\langle B_\theta^2 \rangle$.

We will see that the incoherent dynamo process is capable of generating a well-ordered magnetic field in astrophysical disks. However, in this case, the magnetic field energy density will saturate well below the ambient fluid pressure and by an amount that depends on the disk geometry, i.e., the ratio of h to the disk radius. This leaves open the possibility that this is not the dominant dynamo mechanism in accretion disks. In fact, numerical simulations of magnetic fields in accretion disks (Brandenburg et al. 1996) show no such dependence. On the other hand, a universal value of α_{SS} is inconsistent with phenomenological

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models of disks. Successful models of dwarf novae outbursts and X-ray transients (see Cannizzo 1994 and references therein), the distribution of light in quiescent dwarf novae disks (Minoshige & Wood 1989), and the cooling front speed during the decline from maximum light in these systems (Cannizzo, Chen, & Livio 1995; Vishniac & Wheeler 1996) all imply that the dimensionless viscosity, α_{SS} , varies in accordance with the general law $\alpha_{\text{SS}} \propto (h/r)^{3/2}$. The cooling front results are particularly important, since they are insensitive to any of the complicated physics governing the transition to the cold state or its thermal structure. We will discuss ways in which this contradiction might be eliminated.

In § 2, we discuss the conceptual basis of an incoherent dynamo in a turbulent shearing medium and estimate the growth rate. In § 3, we examine the related case of a dynamo driven by a spatially incoherent, but time-independent helicity and show how a large-scale field can result from the competition between helicity and dissipation. In § 4, we apply this to accretion disks and show that the incoherent dynamo gives a positive growth rate only for axisymmetric magnetic domains. We estimate the saturated state of the field and discuss our results in light of numerical simulations of magnetic fields in a Keplerian disk and phenomenological studies of their behavior. In § 5, we summarize our results and their implications for astrophysical disks and numerical simulations of such disks.

2. THE INCOHERENT DYNAMO

We begin with a brief review of the relevant parts of mean-field dynamo theory (for a general review, see Moffatt 1978; Parker 1979). In a highly conducting medium, the magnetic field obeys the induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}), \quad (1)$$

where we have neglected ohmic diffusion—although this term is important in allowing reconnection and smoothing. Here we assume that these processes take place at a rate determined by turbulent processes. The usual approach to dynamo theory is to define the response of the large-scale magnetic field to small-scale motions as \mathbf{b} and to derive its effects on the large-scale field by substituting \mathbf{b} back into the right-hand side of equation (1). For an incompressible fluid, this yields

$$\partial_t B_i = \epsilon_{ijk} \partial_j (\alpha_{kl} B_l) + \partial_j D_{jk} \partial_k B_i - \partial_j D_{ik} \partial_k B_j, \quad (2)$$

where ϵ_{ijk} is the Levi-Civita tensor and

$$\alpha_{kl} \equiv \left\langle \epsilon_{kij} V_i \partial_l \int_0^t V_j(t') dt' \right\rangle, \quad (3)$$

and

$$D_{jk} = \left\langle V_j \int_0^t V_k(t') dt' \right\rangle. \quad (4)$$

The first term comes from the stretching of large-scale field lines by the local turbulence. The tensor α_{kl} describes the twisting of large-scale field lines into a spiral shape. Reconnection between adjacent spirals produces a large-scale field component at right angles to the original field line, provided that either the degree of twisting or the large-scale magnetic field strength varies in the third direction. The second term is the usual diffusion term, modified by the presence of the third term.

Here the angle brackets denote statistical (ensemble) averages. However, those averages are still fluctuating in space and time (e.g., Hoyng 1988). In the following, we use angle brackets to denote averages over certain spans in time and space. When comparing with local numerical simulations (e.g., Brandenburg et al. 1995), we use averages over the full computational box and over several orbits.

We can see from equation (3) that each component of α_{ij} has either a factor of V_z or ∂_z . If the local velocity field is mirror-symmetric, in the sense that its statistical properties are unchanged under the transformation $z \rightarrow -z$, then the time- and space-averaged value of α_{ij} vanishes. This poses a significant but, as we will see, not insurmountable obstacle to a successful dynamo.

Another problem is that equations (2), (3), and (4) are usually defined kinematically, i.e., the velocity field is assumed to be imposed on the magnetic field. Once the magnetic field becomes sufficiently powerful, it will modify the flow, which is usually taken into account by including a correction term proportional to B^2 . However, in a Keplerian shearing flow, the magnetic field will be unstable and the resulting turbulence will be directly correlated with the magnetic field. Nevertheless, as long as we define V in terms of the motion of the magnetic field lines, equation (2) will remain valid, if difficult to solve. Here we will define our results in terms of the properties of α_{ij} and D_{ij} , regardless of their ultimate source. In this way, we can avoid concerning ourselves with the exact nature of the small-scale field, which in any case is not our principal concern here.

In a Keplerian disk, the dynamo equations can be simplified as

$$\partial_t B_r = -\partial_z (\alpha_{\theta\theta} B_\theta) - \partial_z (V_b B_r) + \partial_z (D_{zz} \partial_z B_r), \quad (5)$$

and

$$\partial_t B_\theta = -\frac{3}{2} \Omega B_r - \partial_z (V_b B_\theta) + \partial_z (D_{zz} \partial_z B_\theta) + \partial_r (D_{rr} \partial_r B_\theta), \quad (6)$$

where $\Omega \propto r^{-3/2}$ is the rotation frequency, V_b is the buoyant velocity of the magnetic field lines relative to the surrounding fluid, and B_r and B_θ are the radial and azimuthal components of the magnetic field. Equations (5) and (6) differ from equation (2) in that we have allowed for the presence of global shearing and magnetic field line buoyancy. In addition, we have assumed that the diffusion matrix is diagonal and dropped the effects of helicity on the evolution of B_θ , given that the shearing of B_r should dominate such effects. Also, we have retained only the $\alpha_{\theta\theta}$ term in equation (5), since the critical feedback term in the dynamo equations involves generating radial magnetic flux from the azimuthal component of the field. Finally, given that we are interested in applying these equations to accretion disks whose thickness is a small fraction of their radius, we have assumed that vertical gradients will dominate over radial gradients.

Everything we have said up to this point is part of the standard treatment of magnetic field generation in cylindrical shearing flows. The main point of this paper is that, under these circumstances, fluctuations in the helicity can maintain a large-scale dynamo, even in the absence of any average helicity. We will start by showing how this can arise based on fairly general considerations, then proceed to demonstrate the existence of this effect in a simple but solvable model.

We assume that the turbulence is symmetric under $z \rightarrow -z$, so that $\langle \alpha_{\theta\theta} \rangle = 0$. Although this eliminates any coherent helicity, the value of $\langle B_r^2 \rangle$ can still increase in a random walk. Ignoring diffusion and buoyancy, we see that the formal solution for B_r is

$$B_r = \int^t -\partial_z [\alpha_{\theta\theta}(t') B_\theta(t')] dt'. \quad (7)$$

We remind the reader once more that we are concerned only with the large-scale field, so that even though $\alpha_{\theta\theta}$ may vary considerably on small scales, the spatial averaging employed below will suppress the higher spatial frequencies in equation (7) and average the resulting B_r on all scales smaller than the size of the magnetic domains we wish to consider. By hypothesis, $\alpha_{\theta\theta}$ is uncorrelated over timescales greater than some eddy correlation time τ_{eddy} . If the radial magnetic field is undergoing a random walk, then it will usually be far enough away from zero that it will not change sign every eddy correlation time. Since B_r drives B_θ through coherent shearing, this implies that the correlation time for B_r and B_θ is much greater than τ_{eddy} . Consequently, we can consider the integrand in equation (7) as consisting of a rapidly varying factor, $\alpha_{\theta\theta}$, multiplying a slowly varying function. Multiplying equation (7) times equation (5), ignoring diffusion and buoyancy, as before, and averaging over space, we see that the integral in equation (7) is correlated with $\alpha_{\theta\theta}$ only over the last eddy correlation time τ_{eddy} . Consequently, we can replace the integral in the product with $-\partial_z [\alpha_{\theta\theta}(t) B_\theta(t)] \tau_{\text{eddy}}$. This implies

$$\partial_t \langle B_r^2 \rangle \approx K_z^2 \frac{\langle \hat{\alpha}_{\theta\theta}^2 \rangle}{N} \tau_{\text{eddy}} \langle B_\theta^2 \rangle, \quad (8)$$

where N is the number of independent turbulent eddies in a magnetic domain, K_z is the vertical wavenumber of the magnetic domain, and $\langle \hat{\alpha}_{\theta\theta}^2 \rangle$ is the mean-square helicity associated with a single eddy. Throughout this paper, we will denote the inverse of magnetic domain scales with K , not to be confused with the wavenumber of individual turbulent eddies or waves. In general, the mean-square helicity of a single eddy will be of order V_T^2 , where V_T is the root mean square turbulent velocity.

The factor of N in equation (8) comes from the fact that the large-scale helicity necessary for creating a large-scale B_r is the result of the incoherent addition of the helicity associated with N eddies. In any particular simulation or experiment, the effective value of N can be measured from the correlation length of the velocity and the size of the magnetic domains.

Since B_r is being driven incoherently, we can expect it to undergo frequent reversals. In between such reversals, the shearing of the field will drive B_θ^2 sharply upward. If we allow for the possibility that the dynamo growth rate, τ_{dynamo} , exceeds the correlation time of the magnetic field, τ_{corr} , then equation (6) implies that

$$\partial_t \langle B_\theta^2 \rangle \sim \frac{9}{4} \langle B_r^2 \rangle \Omega^2 \tau_{\text{corr}}. \quad (9)$$

In this equation, there is no factor N because the generation of the azimuthal field is the consequence of a large-scale shear acting on the large-scale radial field. We can combine this with equation (8) to show that

$$\partial_t^2 \langle B_\theta^2 \rangle \sim \frac{9}{4} \Omega^2 \tau_{\text{corr}} K_z^2 \frac{\langle \hat{\alpha}_{\theta\theta}^2 \rangle}{N} \tau_{\text{eddy}} \langle B_\theta^2 \rangle, \quad (10)$$

or

$$\tau_{\text{dynamo}}^{-2} \tau_{\text{corr}}^{-1} \sim \frac{9}{4} \Omega^2 K_z^2 \frac{\langle \hat{\alpha}_{\theta\theta}^2 \rangle}{N} \tau_{\text{eddy}} \langle B_\theta^2 \rangle. \quad (11)$$

At the same time, since equation (8) describes a random walk for B_r , its correlation time is given roughly by

$$\tau_{\text{corr}}^{-1} \approx \partial_t \ln \langle B_r^2 \rangle \sim K_z^2 \frac{\langle \hat{\alpha}_{\theta\theta}^2 \rangle}{N} \tau_{\text{eddy}} \frac{\langle B_\theta^2 \rangle}{\langle B_r^2 \rangle}. \quad (12)$$

However, since equation (9) implies that $\langle B_\theta^2 \rangle \sim \langle B_r^2 \rangle \Omega^2 \tau_{\text{corr}} \tau_{\text{dynamo}}$, we can combine equations (11) and (12) to show that

$$\tau_{\text{corr}}^{-1} \sim \tau_{\text{dynamo}}^{-1} \sim \left(\frac{K_z^2 V_T^2 \Omega^2 \tau_{\text{eddy}}}{N} \right)^{1/3}. \quad (13)$$

We note that τ_{corr} has to be greater than τ_{eddy} in order for this estimate to be internally self-consistent, i.e., the magnetic field must be correlated over longer times than the turbulence itself. Since a field reversal in B_θ requires that B_r not only reverse its sign but maintain it long enough to push B_θ through zero, it is clear that the correlation time for B_θ may be somewhat larger than the correlation time for B_r . We will return to this point later.

By itself this argument does not show that a succession of random twists in a shearing background can drive an exponential increase in the magnetic field; it merely establishes the scaling laws for the timescales, assuming that this process works. In order to demonstrate that this is a viable dynamo mechanism, we need to show that the growth experienced between field reversals dominates over the abrupt cancellation of the field as B_r reverses itself. We also need to show that our estimate of the growth rate given in equation (13) will dominate over turbulent diffusion for some range of magnetic domain sizes.

We can test the assertion that a series of random changes in B_r can drive a dynamo by constructing a simple toy model of the process that ignores the spatial structure of the field but includes its dynamical evolution. Assuming that $\alpha_{\theta\theta}$ has a stochastic component, and ignoring buoyancy, we can rewrite equations (5) and (6) as

$$\partial_t B_r = [\eta(t) - \alpha_{\text{coh}}] B_\theta - D B_r, \quad (14)$$

and

$$\partial_t B_\theta = -\frac{3}{2} \Omega B_r - D B_\theta, \quad (15)$$

where $\eta(t)$ is a stochastic variable with a correlation time τ_{eddy} , and α_{coh} is the coherent component of $\partial_z \alpha_{\theta\theta}$. Here we have subsumed spatial derivatives into the definitions of η and D and ignored the $-\alpha_{\theta\theta} \partial_z B_\theta$ term that would normally appear in the mean-field dynamo equations. We have also assumed that turbulent damping is the same for each component of the magnetic field, which is not generally true but simplifies the analysis without losing any essential physics. Equations (14) and (15) can be rewritten in a more convenient form by defining $A \equiv (B_r/B_\theta)$. Then

$$\partial_t A = \eta(t) - \alpha_{\text{coh}} + \frac{3}{2} \Omega A^2 \quad (16)$$

and

$$\partial_t \ln B_\theta^2 = -3\Omega A - 2D. \quad (17)$$

The magnetic field will grow exponentially if $\langle A \rangle$ is negative and $-3\Omega \langle A \rangle > 2D$.

We can find $\langle A \rangle$ by solving equation (16) in terms of an unnormalized probability distribution function $P(A)$ and evaluating

$$\langle A \rangle \equiv \frac{\int_{-\infty}^{\infty} AP(A)dA}{\int_{-\infty}^{\infty} P(A)dA}. \quad (18)$$

The distribution function $P(A)$ satisfies the stationary one-dimensional Fokker-Planck equation (see, e.g., Risken 1984)

$$\partial_A [\dot{A}P(A) - \langle \eta^2 \rangle \tau_{\text{eddy}} \partial_A P(A)] = 0, \quad (19)$$

or

$$P(A)(\frac{3}{2}\Omega A^2 - \alpha_{\text{coh}}) - \langle \eta^2 \rangle \tau_{\text{eddy}} \partial_A P(A) = \langle \eta^2 \rangle \tau_{\text{eddy}}, \quad (20)$$

where we have taken advantage of the unnormalized nature of $P(A)$ to set the constant of integration to $\langle \eta^2 \rangle \tau_{\text{eddy}}$. The correlation time τ_{eddy} is defined here using forward averaging in time, that is

$$\tau_{\text{eddy}} \equiv \frac{\langle \int_0^{\infty} \eta(t)\eta(t+t')dt' \rangle}{\langle \eta^2 \rangle}. \quad (21)$$

Equation (20) can be solved to yield

$$P(A) = \exp\left(\frac{\Omega A^3 - 2\alpha_{\text{coh}}A}{2\langle \eta^2 \tau_{\text{eddy}} \rangle}\right) \times \int_A^{\infty} \exp\left(\frac{-\Omega r^3 + 2\alpha_{\text{coh}}r}{2\langle \eta^2 \tau_{\text{eddy}} \rangle}\right) dr. \quad (22)$$

Consequently,

$$\langle A \rangle = \left(\frac{2\langle \eta^2 \tau_{\text{eddy}} \rangle}{\Omega}\right)^{1/3} \times \frac{\int_{-\infty}^{\infty} dy \int_y^{\infty} ds y \exp[\gamma y^3 - s^3 - \gamma(y-s)]}{\int_{-\infty}^{\infty} dy \int_y^{\infty} ds \exp[\gamma y^3 - s^3 - \gamma(y-s)]}, \quad (23)$$

where

$$\gamma \equiv \frac{\alpha_{\text{coh}}}{\langle \eta^2 \tau_{\text{eddy}} \rangle} \left(\frac{2\langle \eta^2 \tau_{\text{eddy}} \rangle}{\Omega}\right)^{1/3}. \quad (24)$$

Equation (23) can be rewritten by defining new variables $w \equiv y + s$ and $x \equiv s - y$ and integrating over w . We obtain

$$\langle A \rangle = \frac{-1}{2} \left(\frac{2\langle \eta^2 \tau_{\text{eddy}} \rangle}{\Omega}\right)^{1/3} \frac{\int_0^{\infty} x^{1/2} \exp[x(\gamma - x^2/4)] dx}{\int_0^{\infty} x^{-1/2} \exp[x(\gamma - x^2/4)] dx}. \quad (25)$$

When γ is small, we can expand $e^{w\gamma} \approx 1 + \gamma w$ and obtain

$$\langle A \rangle \approx -0.32 \left(\frac{\langle \eta^2 \tau_{\text{eddy}} \rangle}{\Omega}\right)^{1/3} (1 + 0.51\gamma), \quad (26)$$

which implies that

$$\partial_t \ln B_{\theta}^2 \approx 0.96(\langle \eta^2 \tau_{\text{eddy}} \rangle \Omega^2)^{1/3} + 0.61\alpha_{\text{coh}} \left(\frac{\Omega}{\langle \eta^2 \tau_{\text{eddy}} \rangle}\right)^{1/3} - 2D. \quad (27)$$

In other words, the magnetic field will grow exponentially roughly as fast as the estimate given in equation (13). This will be suppressed by turbulent diffusion only when the damping rate due to diffusion is comparable to the growth rate.

It should be remembered that this derivation is strictly valid only for δ -correlated noise. However, since we have assumed that τ_{eddy} is the shortest timescale in the problem, this should be approximately valid. We expect this assumption to fail only when the stochastic term is important and the evolution rate, $3\Omega A$, is small compared to τ_{eddy}^{-1} . That is to say,

$$\langle \eta^2 \rangle > \frac{9}{4} A^4 \Omega^2, \quad (28)$$

so that

$$\langle \eta^2 \rangle \tau_{\text{eddy}} > \frac{1}{\Omega^2 \tau_{\text{eddy}}^3}. \quad (29)$$

However, from equation (27) we see that this is only possible when the incoherent dynamo growth rate is greater than τ_{eddy} . In the end, restricting ourselves to the case in which the diffusion approximation is valid is just a more formal version of the restriction we mentioned following equation (13).

The existence of an incoherent dynamo emerges from the fact that the distribution function $P(A)$ given in equation (22) is biased toward negative values of A . This bias comes, paradoxically enough, from the coherent, positive definite term in equation (16). When A is sufficiently positive, it evolves deterministically through $+\infty$ into negative values. (Actually, B , does not change during this phase. This deterministic trajectory is merely a field reversal for B_{θ} .) The end result is that whenever A becomes large and positive it rapidly switches to being large and negative. Ultimately, the sign of the bias is determined by the sign of $\partial_t \Omega$. The frequency of such field reversals is given by examining the probability distribution at large A when the evolution of $P(A)$ is deterministic. If we define $\tau(A)$ as the time it takes for the field to move from some large positive value of A to $A = \infty$, then from equation (16) we see that

$$\tau(A)^{-1} = \frac{3}{2} \Omega A, \quad (30)$$

where we have neglected α_{coh} since for A sufficiently large its effects can be ignored. The field reversal rate is just the limit of this rate times the statistical weight of the distribution between A and ∞ . In other words,

$$\tau_{\text{rev}}^{-1} = \lim_{A \rightarrow \infty} \tau(A)^{-1} \frac{\int_A^{\infty} P(s) ds}{\int_{-\infty}^{\infty} P(s) ds}. \quad (31)$$

Substituting equations (22) and (30) into this result, and making the change of variables to x and w , as before, we have

$$\tau_{\text{rev}}^{-1} = \lim_{A \rightarrow \infty} \frac{3}{2} \Omega A \times \frac{\int_{2A(\Omega/2\langle \eta^2 \tau_{\text{eddy}} \rangle)^{1/3}}^{\infty} \exp\{w[\gamma - 1/4(w^2 + 3x^2)]\} dw dx}{\int_{-\infty}^{\infty} \int_0^{\infty} \exp\{w[\gamma - 1/4(w^2 + 3x^2)]\} dw dx}. \quad (32)$$

Both the numerator and the denominator can be simplified by integrating over x to obtain

$$\tau_{\text{rev}}^{-1} = \left(\frac{3}{\pi}\right)^{1/2} \frac{(2\langle \eta^2 \tau_{\text{eddy}} \rangle \Omega^2)^{1/3}}{\int_0^{\infty} w^{-1/2} \exp[w(\gamma - w^2/4)] dw}. \quad (33)$$

When γ is small, this becomes

$$\tau_{\text{rev}}^{-1} = 0.53(\langle \eta^2 \tau_{\text{eddy}} \rangle \Omega^2)^{1/3} (1 - 0.53\gamma), \quad (34)$$

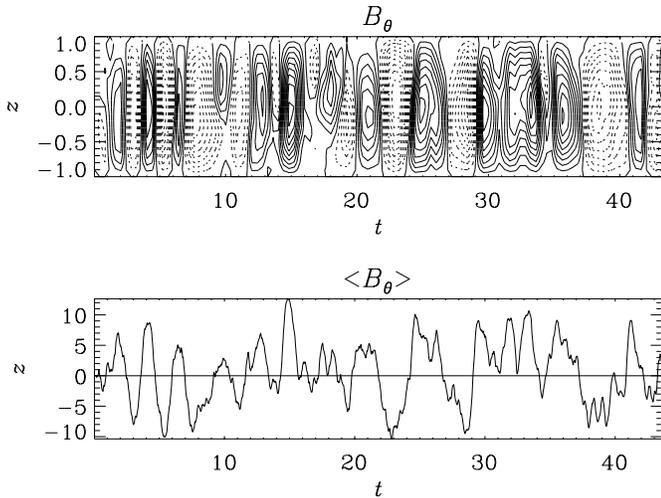


FIG. 1.—Contours of the B_θ field in a spacetime diagram for the one-dimensional, spatially and temporally incoherent dynamo model.

i.e., a rate that is roughly half the e -folding rate for the magnetic field energy.

When α_{coh} is large and positive, we can evaluate the integrals in equation (25) by expanding around the maximum of $x(\gamma - x^2/r)$. We obtain

$$\langle A \rangle \approx - \left(\frac{2\alpha_{\text{coh}}}{3\Omega} \right)^{1/2}, \quad (35)$$

so that

$$\partial_t \ln B_\theta^2 \approx 2 \left(\frac{3}{2} \alpha_{\text{coh}} \Omega \right)^{1/2} - 2D, \quad (36)$$

which is the expected result for a coherent α - Ω dynamo. In this limit, the magnetic field reversal rate becomes

$$\tau_{\text{rev}}^{-1} = 0.68 \alpha_{\text{coh}}^{1/4} \langle \eta^2 \tau_{\text{eddy}} \rangle^{1/6} \Omega^{7/12} \exp \left[\frac{-1.1 \alpha_{\text{coh}}}{\langle \eta^2 \tau_{\text{eddy}} \rangle^{2/3} \Omega^{1/3}} \right]. \quad (37)$$

As expected, field reversals are exponentially suppressed as we go to the usual α - Ω dynamo. Given $\alpha_{\text{coh}} > 0$, then as α_{coh} becomes significant we expect it to enhance the dynamo growth rate and reduce the rate of spontaneous field reversals.

When α_{coh} is large and negative, we can evaluate equation (25) by integrating the denominator by parts and remembering that the bulk of the contribution to the integral comes from $w < -1/\gamma$, so that $w^2 \ll -\gamma$. We obtain

$$\langle A \rangle \approx \frac{\langle \eta^2 \tau_{\text{eddy}} \rangle}{4\alpha_{\text{coh}}}, \quad (38)$$

and

$$\partial_t \ln B_\theta^2 \approx \frac{-3\Omega \langle \eta^2 \tau_{\text{eddy}} \rangle}{4\alpha_{\text{coh}}} - 2D. \quad (39)$$

In this limit, field reversals occur at a rate given by

$$\tau_{\text{rev}}^{-1} = 0.78 (-\alpha_{\text{coh}} \Omega)^{1/2}. \quad (40)$$

We note that in this case the coherent component of the helicity does not completely shut off the incoherent dynamo, even though by itself it is incapable of driving a dynamo. Instead, we find that as $|\gamma|$ increases past 1, the

dynamo growth rate decreases inversely with $|\gamma|$. Eventually, turbulent diffusion will stop the dynamo. In the limit where γ is of order -1 , we anticipate that the dynamo growth rate will be less than expected from the incoherent dynamo alone and the rate of field reversals will be larger.

This concludes our discussion of the stochastic dynamo in the limit where spatial structure can be ignored. These analytic results have the advantage of being based on a solvable model, but they do not include the effects of spatial structure or saturation. We will conclude this section with a brief demonstration of the combined effects of a random electromotive force and shear in a one-dimensional model. We consider the mean-field equations for a uniform disk with Keplerian rotation and half-thickness H ,

$$\partial_t B_r = -\partial_z(\alpha B_\theta) + D_t \partial_z^2 B_r, \quad (41)$$

$$\partial_t B_\theta = -\frac{3}{2}\Omega B_r + D_t \partial_z^2 B_\theta, \quad (42)$$

with $-H \leq z \leq H$ and $B_r = B_\theta = 0$ at $z = \pm H$. We first consider the incoherent α -effect, so we take α to be random in space and time. When the rms value of α is large enough, we find self-excited solutions that grow without bound. In reality, there must eventually be some quenching mechanism, which we model using

$$\alpha = \alpha_0 \frac{\eta(z, t)}{(1 + B_\theta^2)}, \quad (43)$$

where η is a random function in space and time with zero mean and an rms value of unity. Without loss of generality, we put $H = \Omega = D_t = 1$.

In Figure 1, we plot contours of the B_θ field in a spacetime diagram for a dynamo number $\alpha_0 \Omega H^3 / D_t^2$ of 10^4 . (The critical dynamo number for dynamo action depends on the coherence time and length scales λ and τ , respectively. In the present case, we adopt $\lambda = 0.05$ and $\tau = 0.002$ and find the critical dynamo number to be around 2000. At this dynamo number, the ratio of the growth rate given in equation (13) to D_t/H^2 is ~ 7 . The remarkable result is that the B_θ field shows a great deal of spatiotemporal coherence with variations comparable to the diffusion time and diffusion length. Experiments with different dynamo numbers suggest that the degree of coherence is more pronounced for larger dynamo numbers.

Finally, we note that in order for the magnetic field to grow, the growth rate given in equation (13) has to be greater than the dissipation rate. In general, the dissipation rate will depend on the wavenumber of the magnetic domain as K^2 , while the growth rate goes as $(K^2/N)^{1/3}$. Clearly, whether or not there is a self-excited dynamo will depend in large part on the geometry of the fluid.

3. THE SPATIALLY INCOHERENT DYNAMO

Although the dynamo discussed in the previous section is based on a helicity that varies incoherently in time and space—but whose instantaneous large-scale average is nonzero—it turns out that there is a related dynamo in which the helicity is constant in time, but with a spatial average that is strictly zero, and with power concentrated on some very large wavenumber. Here we discuss this dynamo process, not because we have some specific physical application in mind, but because it provides an interesting example of a way in which a fluctuating helicity can give rise to an ordered magnetic field in a system whose global symmetry is preserved. More precisely, we have

taken

$$\alpha = \alpha_0 \sin(n\pi z), \tag{44}$$

where z lies in the interval $[-1, 1]$, and evolved equations (41) and (42). For large values of n , the critical dynamo number is proportional to n^2 . Thus, although the rms value of the α -effect is unchanged, the dynamo becomes harder to excite if α is chopped into many domains of different sign. The magnetic field is steady, and the radial component is of alternating sign. However, more surprisingly, the toroidal magnetic field has the same sign for all values of z (see Fig. 2). This is very similar to the simulation of a random incoherent α -effect mentioned before. There is one difference in that the magnetic field shows global reversals in time when the α -effect is incoherent in time. Here it is constant.

We can understand this result by combining equations (41) and (42). This is

$$(\partial_t - D\partial_z^2)^2 B_\theta = \frac{3}{2}\Omega\partial_z(\alpha B_\theta), \tag{45}$$

with

$$B_r = -\frac{2}{3}\Omega^{-1}(\partial_t - D\partial_z^2)B_\theta, \tag{46}$$

and α is defined in equation (44). A general solution of equation (45) is rather hard to find, but we can understand our numerical results by assuming a solution of the form

$$B_\theta \approx Ae^{\Gamma t} \cos(\pi z/2) \tag{47}$$

and looking for cases in which the corrections to this are small and Γ is positive. Substituting this form into the right-hand side of equation (45), we obtain a correction to B_θ .

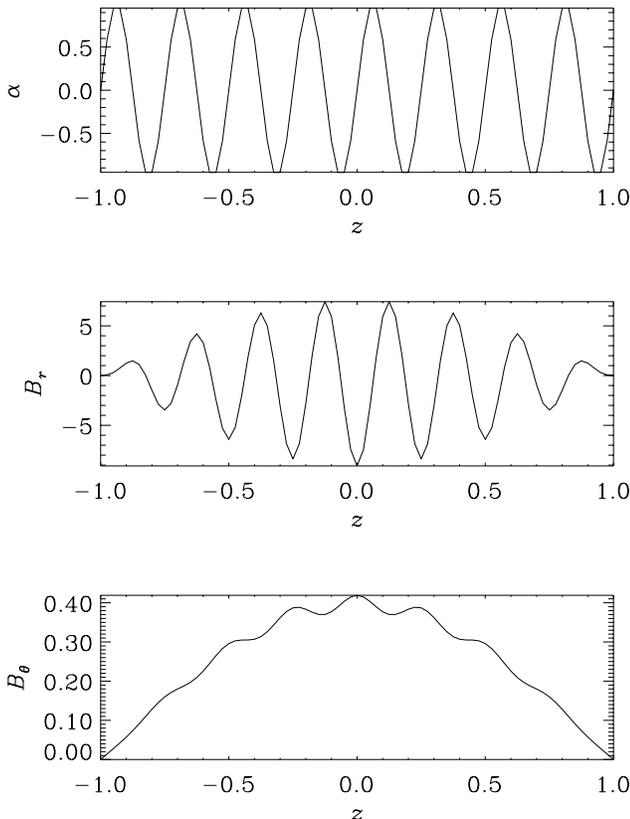


FIG. 2.—Snapshot of the magnetic field and helicity for the one-dimensional, spatially incoherent dynamo model.

This is

$$\Delta B_\theta = \frac{3\pi}{4} \Omega A \alpha_0 e^{\Gamma t} \left\{ \frac{(n+1/2) \cos \left[\frac{(n+1/2)\pi z}{2} \right]}{\Gamma + D\pi^2(n+1/2)^2} + \frac{(n-1/2) \cos \left[\frac{(n-1/2)\pi z}{2} \right]}{\Gamma + D\pi^2(n-1/2)^2} \right\}. \tag{48}$$

If we put this expression back through the right-hand side of equation (45), then the product of α and ΔB_θ gives rise to more large wavenumber terms, but also a long-wavelength piece that can be set equal to the long-wavelength term arising from substituting our original guess for B_θ into the left-hand side of this equation. This allows us to solve for Γ . We find that

$$\left[\Gamma + \left(\frac{\pi}{2} \right)^2 D \right]^2 = \left(\frac{3}{4} \pi \alpha_0 \Omega \right)^2 \frac{1}{2} \times \left\{ \frac{(n-1/2)}{[\Gamma + (n-1/2)^2 \pi^2 D]^2} - \frac{(n+1/2)}{[\Gamma + (n+1/2)^2 \pi^2 D]^2} \right\}. \tag{49}$$

We can check for the presence of a “fast” dynamo by taking $D = 0$ in equation (49). We find

$$\Gamma^4 = -\frac{1}{2} (\Omega \alpha_0)^2 \left(\frac{3\pi}{4} \right)^2. \tag{50}$$

At first glance, this seems to describe a growing oscillatory solution. However, we note that in this limit the ratio of the rms value of ΔB_θ to the rms value of B_θ is

$$\frac{\langle \Delta B_\theta^2 \rangle}{\langle B_\theta^2 \rangle} = \frac{3\pi}{4} \Omega \alpha_0 |\Gamma|^{-2} \left[\frac{1}{2} \left(n + \frac{1}{2} \right)^2 + \frac{1}{2} \left(n - \frac{1}{2} \right)^2 \right]^{1/2}. \tag{51}$$

For large n this is

$$\frac{\langle \Delta B_\theta^2 \rangle}{\langle B_\theta^2 \rangle} \sim \frac{3\pi}{4} \Omega \alpha_0 |n| |\Gamma|^{-2}. \tag{52}$$

Since this is of order n , our assumption that the field is dominated by its large-scale component fails. No conclusion can be drawn regarding the existence of a “fast” dynamo using the derivation sketched above.

If instead we look for a “slow” dynamo with a growth rate much greater than $(\pi/2)^2 D$ and much less than $(n\pi/2)^2 D$, then equation (49) implies that

$$\Gamma^2 \approx \frac{27}{32} \left(\frac{\Omega \alpha_0}{n^2 D} \right)^2, \tag{53}$$

demonstrating the existence of growing, well-ordered magnetic field. The condition that $\Gamma^2 > 0$ for n large and $\Gamma \ll n^2 D$ is

$$\frac{\Omega \alpha_0}{[D(\pi/2)^2]^2} > n^2 \left(\frac{128}{27} \right)^{1/2}, \tag{54}$$

which explains the dependence of the critical dynamo number on n described above. The condition that $\Gamma \ll n^2 \pi^2 D$ is satisfied if

$$n^4 \gg \frac{\Omega \alpha_0}{D^2 \pi^3} \left(\frac{27}{32} \right)^{1/2}. \tag{55}$$

However, a more stringent condition is that the ratio of the rms value of ΔB_θ to the rms value of B_θ be less than 1. Using

equation (48) and the condition that $\Gamma \ll n^2\pi^2 D$, we find that this implies that

$$n^3 \gg \frac{3}{4\pi} \frac{\Omega\alpha_0}{D^2}, \quad (56)$$

which will be harder to satisfy than the previous condition for any choice of system parameters that produce a successful dynamo. Finally, we note that we can derive B_r from equation (46). For the successful dynamo, it is

$$B_r \approx Ae^{\Gamma t} \left[-\frac{2\Gamma}{3\Omega} \cos\left(\frac{\pi z}{2}\right) - \frac{\alpha_0}{2} \times \left\{ \frac{\cos[(n-1/2)\pi z]}{(n-1/2)\pi^2 D} + \frac{\cos[(n+1/2)\pi z]}{(n+1/2)\pi^2 D} \right\} \right]. \quad (57)$$

This implies a ratio of the rms ordered component of B_r to the disordered component that is approximately $n\pi^2 D\Gamma/\Omega\alpha_0$ or, using equation (53), of order n^{-1} . The successful dynamo has an ordered B_θ and a disordered B_r , consistent with the numerical experiment described in the beginning of this section.

4. THE INCOHERENT DYNAMO IN ACCRETION DISKS

Now we are ready to consider an application of the incoherent dynamo, turbulence, and magnetic field generation in accretion disks. We start by considering the nature of turbulence in accretion disks. We will defer for now any discussion of whether the incoherent dynamo is the dominant dynamo mechanism, but we will return to this point toward the end of this section.

A Keplerian accretion disk with a root mean square Alfvén speed of V_A will be subject to a local instability first described by Velikhov (1959). Its pivotal role in transporting angular momentum outward in accretion disks was recognized considerably later (Balbus & Hawley 1991). In the context of accretion disks, this instability is normally referred to as the Balbus-Hawley instability. Its maximum growth rate is of order Ω and occurs at an azimuthal wavelength of $\sim V_A/\Omega$. In three dimensions, the instability saturates in turbulence with a typical turbulent velocity comparable to V_A and a typical eddy size of $\sim V_A/\Omega$. This turbulence is not expected to be isotropic, but the typical eddies are expected to have axis ratios of order unity, which in this context means only that no axis should be more than an order of magnitude larger than another (Vishniac & Diamond 1992). Numerical simulations (Hawley, Gammie, & Balbus 1995; Brandenburg et al. 1995; Stone et al. 1996; Hawley, Gammie, & Balbus 1996; Brandenburg et al. 1996) indicate that the azimuthal scale of the typical eddies is several times the vertical and radial scales, which is expected in light of the large local shear. The azimuthal velocity is also larger, although only by a factor of roughly 2. Neglecting such factors, these scaling laws imply a turbulent diffusivity of $\sim V_A^2/\Omega$.

Zweibel & Kulsrud (1975) have shown that sufficiently strong turbulence will suppress the Parker instability. Subsequent work (Vishniac & Diamond 1992) has shown that the scaling of turbulence due to the Balbus-Hawley instability implies that the Parker instability is always suppressed in Keplerian accretion disks. However, residual buoyant effects lead to a typical buoyant velocity of the magnetic field of order V_A^2/c_s (Vishniac & Diamond 1992; Vishniac 1995b), where c_s is the local sound speed. The

angular momentum flux induced by the turbulence is approximately $\langle V_\theta V_r \rangle \sim V_A^2$, which implies a dimensionless viscosity α_{SS} of order $(V_A/c_s)^2$. Since $h\Omega \sim c_s$, this implies that magnetic flux is lost from the disk at a rate that is some fraction of order unity times $\alpha_{SS}\Omega$. The qualitative nature of this argument makes it difficult to compare this quantitatively with current simulations, but they do show an absence of the kind of large-scale coherent motions predicted from the linear theory of the Parker instability. The simulations also show an outward-directed turbulent-transport velocity (specifically an $r\theta$ component of the α tensor) that can be interpreted as a result of buoyancy. This velocity is only 3% of the turbulent rms velocity, supporting the claim that buoyancy cannot be very strong in disks.

It is by no means obvious that in real disks the magnetically induced turbulence possesses the kind of symmetry that would make $\langle \alpha_{\theta\theta} \rangle = 0$. On the other hand, calculations done without vertical structure or any imposed large-scale field (Hawley et al. 1996) give results that are qualitatively similar to calculations that include vertical structure (Brandenburg et al. 1995). By construction, the former calculations are symmetric under the transformation $z \rightarrow -z$, even though the latter are not. We can estimate $\alpha_{\theta\theta}$ using data from the simulation of Brandenburg et al. (1995). From equation (3) it is clear that a time integration has to be carried out. However, video animations of those data suggest that the lifetime of turbulent eddies is shorter than the lifetime of magnetic structures, which, in turn, is shorter than the eddy turnover time. In other words, the Strouhal number—i.e., the ratio of correlation time to turnover time (e.g., Krause & Rädler 1980)—is small. As a rough approximation, we may therefore replace the time integration by a multiplication with a relevant timescale. We adopt the natural timescale Ω^{-1} , which is sufficient since we are only interested in relative variations. We adopt volume averages and note that, because of the periodic boundary conditions in the toroidal direction, $\langle V_r V_{z,\theta} \rangle = -\langle V_z V_{r,\theta} \rangle$, so we can compute

$$\alpha_{\theta\theta} \approx \frac{2}{r} \langle V_r V_{z,\theta} \rangle \Omega^{-1}. \quad (58)$$

In Figure 3, we plot the evolution of $\alpha_{\theta\theta}$ using the data from run C of Brandenburg et al. (1995), which has now been carried out for an additional 200 orbits (see also Torkelson

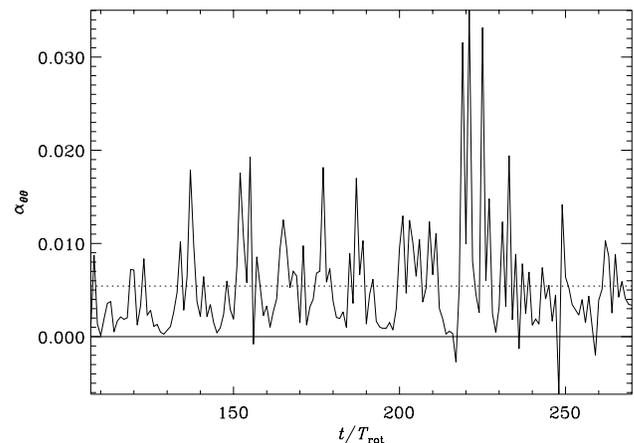


FIG. 3.—Evolution of $\alpha_{\theta\theta}$ [normalized by the product of the rms velocity and $(\Omega\tau_{\text{eddy}})^{-1}$] in the upper half-plane of run C from Brandenburg et al. (1995).

et al. 1996). This average was computed for the upper half-plane of the simulation. We note that $\alpha_{\theta\theta}$ is positive, in agreement with the expected effect for bubbles that expand as they rise in a Keplerian disk. However, the sign of $\alpha_{\theta\theta}$ suggested by the correlation between the azimuthal magnetic and electric fields is negative (Brandenburg et al. 1995). The source of this discrepancy is not yet clear, but it is possibly related to the effect of shear twisting magnetic field loops by almost 180° after they are formed (Brandenburg & Donner 1996). In any case, the spatially averaged helicity shows large variations from its long-term average, although the variations in the electromotive force are much larger.

The size of the fluctuations in the electromotive force, as well as the persistence of the dynamo in the absence of any \hat{z} symmetry breaking, implies that any preferred helicity resulting from vertical structure is not strong enough to completely dominate the simulations. In what follows, we will assume that real disks lack any significant $\langle\alpha_{\theta\theta}\rangle$. At a minimum, our results can be taken as demonstrating that there is an incoherent dynamo operating in the simulations and in real accretion disks, whose effects need to be understood in order to be clearly distinguished from any other dynamo mechanisms that might be present.

How will the incoherent dynamo work in such an environment? We consider a magnetic domain characterized by the wavenumbers (K_r, K_θ, K_z) . Ignoring the anisotropies in the turbulence, we find that the number of turbulent eddies per domain is roughly $\sim (K_r K_\theta K_z V_A^3)^{-1} \Omega^3$. Consequently, the growth rate for the dynamo is

$$\tau_{\text{dynamo}}^{-1} \sim \left(\frac{V_A^5 K_z^3 K_r K_\theta}{\Omega^2} \right)^{1/3} \quad (59)$$

(see eq. [13]). We have ignored the distinction between Ω and τ_{eddy}^{-1} in this expression. The latter is smaller by some constant factor, but the inaccuracy introduced by ignoring the difference is comparable to other uncertainties in the problem. We note that in a shearing environment we are not free to specify K_r and K_θ separately. The shear implies a minimum K_r for any K_θ , since in a time $\sim \tau_{\text{dynamo}}$ the shear will increase K_r by an amount $(3/2)K_\theta \Omega \tau_{\text{dynamo}}$. If we choose a value of K_r above this minimal value, then τ_{dynamo} will vary as $K_r^{1/3}$ while the dissipation rate scales as K_r^2 . Clearly, our chances for a successful dynamo will be maximized by taking $K_r \sim K_\theta \Omega \tau_{\text{dynamo}}$; this gives us

$$\tau_{\text{dynamo}}^{-1} \sim \left(\frac{V_A^5 K_z^3 K_r^2}{\Omega^3} \right)^{1/2} \quad (60)$$

This analysis only makes sense in the limit where the magnetic domains encompass at least one eddy, or $K_z V_A < \Omega$ and $K_r V_A < \Omega$. The dissipation rate is roughly

$$\tau_{\text{dissipation}}^{-1} \approx (K_z^2 + K_r^2) \frac{V_A^2}{\Omega} \quad (61)$$

By comparing equations (60) and (61) we see that the incoherent dynamo would be incapable of generating large-scale magnetic fields. The dissipation rate of such domains exceeds the generation rate for all domain sizes greater than a single eddy because $V_A K_z / \Omega < 1$.

This would seem to rule out a successful incoherent dynamo driven by the Balbus-Hawley instability. However, there is a flaw in the preceding argument. We have assumed

that the azimuthal scale of the magnetic domains can be taken to be arbitrarily large. In fact, the azimuthal domain size cannot be greater than $2\pi r$ and axisymmetric domains are not subject to shearing effects. In other words, the number of eddies in a magnetic domain does not increase indefinitely as $K_\theta \rightarrow 0$. The finite circumference of the disk implies that for axisymmetric domains

$$N \sim \frac{r\Omega^3}{K_z K_r V_A^3} \quad (62)$$

Consequently, we can rewrite equation (59) as

$$\tau_{\text{dynamo}}^{-1} \sim \left(\frac{V_A^5 K_z^3 K_r}{r\Omega^2} \right)^{1/3} \quad (63)$$

At a fixed wavenumber and, therefore, at a fixed dissipation rate, this rate is maximized for $K_z = K_r 3^{1/2}$. Assuming this ratio, we see that the dynamo growth rate for axisymmetric domains varies as $K^{4/3}$, which implies that at some sufficiently small K the dynamo will work. More exactly, the incoherent dynamo caused by the Balbus-Hawley instability will drive an increase in the magnetic field strength if

$$K^2 < \frac{\Omega}{rV_A} \quad (64)$$

In other words, the incoherent dynamo only works for

$$V_A < \frac{\Omega}{rK^2} \quad (65)$$

Ultimately, K_z is limited by the height of the disk, i.e., $K_z h > 1$. Moreover, as we approach this limit the buoyant loss of magnetic flux becomes significant. The buoyant loss rate from a single magnetic domain goes as

$$\tau_{\text{buoyant}}^{-1} \sim K_z V_b \sim K_z \frac{V_A^2}{c_s} \quad (66)$$

so when $K_z h \sim 1$, buoyant losses are as important as turbulent diffusion. Of course, the only limit on the radial extent of a magnetic domain is $K_r r > 1$, but lowering K_r past h^{-1} will lower the growth rate without affecting the dissipation rate. From equation (65), we see that the magnetic field associated with scales of order the disk thickness will be the strongest and will be given by

$$V_A \sim c_s \frac{h}{r} \quad (67)$$

This in turn implies that the dimensionless viscosity associated with this dynamo mechanism is

$$\alpha_{\text{ss}} \sim \left(\frac{V_A}{c_s} \right)^2 \sim \left(\frac{h}{r} \right)^2 \quad (68)$$

We expect this scaling law to hold only in the limit $h \ll r$. As $V_A \rightarrow c_s$, corrections of order V_A/c_s will become important in our formula for buoyancy. Since the saturation limit for the magnetic field involves the small difference between the growth rate dependence on V_A , which has an exponent of $5/3$, and the buoyant loss rate dependence, which varies as V_A^2 , we expect the saturation strength of the magnetic field to be extremely sensitive to such corrections unless $V_A \ll c_s$. We also note that the disk radius enters into this result only

through its role as the circumference of an annulus. Computer simulations typically involve a short arc in place of a full annulus. In this case, the azimuthal length of the simulation has to be used in place of $2\pi r$ in equation (68). Finally, we note that equations (63) and (66) involve unknown coefficients of order unity. If the ratio of these coefficients is greater than 1, in the sense that the coefficient in front of the scaling law for $\tau_{\text{dynamo}}^{-1}$ is greater than the coefficient in front of the scaling law for $\tau_{\text{buoyant}}^{-1}$, then equation (68) will become valid only for h/r less than this ratio to the third power. For larger h/r , the magnetic field saturation will be controlled by magnetic quenching instead.

What can we learn from a comparison between numerical simulations and this model? We start by noting that the numerical simulations invariably show an initial rise in magnetic field energy at a rate comparable to Ω . This has been used as the basis for the claim that the Balbus-Hawley mechanism leads to a dynamo growth rate $\sim \Omega$ (Balbus & Hawley 1991). However, we note that such a rise is the inevitable result of beginning with a uniform magnetic field. The situation is similar to a simulation in which a field is inserted in a turbulent medium. This will amplify the field at the eddy turnover rate as the field becomes concentrated in intermittent structures and is folded by the surrounding eddies (see, e.g., Meneguzzi, Frisch, & Pouquet 1981). This effect is probably the result of the ability of the turbulence to produce a negative effective diffusion coefficient (Moffatt 1978). The fact that in this case the turbulence is induced by the magnetic field itself does not change this result in any essential way. We expect a rapid but transient rise in the magnetic field energy even in the absence of any large-scale dynamo effect. After this rise, the magnetic field has not yet lost its memory of its initial conditions (see, for example, the results of Brandenburg et al. 1995), but the large-scale field present in the initial conditions is dwarfed by the small-scale components of the magnetic field. However, on somewhat longer timescales—perhaps as long as a few dozen rotational periods—the system approaches a steady state with an α_{SS} somewhat less than 1% and a large-scale magnetic field, albeit one that can undergo reversals on long timescales. In a more recent paper, Brandenburg et al. (1996) established three results of particular relevance to this work. First, they found that the value of α_{SS} was strongly correlated with $\langle B_\theta \rangle$, the large-scale average toroidal field, with no noticeable phase lag. Second, they found that the simulation results had not yet converged. Doubling the resolution increased α_{SS} by a factor of between 1.4 and 1.6. Third, they found no significant difference in α_{SS} when they increased the azimuthal extent of the computational box from 2π to 8π times the vertical scale height.

We can see from this that our emphasis on the generation of a large-scale field is justified. Although correlation does not necessarily imply causality, the lack of a time lag between $\langle B_\theta \rangle$ and α_{SS} tends to support the notion, implicit in our estimates of α_{SS} , that the former determines the latter in accretion disks. In addition, the long timescale associated with the generation of a large-scale field implies the presence of a dynamo with a growth rate well below Ω . We have already noted that this cannot be a standard α - Ω dynamo, since it arises regardless of any global symmetry breaking in the simulations.

On the less encouraging side, the lack of numerical convergence implies that the simulation results should be interpreted with caution. The last point, that the simulations do

not show the predicted decrease in α_{SS} as box height-to-length ratio decreases, could be a sign that some other dynamo process (e.g., a coherent α - Ω dynamo) might be important in the simulations. In fact, the simulations (Brandenburg et al. 1995) do show a coherent component of α (see also Brandenburg & Donner 1996). In that case, the dynamo would continue to grow until the magnetic energy of the large-scale field approaches the kinetic energy of the turbulence. Another source of discrepancy could be that the simulations are still strongly affected by numerical resolution. The latter could be the result of the role of very high wavenumber modes in the instability (cf. Terquem & Papaloizou 1996; Ogilvie & Pringle 1996) or some critical role for small-scale magnetic field features in MHD turbulence (e.g., Cattaneo & Vainshtein 1991; Tao, Cattaneo, & Vainshtein 1993; Gruzinov & Diamond 1994; Vishniac 1995a). In particular, we note that if the buoyant velocity does not scale as V_A^2 (cf. eq. [66]) but with an exponent less than $5/3$, then the saturation strength of the magnetic field will not depend on h/r . Any effects that tend to soften the dependence of the buoyant velocity on V_A in the simulations (or in reality) will have dramatic consequences for the final value of V_A and therefore for α_{SS} as well. Finally, decreasing h/r by a factor of 4 reduces the dynamo growth rate by a factor of only $4^{1/3}$, which may be insufficient to move the simulations from the regime where saturation is determined by magnetic quenching to one where it is determined by buoyant magnetic flux loss and turbulent dissipation. Difficulties with the numerical simulations should eventually be overcome through increased resolution, but models of dwarf novae and X-ray transients may give us more immediate guidance. We will return to this point below.

The rate of spontaneous magnetic reversals expected in the absence of any coherent component to $\alpha_{\theta\theta}$ is comparable to dynamo growth rate. However, while current simulations seem to show a significant reversal rate in the presence of vertical structure (Brandenburg et al. 1995), in its absence the field can evolve for 100 orbital times without reversing (Torkelsson et al. 1996). The exact relationship between the dynamo growth rate and the field reversal rate dependent on the particular model for the process and the zero-dimensional model used in this paper may well overestimate the rate of spontaneous field reversals. Nevertheless, such reversals are an intrinsic part of the model and should occur if the simulation is run for several growth times. The sharp rise in the field reversal rate when vertical structure is included suggests that a significant coherent $\alpha_{\theta\theta}$ is present in such simulations. (In the zero-dimensional model this would argue for a negative vertical gradient in the coherent $\alpha_{\theta\theta}$. The situation is less clear in three spatial dimensions.) In order for this helicity to allow the buildup of a coherent field in these simulations, as well as in accretion disks, it has to scale with the local rms turbulent velocity more steeply than the square of the incoherent dynamo growth rate, or $(V_T/c_s)^{10/3}$. If this helicity is the result of the Parker instability, and, if, as has been argued elsewhere (Vishniac & Diamond 1992), the Balbus-Hawley instability reduces the Parker instability to vertical motions of order V_T^2/c_s , then we can estimate the magnitude of the helicity as

$$V_r k_\theta V_z \tau_{\text{buoyant}}, \quad (69)$$

where τ_{buoyant} is the correlation time for these buoyant motions. Since shearing imposes the requirement that

$k_\theta \Omega < k_r \tau_{\text{buoyant}}^{-1}$ and since these motions are approximately incompressible, i.e., $k_r V_r \sim k_z V_z$, this gives a helicity less than

$$\frac{V_z^2 k_z}{\Omega} \sim \frac{V_A^4}{c_s^3}. \quad (70)$$

If the coherent helicity has this dependence, then it becomes important only as the dynamo saturates because of turbulent mixing and buoyancy. In this case, it will not suppress the incoherent dynamo in simulations with smaller h/r or in real accretion disks, but it will remain significant in the saturated state.

The fact that the buoyancy does not significantly enhance the loss of magnetic flux is a critical element in the derivation of equation (68). Consequently, environments that increase magnetic buoyancy will saturate at much lower field strengths. As an example, we can consider magnetic flux tubes in a radiation-pressure-dominated environment. In this case, we have

$$V_b \sim \frac{P_{\text{radiation}}}{P_{\text{gas}}} \frac{V_A^2}{c_s} \quad (71)$$

(Vishniac 1995b). Combining this result with equation (63) for $K_r \sim K_z \sim h^{-1}$ yields

$$\alpha_{\text{SS}} \sim \left(\frac{V_A}{c_s}\right)^2 \sim \left(\frac{P_{\text{gas}}}{P_{\text{radiation}}}\right)^6 \left(\frac{h}{r}\right)^2. \quad (72)$$

These scaling laws will be generally true of thin disks if we write P_{total} in place of $P_{\text{radiation}}$.

Can observations tell us anything about the role of the incoherent dynamo in accretion disks? We have already mentioned that numerical simulations suggest α_{SS} reaches a constant value, with no dependence on h/r . The incoherent dynamo gives $\alpha_{\text{SS}} \propto (h/r)^2$, although with no guarantee that this is the dominant dynamo mechanism at work. Observations of accretion disk systems favor neither answer. Successful models of dwarf novae outbursts and X-ray transients (Smak 1984a, 1984b; Meyer & Meyer-Hofmeister 1984; Huang & Wheeler 1989; Mineshige & Wheeler 1989; Cannizzo 1994), as well as the distribution of light in quiescent dwarf novae disks (Mineshige & Wood 1989), all imply that α_{SS} varies spatially and with time. These variations can be reproduced using models with $\alpha_{\text{SS}} \propto (h/r)^n$, where n is a constant lying somewhere between 1 and 2. More recently, Cannizzo et al. (1995) have shown that n must be close to 1.5 to reproduce the observed exponential decay of soft X-ray transients from maximum light. Vishniac & Wheeler (1996) have shown that this result follows from the scaling of α in the hot state alone and represents an independent, and substantially more precise, argument for α_{SS} . We conclude that the exponent of h/r given in equations (3) and (72) is too large in comparison to the value suggested by the phenomenology of disks. If $n = 2$, then the cooling front velocity will drop too quickly as the front progresses to small radii and the disk luminosity will drop too slowly. This presents a problem for the incoherent dynamo, but it is less of a problem than if the predicted value of n were too small. Competing dynamo mechanisms and/or hydrodynamic angular momentum transport mechanisms could be driving α_{SS} up. Similarly, there seems to be no room for a constant α_{SS} unless it rarely dominates in the mix of competing processes.

What other dynamo mechanisms can lead to viscosity in disks, and how important is the incoherent dynamo in a real disk? The prediction given in equation (68) has an extremely uncertain coefficient. Current numerical simulations give α_{SS} of order 10^{-2} or less, which would suggest that this coefficient is very small. On the other hand, these simulations have $V_A \sim c_s$, whereas we have based our discussion on the assumption that $V_A \ll c_s$. In particular, we have already noted that the small difference in the exponent of V_A in the dynamo growth rate and the dissipation rate, coupled to the presence of corrections to both these rates of order (V_A/c_s) , makes it difficult to extrapolate from current results. If the saturation value of V_A/c_s approaches its asymptotic dependence on h/r gradually as $h/r \rightarrow 0$, then the final value of the coefficient will be much larger than 10^{-2} . Bearing in mind the sensitivity of simulation results to their resolution, it seems prudent to regard the coefficient as an unknown numerical constant.

Since $h \ll r$ for many realistic disks, we can compare this dynamo mechanism to others based purely on the value of the exponent in the scaling relationship. Internal waves, excited by tidal instabilities in binary system disks (Goodman 1993), will produce an effective α_{SS} that scales as $(h/r)^2$ (Vishniac & Diamond 1989). This will be a competing mechanism for angular momentum transport in gas pressure-dominated disks, and, potentially, the dominant one in radiation pressure-dominated disks. (Although such conditions are most likely in active galactic nuclei disks, where the potential for the tidal excitation of waves is less certain.) Given the nonlocal nature of the angular momentum transport mediated by internal waves, the existence of a purely local mechanism might be important, even if it does not clearly dominate. When the disk is ionized and when internal waves are present, then the waves are capable of driving a dynamo with a growth rate $\sim (h/r)^{3/2} \Omega$ (Vishniac, Jin, & Diamond 1990) and perhaps faster, depending on the nature of the turbulent cascade of wave energy (Vishniac & Diamond 1992). The resulting value of α_{SS} will be $\sim (h/r)^{3/2} (P_{\text{gas}}/P)$ (Vishniac & Diamond 1992; Vishniac 1995b), since the saturation level of the magnetic field is set by the balance between buoyant losses and the dynamo growth rate. When these conditions are met, this would appear to be a more important dynamo mechanism, although, once again, we note that nonlocal effects on the wave-driven dynamo make the two processes somewhat incommensurate. An equivalent estimate based on purely local physics was given by Meyer & Meyer-Hofmeister (1983). However, this estimate is based on using large-scale buoyant cells driven by magnetic buoyancy, a picture that is inconsistent with turbulence in the disk (Zweibel & Kulsrud 1975; Vishniac & Diamond 1992). In addition, they assumed approximate isotropy of the helicity tensor and offered a calculation of α_{rr} instead of $\alpha_{\theta\theta}$. This assumption of isotropy is inconsistent with the notion that the motions are driven by magnetic buoyancy, which for $h \ll r$ will have a timescale much longer than the local shearing timescale. Finally, we note that our result for the incoherent dynamo in an accretion disk is sensitive to our assumption that the process must be self-exciting.

5. CONCLUSIONS

In this paper, we have shown that mean-field dynamo theory allows for the existence of a new kind of α - Ω dynamo, which we have named the incoherent α - Ω dynamo,

in which there is no coherent helicity whatsoever. In this class of dynamos, the magnetic field is driven by a combination of a random walk for B_r and its shearing, which creates B_θ . The resultant large-scale field derives its organization from coherent shearing effects, rather than any loss of mirror symmetry in the turbulence. Although this kind of dynamo necessarily includes spontaneous field reversals, such reversals may occur at a rate that is some fraction of the dynamo growth rate. The existence of a mean-field dynamo in a flow with a mean helicity of zero is interesting for its own sake, since it provides an example of how large-scale order in the magnetic field can arise from the interaction between a large-scale shear and statistically symmetric local motions. In this sense, it represents an alternative to models that seek to explain dynamo activity through asymmetric turbulence and a coherent helicity. It differs from previous attempts to do without a coherent helicity (e.g., Montgomery & Hatori 1984; Gilbert, Frisch, & Pouquet 1988) in that it does so without appealing to the other terms in equation (2). This model is also interesting in light of previous claims (Moffatt 1979) that the coherent α -effect does not converge (cf. Kraichnan 1979).

The most obvious application of this dynamo is to simulations of magnetic field instabilities in accretion disks. We have suggested that this dynamo can operate successfully in accretion disks, but only to produce axisymmetric large-scale fields. Comparing the growth rate for this dynamo with the buoyant loss rate for magnetic flux, we see that *if* this is the only dynamo associated with magnetic shearing instabilities, and *if* the buoyant loss rate of magnetic field from the disk is proportional to V_A^2 , then the large-scale magnetic field will saturate when $V_A \sim (h/r)c_s$ and $\alpha_{SS} \sim (h/r)^2$. This result may seem somewhat odd, since the dynamic equations do not depend on r at all. However, the factor of r comes in through geometrical considerations, i.e., from considering the number of independent eddies in an axisymmetric magnetic field domain. In that sense, it refers to the circumference of such an annulus rather than its radius. Consequently, when comparing numerical simulations to the predictions of this model one should substitute the azimuthal extent of the simulations for $2\pi r$. For current simulations, this gives $h/r \sim 1$. This agrees with the observation that the number of independent eddies that can be stacked end to end in most current simulations is one or two. However, simulations with smaller values of h/r do not show the expected drop in α_{SS} . It remains to be seen whether this is a shortcoming of the numerical models, the result of a softening of the relationship between the average buoyant velocity of magnetic flux and the average magnetic field strength at large field strengths, as a result of our failure to explore h/r small enough that buoyancy and turbulent dissipation dominate magnetic quenching, or whether it reflects the presence of another, more powerful dynamo mecha-

nism. The last possibility appears to be inconsistent with successful phenomenological models of real accretion disks.

It is interesting to note that the only numerical simulation with no imposed field and with disk vertical structure does seem to have a coherent component to the helicity, which may have the wrong sign to drive a conventional dynamo. This may account for the large rate of spontaneous field reversals when vertical structure is put in the simulations. The existence of such a component is consistent with the existence of an incoherent α - Ω dynamo, but only if its amplitude scales steeply with the strength of turbulence in the disk.

There may be other applications of the incoherent dynamo, but it is worth remembering that this mechanism is unusually sensitive to dissipative effects. Our treatment was based on the assumption that the dynamo growth time is longer than the eddy turnover time in the underlying turbulence. Consequently, any situation in which turbulence and shear are present, but where the eddies are as large as any reasonable magnetic domain size, will fail to show an incoherent dynamo effect. Turbulent mixing will always destroy any large-scale magnetic field faster than it can be created. Examples of this would include convection, when the convective zone is only one pressure scale height deep, or a linear or nonlinear hydrodynamic shearing instability. In addition, we found a successful dynamo here only by invoking periodicity in the azimuthal direction and considering axisymmetric magnetic domains. While it is difficult to prove that this is a necessary part of any successful model, it is clear that only axisymmetric magnetic domains can avoid the destructive effects of shearing.

We note that an alternative treatment of mathematical aspects of the incoherent dynamo has been done by Sokoloff (1996) in response to an earlier version of this paper. That paper contains a more detailed treatment of the spatial structure of the field generated by an incoherent dynamo in a shearing system.

Assuming that the incoherent dynamo is the dominant dynamo mechanism in the numerical simulations and that the failure of such simulations to show the expected scaling with geometry reflects the role of numerical viscosity in the simulations, this model successfully reconciles phenomenological models of stellar accretion disks and the existence of a dynamo effect in a magnetized disk. Unfortunately, this model gives a relationship between α_{SS} and h/r that is probably too steep, implying the existence of other, more efficient dynamo mechanisms in accretion disks in binary systems.

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