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ASTRONOMICAL REFRACTION: COMPUTATIONAL METHOD FOR ALL ZENITH ANGLES

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ABSTRACT

It is shown that the problem of computing astronomical refraction for any value of the zenith angle may be reduced to a simple, nonsingular, numerical quadrature when the proper choice is made for the independent variable of integration. The angle between the radius vector and the light ray is such a choice. The implementation of the quadrature method is discussed in its general form and illustrated by means of an application to a piecewise polytropic atmosphere. The flexibility, simplicity, and computational efficiency of the method are evident.

Key words: methods: numerical

1. INTRODUCTION

In the past, the problem of refraction has been approached analytically. The simple relation

$$R = A \tan z - B \tan^3 z$$

is commonly used for small zenith angles, $z \leq 75°$ (Woolard & Clemence 1966). The analytical theory (Garfinkel 1944, 1967), valid for large zenith angles, is much more cumbersome and in fact requires the use of electronic computers for evaluation. We find that for cases in which the simple formula cannot be used, direct numerical quadrature using a proper choice for the variable of integration offers the following practical advantages: (1) flexibility, permitting arbitrary choice of an atmospheric model, (2) far greater ease of implementation, as the formulae are simpler and more closely related to physical theory, and (3) decreased computational time and program size.

This paper presents a transformation of the classical equations to a form that is readily adaptable to numerical quadrature. Relevant details of the computational scheme are discussed, and the method is applied to the piecewise polytropic atmospheric model used by Garfinkel (1944).

2. BASIC EQUATION

The astronomical refraction, R, for a spherically symmetric atmosphere is given by the integral

$$R = \int_0^{\ln\mu_o} \tan \psi \, d(\ln\mu) , \tag{1}$$

subject to the invariant relation

$$\mu r \sin \psi = \mu_o r_o \sin \psi_o , \tag{2}$$

where μ is the index of refraction, r is the distance from the center of Earth, and ψ is the angle between the light ray and the radius vector, as shown in Figure 1. The subscript o denotes the values at the observer's station. Thus, ψ_o is the apparent zenith angle, often denoted by z or ζ.

In principle, R could be calculated directly from equation (1) by numerical quadrature, but because of numerical difficulties (for $\psi \approx 90°$), it is preferable to use ψ itself as the variable of integration. Taking the logarithmic derivative of equation (2) and substituting into equation (1), we find

$$R = -\int_0^{\psi_o} \frac{d(\ln\mu)}{d(\ln r\mu)} \, d\psi = -\int_0^{\psi_o} \frac{d(\ln\mu)/d(\ln r)}{1 + d(\ln\mu)/d(\ln r)} \, d\psi , \tag{3}$$

where the value of the integrand as a function of ψ is given by the solution of equation (2).

The integrand of equation (3) is a well-behaved function. It would become singular only for the unlikely atmospheric model given by the relation $\mu \propto 1/r$. Even so, in such a case, $\psi = \text{const}$, as seen by equation (2), and the integral of equation (1) would be trivial.

3. REMARKS ON IMPLEMENTATION

The atmosphere is specified by the function $\mu(r)$. It is also necessary to have the function $d(\ln \mu)/d(\ln r)$, which may be easily obtained, even for empirical atmospheres, since $\ln \mu$ is a smooth function of $\ln r$.

The most time-consuming part of the computation is the solution of equation (2) for r, given ψ. When $\mu(r)$ and $d\mu/dr$ are analytically known, it appears advantageous to use a Newton-Raphson iterative scheme. The successive approximations to r are given by

$$r_{i+1} = r_i - F(r_i)/F_r(r_i) ,$$

where

$$F(r) = \mu r - \mu_o r_o \sin \psi_o/\sin \psi ,$$

$$F_r(r) = \frac{d\mu}{dr} r + \mu .$$

An excellent initial approximation may be found by fitting the function $y = r - 1$ with a quadratic in $x = \mu r - 1$, where r is in units of the Earth's radius. When r has converged to suitable accuracy, one has the values of $r(\psi)$, $\mu(\psi)$, and $d\mu/dr$ from which $d(\ln \mu)/d(\ln r)$ is computed. For the atmosphere considered in § 4, no more than two iterations were ever required.

The formal lower limit in equation (3) is $\psi = 0$, corresponding to $r = \infty$; however, it is unnecessary to integrate past the point where μ is not significantly different from unity, for at this point the integrand is negligibly small. The practical lower limit, $\psi_{\min}$, may be determined from equation (2). Terrestrial refraction, where the object being observed is inside the atmosphere, may be treated by using

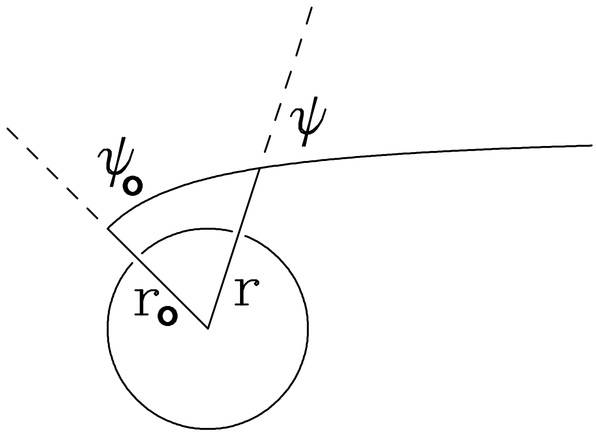

FIG. 1.—Light path as a function of radius, r, and angle, ψ. The observer at r_o measures the star as having a zenith distance ψ_o.

the proper lower limit for integration. The value of $\psi_{\min}$ is found by solving equation (2), with r taken as the height of the object.

In the case of a piecewise atmosphere, the integrand of equation (3) is not, in general, a continuous function across the boundaries of each region. It is therefore necessary to integrate each region separately and then sum the parts. The limits on the integrals are then the corresponding values of ψ at each boundary. These may be found directly by solving equation (2) as a function of r.

TABLE 1

PHYSICAL CONSTANTS

Symbol	Definition	Value
α		0.0029241
$r_\oplus$	Earth's radius	6,378,390 m
g	Gravitational constant	9.80655 m s^{-2}
$\mathscr{R}$	Gas constant	287.053 m^2 s^{-2} °C^{-1}
n	Polytropic index for the troposphere	5
h_B	Altitude of troposphere	11,019 m

4. APPLICATION TO PIECEWISE POLYTROPIC ATMOSPHERES

One model for Earth's atmosphere that has been used previously is the piecewise polytropic model of Garfinkel (1944, 1967). In this model, the density of the atmosphere is described by a polytrope of index n in the troposphere and by another polytrope of index ∞ (i.e., isothermal) in the stratosphere. The two parts are connected by assuming the continuity of the temperature and density across the boundary (tropopause), defined by the altitude h_B above the surface of Earth. This model has the advantages that it is simple to treat and permits adjustment according to present weather conditions.

The index of refraction, μ, is related to the density, ρ, by the Gladstone-Dale relation,

$$\mu = 1 + \alpha\rho ,$$

where the density is equal to unity for the standard weather conditions of 273.15 K and 760 mm of Hg. The relations for the density as a function of altitude, h, are for the troposphere ($h < h_B$),

$$\rho(r) = \rho_w\left[1 + \beta_w\left(\frac{1}{r} - \frac{1}{r_w}\right)\right]^n ,$$

and for the stratosphere ($h > h_B$),

$$\rho(r) = \rho_w \exp\left[\gamma_w\left(\frac{1}{r} - \frac{1}{r_w}\right)\right] ,$$

where

$$\rho_w = \frac{p_w}{760}\frac{273.15}{T_w} ,$$

$$\beta_w = gr_\oplus/[\mathscr{R}T_w(1 + n)], \quad \gamma_w = gr_\oplus/(\mathscr{R}T_w) ,$$

$$r = (r_\oplus + h)/r_\oplus, \quad r_w = (r_\oplus + h_w)/r_\oplus .$$

The physical constants are defined in Table 1.

The scale of the model is adjusted according to the observed values of the temperature, T_w, and the pressure, p_w, at an altitude h_w. This observation may be in either the

TABLE 2

COMPUTED REFRACTION

ZENITH ANGLE (deg)	ANALYTICAL FORMULA: $h_w = 0$ m, $T_w = 273.15$ K, $P_w = 760$ mm, $h_o = 0$ m	NUMERICAL METHOD: $h_w = 0$, $T_w = 273.15$, $P_w = 760$, $h_o = 0$	0, 273.15, 780, 0	0, 303.15, 760, 0	0, 273.15, 760, 2000	0, 273.15, 760, 15,000
15......	16″.14	16″.14	16″.56	14″.54	13″.05	2″.3
30......	34.77	34.77	35.68	31.32	28.10	4.97
45......	60.17	60.17	61.76	54.20	48.64	8.60
60......	103.99	103.99	106.73	93.65	84.07	14.87
75......	221.34	221.49	227.33	199.15	179.09	31.73
80......	329.46	330.52	339.25	296.52	267.34	47.46
85......	588.87	614.56	630.96	546.76	497.75	89.20
86......		732.77	752.42	649.25	593.86	106.99
87......		899.23	923.52	791.88	729.38	132.53
88......		1145.51	1176.89	999.39	930.14	171.49
89......		1532.65	1575.47	1317.72	1245.89	235.77
90......		2189.42	2253.01	1838.65	1780.59	353.36
91......					2777.33	600.62
92......						1187.87
93......						2316.43

troposphere or the stratosphere. These values are used to compute the coefficients for only that associated region of the atmosphere. In order to ensure continuity, the temperature and the density must be extrapolated to the boundary, defined by the altitude h_B. For the extrapolation from the troposphere, we have

$$T_B = T_w\left[1 + \beta_w\left(\frac{1}{r_B} - \frac{1}{r_w}\right)\right],$$

$$\rho_B = \rho(r_B) .$$

For extrapolation from the stratosphere, we have

$$T_B = T_w, \quad \rho_B = \rho(r_B) .$$

The values T_B and ρ_B then become the values used to compute the coefficients in the new region, replacing T_w and ρ_w.

The method has been used to compute the refraction at various altitudes for different values of the zenith angle and weather conditions. These results are presented in Table 2. Also included, for the purpose of comparison, is the refraction from the analytical formula,

$$R = A \tan \psi_o - B \tan^3 \psi_o ,$$

where A and B as given in Woolard & Clemence (1966) were adjusted for the present values of α and r from Table 1.

The average computing time per case with the IBM 7094 is 0.12 s.

5. DISCUSSION

The sensitivity of the refraction with respect to various weather features is evident in Table 2. Though the method readily allows weather observations at a high altitude, such observations may not be as accurate as those based on the ground. This could seriously affect the refraction computed for a light ray that passes through a major portion of the atmosphere.

The coefficients of the analytical formula may be adjusted for weather conditions and altitude as is done with the present method. We would expect similar agreement between the formulae as is shown in the standard case in Table 2.

The analytical formula is, of course, by far the fastest and most simple to use. For zenith angles greater than 75°, however, its accuracy begins to deteriorate seriously. For these higher angles, the method presented here surpasses any other presently known with respect to simplicity, generality, and computational speed.

Postscript.—This paper and the method presented in it were submitted for publication in 1970 July. Unfortunately, the referee did not understand the utility of our new approach, and for personal reasons we did not have the time to argue the point sufficiently. We did distribute preprints, and the method has become, with improved atmospheric models, the technique of choice for the computation of refraction (see, e.g., Seidelmann 1992). Further, in addition to the computation of refraction near the horizon, this method can easily be extended to include the effects of atmospheric inhomogeneities. We hope that our work will promote investigations of such topics as the variability of refraction near the limb.

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