

COLD + HOT AND COLD DARK MATTER COSMOLOGIES: ANALYSIS OF NUMERICAL SIMULATIONS

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ABSTRACT

We present a series of four simulations of cold dark matter (CDM) and cold + hot dark matter (CHDM) cosmologies, which we analyze together in this and subsequent papers. These dissipationless simulations were done using the particle mesh method with a 512^3 mesh, corresponding to a resolution of approximately 200 kpc for an assumed Hubble parameter of $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and with approximately 17×10^6 cold and (for CHDM) an additional 34×10^6 hot particles. In this paper we discuss the power spectrum and correlation functions in real and redshift space, with comparisons to the CfA2 and IRAS redshift data, the pairwise velocity of galaxies in real space, and the distribution of hot and cold particles in CHDM simulations. We confirm that CHDM with cold/hot/baryon density ratios $\Omega_c/\Omega_v/\Omega_b = 0.6/0.3/0.1$ is a good fit to a wide variety of present-epoch data, much better than CDM. In particular, with reasonable assumptions about identification of galaxies and biasing, we find that the power spectrum from our CHDM simulations agrees rather well with both the CfA2 and IRAS power spectra in both the nonlinear and linear regimes.

New variants of the CHDM scenario (e.g., with 20% of the mass in hot particles or with two massive neutrinos) predict a significantly larger rate of formation of galaxies at high redshift, which may be needed to explain some observational data. At the same time, the difference between the variants is rather small at $z = 0$. The results presented in this paper are interesting for two purposes: (i) For a rough comparison with other classes of models (like CDM or models with cosmological constant Λ CDM) at $z = 0$ —indeed, we have used the simulations described here as a test bed for developing a number of new statistics for quantifying large-scale structure and comparing it to observations; (ii) As a reference point for comparison between different variants of the CHDM model. In addition, we explain here how we modify the usual Zeldovich approximation used to set up the initial conditions for both cold and hot particles in our simulations, taking into account that the growth rates of both kinds of fluctuations are different from the usual CDM case. We have also added an Appendix on issues of resolution in particle mesh simulations.

Subject headings: cosmology: theory — dark matter — galaxies: formation — large-scale structure of the universe

1. INTRODUCTION

Cold dark matter (CDM) was perhaps the simplest cosmological theory proposed in the 1980s that had a chance to be true. Standard CDM is based on the hypotheses that the primordial fluctuations are Gaussian and adiabatic with a Zeldovich spectrum, as predicted by simple inflation models, and that the dark matter is “cold,” i.e., preserves fluctuations on all cosmologically relevant scales. Standard CDM is also usually assumed to have a total density that is critical ($\Omega = 1$). Aside from the Hubble parameter $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$ and the fraction of critical density in ordinary matter (baryons), Ω_b , the only free parameter in CDM is then the overall normalization of the spectrum. Since for $\Omega = 1$ the age of the universe $t_0 = (\frac{2}{3})H_0^{-1} = 6.52 h^{-1} \text{ Gyr}$ is shorter than the age of the oldest globular clusters unless $h \lesssim 0.5$, it is usual to assume for $\Omega = 1$ that $h = 0.5$, and we make that assumption in this paper.

In the earliest work on CDM (Peebles 1982; Blumenthal

et al. 1984), the spectrum was normalized so that $\sigma_8 = 1$, where

$$\sigma_8 \equiv \left(\frac{\Delta M}{M} \right)_{\text{rms}} \equiv \frac{1}{b}; \quad (1)$$

the normalization is also sometimes specified by quoting the linear bias parameter b . The first CDM N -body simulations (Davis et al. 1985) indicated that CDM with this normalization was not a good match to the galaxy redshift data on scales of a few Mpc, in particular the galaxy pairwise velocities $\sigma_{12}(1 h^{-1} \text{ Mpc}) \approx 340 \text{ km s}^{-1}$ (Davis & Peebles 1983, hereafter DP83). CDM is a better fit to this data for $\sigma_8 \approx 2.5$.

As more data became available on larger scales, however, it became increasingly clear that this data favored a higher normalization (see, e.g., the reviews by Davis et al. 1992; Ostriker 1993). Thus, the problem with CDM is, at least, that the spectrum shape is not quite right. It is, of course, also possible that the problem is much more serious: that

the dark matter in the universe is *not* mostly cold plus a little baryonic dark matter (presumably relatively cool ionized gas and massive astrophysical compact halo objects [MACHOs]). But since there are at least two well-motivated candidate cold dark matter particles (axions and lightest superpartner particles; see, e.g., Primack, Seckel, & Sadoulet 1988, Griest 1996), and since it is remarkable that the simple a priori CDM model came so close to fitting all the data, we will assume that the dark matter is indeed mostly cold, but that there is some aspect of the theory that needs to be modified slightly. Many variants of CDM were studied by Holtzman (1989) and Holtzman & Primack (1993) and others (for a review, see Liddle & Lyth 1993). Of all the variants of CDM with $\Omega = 1$, the best fit to the data appears to be a model in which, along with mostly cold dark matter, there is also some hot dark matter (light neutrinos).

We call this theory cold + hot dark matter (CHDM). The first detailed simulations of CHDM showed that it indeed appears to be a good fit to all available present-epoch data, including both large- and small-scale galaxy velocities (Klypin et al. 1993, hereafter KHPR; cf., Davis, Summers, & Schlegel 1992, Jing et al. 1994; these papers give references to earlier papers on CHDM).

In the present paper we describe a series of higher resolution simulations that we have performed in order to understand better the differences between CDM and CHDM, in particular the details of the distribution and velocities of dark matter halos. In this paper, we describe the simulations including the evolution of the power spectra (§ 3) and our simplest scheme for identifying “galaxies” in these simulations (§ 4). We show then that the corresponding CHDM galaxy power spectra are in good agreement with recent observations of both CfA2 optical and *IRAS* galaxy redshifts on both large and small scales (§ 5), and we compare correlation functions of galaxies and cold dark matter particles in real space and of galaxies compared to observational data in redshift space (§ 6). Finally, we discuss velocity dispersion and velocity bias in our simulations (§ 7) and summarize our conclusions (§ 8).

In a series of related papers, we have analyzed these same simulations further in a variety of ways. In Yepes et al. (1994), the results were compared to observations of the galaxy angular correlation function; this paper also includes a discussion of the time evolution of correlations in the simulations. In Bonometto et al. (1995) and Ghigna et al. (1996a), we showed that higher order correlations of the halos in both CDM and CHDM show hierarchical clustering, although the dark matter particles do not do so because of sampling effects. We have also analyzed these simulations with several varieties of shape statistics (filament statistics: Davé et al. 1996a; moment statistics: Davé et al. 1996b) and also a version of the Davis & Peebles (1983) pairwise velocity statistic (Somerville, Nolthenius, & Primack 1996), all of which we have been developing using the suite of simulations described in the present paper as a test bed.

In Nolthenius, Klypin, & Primack (1994, hereafter NKP94), we presented the highlights of our analysis of the velocity structure of the galaxy groups in these simulations, compared to the CfA1 data. We merged the CfA1 galaxies that we could not have resolved, determined the corresponding Schechter luminosity function, assigned Schechter-distributed luminosities in rank order to the halos in our simulation volume, and constructed

magnitude-limited redshift-space catalogs that we analyzed in parallel with the merged CfA1 catalog. Our conclusion was that CDM groups have systematically higher median velocity dispersion than the CfA1 data, as a function of either the redshift-link parameter or the fraction of galaxies in groups. Visualization of these simulations, including a videotape, show that the CHDM simulations are much more filamentary than the CDM simulations (Brodbeck et al. 1996, hereafter BHNPK). Nolthenius, Klypin, & Primack (1996, hereafter NKP96) present a more detailed group analysis, including a number of alternative methods of assigning luminosities and identifying galaxies, and several schemes for breaking up overmerged halos. These papers show that CHDM with $\Omega_c/\Omega_v/\Omega_b = 0.6/0.3/0.1$ is a much better match to this data than CDM, but the more detailed analysis of NKP96 suggests that the fit would be somewhat better with a little lower Ω_v (although with considerable uncertainty connected with overmerging and breakup). This also appears to be indicated by the analysis of the void probability function (VPF) for these same simulations by Ghigna et al. (1994), which finds that our CDM simulations are a better match to the Perseus-Pisces Survey (PPS) data than our CHDM simulations. Analysis by Ghigna et al. (1996b) of a more recent CHDM simulation with $h = 0.5$, $\Omega = 1$, $\Omega_b = 0.075$, and $\Omega_n u = 0.2$ shared equally between two species of neutrinos (Primack et al. 1995) shows that the predicted VPF agrees well with that from the PPS and CfA2 data.

Several papers (Kauffmann & Charlot 1994; Mo & Miralda-Escude 1994; Ma & Bertschinger 1994) pointed out that CHDM with $\Omega_v = 0.3$ does not produce enough dark matter halos to account for the data on damped Lyman- α systems at redshifts $z > 3$. We (Klypin et al. 1995, hereafter KBHP) agree with this (although our calculations disagree in some details with those of the above papers), but we show that if Ω_v is lowered a little—to $\Omega_v \approx 0.2$, corresponding to ~ 5 eV of neutrino mass for $h = 0.5$ —there are enough halos with characteristic velocities $\lesssim 100 \text{ km s}^{-1}$ to correspond to the observations, as long as the number of these damped Ly α systems decreases above about $z \gtrsim 3.5$. Recent neutrino experimental data suggest that ~ 5 eV of neutrino mass is shared about equally among two species of neutrinos, ν_μ and ν_τ , and linear calculations suggest that this will be a better fit to the available data than CHDM with one 5 eV neutrino (Primack et al. 1995).

2. COSMOLOGICAL MODELS

We describe simulations of two cosmological models in this paper: cold + hot dark matter (CHDM) and cold dark matter (CDM). Both models assume that the universe has critical density ($\Omega = 1$), that the Hubble parameter is $h = 0.5$ ($H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$), and that the cosmological constant is zero. The CHDM model has the following parameters. The mass density of the universe in the form of light neutrinos is $\Omega_v = 0.30$, and the density of baryons is $\Omega_b = 0.10$. We use the analytical approximations from equation (1) of KHPR for the “cold” and “hot” spectra for the CHDM₁ simulation, and the spectra given in the KHPR notes added in proof (eq. [A1-2]) for the CHDM₂ simulation. The amplitude of fluctuations is normalized so that our realizations are drawn from an ensemble producing the quadrupole in the angular fluctuations of the cosmic microwave background at the $Q_{\text{ps-norm}} = 17 \mu\text{K}$ level measured by *COBE* (Smoot et al. 1992). In linear theory, this corresponds to rms fluctuations of mass in a sphere of $8 h^{-1}$

Mpc radius $\sigma_8 = 0.667$ and the rms velocity of the “cold” dark matter relative to the rest frame $\sigma_v = 750 \text{ km s}^{-1}$.

Our CHDM model perhaps assumes an unrealistically large Ω_b . A lower value $\Omega_b = 0.05$ is better consistent with standard big bang nucleosynthesis (BBN) (Walker et al. 1991), although the large X-ray luminosity of rich clusters ($\propto \Omega_b^2$) suggests higher Ω_b (e.g., White et al. 1993) if $\Omega = 1$. Recent BBN studies also suggest a higher upper limit $\Omega_b \lesssim 0.1$ for $h = 0.5$ (Copi, Schramm, & Turner 1995). The power spectrum of fluctuations in the CHDM model increases by approximately on 20% scales of $\lesssim 10 h^{-1} \text{ Mpc}$ if we change Ω_b from 0.1 to 0.05 and keep the same normalization on very large scales. Thus, the results presented in this paper are also approximately valid for a CHDM model with $\Omega_b = 0.05$ and the amplitude of the quadrupole $Q_{\text{ps-norm}} = 14 \mu\text{K}$.

Another possible interpretation is a model with $\Omega_b = 0.05$, $Q_{\text{ps-norm}} = 19 \mu\text{K}$ as indicated by the latest *COBE* analyses (Gorski et al. 1994; Bunn, Scott, & White 1995), with a 10% contribution of gravity waves to the $\Delta T/T$ fluctuations on *COBE* scales and a 20% further reduction of power on scales of $\sim 20 h^{-1} \text{ Mpc}$ due to a small amount of tilt—i.e., a primordial power spectrum $P(k) \propto k^n$ with $n \approx 0.95$ instead of the Zeldovich value $n = 1$. This amount of gravity waves and tilt is consistent with expectations for simple inflationary models with inflation potential $V(\phi) = \lambda\phi^\alpha$ with $\alpha = 4$ (see, e.g., Turner 1993). Our CHDM normalization is in any case within the *COBE* errors.

We use the spectrum given by Bardeen et al. (1986, hereafter BBKS) for the CDM model. Two normalizations are used for CDM: a “biased” model—which we call CDM1.5—with linear bias factor $b \equiv \sigma_8^{-1} = 1.5$ or $\sigma_8 = 0.667$ (which leads in linear theory to $\sigma_v = 660 \text{ km s}^{-1}$ and $Q_{\text{ps-norm}} = 8.5 \mu\text{K}$), and an “unbiased” CDM model—called CDM1—with $\sigma_8 = 1$, $\sigma_v = 990 \text{ km s}^{-1}$, and $Q_{\text{ps-norm}} = 12.8 \mu\text{K}$. This CDM1 model normalization was compatible with the lowest normalization quoted from the first-year *COBE* measurements (Smoot et al. 1992), but in view of the more recent *COBE* results it would perhaps be more reasonable to interpret this model as corresponding to about 35% contribution of gravity waves to the *COBE* amplitude but with no tilt in the spectrum. This is not what is expected in most cosmic inflation models, but “natural inflation” models (e.g., Adams et al. 1993; Turner 1993) do produce gravity waves without tilt.

It is perhaps worth noting that various CDM spectra are not entirely equivalent (see also Liddle & Lyth 1993). For example, Efstathiou, Bond, & White (1992), using the spectrum of Bond & Efstathiou (1984), quote $Q_{\text{ps-norm}} = 15.3 \mu\text{K}$ as implying $\sigma_8 = 1.10$ for CDM; thus, $Q_{\text{ps-norm}} = 17 \mu\text{K}$ would correspond to $\sigma_8 = 1.22$. With the BBKS spectrum, $Q_{\text{ps-norm}} = 17 \mu\text{K}$ corresponds to $\sigma_8 = 1.33$, or $b = 0.75$, for CDM.

3. N-BODY SIMULATIONS

Numerical simulations were done using a standard particle-mesh (PM) code (Hockney & Eastwood 1981). The equations we actually solve are given by Kates, Kotok, & Klypin (1991). The code is very fast, and it is especially suitable for analysis of *statistical* properties of cosmological objects (galaxies, groups, and clusters). Two conditions should be satisfied at the same time for this kind of analysis: the volume of the computational box should be representative for the objects under investigation, and the resolution

should be good enough to allow the identification of the objects and to allow us to estimate a few basic parameters of the objects (e.g., the mass distribution and the velocity relative to the rest frame) in the simulation. There are numerical methods that can provide better resolution of the gravitational forces (e.g., P³M or tree codes; for a review, see Sellwood 1987), but the number of objects and the mass resolution—and thus the resolution of the initial spectrum of fluctuations—is worse compared to what the PM code provides. Those methods are more suitable, and at present are used more often, to study separate objects or a few objects (e.g., the collision of galaxies, the formation of a galaxy cluster). We have had experience with such codes, and also with hydrodynamics codes, but for our present purpose the PM code is superior. This is especially true for CHDM, for which the high velocities of the hot dark matter require that we have many hot particles to sample adequately the six-dimensional phase space.

This paper discusses three runs: two for the CHDM model (CHDM₁ and CHDM₂) and one for the CDM model, but we analyze results from the CDM run at two different expansion factors, corresponding to linear bias parameters $b = 1.5$ (CDM1.5) and $b = 1$ (CDM1). All simulations were done using a 512^3 force mesh. The CHDM₁ and CDM runs were started with fluctuations generated from the same set of random numbers.

Each simulation has 256^3 “cold” particles. The CHDM simulation has two additional sets of 256^3 particles to represent the “hot” neutrinos. We use the same prescription as KHPR to simulate random thermal velocities of the “hot” particles. “Hot” particles are generated in pairs, particles of each pair having random “thermal” velocities of exactly equal magnitude but pointing in opposite directions. The directions of these “thermal” velocities are random. The magnitudes of the velocities are drawn from relativistic Fermi-Dirac statistics (see KHPR). The CHDM spectra we use were calculated treating cold dark matter, hot dark matter (neutrinos of 7 eV mass), and baryonic matter separately. However, as usual in CDM simulations, when we switch from linear to nonlinear evolution, the cold particles represent both cold and baryonic dark matter. The particles have different relative masses: each “cold” particle has a relative mass of 0.7, and each “hot” particle has relative mass $0.3/2 = 0.15$.

The size of the computational box for both the CHDM and the CDM simulations is 100 Mpc (i.e., $50 h^{-1} \text{ Mpc}$ for $h = 0.5$). The smallest resolved comoving scale in these simulations is $97.6 h^{-1} \text{ kpc}$, and the mass of a “cold” particle was $1.45 \times 10^9 h^{-1} M_\odot$ for the CHDM simulation and $2.07 \times 10^9 h^{-1} M_\odot$ for the CDM simulation. Both CHDM simulations were started at redshift 15 and were run to redshift zero with a constant step Δa in the expansion parameter a . The CDM simulation was started at $z = 18$ for the biased model and $z = 27.5$ for the unbiased model. The step for CHDM₁ and CDM simulations was $\Delta a = 0.01$; for the CHDM₂ simulation, it was a factor of 2 smaller, $\Delta a = 0.005$.

After running the CHDM₁ simulation, we found that there were two mistakes in our initial conditions (see KHPR, Note Added in Proof): the fitting formula for the cold spectrum was too small and the velocities were too large, both by about 20% on small scales. However, these effects are in phase and largely cancel. We have run another CHDM₂ (rev) simulation with both errors corrected and

fond that the power and velocity differences between this and CHDM₂ (old) on small scales declined to 5% by $z = 7$ and remained at this level. All results discussed here are from CHDM₂ (rev), but the agreement with CHDM₂ (old) is well within the 1σ error bars; thus, the CHDM₁ simulation should also be reliable. (This is discussed further below in § 5.)

Initial positions and velocities of particles were set using the Zeldovich (1970) approximation. The displacement vector was simulated directly. Phases of fluctuations were exactly the same for “hot” and “cold” particles. When generating velocities of “hot” particles, the “thermal” component, as described above, was added to the velocity produced by the Zeldovich approximation. In the case of the CHDM₂ (rev) simulation (referred to hereafter as CHDM₂), the Zeldovich approximation was modified slightly to take into account the fact that the growth rate of fluctuations is different for different wavelengths (Ma 1993). The correction was made as follows. If Φ_k is the spectrum of the velocity potential at the initial moment and α is a normalization constant, then the relations between the Lagrangian (unperturbed) position q , Eulerian position x , and peculiar velocity v are given by

$$x = q - \alpha a(t) \sum k \Phi_k \exp(-ikq),$$

$$v \equiv a dx/dt = -\alpha \dot{a} \sum f_k k \Phi_k \exp(-ikq), \quad (2)$$

where $f_k = d \ln \delta_{\text{cold}}(k) / d \ln a$ is given by equation (A2) of KHPR (for $z = 15$, f_k changes from 1 at small k to 0.805 around $k = 0.3 \text{ Mpc}^{-1}$). When setting up initial conditions for CDM and CHDM₁ we used $f_k = 1$.

When we completed the CHDM₁ and the CDM simulations, we realized that there was a statistical fluke of probability $\sim 10\%$ affecting the longest waves in the simulations: the amplitude of the waves was a factor of 1.3–1.4 larger (so the power was about a factor of 2 larger) than that expected for the ensemble average. Nevertheless, the waves ($\lambda = 100 \text{ Mpc}$) are still in the linear regime even at the end of the simulations, and it is relatively easy to make corrections to the correlation functions and the power spectrum. Moreover, as we argue below, this small additional power could be considered as a compensation for the finite size of the simulations. It brings the level of fluctuations inside our 100 Mpc box close to that expected for a much larger box. The CHDM₂ simulation has power typical for a box size of 100 Mpc. Differences between the CHDM simulations, which are readily apparent in our BHNPK visualizations, thus indicate to what extent various statistics that we use in this and other papers are affected by cosmic variance.

The CHDM₂ simulation does not have that excess power on long waves and, as a result, it does not have a structure comparable with the Great Wall, which to some extent dominates the CfA2 ($m < 15.5$) catalog. Nevertheless, not all statistics are affected strongly by the presence of the large structure. We found that the pairwise velocities and fraction in groups of galaxies were affected the most (because of a larger fraction of galaxies being in galaxy groups with higher velocity dispersion). The power spectra and group velocities were less sensitive to the effect (NKP96).

4. BIASING SCHEMES AND GALAXY FINDING ALGORITHMS

We use the following procedures to identify “galaxies” in

our simulations in this paper. The density field is produced on the 512^3 simulation mesh. We find maxima of the total (“cold” plus “hot”) density. Then all local maxima above the density threshold $\rho_{\text{thr}} > 30\rho_{\text{crit}}$ are found. This list played the role of a pool from which more dense objects were chosen for data analysis. The overdensity threshold corresponds to a mass of $7.76 \times 10^9 h^{-1} M_\odot$ in a cell. The number of maxima depends on the simulation considered and the moment of time at which we consider it. For CHDM₁ and for the “unbiased” CDM simulations at the final moment, there are 29,151 and 37,164 dark halos, respectively. Usually we use more massive objects selected from the list of dark halos, with the mass limits $1.27 \times 10^{10} h^{-1} M_\odot$ and $2.5 \times 10^{10} h^{-1} M_\odot$ being typical choices. Because the mass function of dark halos $n(M)$ decreases with mass M approximately as $n(M) \propto M^{-2}$ [implying that the cumulative number $N(>M) \propto M^{-1}$], doubling the mass limit reduces the number of “galaxies” by about a factor of 2. (For more details, and results using several alternative galaxy identifications and luminosity assignment schemes, see NKP96.)

We use two estimates of the dark halo mass: the mass m_1 in the cell of the density maximum, or the mass in the cube of $3 \times 3 \times 3$ cells centered on the maximum. Effectively these definitions correspond to (i.e., have the same volume as) spherical “galaxies” of radii $61 h^{-1} \text{ kpc}$ and $182 h^{-1} \text{ kpc}$, respectively. In order to find the velocities of these dark halos, the velocity field was constructed on three 512^3 grids (one for each of the vector components) using the cloud-in-cell technique. The velocity at the position of the density maximum was assigned to the dark halo.

Figures 1 and 2 present examples of two groups of galaxies identified in the CHDM₂ simulation. Groups were found as local maxima of “cold” mass within a sphere of

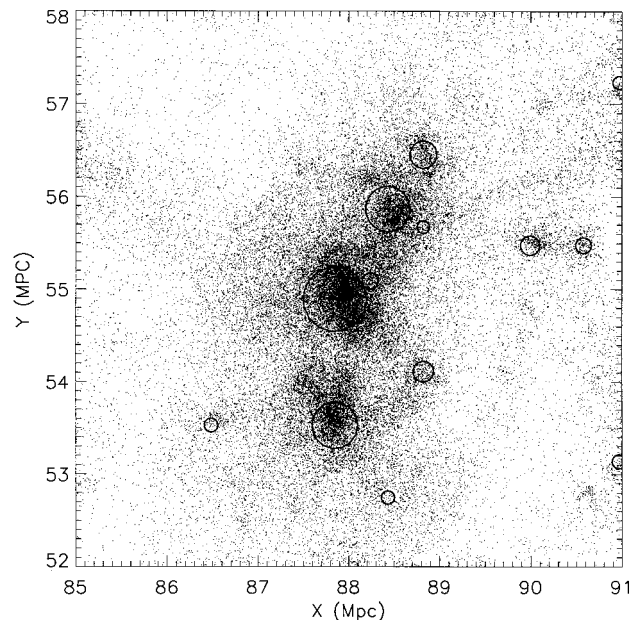


FIG. 1.—Example of a large group of galaxies identified in the CHDM₂ simulation. Coordinates are given in units of Mpc for $h = 0.5$. Positions of all cold particles inside a $3 h^{-1} \text{ Mpc}$ cube are shown together with positions of dark halos with central overdensity larger than $\delta\rho/\rho > 150$. Each circle on the plots corresponds to a dark halo, with the area of the circle being proportional to the mass of the dark halo. The group has mass $M = 7.5 \times 10^{13} h^{-1} M_\odot$, with nine “galaxies” exceeding the density threshold. The rms three-dimensional velocity of cold particles in the group is 644 km s^{-1} .

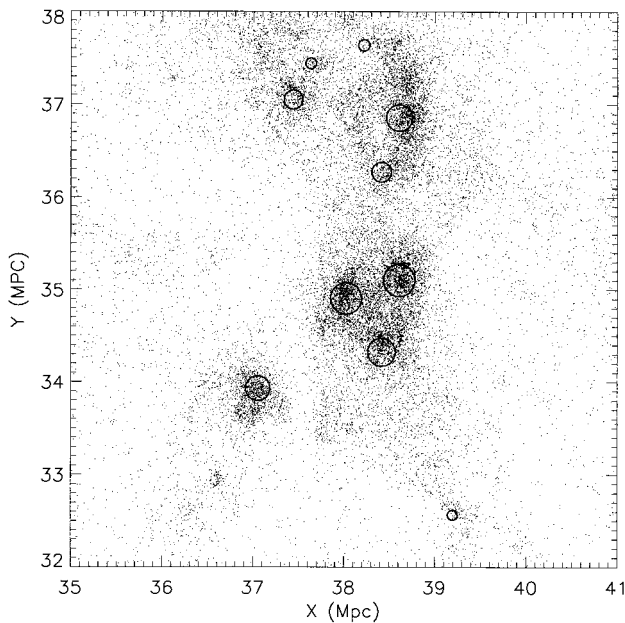


FIG. 2.—Same as Fig. 1, but for a small group of galaxies: the cold particle rms three-dimensional velocity is 310 km s^{-1} , and $M = 1.85 \times 10^{13} h^{-1} M_{\odot}$. The group has four members of about equal mass. Five “galaxies” on the upper part of the plot and one small dark halo at the bottom are not members of the group.

$0.75 h^{-1} \text{ Mpc}$. Positions of all cold particles inside a $3 h^{-1} \text{ Mpc}$ cube are shown together with positions of dark halos with central overdensity larger than $\delta\rho/\rho > 150$. Each circle on the plots corresponds to a dark halo, with the area of the circle being proportional to the mass of the dark halo. The group in Figure 1 is quite massive: $M = 7.5 \times 10^{13} h^{-1} M_{\odot}$. There are about 66,000 cold particles in the frame. The group has nine “galaxies” with $\delta\rho/\rho > 150$. (“Galaxies” with $X > 89.5 \text{ Mpc}$ do not belong to the group). The cold particle rms three-dimensional velocity for that group is 644 km s^{-1} . Figure 2 shows a smaller group: the cold particle rms three-dimensional velocity is 310 km s^{-1} , and $M = 1.85 \times 10^{13} h^{-1} M_{\odot}$. The group has four members of about equal mass. Five “galaxies” on the upper part of the plot and one small dark halo at the bottom are not members of the group.

5. POWER SPECTRA AND CORRELATION FUNCTIONS

Initial spectra for the simulations are shown in Figure 3. The spectra are given at $z = 15$. The CDM simulation was started earlier ($z = 18$), but its initial spectrum was scaled up by linear theory to the moment $z = 15$. We present spectra up to the wavenumber corresponding to the Nyquist frequency of particles: $k = (2\pi/L) \times (N_{\text{part}}/2) = 16 h \text{ Mpc}^{-1}$, where $L = 50 h^{-1} \text{ Mpc}$ and $N_{\text{part}} = 256$. (Recall that we have 256^3 cold particles in each simulation.) Because of the small number of harmonics with long waves, statistical fluctuations are apparent relative to theoretical spectra at $k < 1 h \text{ Mpc}^{-1}$. Nevertheless, the overall accuracy of normalization is better than 5%.

At $k > 1 h \text{ Mpc}^{-1}$ there was 10%–20% difference in the power spectra between the CHDM₁ and CHDM₂ models. The difference originates in different approximations used for the simulation. The CHDM₁ simulation was done for the spectrum given by equation (1) of KHP. The spectrum actually was the power spectrum of *baryons*, not cold particles. As one might expect, the difference between the

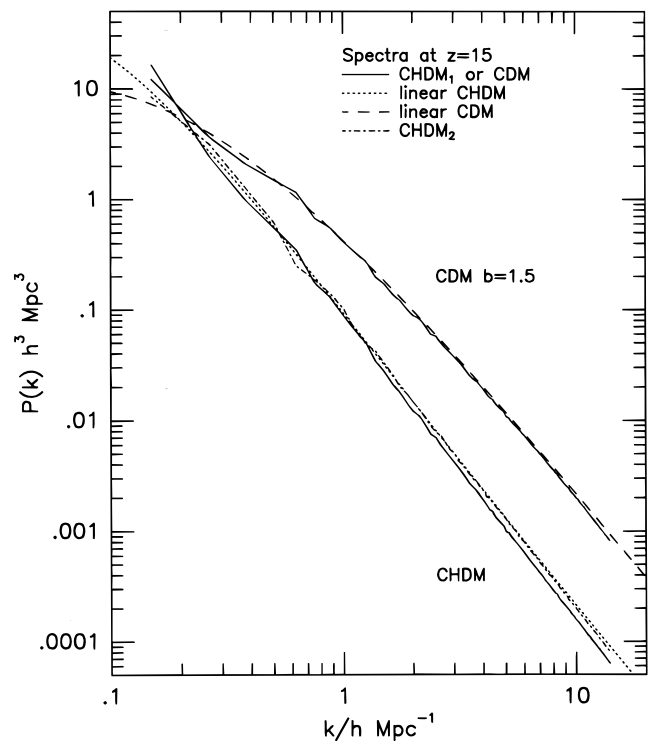


FIG. 3.—The initial power spectra at $z = 15$. Because of the small number of harmonics with long waves, statistical fluctuations are apparent relative to theoretical spectra at $k < 1 h \text{ Mpc}^{-1}$.

spectra is small because baryons had enough time from the recombination to $z = 15$ to fall in the potential well defined by the dark matter, but 10%–20% on the smallest scales is not negligible nowadays. Another effect on the same level basically compensates the difference in the spectra. When setting initial conditions for the CHDM₁ simulation, we forgot to take into account that the growth rates of fluctuations are slightly smaller as compared with pure CDM case on scales smaller than the effective Jeans length in neutrinos (see eq. [2]). Thus, velocities were set 10%–20% higher than they should be. Both effects—smaller densities and larger velocities—have the same phases and amplitudes, but opposite signs. So after few expansions, the system (still in linear regime) arrives at almost the same configuration as if it started from correct initial conditions.

The CHDM₂ simulation was done for the spectrum (A1-2) of KHP, which does not have the problems just described for CHDM₁. In order to check the effect of the difference in spectra and velocities, two tests were done. One was discussed in detail by KHP in their note added in proof. Two models with the same realization of random numbers were run up to $z = 1.48$, one model having 10% higher initial density amplitude and 20% lower initial velocities than the other. At the final moment, positions and velocities of particles in the two simulations were compared one by one. The rms difference in particle positions was 0.11 cell units and the rms velocity difference was 1.6%, i.e., truly negligible. Another test involved comparing the CHDM₂ simulation with one run with the same initial random numbers, but with the initial conditions set exactly as for the CHDM₁ simulation (that is, eq. [1] of KHP for the power spectrum and no corrections for the growth rates). By redshift $z = 7$, the difference in power spectra of the two simulations became smaller than 5%. The difference

between the runs at the end of the simulations ($z = 0$) was also small. For example, the number of dark halos with mass larger than $M > 1.8 \times 10^{10} h^{-1} M_\odot$ was 29,662 in the “uncorrected” simulation and 29,795 for the CHDM₂ simulation. Even the numbers of the most massive, rare dark halos were quite close: 88 and 81 for $M > 5 \times 10^{12} h^{-1} M_\odot$. (All masses are estimated for 3^3 cells centered on density maximum.)

In Figure 4, we present the evolution of the power spectrum in the CHDM₁ model. The three solid curves show the power spectra of the total density (“cold” plus “hot”) in the simulation at $z = 15$ (bottom curve: initial moment), at $z = 3.70$ (middle curve), and at the present moment ($z = 0$, top curve). The dot-dashed curves present the linear theory predictions. Effects of nonlinearity are obviously seen in the spectrum at $z = 0$ for $k > 0.2 h \text{ Mpc}^{-1}$. First, there is a long tail with slope $n \approx -1.2$ from $k = 0.2 h \text{ Mpc}^{-1}$ to $k = 2 h \text{ Mpc}^{-1}$, which is responsible for the power-law behavior of the correlation function. Second, at higher wavenumbers ($k > 2 h \text{ Mpc}^{-1}$) the power spectrum has a tail $P(k) \propto k^{-2.5}$, which agrees with the BBGKY similarity solution for clustering in statistical equilibrium (Peebles 1980, 275–278). [The BBGKY solution predicts the slope $-6/(5+n)$, where n is the initial slope of the power spectrum, which for the CHDM model in this range of wavenumbers is $n = -2.6$.] As we mentioned above, due to a statistical fluke the first harmonic in the CHDM₁ and CDM simulations ($k = 0.125 h \text{ Mpc}^{-1}$) is above the linear prediction just at the beginning of simulations. But it stays in the linear regime even until $z = 0$. In Figure 5 we compare the power spectra in the CHDM₁ and the CDM simulations. The spectrum of the CHDM₁ and that of the biased CDM ($\sigma_8 = 0.66$) simulations are quite close in the range $k = (0.1-1) h \text{ Mpc}^{-1}$, but their long waves are very different: the biased CDM spectrum is higher by more than a factor of 3. Also the slope of the power spectrum in the nonlinear regime $k = (0.2-2.0) h \text{ Mpc}^{-1}$ is shallower for the CDM models ($n \approx -1$) as com-

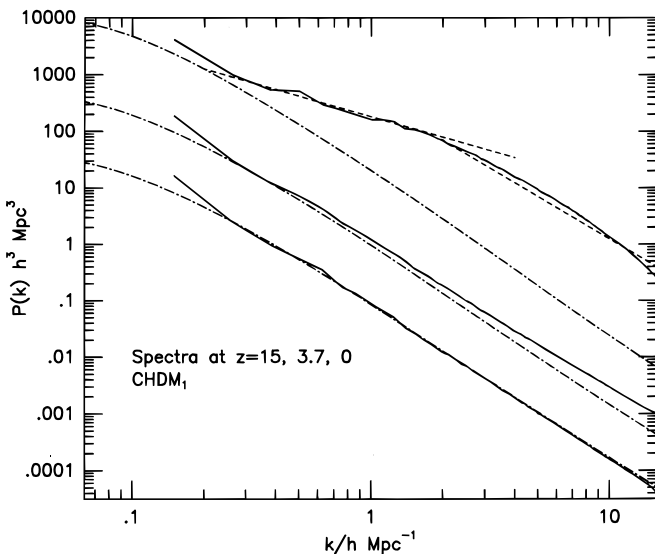


FIG. 4.—The evolution of the power spectrum in the CHDM₁ model. Solid curves show the power spectra of the total density (“cold” plus “hot”) in the simulation at different redshifts. Dot-dashed curves present the linear theory predictions. Due to a statistical fluke, the first harmonic in the simulation ($k = 0.15 h \text{ Mpc}^{-1}$) is above the linear prediction just at the beginning of simulations. Dashed lines correspond to the power-law slopes $n = -1.2$ and $n = -2.5$.

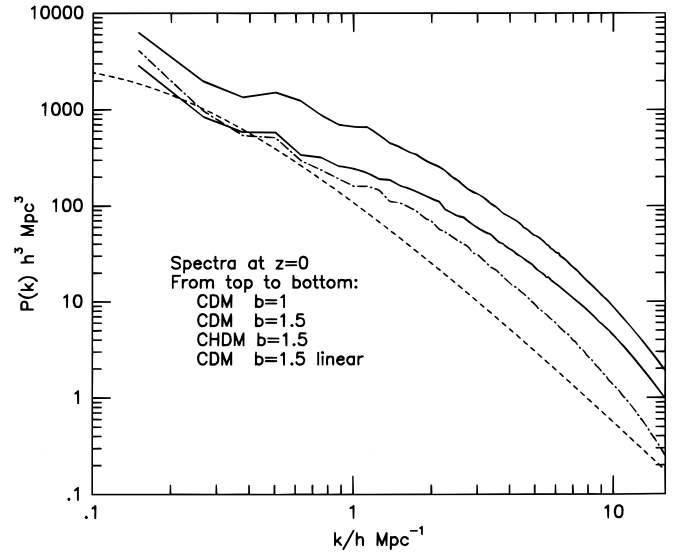


FIG. 5.—Comparison of power spectra for CDM and CHDM₁ simulations in the nonlinear regime at the final moment ($z = 0$). The spectrum of the CHDM and that of the biased CDM ($\sigma_8 = 0.66$) models are quite close in the range $k = (0.1-1) h \text{ Mpc}^{-1}$, but the slope of the power spectrum in the nonlinear regime $k = (0.2-2.0) h \text{ Mpc}^{-1}$ is shallower for the CDM models ($n \approx -1$) as compared with the CHDM spectrum, which results in a steeper correlation function at small radii for the CDM model. Dashed curve shows the linear spectrum for the biased CDM model.

pared with the CHDM₁ spectrum, which results in a steeper correlation function at small radii for CHDM₁.

The power spectra obtained in the simulations are compared with observational data in Figures 6 and 7. Because observational spectra were measured in redshift space, we simulated the distribution of “galaxies” in redshift space s by placing an “observer” in a corner of the simulation box and by displacing dark halos along the line of sight in accordance with their velocity v and distance r from the

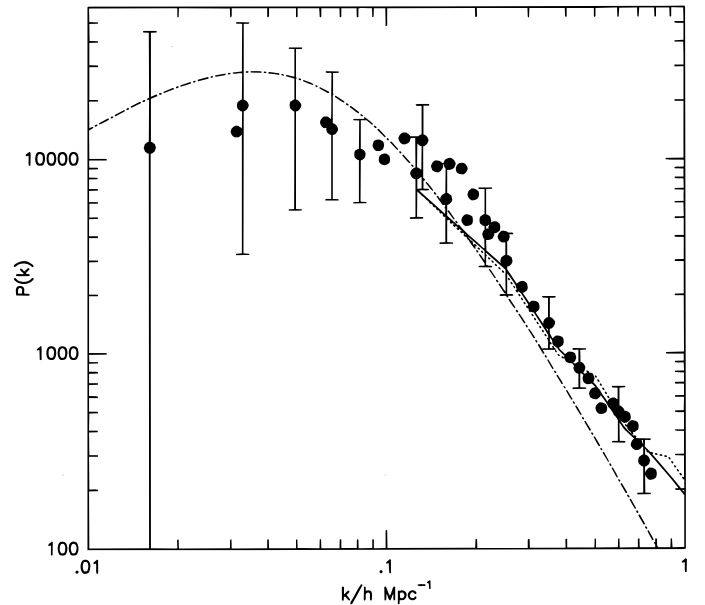


FIG. 6.—Comparison of the power spectrum of the galaxies in the CfA2 catalog (dots with error bars; Vogeley et al. 1992) with the spectrum predicted by the CHDM model in redshift space. Solid and dotted curves represent results for CHDM₁ and CHDM₂ simulations respectively. Dashed curve is the linear power spectrum scaled upward by a factor of 2.75 (see text).

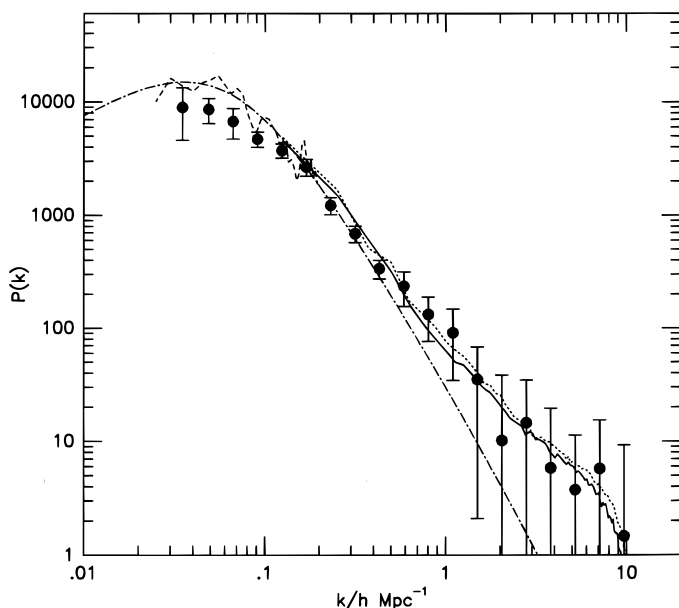


FIG. 7.—Same as Fig. 6, but for *IRAS* galaxies. (Dots are data of Fisher et al. 1993 (note that these were derived with a cylindrical window function in redshift space); broken dashed curve is the same data as analyzed by Feldman et al. (1993). The linear CHDM spectrum (dot-dashed curve) was shifted upward by a factor of 1.47.

observer: $s = r[1 + (vr)/Hr^2]$. Dark halos with central overdensity more than 100 were selected as “galaxies.” Redshift distortions produce an effect that is analogous to Malmquist bias: some of the halos close to boundaries of the simulation box *and* moving away from the observer can disappear from the box, reducing the mean density at the boundaries. (Note that we cannot apply periodic boundary conditions for those receding halos.) In order to avoid this Malmquist-like bias, we periodically replicated all halos before going to redshift space, but then we retained only the halos inside the box. The mean number density of “galaxies” in real and redshift space was almost the same: $0.0525 h^3 \text{ Mpc}^{-3}$ in redshift space and $0.0508 h^3 \text{ Mpc}^{-3}$ in real space. The density field of the halos was constructed using the cloud-in-cell technique on our 512^3 mesh and was smoothed using a Gaussian filter with 1 cell width to reduce the shot noise. This did not remove all the noise: there was apparent flattening of the spectrum at $k > 2k \text{ Mpc}^{-1}$. A constant $P = 12$ was subtracted from the power spectra to approximately remove the shot noise. In addition, the power in the first k bin (longest waves in the simulations) in the CHDM₁ simulation was reduced by two to take into account the excess of power in the initial spectrum for this simulation. Besides that difference in the first bin, both CHDM₁ and CHDM₂ simulations produce remarkably similar power spectra.

Optically selected CfA and IR-selected *IRAS* galaxies have different bias relative to each other and thus sample the underlying power spectrum of the dark matter differently. So we clearly cannot compare both these data sets the same way to dark matter particles or halos. Instead, we adopt what we think are simple and reasonable prescriptions. When estimating the CHDM power spectrum (*solid curve*: CHDM₁; *dotted curve*: CHDM₂) to compare to that of the CfA2 galaxies (Vogeley et al. 1992) presented in Figure 6, each dark halo in the numerical simulations was weighted by its mass (defined as mass within 3^3 cells cen-

tered on the peak of density). This weighting procedure was designed to compensate for the lack of resolution in the central parts of clusters, which leads to “overmerging” at the centers of clusters—a very massive dark halo representing in reality several “galaxies,” not one. To the extent that CfA-selected galaxies have the same mass-to-light ratio, by mass-weighting the halos we are mimicking this in the solid curve. It is less clear how to mimic the *IRAS* galaxies, presented in Figure 7 (*dots*: results of Fisher et al. 1993; *dashed broken curve*: results of Feldman, Kaiser, & Peacock 1993). The solid and dotted curves attempt to mimic *IRAS* galaxies by weighting equally all dark halos with density greater than $100\rho_c$. This prescription underestimates the central regions of clusters and large groups, just as *IRAS* galaxies do.

Because of the quite limited size of our computational box, we cannot confront the results of our simulations with observational data on the power spectrum at small k . Nevertheless, it seems quite reasonable to suggest that on long waves the redshift distortions do not depend on the wavelength (Kaiser 1987). As we discussed earlier, the longest waves in our simulations still grow in the linear regime. Thus, we can use the power spectrum given by the linear theory to make a prediction for the spectrum of galaxies in redshift space. The normalization constant for the power spectrum in redshift space was found by matching powers at the longest waves in the simulations. The power spectra estimated in this way are shown as dot-dashed curves in Figures 6 and 7. The linear spectrum was multiplied by the factor 2.75 in the case of CfA2 galaxies and by the factor 1.47 for the case of *IRAS* galaxies. If we assume that the relation between the redshift P_s and real-space P_r spectra is given by (Kaiser 1987)

$$P_s(k) = P_r(k)b^2(1 + 2/3b + 1/5b^2), \quad (3)$$

where b is the biasing parameter, then biasing parameters for CfA2 and *IRAS* galaxies are $b_{\text{CfA}} = 1.3$ and $b_{\text{IRAS}} = 0.85$, which are close to other estimates of the biasing parameters for those types of galaxies. Unfortunately, these estimates are of uncertain accuracy both because of uncertainties in the matching procedure and because it is not clear whether the redshift correction is truly scale independent at k around 0.1–0.2, where we adjust the spectra. Nevertheless, we think it is quite plausible that these CHDM spectra agree reasonably well with power spectra determined from both CfA2 and *IRAS* galaxy catalogs.

6. CORRELATIONS

In Figure 8, the correlation functions ξ of different populations in the CHDM₁ model are shown at the present moment $z = 0$. The lower solid curve presents the correlation function of the “cold” dark matter in the simulation. In the range $R = (0.5\text{--}5) h^{-1} \text{ Mpc}$, the correlation function follows the power law $\xi(R) = (R/4.0 h^{-1} \text{ Mpc})^{-1.8}$ indicated by the lower dot-dashed line. At larger radii, the correlation function goes above the power law but reasonably close to the correlation function predicted by the linear theory for an infinite box (*dashed curve*). Actually on scales around $10 h^{-1} \text{ Mpc}$, the level of fluctuations in the simulation is closer to that expected for much larger box size. The correlation function crosses zero at $25 h^{-1} \text{ Mpc}$ due to the finite size of the computational box.

The correlation function of dark halos in the CHDM model is larger than the correlation function of the dark

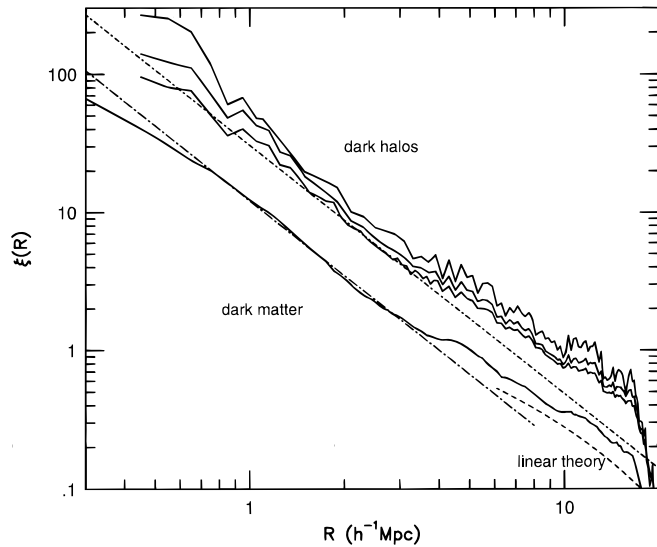


FIG. 8.—The correlation functions ξ of different populations in the CHDM₁ model at the present epoch ($z = 0$). Lower solid curve presents the correlation function of the “cold” dark matter in the simulation. In the range $R = (0.5\text{--}5) h^{-1} \text{ Mpc}$, the correlation function follows the power law $\xi(R) = (R/4.0 h^{-1} \text{ Mpc})^{-1.8}$ indicated by the lower dot-dashed line. At larger radii the correlation function goes above the power law, but it remains reasonably close to the correlation function predicted by the linear theory for an infinite box (dashed curve). The three upper solid curves on the plot show the correlation functions of dark halos chosen for different mass limits. From the bottom to the top, the curves are for the following mass limits and numbers of “galaxies”: $m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$ (corresponding to $\rho > 50\rho_c$), $N = 14606$; $m_1 > 2.56 \times 10^{10} h^{-1} M_\odot$, $N = 6172$; $m_1 > 7.67 \times 10^{10} h^{-1} M_\odot$, $N = 1410$. Here m_1 indicates that the mass corresponds to the mass in the single cell at the density maximum.

matter. The three upper solid curves on the plot show the correlation functions of dark halos satisfying three different mass limits. From the bottom to the top, the curves are for the following mass limits and numbers of “galaxies”: $m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$, $N = 14606$; $m_1 > 2.56 \times 10^{10} h^{-1} M_\odot$, $N = 6172$; and $m_1 > 7.67 \times 10^{10} h^{-1} M_\odot$, $N = 1410$. Here m_1 indicates that the mass corresponds to the mass in one cell at the density maximum. The mass of a halo increases typically by 7–10 times when one goes from m_1 to the mass defined in 3^3 cells. When we estimated the correlation function, a dark halo was weighted proportionally to its mass found in the 3^3 cells volume centered on the density maximum. The reason we use this weighting of dark halos is to compensate for the effects of “overmerging” of dark halos in central regions of groups and clusters.

The growth of correlations when the mass limit increases is apparent, but it is not large. At radii smaller than $5 h^{-1} \text{ Mpc}$, the correlation function is approximated by the power law $\xi(R) = (R/6.7 h^{-1} \text{ Mpc})^{-1.8}$. This corresponds to the bias parameter $b = (\xi_{\text{gal}}/\xi_{\text{dm}})^{1/2} \approx 1.6$, in a reasonable agreement with the linear bias $b_{\text{linear}} = 1/\sigma_8 = 1.5$. The correlation length increases by about 1 Mpc when the number of “galaxies” drops by a factor of 10 because we have selected a higher mass limit.

Any types of N -body simulations do not include waves longer than the size of the initial cloud of particles. We hope that those long waves do not change our results too much. Different statistics are sensitive in different degree to those waves. By comparing results from simulations with different sizes or variations from one realization to another, one can obtain an estimate of the effect. Sometimes it is possible to make corrections that take into account effects of the waves.

Because the waves of the size of the computational box and larger are still in the linear regime, it is possible to make corrections for those waves and for the statistical fluctuations of the largest waves in the box. The relation between the power spectrum and the correlation function could be split into two parts: the contribution of waves in the simulation box and the contribution of waves longer than the length of the box. We have

$$\xi(r) = \frac{1}{2\pi^2} \int_{k_{\text{box}}}^{k_{\text{max}}} k^2 P(k) \frac{\sin(kr)}{kr} dk + \frac{1}{2\pi^2} \int_0^{k_{\text{box}}} k^2 P(k) \frac{\sin(kr)}{kr} dk, \quad (4)$$

where $k_{\text{box}} = 2\pi/L$ and $k_{\text{max}} = 256k_{\text{box}}$ are the smallest and the largest wavenumbers in the simulations ($L = 50 h^{-1} \text{ Mpc}$ is the box size). The first integral in this expression is estimated by the usual procedure of counting pairs of objects at given separation. The second part is estimated analytically and added to the first one. In order to make the correction due to the fluctuation in the initial spectrum, we estimate the additional power due to the fluke in the initial spectrum, extrapolate it by linear theory to the present moment of time, and subtract the contribution due to the power from the correlation function. This procedure is applied to radii smaller than $20 h^{-1} \text{ Mpc}$. At larger radii, we use the linear theory. A test for the procedure is provided by comparing predictions of the linear theory with the corrected correlation function at radii $(10\text{--}20) h^{-1} \text{ Mpc}$, where the correlation function is small and one can hope that the linear theory predictions are still reasonably accurate.

In Figure 9, the correlation function of the “cold” dark matter in the CHDM₁ simulation is shown as the lower solid curve. The dashed curve presents the linear theory prediction. The agreement with the corrected correlation function in the simulation looks quite reasonable. The same procedure was applied to the correlation function of dark

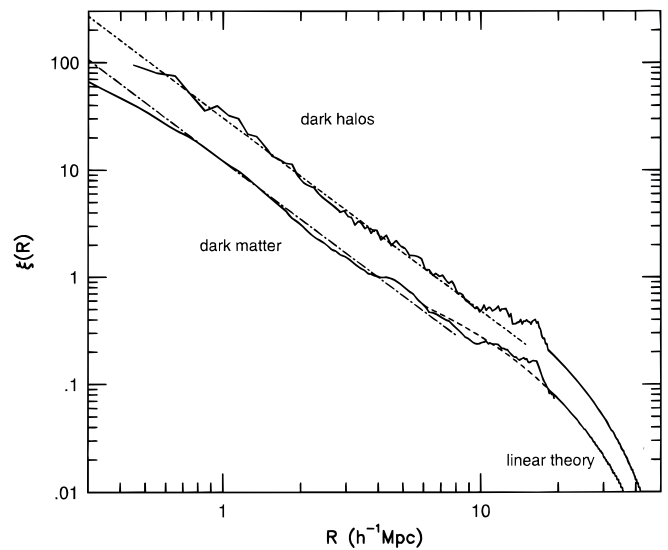


FIG. 9.—The correlation function of the “cold” dark matter in the CHDM₁ simulation is shown as the lower solid curve. Dashed curve represents the linear theory prediction. The same procedure was applied to the correlation function of dark halos with $m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$. In this case the power spectrum predicted by the linear theory was multiplied by a factor of 2.3 to account for the biasing of dark halos observed in Fig. 8. Dot-dashed lines on the plot are the same power laws as in Fig. 8.

halos with $m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$. In this case, the power spectrum predicted by the linear theory was multiplied by a factor 2.3 to account for the biasing of dark halos observed in Figure 8. The dot-dashed lines on the plot are the same power laws as in Figure 8. The CHDM model predicts that in real space the correlation function of both the dark matter and the galaxies is a power law on scales smaller than $\approx 20 h^{-1}$ Mpc with the slope $\gamma = -1.8$. The correlation length of the dark halos is $r_{\text{gal}} = 6.7 h^{-1}$ Mpc. The correlation function crosses zero at $r_0 = 50 h^{-1}$ Mpc and becomes negative at larger radii.

For comparison, Figure 10 presents the correlation function in real space for the unbiased CDM simulation. The correlation function of the dark matter particles is not a power law, as was well known (e.g., Suto & Sugimoto 1991; Bahcall, Cen, & Gramann 1993). Nevertheless, the correlation function of dark halos does show power-law behavior (*dot-dashed line*) with -1.8 slope and the correlation length $r_{\text{gal}} = 5.7 h^{-1}$ Mpc. The same mass threshold ($m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$; $N = 22,275$) and weighting scheme was used as for the CHDM halos. No corrections were applied in this case.

In order to estimate the correlation function in redshift space $\xi(s)$, two “observers” were placed in opposite corners of the computational cube. Then objects of interest (dark halos or a small random fraction of the dark matter particles) were replicated periodically 27 times in space in such a way that the original box was surrounded by 26 replicated boxes. For each observer, the objects were displaced along the line of sight in accordance with their peculiar velocities. Then neighbors at a given distance were counted for all objects inside the original box. Results of the two observers were averaged.

Figure 11 shows the correlation function for the dark matter and dark halos in the CHDM₁ simulation. Note that at small radii the difference between halos and dark

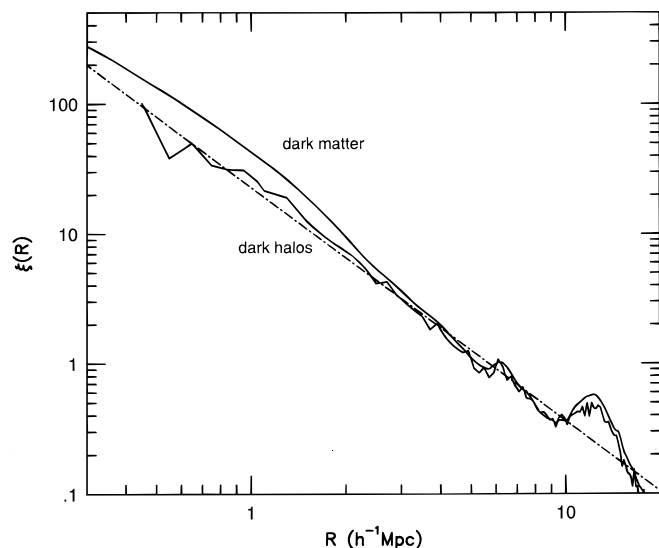


FIG. 10.—The correlation function in real space for the unbiased CDM simulation. The correlation function of the dark matter deviates significantly from a power law, but that of the dark halos—which lies below that of the dark matter, i.e., it is “antibiasing”—does show power-law behavior (*dot-dashed line*) with -1.8 slope and correlation length $r_{\text{gal}} = 5.7 h^{-1}$ Mpc. The same mass threshold ($m_1 > 1.28 \times 10^{10} h^{-1} M_\odot$; $N = 22,275$) and weighting scheme was used as for the CHDM halos. No corrections were applied.

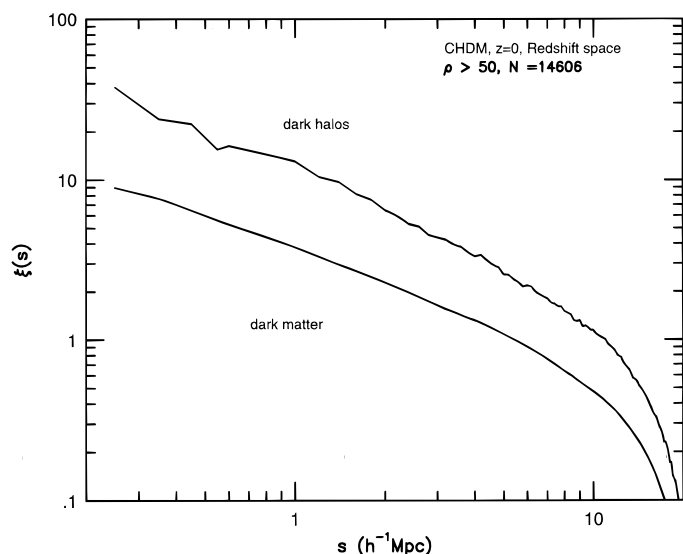


FIG. 11.—The correlation function in the redshift space for the dark matter and dark halos in the CHDM₁ simulation. Note that at small radii the difference between halos and dark matter becomes larger.

matter becomes larger, which indicates that the peculiar velocities of the halos are smaller than those of the dark matter, i.e., they exhibit a global velocity bias. (We saw this directly in KHP; see also NKP96). These peculiar velocities have two opposite effects on the correlation function: at small radii it becomes smaller, and at large radii it is increased. This agrees with theoretical expectations. Peculiar velocities on small scales produce a “finger of god” effect, which results in less compact clusters and groups. On large scales, where fluctuations are still in the linear regime, peculiar velocities displace particles toward collapsing regions, thus increasing apparent clustering.

In redshift space, corrections to the correlation function due to waves longer than our simulation box or due to statistical fluctuations of the longest waves in the simulations can be done in the same way as for $\xi(r)$ in real space. In order to take into account the effect of peculiar velocities at large radii on $\xi(s)$, we apply the Kaiser (1987) correction to the linear power spectrum given by equation (3) (note that the biasing parameter is $b = 1$ for the dark matter; it is unity even for “biased” models). For dark halos in the CHDM model, the parameter is $b = 1.5$ in accordance with the ratio of correlation functions of “galaxies” and the dark matter in real space. For the unbiased CDM model, we take $b = 1$. The corrected correlation functions for the CHDM₁ model are shown in Figure 12 as the solid curves. The dashed curve presents the linear theory prediction amplified by the velocity effects. Triangles show the estimates of the correlation function for the CfA Redshift Survey (Vogeley et al. 1992). The agreement between the observational data and the model is reasonably good on all scales. There is a slight indication that the model predicts larger $\xi(s)$, but taking into account all uncertainties involved on both the theoretical and the observational sides, the difference is hardly significant.

The correlation function in redshift space for the unbiased CDM model is shown in Figures 13 (with no corrections) and 14 (corrected for the box size and Kaiser’s correction for $b = 1$, with no corrections for the additional fluctuations in the box). The dashed curve in the last plot is

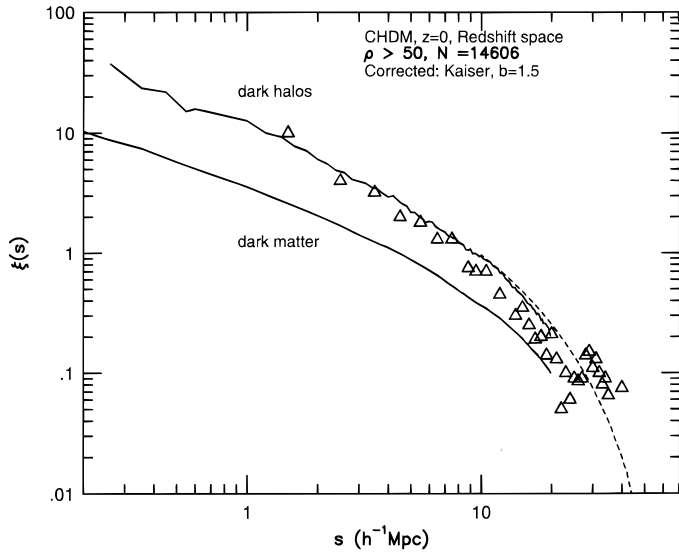


FIG. 12.—Comparison of predicted correlation function of “galaxies” in the CHDM model with the CfA2 data. The correlation functions in redshift space for the CHDM₁ model are shown as the solid curves. Corrections for the finite box size and the statistical fluctuations of few longest waves in the simulations were applied. Dashed curve represents the linear theory prediction amplified by the velocity effects. Triangles show the estimates of the correlation function for the CfA2 Redshift Survey (Vogeley et al. 1992). The agreement between the observational data and the model is reasonably good on all scales.

for linear theory in redshift space. It really does not match well with $\xi(s)$ from the simulation. Many different parameters were tested, and this is the best match we can obtain. The triangles again show the results of Vogeley et al. (1992). The correlation functions of the dark matter and the dark halos (“galaxies”) match very well at scales smaller than $s < 3 h^{-1}$ Mpc, but ξ_{gal} is about 1.5 times higher than ξ_{dm} for $s < 3 h^{-1}$ Mpc. This is probably the reason why the linear theory prediction does not match the results of the simulations: larger b would increase $\xi_{\text{linear}}(s)$, but unfor-

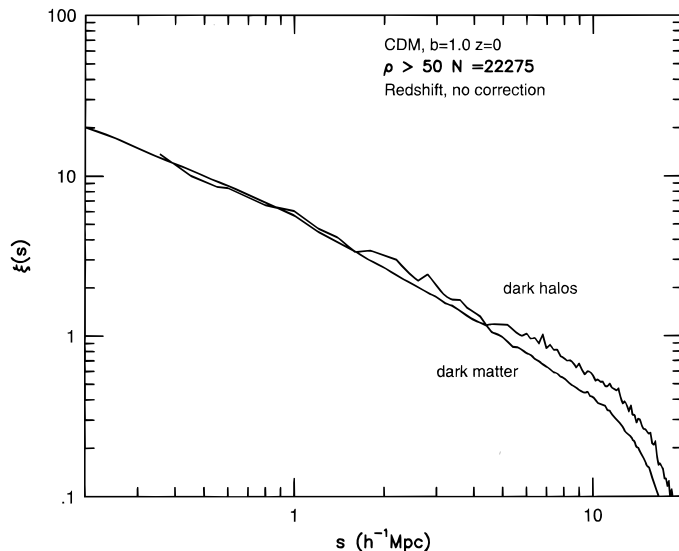


FIG. 13.—The correlation function in redshift space for the unbiased CDM model. No corrections were applied.

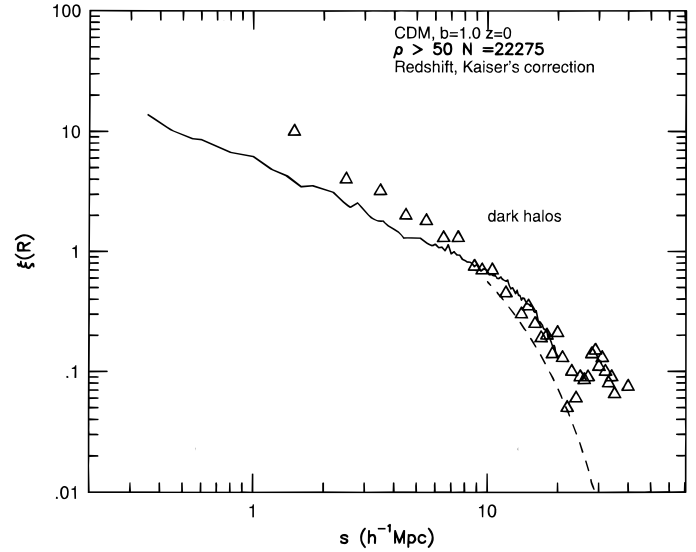


FIG. 14.—The correlation function in redshift space for the unbiased CDM model corrected for the box size and Kaiser's correction for $b = 1$. Dashed curve, representing linear theory in redshift space, does not match $\xi(s)$ from the simulation very well. We tried many parameters, and this is the best match we could obtain. Triangles show the CfA2 data of Vogeley et al. (1992). The correlation function of the dark matter and the dark halos (“galaxies”) match very well at scales smaller than $s < 3 h^{-1}$ Mpc, but ξ_{gal} is about 1.5 times higher than ξ_{dm} for $s < 3 h^{-1}$ Mpc.

tunately that would contradict the very tight match of ξ_{dm} and ξ_{gal} in real space at this radius. In any case, the unbiased CDM model predicts too small $\xi_{\text{gal}}(s)$ at $s < 5 h^{-1}$ Mpc, and it fails to reach the observed level of the correlation function at scales around $30 h^{-1}$ Mpc. As a matter of fact, no matter what amplitude one assumes or what correction one applies, the CDM model cannot produce the observed correlation function at $\sim 30 h^{-1}$ Mpc because at that radius the CDM correlation function crosses zero.

7. VELOCITIES

Velocities provide very important but somewhat controversial information. One of the most important issues is the pairwise velocities on small scales. For a long time the results of DP83, who found that the rms galaxy pairwise velocity [which they designated as $\sigma(r_p)$] is $\sigma_{12}(1 h^{-1} \text{ Mpc}) = 340 \text{ km s}^{-1}$, have been a stumbling block for cosmological models. One should mimic all the steps of DP83 in numerical simulations in order to obtain directly comparable results. In KHPR we mimicked only partly the DP83 analysis; in particular, their infall model was not simulated. We showed that the CHDM model predicts smaller pairwise velocities than the CDM model, and that although the dark matter particles have higher velocities, the pairwise velocities of dark halos are consistent with DP83. In a similar calculation, Cen & Ostriker (1994) found somewhat higher pairwise velocities in simulations with the same CHDM parameters and also found that the velocities depend on the local density. In fact, because the pairwise velocity statistic σ_{12} is pair weighted and clusters not only have many close pairs but also high relative velocities, σ_{12} is extremely sensitive to the presence of clusters in the volume covered by the redshift survey (da Costa, Vogeley, & Geller 1994) or simulation; thus, it is not a very robust statistic. Moreover, other authors who have recently tried to repro-

duce DP83 σ_{12} results from the same CfA1 data have been unable to do so (Mo, Jing, & Borner 1993; Zurek et al. 1994; Somerville et al. 1996). It turns out that, due to a typographic mistake in the computer code used in the σ_{12} calculation, the entire core of the Virgo Cluster was inadvertently omitted; when it is included, $\sigma_{12}(1 h^{-1} \text{ Mpc}) \approx 600 \text{ km s}^{-1}$. The fact that *IRAS* galaxies give smaller σ_{12} (Fisher et al. 1994) doubtless reflects the fact that they are not found in clusters.

We do not repeat the DP83 analysis in this paper, and we postpone detailed comparison of pairwise velocities with observation for a later publication. (As mentioned in § 1, we have compared another velocity-related statistic, the median group velocity from the CfA1 data, with the present CDM and CHDM simulations in some detail in NKP94 and NKP96, and we have concluded that CHDM with $\Omega_c/\Omega_b/\Omega_d = 0.6/0.3/0.1$ probably has velocities that are a little smaller than the data.) Instead we study here another important and controversial statistic: the velocity bias and real-space pairwise velocities.

Figure 15 presents one of the components of the real-space pair-wise velocity tensor: rms velocities along the line joining the pair of objects or $\sigma_{\parallel}$. The top panel shows $\sigma_{\parallel}(R)$ for dark halos with central overdensity larger than 100. The mass inside the cell centered on the maximum is $m_1 > 2.5$

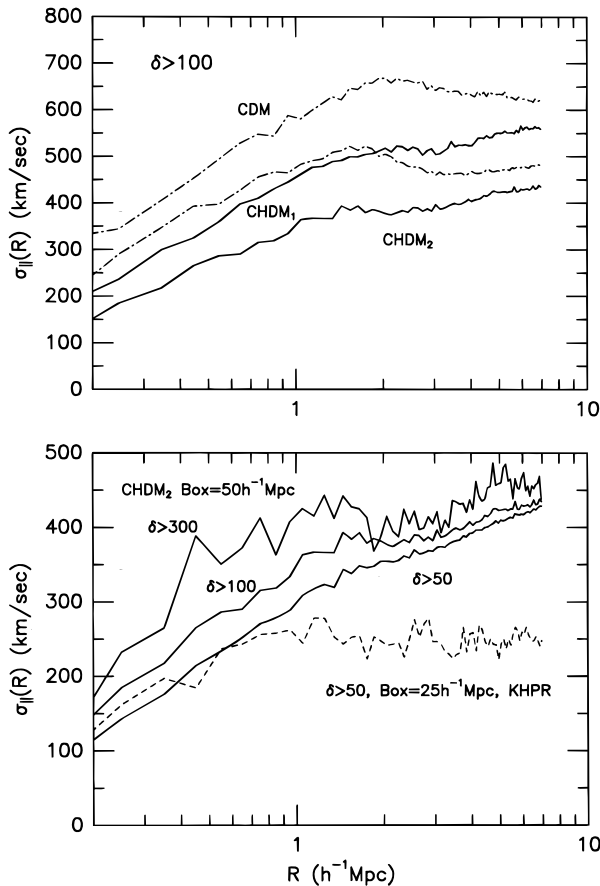


FIG. 15.—Real space rms velocity along the line joining a pair of objects. *Top:* $\sigma_{\parallel}(R)$ for dark halos with central overdensity larger than 100. Dot-dashed curves are for the CDM simulations (*higher curve:* $b = 1.0$; *lower:* $b = 1.5$). *Bottom:* shows results for the CHDM₂ simulation (*solid curves*). Different curves correspond to different mass thresholds: the larger the mass of both “galaxies” in the pair, the larger their relative velocity. For comparison, we show the pairwise rms velocity from the KHPR simulations for the CHDM model (*dashed curve*).

$\times 10^{10} h^{-1} M_{\odot}$. The dot-dashed curves are for the CDM simulations (the higher one corresponds to $b = 1.0$, and the lower one is for $b = 1.5$). The CDM $b = 1.5$ results are close to CHDM₁ on scales around $1 h^{-1} \text{ Mpc}$, but $b = 1$ CDM shows significantly larger pairwise velocities. Note that the CHDM and CDM models have different trends. The CDM curves have maxima at $1.5\text{--}2 h^{-1} \text{ Mpc}$, and at larger radii $\sigma_{\parallel}$ decreases. The CHDM curves do not show maxima in $\sigma_{\parallel}(R)$, but the slope of the curves is significantly smaller at $R > 1.5 h^{-1} \text{ Mpc}$ than at lower R . This difference is probably related to the difference in the large-scale power in the two models.

The bottom panel shows results for the CHDM₂ simulation. The three solid curves correspond to different mass thresholds: the larger the mass of both “galaxies,” the larger their relative velocity. This trend can be explained if a large fraction of mass on these scales is associated with the dark halos. Then the more massive the halos that are selected, the deeper the potential well and the larger the relative velocities. Artificial “overmerging” due to the lack of either force or mass resolution would give another trend, with more massive halos moving slower. We do not see this trend in our simulations. A significant “overmerger” problem appears in our case at overdensities larger than 1000. For comparison, we show also in Figure 15 the pairwise velocity dispersion from KHPR simulations for the same CHDM model. The lack of long waves in the smaller box used by KHPR ($L = 25 h^{-1} \text{ Mpc}$) resulted in smaller $\sigma_{\parallel}$, but the agreement on scales less than $1 h^{-1} \text{ Mpc}$ is good.

The tendency for more massive halos to move faster in pairs is *not* the result of all massive halos moving faster; there is such a tendency, but it is far too weak. The effect appears only in pairs of “galaxies.” In Table 1 we present three-dimensional rms velocities relative to the rest frame, σ_v , for different populations of objects. For the CDM simulation, there is visible difference between σ_v of the dark matter and that of dark halos: “galaxies” are moving slower than dark matter particles in the CDM model. Nevertheless, the difference between galaxies with different masses is barely noticeable. The same is true for CHDM simulations, but in this case there is almost no difference between the rms velocities of cold dark matter particles and those of “galaxies.”

The ratio of rms velocities of dark halos to that of dark matter (both measured relative to the rest frame) can be called the global velocity bias:

$$b_{v,\text{glob}}^2 \equiv \frac{\sigma_v^2(\text{galaxies})}{\sigma_v^2(\text{dark matter})}. \quad (5)$$

Table 1, column (5), gives $b_{v,\text{glob}}$ for our simulations. It is quite straightforward to add corrections to b_v due to waves longer than the box size because those waves are still in the linear regime and they simply move our computational box as a whole. Columns (7) and (8) give corrected σ_v and $b_{v,\text{glob}}$. The CDM model has global velocity bias $b_{v,\text{glob}} \approx 0.8$, while both CHDM simulations give numbers very close to unity: $b_{v,\text{glob,CHDM}} \approx 1.0$.

In Table 1, columns (4) and (6), we present $\sigma_{\parallel}$ measured at $R = 1 h^{-1} \text{ Mpc}$ and relate to it the pairwise velocity bias:

$$b_{v,\text{pairs}}^2(R = 1 h^{-1} \text{ Mpc}) \equiv \frac{\sigma_{\parallel}^2(\text{halo-halo})}{\sigma_{\parallel}^2(\text{dark matter-dark matter})}. \quad (6)$$

TABLE 1
VELOCITY DISPERSIONS AND VELOCITY BIASES

Model (1)	Objects (2)	σ_v (km s ⁻¹) (3)	$\sigma_{ }^a$ (km s ⁻¹) (4)	$b_{v, glob}$ (5)	$b_{v, pairs}$ (6)	σ_v (corrected) ^b (km s ⁻¹) (7)	$b_{v, glob}$ (corrected) ^b (8)	n $h^3 \text{ Mpc}^{-3}$ (9)
CDM $b = 1.5$	Dark matter	872	923	...	...	954	...	...
	Halos, $m_1 > 1.3$	675	413	0.77	0.45	778	0.81	0.217
	Halos, $m_1 > 2.5$	680	473	0.78	0.51	782	0.82	0.108
	Halos, $m_1 > 7.6$	687	571	0.79	0.62	789	0.83	0.032
CDM $b = 1.0$	Dark Matter	992	1160	...	...	1150	...	...
	Halos, $m_1 > 2.5$	806	584	0.81	0.50	994	0.86	0.091
CHDM ₁	Cold dark matter	798	901	...	...	980 ^d	...	...
	Halos, $m_1 > 2.5$	786	452	0.98	0.50	970 ^d	0.99	0.049
CHDM ₂	Cold dark matter	581	673	...	...	873	...	...
	Halos, $m_1 > 1.3$	552	298	0.95	0.44	854	0.98	0.121
	Halos, $m_1 > 2.5$	560	348	0.96	0.52	859	0.98	0.052
	Halos, $m_1 > 7.6$	565	416	0.97	0.62	863	0.99	0.012

^a At $1 h^{-1}$ Mpc.

^b Corrected for the finite box size: $\sigma_v^2 = \sigma_{v, simulation}^2 + \sigma_{v, \lambda > \text{Box}}^2$.

^c One-cell mass, in units of $10^{10} h^{-1} M_\odot$.

^d Corrected for the excess large-scale power in the initial conditions.

The pairwise velocity bias is significantly smaller than the global velocity bias and, for halos with the same mass, it does not depend on either the model or the large-scale power (compare CHDM₁ vs. CHDM₂, and present paper vs. KHPR). Nevertheless, there is significant dependence of $b_{v, pairs}$ on the mass of dark halos.

Thus, on small scales ($r < 2 h^{-1}$ Mpc) the CDM model predicts pairwise velocities larger than in CHDM. CDM normalized to COBE ($b = 1.0$) gives pairwise velocities that may be too high ($\approx 600 \text{ km s}^{-1}$ at $1 h^{-1}$ Mpc) to be compatible with observations. On large scales $r > 2 h^{-1}$ Mpc, the CHDM model predicts $\sigma_{||}$ that is between the CDM results for the model with the same biasing parameter $b = 1.5$ and the CDM with $b = 1$. It is interesting to note that for the CHDM $\sigma_{||}$ is a rising function (at least up to $10 h^{-1}$ Mpc), which agrees at least qualitatively with observations (DP83; Mo et al. 1993). As we remarked above, in order to compare with observational results one needs to mimic the observations and the procedure for estimating the “observational” pairwise velocities, which we are in the process of doing. Nonetheless, we can naively compare $\sigma_{||}$ with observational results as other authors have done. According to Mo et al. (1993), the relative one-dimensional velocity in pairs with $r_p = 1 h^{-1}$ Mpc is in the range 250–450 km s^{-1} . Here we take into account only optical samples (IRAS galaxies should be mimicked in a different way), and we do not consider the CfA2 sample with Coma and other clusters for two reasons: (i) The sample is dominated heavily by the Great Wall and is not a fair sample; the correlation function is far too big. (ii) In our numerical simulations, there are not such large clusters as Coma. Thus, it is not fair to compare the simulations, which do not have a cluster of that size, with a piece of the universe, which does have a very large cluster. In the simulations, $\sigma_{||}$ is the range 350–450 km s^{-1} for the CHDM simulations, which probably indicates that CHDM is compatible with observations. The COBE-normalized CDM with $b = 1$ gives $\sigma_{||} \approx 600 \text{ km s}^{-1}$, which as we said is probably too large.

Our results should be compared with results of other publications. Gelb, Gradwohl, & Frieman (1993) give $\sigma_{||} = 1100 \text{ km s}^{-1}$ at $1 h^{-1}$ Mpc for dark matter particles in

CDM with $b = 1$, which is practically the same as we found in this paper (1160 km s^{-1}). Because of their low resolution, Gelb, Gradwohl, & Frieman do not give estimates for dark halos. Gelb & Bertschinger (1994) give a slightly larger number for the same CDM model: $\sigma_{||} = 1200 \text{ km s}^{-1}$. Their resolution was significantly better than that of Gelb, Gradwohl, & Frieman, and they also present estimates for pairwise velocities of dark halos: 450–650 km s^{-1} at $1 h^{-1}$ Mpc for “galaxies” with different circular velocities. Unfortunately, we cannot give reliable estimates for rotational velocities, but pairwise velocities for our dark halos in CDM with $b = 1$ are in the same range as found by Gelb & Bertschinger. Nevertheless, Gelb & Bertschinger find the same trend as we do, for pairwise velocities to be larger for more massive dark halos. There is also a reasonable agreement on the velocity bias in pairs: $b_{v, pairs}(R = 1 h^{-1} \text{ Mpc}) \approx 0.4\text{--}0.6$.

Comparison with KHPR shows how the size of the computational box and the presence of long waves affects the results on the pairwise velocities. If we compare results for dark halos in the CHDM₂ with those given by KHPR (bottom panel in Fig. 15), then we find the difference of about 15% at $1 h^{-1}$ Mpc. The smaller boxes used by KHPR give progressively lower estimates at larger radii, and the KHPR results for CDM were affected equally by the box size. If we take the same mass threshold for dark halos, $m_1 > 1.3 \times 10^{10} h^{-1} M_\odot$, the difference between CDM $b = 1.5$ simulations is about 20% at $1 h^{-1}$ Mpc. We find the same difference for dark halos (about 20%).

8. DISTRIBUTION OF DENSITY

In CHDM, we of course have two different kinds of dark matter, cold and hot. Here we consider the distribution of each kind of particle as a function of total density. See also BHNPK, where the distributions are discussed further and visualized, both by showing isodensity surfaces for cold and hot densities and by showing the relative densities on cutting planes, for the CHDM₂ simulation.

The top panel in Figure 16 presents the ratio of the density of hot particles, ρ_{hot} , to the total density, $\rho_{\text{total}} =$

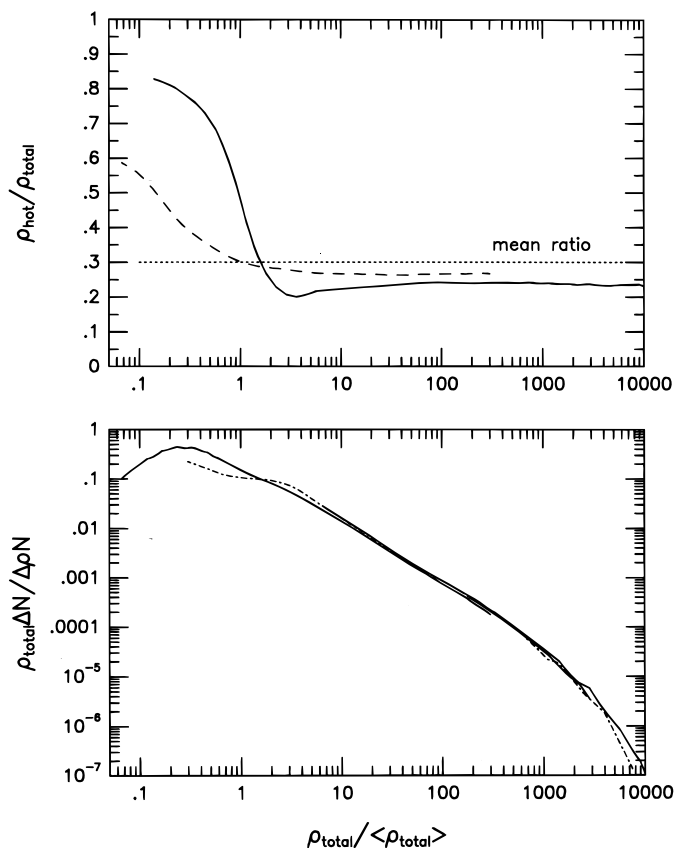


FIG. 16.—Distribution of hot and cold particles in the CHDM₂ simulation. *Top*: The ratio of the density of hot particles to the total density as a function of the total local density. *Bottom*: The density distribution function multiplied by the total density ρ for the CHDM₂ simulation (solid curve) compared to the same quantity from KHPR. In order to make more reliable estimates for the low-density tail of $P(\rho)$, we smoothed the total density field with a filter that weights the density of the central cell with weight 3 and the six nearest cells with unit weight. The filter reduces discreteness effects due to small number of points in low-density areas (“voids”). The dashed curve in the top panel and both curves in the bottom panel show results for the smoothed density; the unsmoothed density (obtained with the usual cloud-in-cell method) is shown as the solid curve in the top panel.

$\rho_{\text{hot}} + \rho_{\text{cold}}$, as a function of the total local density. The solid curve is for unsmoothed density, and the dashed curve is for the density effectively smoothed with a Gaussian filter with $\sigma = 0.04$ in units of our PM cell size (see below). It is interesting to note that cold and hot particles are distributed differently. In high-density areas ($\rho_{\text{total}} > (1-2) \langle \rho_{\text{total}} \rangle$), the density of hot particles is systematically lower by about 20% ($\rho_{\text{hot}}/\rho_{\text{total}} \approx 0.25$) than the mean value $\Omega_b = 0.30$. Perhaps more surprising is the fact that the fraction of mass in hot particles does not change with density, for overdensity greater than about 10. (NKP96 shows that a related quantity, $\rho_{\text{cold}}/\rho_{\text{hot}}$, does rise near the centers of groups.) One might expect naively that the higher the density of an object, the larger the fraction of neutrinos it captures because it has a deeper potential well. This is not what we see in our simulations. It seems that another effect plays a significant role: the larger the density, the earlier it was formed and the larger the velocity dispersion of neutrinos it accreted. At the same time, low-density areas (lower than the mean density) have an excess of neutrinos. The enhancement is very large—a factor of 2 or more. It is readily appar-

ent in the BHNPK visualizations. The much higher abundance of hot particles in low-density regions no doubt impedes the gravitational collapse of galaxies and the development of larger scale structures in such regions.

An immediate implication of this difference between hot and cold particles is that one *must* include hot and cold particles in numerical simulations in order to make accurate predictions for CHDM models. It is not enough to have only the initial CHDM power spectrum, but evolve it with cold particles as in CDM models. Not only would the rate of evolution in the linear regime be wrong, but also the dynamics of both high- and low-density regions in the non-linear regime would not be simulated properly. We note that in a number of publications (e.g., Mo et al. 1993; Croft & Efstathiou 1994; Gelb et al. 1993; and part of Jing et al. 1994) hot particles were not simulated and, thus, results of those publications do not represent accurately the CHDM model.

The bottom panel of Figure 16 presents the density distribution function, $P(\rho)$, multiplied by the total density ρ for the CHDM₂ simulation (solid curve). For comparison, we also present results from KHPR (dot-dashed curve). The agreement between the two simulations is very good in the density range $\rho/\langle \rho \rangle = 10-1000$. There is a lack of high-density areas in KHPR simulations for $\rho/\langle \rho \rangle > 1000$, but the difference is not that large, and is quite understandable taking into account the much better resolution in our present simulations. Effects of particle discreteness are seen as wiggles in the KHPR curve.

9. CONCLUSIONS AND DISCUSSION

In the present paper we have explained how we simulated a suite of cosmological models with $\Omega = 1$ and $h = 0.5$: cold dark matter (CDM) with linear bias factor $b \equiv \sigma_8^{-1} = 1$ and 1.5, and two simulations of cold + hot dark matter (CHDM) with $\Omega_c/\Omega_{\nu}/\Omega_b = 0.6/0.3/0.1$, one started with the same random numbers as the CDM simulations (with large waves somewhat higher than typical) and the second with different random numbers corresponding to lower (more typical) amplitude for the largest waves. These simulations are a more detailed examination of the models simulated in KHPR in a larger range of boxes but at lower resolution. Here we simulated periodic boxes of linear size 100 Mpc. We have analyzed these (512^3 particle mesh) simulations both in the present paper and in a number of related papers, in particular NKP94, NKP96, and our visualization paper BHNPK; also Yepes et al. (1994), Ghigna et al. (1994), Bonometto et al. (1995), Ghigna et al. (1996a), Davé et al. (1996a, 1996b).

In the present paper, we have mainly analyzed the evolution of the power spectrum of these models in real space, compared the power spectrum to observational data in redshift space, compared correlation functions of various populations in these simulations in both real and redshift space and with relevant data, analyzed the real-space rms velocity of pairs of galaxies $\sigma_{\parallel}$ in various simulations and compared to data, and analyzed the density of hot and cold particles as a function of total density. Our conclusions are as follows:

Evolution of simulations.—The evolution of our simulations is well described by linear theory on the largest scales, and by standard clustering analyses on the smallest scales we can resolve.

Power spectra.—We find that the power spectrum of the CHDM model in the nonlinear regime agrees rather well both with CfA2 data (if we compare this optical data with the mass-weighted power spectrum, to compensate for overmerging in dense regions) and with IRAS data (if we compare with the number-weighted power spectrum, since IRAS galaxies avoid dense regions). Moreover, with appropriate rescaling to account for the biasing of these two different galaxy data sets, we find reasonable agreement also in the linear regime.

Correlation functions.—Real-space autocorrelation functions $\xi(r)$ were presented for CHDM dark matter halos (“galaxies”) having masses greater than various lower limits. As expected, the higher mass “galaxies” are more correlated. Linear and quasi-linear analyses appear to account for the behavior of $\xi(r)$ on large scales. For CDM, $\xi(r)$ for the dark matter is not well fit by a power law, but the dark halos were better fit by a power law and were antibiased with respect to the dark matter. Redshift-spaced CHDM $\xi(s)$, corrected on large scales by the method of Kaiser (1987), agrees well with the CfA2 data on all scales. But the CfA2 data do not agree as well with $\xi(s)$ from the CDM simulation; $\xi_{\text{CDM}}(s)$ is too low especially on small scales $s < 5 h^{-1}$ Mpc, and also on large scales $s > 10 h^{-1}$ Mpc.

Velocities.—Here we concentrated on two statistics in real space: the rms velocity along a line joining a pair of halos $\sigma_{\parallel}(R)$, and rms velocities σ_v of dark matter and dark halos. We found $\sigma_{\parallel}(R)$ to be systematically higher in CHDM₁ than CHDM₂, reflecting the greater large-scale power in CHDM₁, but lower in both CHDM simulations than in the unbiased CDM one. We see a fairly strong tendency for more massive pairs of “galaxies” to have larger pairwise velocities, although the rms velocities of the more massive halos are only slightly higher than that of the lower-mass halos. We measure a global velocity bias of

about 0.8 for CDM and 1.0 for both CHDM simulations, but the pairwise velocity bias is $b_{v,\text{pairs}} = 0.44\text{--}0.62$ in both CDM and CHDM simulations, with $b_{v,\text{pairs}}$ smaller for smaller mass pairs of galaxies.

Hot versus cold particle distribution.—We find that the ratio of hot/total density is remarkably constant for overdensities exceeding about 10 and reduced compared to the average density in the box by about 20%. While hot particles are thus relatively depleted in the dense regions, we find that their abundance is enhanced by a factor of 2 or more in the low-density regions. Thus, the voids are hot dark matter balloons! It follows from these strikingly different distributions of cold and hot dark matter that simulating CHDM accurately requires treating the two different types of dark matter particles differently, as we have done.

On the basis of the results discussed above, CHDM with $\Omega_c/\Omega_v/\Omega_b = 0.6/0.3/0.1$ (i.e., with the hot dark matter consisting of neutrinos of mass 7 eV) appears to be a good fit to the data on the distribution and velocities of galaxies at the present epoch. We explained in NKP46, Ghigna et al. (1994), and especially in KBHP and Primack et al. (1995) why we think that a slightly lower neutrino mass ~ 5 eV may be preferred (for $h = 0.5$).

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APPENDIX

EFFECTS OF RESOLUTION

The method we use to simulate the evolution of clustering in the expanding universe is essentially the same as the particle-in-cell codes used in plasma physics; it is well known and well tested. It is called PM (for particle-mesh) in the astrophysical literature (e.g., Hockney & Eastwood 1981; Kates et al. 1991). The code is very fast and the most suitable for analysis of *statistical* properties of cosmological objects (galaxies, groups, and clusters).

In order to clarify to what degree our results might be affected by effects of finite resolution, we ran a number of tests that addressed different aspects of the problem.

Figure 17 presents results of the first test: the force resolution at a given distance. We place two particles at some distance apart and give them velocities in such a way that in the case of a perfect solution they would stay on a circular orbit. The full curves in this figure are for integration with 300 time steps per period of rotation (or 75 steps per free-fall time), and the dashed curves are for 10 times larger time step—30 steps per period of rotation. Because of different numerical errors (mainly the errors in the force), the actual trajectory of a particle is not a circle. The top panel of Figure 17 shows the trajectories of one of the particles with the radius of the “true” orbit being one cell of the PM code, and the bottom panel is for a three cell radius. The deviation of the diameter of the orbit from its true value is a measure of the accuracy of the code. For both orbits simulated with a small time step, the radius of trajectory was *smaller* than the “true” radius with maximum error in diameter 12% and 4% for one cell and three cells “true” trajectories. A large time step results in a drift of the center of the trajectory, but it does not produce catastrophic break of the trajectory.

In our PM code, the resolution (a cell size) is constant in *comoving*, not proper, coordinates. This means that the resolution actually decreases with time. (Looking at the situation from a positive angle, the resolution becomes much better if we go to high redshifts.) For example, our comoving resolution is 100 kpc, and we have a galaxy with radius 100 kpc collapsing at $z = 4$, $a = 0.2$. The proper resolution at that redshift was 20 kpc, enough to resolve the object reasonably. Now, our resolution drops very severely. If this happened quickly (on a dynamical scale), the galaxy would explode: its total energy would be

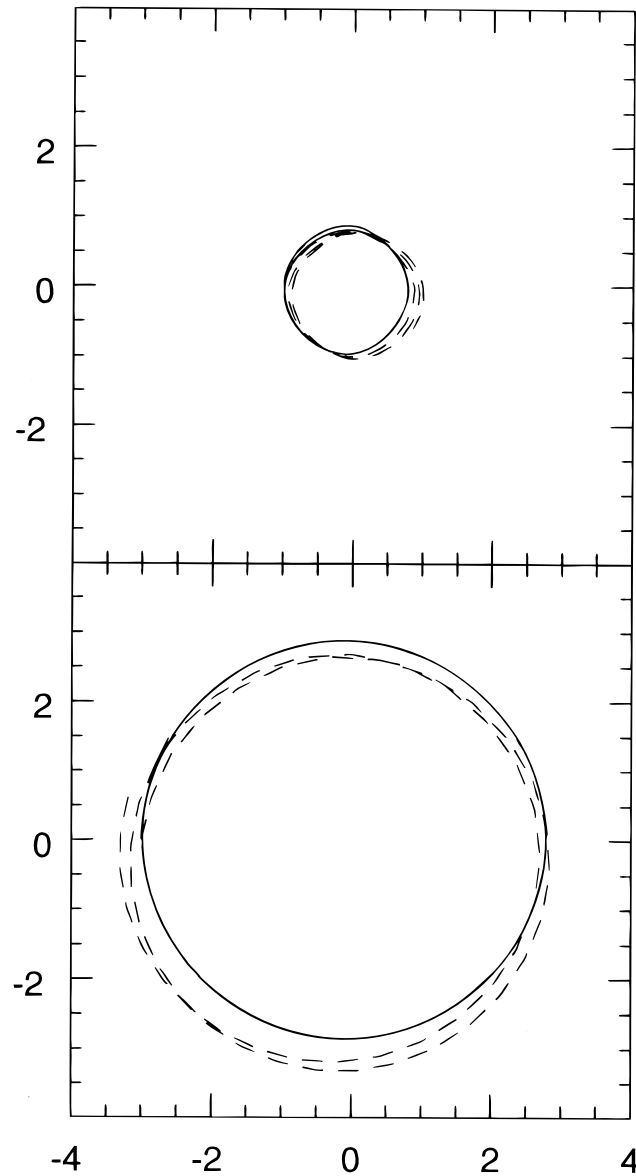


FIG. 17.—Test of the force resolution in PM code. Initially two particles are set in circular orbits with radius one cell (*top*) and three cells. Trajectory of one of them is shown. The full curves in this figure are for integration with 300 time steps per period of rotation (or 75 steps per free-fall time), and the dashed curves are for 10 times larger time step, 30 steps per period of rotation.

positive. Fortunately, this does not happen because the timescale of growth of the softening parameter is equal to the expansion rate of the universe, and the latter is always longer than the dynamical time. Thus, the galaxy will expand adiabatically once the softening length comes close to its radius.

Figure 18 illustrates this behavior. A system of 300 particles with initial velocities slightly below virial velocities was simulated using a PP code. Initially the particles were distributed randomly in a sphere with 100 kpc proper radius. Parameters were scaled in such a way that the total mass of the system is $10^{12} M_{\odot}$, and the initial moment corresponds to $a = 0.2$, $z = 4$. Collapse happens at $a = 0.25$, $z = 3$. The radius of half-mass is 50 kpc after collapse. The system was run once with softening parameter $\epsilon = 10$ kpc in *proper* coordinates, and another time with $\epsilon = 10$ kpc in *comoving* coordinates. Figure 18 shows snapshots of the simulations. Comoving coordinates are chosen to display the particles. The coordinate scale is in units of 100 kpc. The bottom row shows the evolution of the system simulated with constant proper resolution, while the top row is for constant comoving resolution. The circles show the size of the resolution: the radius of each circle is 1.5ϵ . At this radius, the error in force at one cell radius in our PM code is equal to the error in the PP code. As time goes on, the system shrinks in comoving coordinates. But once its size is close to the resolution, the shrinking stops. This just confirms what is found in numerical simulations: isolated unresolved “galaxies” are small, almost spherical balls with two grid cells across their diameters.

In order to obtain a better insight on the effects of the resolution and to understand what should be expected if we increase the resolution significantly, two simulations were run with exactly the same initial conditions, but with resolution differing by

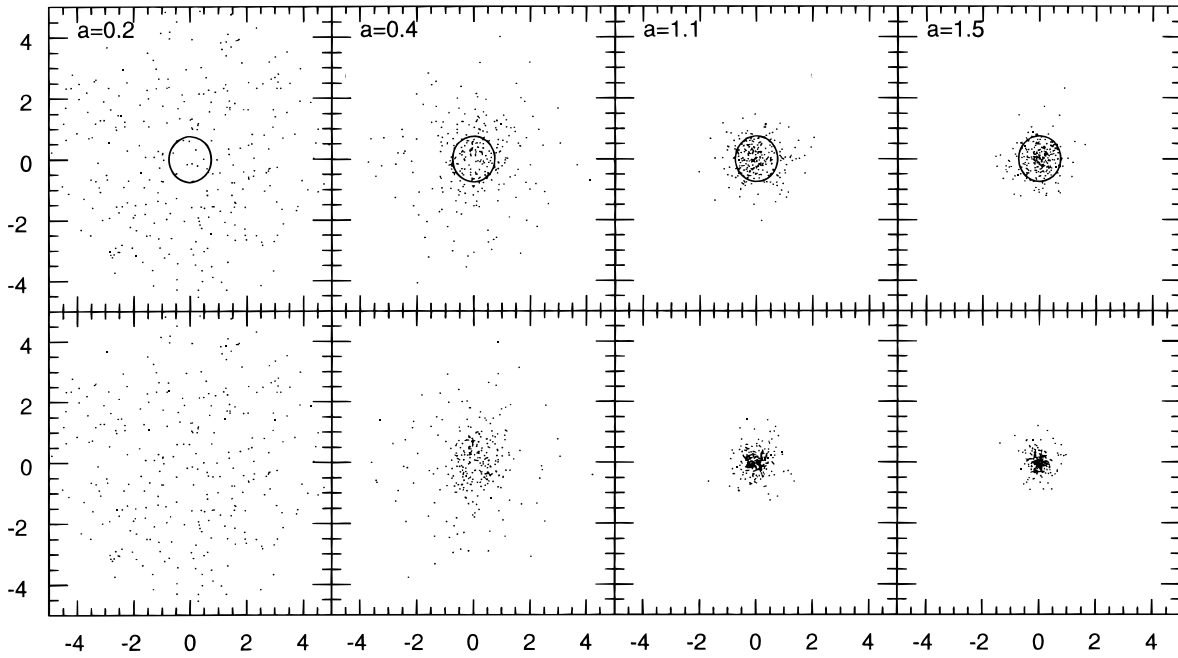


FIG. 18.—Evolution of a system of 300 particles in comoving coordinates. Initially the particles were distributed randomly in a sphere with 100 kpc radius. Collapse happens at $a = 0.25$, $z = 3$. The system was run once with softening parameter $\epsilon = 10$ kpc in *proper* coordinates (*bottom row*), and another time with $\epsilon = 10$ kpc in *comoving* coordinates (*top row*). The coordinate scale is in units of 100 kpc, and circles (radius 1.5ϵ) show the resolution.

a linear factor of 2.1. The cosmological model was Λ CDM with $\Omega_0 = 0.3$, $h = 0.70$. A PM code with 128^3 particles, mesh 450^3 and 216^3 , box size $20 h^{-1}$ Mpc was used. This corresponds to a resolution of $44 h^{-1}$ kpc and $92 h^{-1}$ kpc for the two runs, and mass per particle $m_1 = 3.1 \times 10^8 h^{-1} M_\odot$. Figure 19 shows particles in a small 3×3 Mpc window in the simulations at $z = 3$. Only particles with estimated overdensity 150 (low resolution, *top panel*) and 300 (high resolution, *bottom panel*) are shown, and only $\frac{1}{4}$ of all particles are displayed. Because the same isolated clump would have smaller radius and, thus, higher density with twice better resolution, density limits were adjusted to take into account the difference in the resolution. Figure 20 shows the same window at $z = 0$. While plots for higher resolution show more small clumps, all large clumps are found in both plots. The differences are as expected: higher resolution results in more compact and dense objects. The most significant result lies in what was feared, but not found. The largest object in the $z = 0$ plot has a radius of about 300 kpc. The usual question was as follows: having a low-resolution run with 100 kpc resolution, how does one know that with better resolution it will not split into, say, two objects? Our results indicate that this did not happen: with more than twice better resolution, the object became a bit smaller and a bit rounder and developed a few tiny satellites that were barely seen with the low resolution, but it did not show any tendency to break into large pieces. Figures 21 and 22 show another example of comparison of the two simulations. In this case, we deal with a group of galaxies. The result is just the same: the same large objects, more small objects, and no tendency for splitting of large halos into smaller ones.

While such visual analysis indicates that we might be quite safe with relatively low resolution if only large halos (above $10^{10} M_\odot$) are considered, it is not that easy to make the analysis more quantitative. Comparison of coordinates, velocities, and densities of individual particles in both simulations would be a natural test, but it just fails. The reason is the divergence of trajectories in dynamical systems: small differences in initial coordinates result in large differences after few dynamical times. As the result, deviations of coordinates $x_{44} - x_{92}$ and velocities $V_{x,44} - V_{x,92}$ as functions of density (Figs. 23 and 24) indicate mainly sizes of virialized objects that exist at different redshifts. Even in spite of the divergence of the trajectories, an extremely large fraction of the particles, 99%, indicates a difference in coordinates less than two cell sizes and differences in velocities less than 100 km s^{-1} .

Finally, in Figure 25 we compare the halo mass functions from these two simulations with differing resolution. We find that the high-mass tail of the mass functions ($M > 10^{11} h^{-1} M_\odot$) does not depend on resolution, which indicates convergence of the results. The disagreement occurs only on smaller scales, where it should be expected because of the differing resolution, and it is in the expected direction: more small-mass objects in the higher resolution simulation. The difference between the two simulations starts to be significant (about 20%) at around $M = 10^{11} h^{-1} M_\odot$, which corresponds to mass within resolution element (one to 1.5 cell radii) with mean overdensity 200. Objects with smaller radius are underresolved in the simulation. Both curves flatten out at around $3 \times 10^{10} h^{-1} M_\odot$, which corresponds to 10 particles per halo, a limit imposed by our mass resolution.

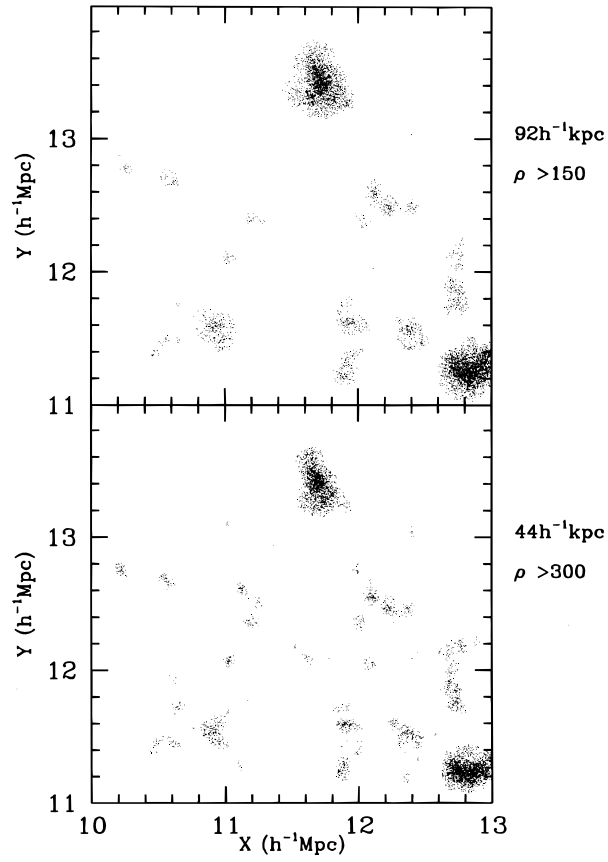


FIG. 19.—Particles in a small 3×3 Mpc window at $z = 3$ in a Λ CDM model with $\Omega_0 = 0.3$, $h = 0.70$. Only particles with estimated overdensities 150 and 300 are shown, and only $\frac{1}{4}$ of all particles are displayed.

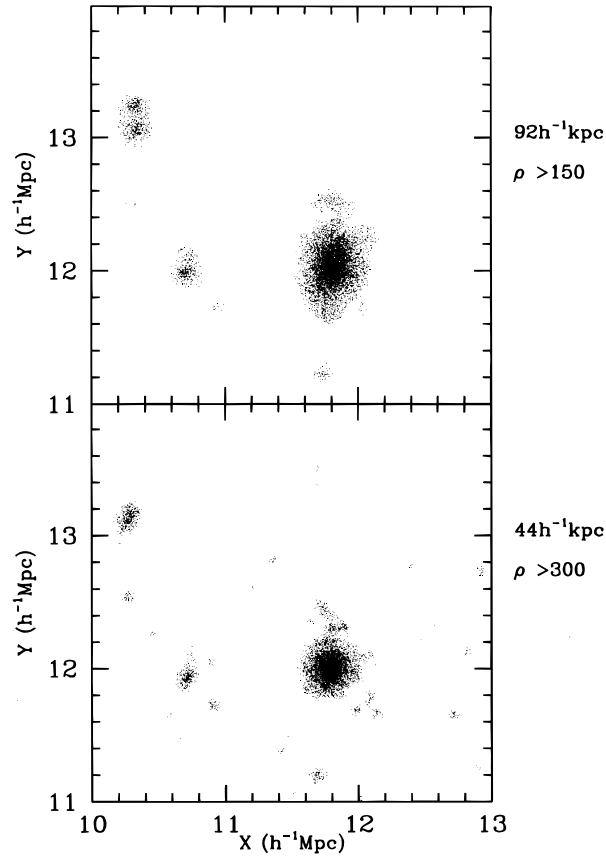


FIG. 20.—Same as Fig. 19, but at $z = 0$

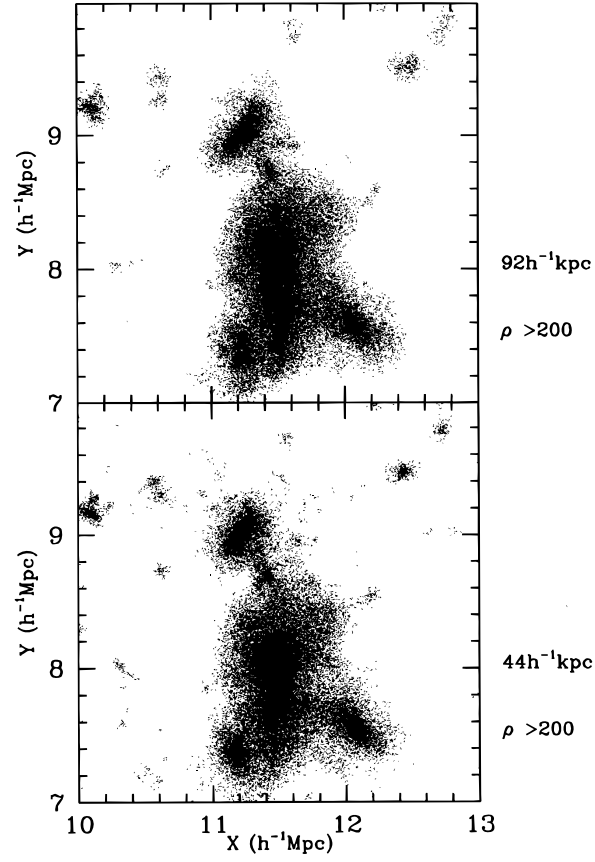


FIG. 21.—An example of a group of galaxies simulated with different resolutions (Λ CDM, $z = 0$). Particles with overdensity larger than 200 are shown.

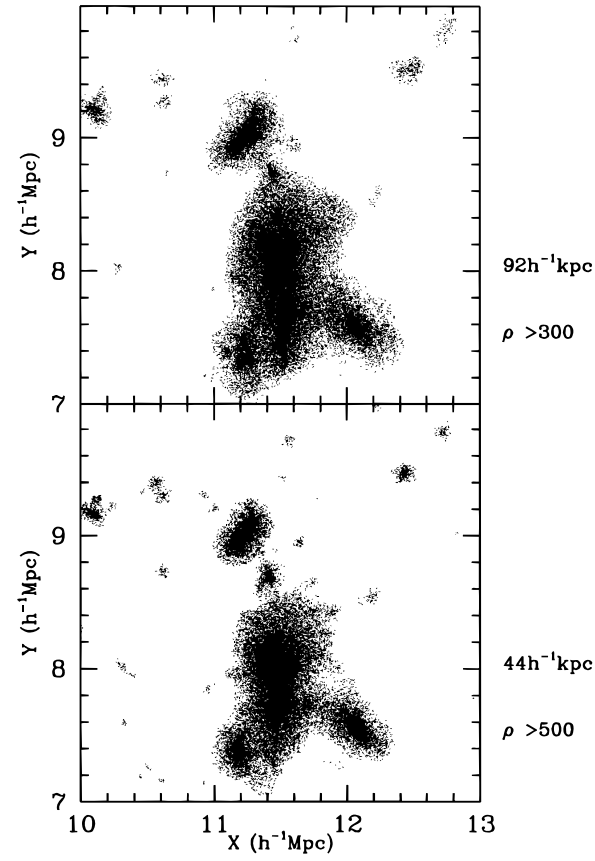
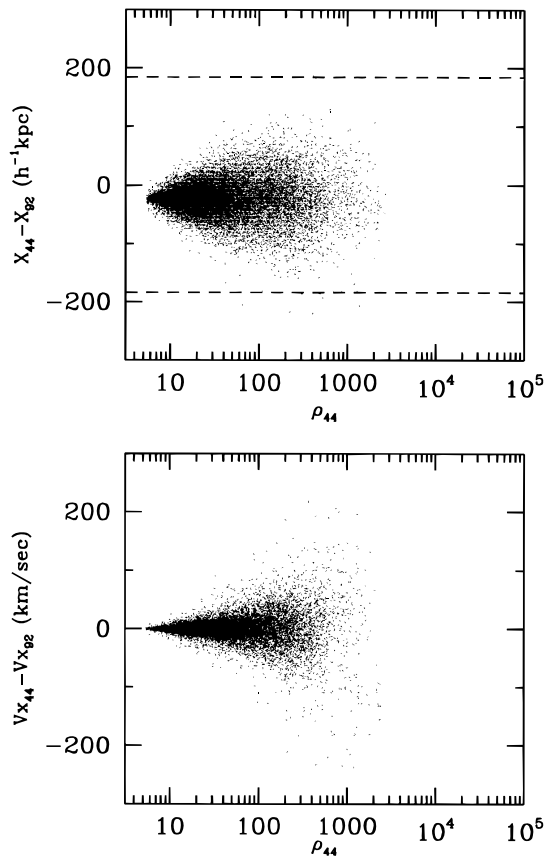
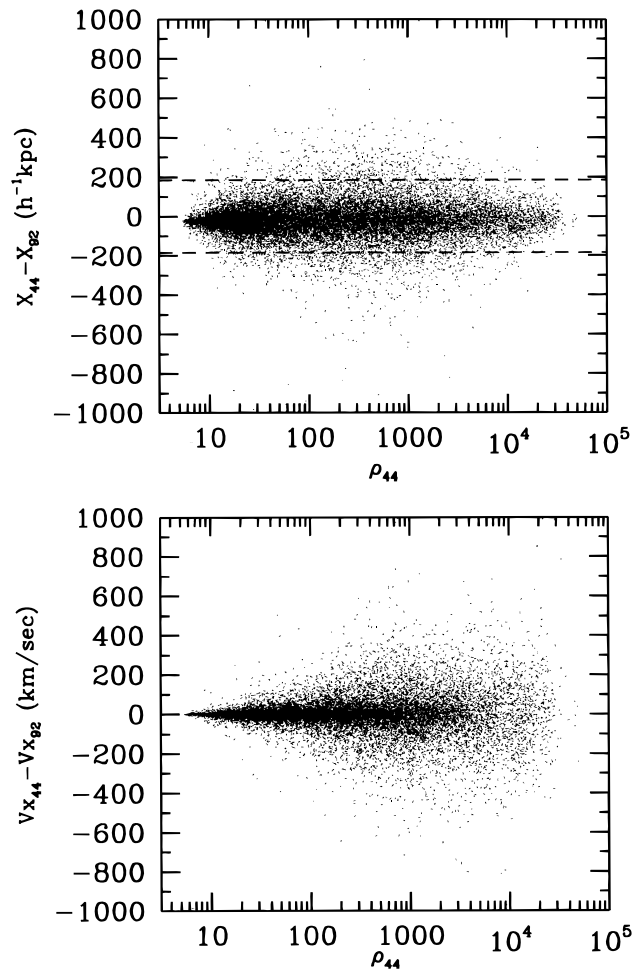
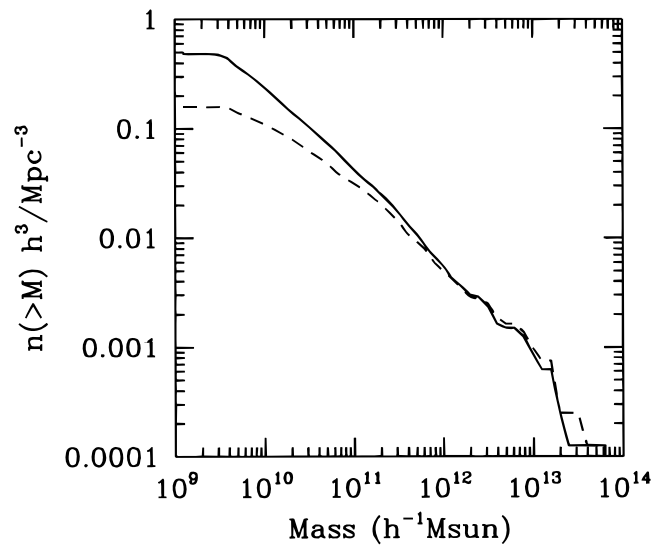


FIG. 22.—Same as Fig. 21, but with overdensity thresholds 300 and 500

FIG. 23.—Deviations of coordinates and velocities in high- and low-resolution simulations at $z = 3$

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FIG. 24.—Same as Fig. 23, but at $z = 0$ FIG. 25.—Comparison of mass functions of dark halos $n(>M)$ in two simulations (44 kpc vs. 92 kpc) with different resolutions. Solid curve shows the mass function for the higher resolution run.

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