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General elephants for threefold extremal contractions with one-dimensional fibres: exceptional case

S. Mori and Yu. G. Prokhorov

Abstract. Let (X,C) be a germ of a threefold X with terminal singularities along a connected reduced complete curve C with a contraction $f\colon (X,C)\to(Z,o)$ such that $C=f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is f-ample. Assume that each irreducible component of C contains at most one point of index >2. We prove that a general member $D\in|-K_X|$ is a normal surface with Du Val singularities.

Bibliography: 16 titles.

Keywords: terminal singularity, extremal curve germ, flip, divisorial contraction, $\mathbb{Q}$-conic bundle.

§ 1. Introduction

This paper is a continuation of a series of papers on the classification of extremal contractions with one-dimensional fibres (see the survey [13] for an introduction). Recall that an *extremal curve germ* is the analytic germ (X,C) of a threefold X with terminal singularities along a reduced connected complete curve C such that there exists a contraction $f\colon (X,C)\to(Z,o)$ such that $C=f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is f-ample. There are three types of extremal curve germs: flipping, divisorial and $\mathbb{Q}$-conic bundles, and all of them are important building blocks in the three-dimensional minimal model program.

The first step in the classification is to establish the existence of a 'good' member of the anticanonical linear system. This is Reid's so-called 'general elephant conjecture' [15]. In the case of an irreducible central curve C the conjecture has been proved.

Theorem 1.1 (see [2], Theorem (2.2), and [11]). *Let (X,C) be an extremal curve germ with irreducible central curve C. Then a general member $D\in|-K_X|$ is a normal surface with Du Val singularities.*

Moreover, all the possibilities for general members of $|-K_X|$ have been classified. Firstly, extremal curve germs with irreducible central curve are divided into two classes: semistable and exceptional. Such a germ (X,C) is said to be *semistable* if the restriction of the corresponding contraction $f\colon (X,C)\to(Z,o)$ to a general

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member $D \in |-K_X|$ has the Stein factorization $f_D : D \to D' \to f(D)$, where the surface D' has only Du Val singularities of type A [2]. Non-semistable extremal curve germs are called *exceptional*. Semistable extremal curve germs are subdivided into two types: (k1A) and (k2A), while exceptional ones are subdivided into the following types: cD/2, cAx/2, cE/2, cD/3, (IIA), ($\mathrm{II}^\vee$), ($\mathrm{IE}^\vee$), ($\mathrm{ID}^\vee$), (IC), (IIB), (kAD) and (k3A) (see [2], [9] and [11]).

The result stated in Theorem 1.1 is very important in three-dimensional geometry. For example, the existence of a good member $D \in |-K_X|$ for flipping contractions is a sufficient condition for the existence of flips (see [1]) and the existence of a good member $D \in |-K_X|$ in the $\mathbb{Q}$-conic bundle case proves Iskovskikh's conjecture about singularities of the base (see [14] and [9]).

Reid's conjecture has also been proved for an arbitrary central curve C in the case of $\mathbb{Q}$-conic bundles over singular base.

Theorem 1.2 (see [10]). *Let (X, C) be a $\mathbb{Q}$-conic bundle germ and let $f : (X, C) \to (Z, o)$ be the corresponding contraction. Assume that (Z, o) is singular. Then a general member $D \in |-K_X|$ is a normal surface with Du Val singularities.*

In this paper we study Reid's conjecture for extremal curve germs with reducible central curve. Our main result is the following theorem.

Theorem 1.3. *Let (X, C) be an extremal curve germ. Assume that (X, C) satisfies the following condition*:

($*$) *each irreducible component of C contains at most one point of index > 2.*

Then a general member $D \in |-K_X|$ is a normal surface with Du Val singularities. Moreover, for each irreducible component $C_i \subset C$ with two non-Gorenstein points or points of type (IC) *or* (IIB), *the dual graph $\Delta(D, C_i)$ has the same form as the irreducible extremal curve germ (X, C_i)* (*see Theorem 5.1*).

Throughout this paper we use the standard notation (IC), (IIB) and so on for types of extremal curve germs (X, C) with irreducible central fibre [2]. Sometimes, we will use subscripts to specify the indices of singular points. For example, ($\mathrm{kAD}_{2,\mathrm{m}}$) means that the indices of points of (X, C) are 2 and m. Some of the subscripts can be omitted if it is not important to our argument, for instance, ($\mathrm{k2A}_2$) means that (X, C) contains a point of index 2 (and another point of index > 1).

According to the classification of birational extremal curve germs, condition ($*$) in Theorem 1.3 is equivalent to saying that an arbitrary component $C_i \subset C$ of type (k2A) has a point of index 2.

Corollary 1.4. *Let (X, C) be an extremal curve germ and let $C_i \subset C$ be an irreducible component.*

(i) *If C_i is of type* (IIB), *then any other component $C_j \subset C$ is of type* (IIA) *or* ($\mathrm{II}^\vee$).

(ii) *If C_i is of type* (IC) *or* (k3A), *then any other component $C_j \subset C$ meeting C_i is of type* (k1A) *or* (k2A).

(iii) *If C_i is of type* (kAD), *then any other component $C_j \subset C$ meeting C_i is of type* (k1A), (k2A), cD/2 *or* cAx/2.

(iv) *If C_i is of type* ($\mathrm{k2A}_2$), *then any other component $C_j \subset C$ meeting C_i is of type* (k1A), (IC) *or* ($\mathrm{k2A}_{\mathrm{n,m}}$), *where $n, m \geqslant 3$.*

There are more restrictions on the combinatorics of the components of C. These will be treated in a subsequent paper. Examples of extremal curve germs satisfying the conditions of Theorem 1.3 can be found in the appendix of the arXiv version of this paper (see arXiv:2002.10693).

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§ 2. Preliminaries

2.1. Recall that a *contraction* is a proper surjective morphism $f\colon X \to Z$ of normal varieties such that $f_*\mathscr{O}_X = \mathscr{O}_Z$.

Definition 2.1. Let (X, C) be the analytic germ of a threefold with terminal singularities along a reduced connected complete curve. We say that (X, C) is an *extremal curve germ* if there is a contraction $f\colon (X, C) \to (Z, o)$ such that $C = f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is f-ample. Furthermore, f is called *flipping* if its exceptional locus coincides with C and *divisorial* if its exceptional locus is two-dimensional. If f is not birational, then Z is a surface and (X, C) is said to be a $\mathbb{Q}$-*conic bundle germ*.

Lemma 2.2. *Let (X, C) be an extremal curve germ. Assume that C is reducible. Then for any proper connected subcurve $C' \subsetneqq C$ the germ (X, C') is a birational extremal curve germ.*

Proof. Clearly, there exists a contraction $f'\colon X \to Z'$ of C' over Z (see [6], Corollary (1.5)). We only need show that f' is birational. Assume that (X, C') is a $\mathbb{Q}$-conic bundle germ. Then there exists the following commutative diagram

$$
\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "f'"] & \\
Z & Z' \arrow[l, "\varphi"']
\end{tikzcd}
$$

where f and f' are $\mathbb{Q}$-conic bundles contracting C and C', respectively. The image $\Gamma := f'(C'')$ of the remaining part $C'' := C - C'$ is a curve on Z' such that $\varphi(\Gamma) = f(C)$ is a point, say $o \in Z$. Hence the fibre $f'^{-1}(\Gamma) = f^{-1}(o)$ is two-dimensional, a contradiction. The lemma is proved.

2.2. Recall the basic definitions of ℓ-structure techniques; see [6], § 8, for details. Let (X, P) be three-dimensional terminal singularity of index m. Throughout this paper $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ denotes its index-one cover. For any object V on X we denote the pull-back of V on $X^\sharp$ by $V^\sharp$.

Let $\mathscr{L}$ be a coherent sheaf on X without submodules of finite length > 0. An *ℓ-structure* of $\mathscr{L}$ at P is a coherent sheaf $\mathscr{L}^\sharp$ on $X^\sharp$ without submodules of finite length > 0, with $\boldsymbol{\mu}_m$-action and endowed with an isomorphism $(\mathscr{L}^\sharp)^{\boldsymbol{\mu}_m} \simeq \mathscr{L}$. An *$\ell$-basis* of $\mathscr{L}$ at P is a collection of $\boldsymbol{\mu}_m$-semi-invariants $s_1^\sharp, \dots, s_r^\sharp \in \mathscr{L}^\sharp$ generating $\mathscr{L}^\sharp$ as an $\mathscr{O}_{X^\sharp}$-module at $P^\sharp$. Let Y be a closed subvariety of X. Note that $\mathscr{L}$ is an $\mathscr{O}_Y$-module if and only if $\mathscr{L}^\sharp$ is an $\mathscr{O}_{Y^\sharp}$-module. We say that $\mathscr{L}$ is an *ℓ-free* $\mathscr{O}_Y$-module at P if $\mathscr{L}^\sharp$ is a free $\mathscr{O}_{Y^\sharp}$-module at $P^\sharp$. If $\mathscr{L}$ is an ℓ-free $\mathscr{O}_Y$-module at P, then an ℓ-basis of $\mathscr{L}$ at P is said to be *ℓ-free* if it is a free $\mathscr{O}_{Y^\sharp}$-basis.

Let $\mathscr{L}$ and $\mathscr{M}$ be $\mathscr{O}_Y$-modules at P with ℓ-structures $\mathscr{L} \subset \mathscr{L}^\sharp$ and $\mathscr{M} \subset \mathscr{M}^\sharp$. Define the following operations $\widetilde{\oplus}$ and $\widetilde{\otimes}$:

- $\mathscr{L} \mathbin{\widetilde{\oplus}} \mathscr{M} \subset (\mathscr{L} \oplus \mathscr{M})^\sharp$ is an $\mathscr{O}_Y$-module at P with ℓ-structure
$$(\mathscr{L} \mathbin{\widetilde{\oplus}} \mathscr{M})^\sharp = \mathscr{L}^\sharp \oplus \mathscr{M}^\sharp;$$
- $\mathscr{L} \mathbin{\widetilde{\otimes}} \mathscr{M} \subset (\mathscr{L} \otimes \mathscr{M})^\sharp$ is an $\mathscr{O}_Y$-module at P with ℓ-structure
$$(\mathscr{L} \mathbin{\widetilde{\otimes}} \mathscr{M})^\sharp = (\mathscr{L}^\sharp \otimes_{\mathscr{O}_{X^\sharp}} \mathscr{M}^\sharp)/\operatorname{Sat}_{\mathscr{L}^\sharp \otimes \mathscr{M}^\sharp}(0),$$
where $\operatorname{Sat}_{\mathscr{F}_1} \mathscr{F}_2$ is the saturation of $\mathscr{F}_2$ in $\mathscr{F}_1$.

These operations satisfy the standard properties (see [6], (8.8.4)). If X is an analytic threefold with terminal singularities and Y is a closed subscheme of X, then the above local definitions of $\widetilde{\oplus}$ and $\widetilde{\otimes}$ match the corresponding operations on $X \setminus \operatorname{Sing} X$. Therefore, they give well-defined operations of global $\mathscr{O}_Y$-modules.

Lemma 2.3. *Let (D, C) be the germ of a normal Gorenstein surface along a proper reduced connected curve $C = \bigcup C_i$, where C_i are irreducible components. Assume that the following conditions hold*:

(i) $K_D \sim 0$;

(ii) *there is a birational contraction $\varphi\colon (D, C) \to (R, o)$ such that $\varphi^{-1}(o)_{\mathrm{red}} = C$*;

(iii) *there is a point $P \in D$ which is not Du Val of type* A.

Then D has only Du Val singularities on $C \setminus \{P\}$.

Proof. Assume that there is a point $Q \in D \setminus \{P\}$ which is not Du Val. If there exists a component $C_i \subset C$ passing through Q but not passing through P, we can contract it: $D \to D'$ over R. The contraction is crepant, so the image of Q is again a non-Du Val point. Replace D with D'. Continuing the process we can assume that P and Q are connected by some component $C_i \subset C$. Moreover, by shrinking C we may assume that $C_i = C$, that is, C is irreducible. Since D is Gorenstein, the point $Q \in D$ is not log terminal and the point $P \in D$ is log terminal only if it is Du Val of type D or E. Hence the pair (D, C) is not log canonical at Q and not purely log terminal at P (see [3], Theorem 4.15). Let H be a general hyperplane section passing through P. For some $0 < \varepsilon$ and $\delta \ll 1$ the pair $(D, (1-\varepsilon)C + \delta H)$ is not log canonical at P and Q. Since $-(K_D + (1-\varepsilon)C + \delta H)$ is φ-ample, this contradicts Shokurov's connectedness lemma [16]. The lemma is proved.

§ 3. Low index cases

Extremal curve germs of index 2 with arbitrary central curve were completely classified in [2], § 4, and [9], § 12. As an easy consequence, we have the following.

Proposition 3.1. *Let (X, C) be an extremal curve germ. Assume that all the singularities of X are of index 1 or 2, that is, $2K_X$ is Cartier. Then a general member $D \in |-K_X|$ is a normal surface with Du Val singularities and D does not contain any component of C.*

Proof. Since the case where X is Gorenstein is trivial, we assume that X has at least one point, say P, of index 2. In the birational case there are no other non-Gorenstein points and all the components $C_i \subset C$ pass through P (see [2], Proposition (4.6)).

By Theorem (2.2) in [2] a general local member $D \in |-K_{(X,P)}|$ is in fact a general member of $|-K_X|$ and this D has only a Du Val singularity (at P) (see [15], (6.3)). For the $\mathbb{Q}$-conic bundle case we refer to the proof of Theorem (12.1) in [9], and [10], Corollary (1.4). The proposition is proved.

Proposition 3.2. *Let (X, C) be an extremal curve germ. Assume that C is reducible and (X, C) contains a point P of one of the types* cD/2, cAx/2, cE/2 *or* cD/3. *Then one of the following holds.*

(i) *P is the only non-Gorenstein point of X, all the components pass through P and do not meet each other elsewhere, and a general member $D \in |-K_X|$ is a normal surface with Du Val singularities. Moreover, $D \cap C = \{P\}$.*

(ii) *There is a component $C_i \subset C$ passing through P such that the germ (X, C_i) is divisorial of type* (kAD). *Moreover, (X, P) is a singularity of type* cD/2 *or* cAx/2.

Proof. Recall that the intersection points $C_i \cap C_j$ of different components $C_i, C_j \subset C$ are non-Gorenstein by [6], Corollary (1.15), [4], Proposition 4.2, and also by [9], Lemma (4.4.2). If P is the only non-Gorenstein point of X, then a general member $D \in |-K_{(X,P)}|$ is in fact a general member of $|-K_X|$ (see [6], (0.4.14)). This D has only a Du Val singularity (at P) (see [15], (6.3)). If there exists a non-Gorenstein point $Q \in X$ other than P, then we may assume that Q lies on some component $C_i \subset C$ passing through P. Thus (X, C_i) is a birational extremal curve germ with two non-Gorenstein points (see Lemma 2.2). According to Theorem 2.2 in [2] and [8] the germ (X, C_i) is divisorial of type (kAD) and (X, P) is a singularity of type cD/2 or cAx/2. This proves the proposition.

§ 4. Extension techniques

Theorem 4.1 (see [6], Theorem (7.3), and [9], Proposition (1.3.7)). *Let $(X, C \simeq \mathbb{P}^1)$ be an irreducible extremal curve germ satisfying condition $(*)$ in Theorem 1.3. Then any general member $S \in |-2K_X|$ satisfies $S \cap C = \{P\}$, where P is the point of index $r > 2$ or a smooth point (if (X, C) is of index 2). Moreover, the pair $(X, \frac{1}{2}S)$ is log terminal.*

Proposition 4.2 (see [2], Lemma (2.5), and [11], Proposition 2.1). *Let (X, C) be an extremal curve germ (C is not necessarily irreducible) and let $S \in |-2K_X|$ be a general member. Assume that the set $\Sigma := S \cap C$ is finite.*

(i) *If (X, C) is birational, then the natural map*

$$\tau \colon H^0(X, \mathscr{O}_X(-K_X)) \to \boldsymbol{\omega}_{(S,\Sigma)} = H^0(S, \mathscr{O}_S(-K_X)) \tag{4.1}$$

is surjective, where $\boldsymbol{\omega}_{(S,\Sigma)}$ is the dualizing sheaf of (S, Σ).

(ii) *If (X, C) is a $\mathbb{Q}$-conic bundle germ over a smooth base surface, then the natural map*

$$\overline{\tau} \colon H^0(X, \mathscr{O}_X(-K_X)) \to \boldsymbol{\omega}_{(S,\Sigma)}/\Omega^2_{(S,\Sigma)} \tag{4.2}$$

is surjective, where $\Omega^2_{(S,\Sigma)}$ is the sheaf of holomorphic 2-forms on (S, Σ).

(iii) *If (X, C) is a $\mathbb{Q}$-conic bundle germ over a base surface and $\Sigma = \Sigma_1 \amalg \Sigma_2$, $\Sigma_i \neq \varnothing$, then*

$$\tau_1 \colon H^0(X, \mathscr{O}_X(-K_X)) \to \boldsymbol{\omega}_{(S,\Sigma_1)} \tag{4.3}$$

is surjective.

Proof. For the proof of (i) we refer to [2], Lemma (2.5).

We now prove (ii). Note that by adjunction $\mathscr{O}_S(K_S) = \mathscr{O}_S(-K_X)$. Let $f\colon (X, C) \to (Z, o)$ be the corresponding $\mathbb{Q}$-conic bundle contraction and let $g = f|_S\colon S \to Z$ be its restriction to S. Since the base surface Z is smooth, by Lemma (4.1) in [9] there is a canonical isomorphism

$$R^1 f^* \boldsymbol{\omega}_X \simeq \boldsymbol{\omega}_Z.$$

Now we apply Proposition 2.1 from [11] to our situation:

$$\begin{tikzcd} H^0(X, \mathscr{O}_X(-K_X)) \arrow[r] \arrow[d, equal] & H^0(S, \mathscr{O}_S(-K_X)) \arrow[d, "\simeq"] & \\ f_*\boldsymbol{\omega}_X(S) \arrow[r] \arrow[dr] & \boldsymbol{\omega}_{(S,\Sigma)} \arrow[d] & \\ & \boldsymbol{\omega}_{(S,\Sigma)}/g^*\boldsymbol{\omega}_{(Z,o)} \arrow[r, two heads] & \boldsymbol{\omega}_{(S,\Sigma)}/\Omega^2_{(S,\Sigma)} \end{tikzcd}$$

and obtain the surjectivity of τ.

To prove (iii) we consider the map $g_i\colon S_i \to Z$ which is the restriction of g to $S_i = (S, \Sigma_i) \subset S$ and the induced exact sequence

$$\begin{tikzcd} f_*\boldsymbol{\omega}_X(S) \arrow[r] & \boldsymbol{\omega}_{(S,\Sigma_1)} \oplus \boldsymbol{\omega}_{(S,\Sigma_2)} \arrow[r] & \boldsymbol{\omega}_Z \arrow[r] \arrow[l, bend left, "g_2^*"] & 0 \end{tikzcd}$$

Then we see that $g_2^*\colon \boldsymbol{\omega}_Z \to 0 \oplus \boldsymbol{\omega}_{(S,\Sigma_2)}$ is a splitting homomorphism. Therefore, the homomorphism

$$f_*\boldsymbol{\omega}_X(S) \to \boldsymbol{\omega}_{(S,\Sigma_1)} \oplus (\boldsymbol{\omega}_{(S,\Sigma_2)}/g_2^*\boldsymbol{\omega}_{(Z,0)})$$

is surjective. The proposition is proved.

Lemma 4.3. *Let $(\overline{X}, \overline{C})$ be an extremal curve germ with reducible central curve $\overline{C}$. Suppose $\overline{X}$ satisfy condition $(*)$ in Theorem 1.3 and that there is a component $C \subset \overline{C}$ of type* (k1A) *which meets $\overline{C} - C$ at a point P of index 2. Then a general member $D \in |-K_{\overline{X}}|$ does not contain C.*

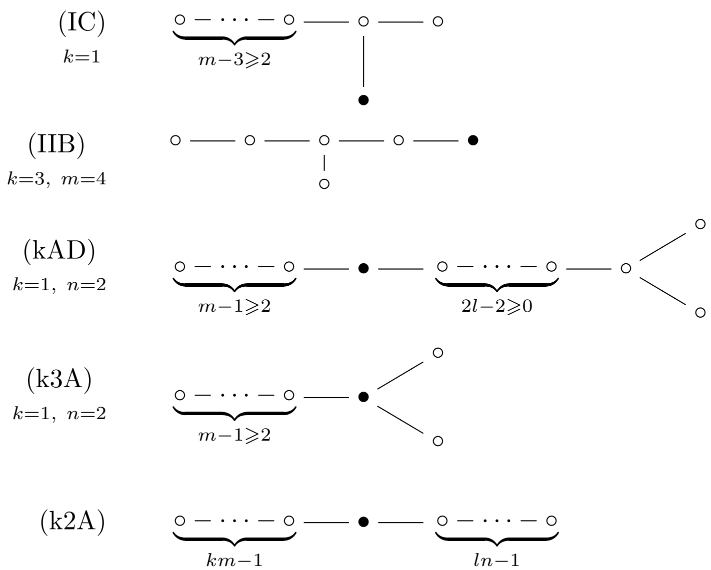
Proof. On each irreducible component C_i of $\overline{C}$ there exists at most one point of index > 2. Let $\{P_a\}_{a \in A}$ be the collection of such points. For each C_i without points of index > 2, choose one general point of C_i. Let $\{P_b\}_{b \in B}$ be the collection of such points. For each $i \in A \cup B$, let $S_i \in |-2K_{(X,P_i)}|$ be a general element on the germ (X, P_i), and set $S = \sum_{i \in A \cup B} S_i$. Then S extends to an element $|-2K_X|$ by [6], Theorem (7.3). A generator σ_b of $\mathscr{O}_{S,b}(-K_X) \simeq \mathscr{O}_{S,b}$ lifts to $s \in H^0(X, \mathscr{O}_X(-K_X))$ by Proposition 4.2, (i) if $(\overline{X}, \overline{C})$ is birational and since $A \neq \varnothing$, and by Proposition 4.2, (ii) otherwise. In either case we have $C \not\subset D$. The lemma is proved.

§ 5. A review of [2], § 2

We need some refinements of some facts on birational extremal curve germs with irreducible central fibre, proved in [2], § 2.

5.1. Below, for a normal surface D and a curve $C \subset D$, we use the usual notation for graphs $\Delta(D, C)$ of the minimal resolution of D near C: each vertex labelled $\bullet$ corresponds to an irreducible component of C and each labelled $\circ$ corresponds to a component $E_i \subset E$ of the exceptional divisor E on the minimal resolution of D. Note that, in our situation, below $E_i^2 = -2$ for all E_i.

Theorem 5.1 (see [2], Theorem (2.2), and [8]). *Let $(X, C \simeq \mathbb{P}^1)$ be a birational extremal curve germ and let $D \in |-K_X|$ be a general member. Then D is a normal surface with Du Val singularities. Moreover, either $D \cap C$ is a point or $D \supset C$ and one and only one of the following possibilities holds for the graph $\Delta(D, C)$:*



where m and k are the index and axial multiplicity (*see Definition-Corollary* (1a.5), (iii) *in* [6]) *of a singular point of X, and n and l are those for the other non-Gorenstein point* (*if any*).

In the cases (IC), (IIB), (kAD), (k3A) and (k2A$_2$), Theorem 5.1 is a consequence of the following.

Theorem 5.2 (cf. [2], § 2, and [8]). *Let (X, C) be a birational extremal curve germ with irreducible central curve of type* (IC), (IIB), (kAD), (k3A) *or* (k2A$_2$). *Let $S \in |-2K_X|$ be a general member* (*so that $S \cap C = \{P\}$, where P is the point of index $r > 2$*). *Let $\sigma_S \in H^0(S, \mathscr{O}_S(-K_X))$ be a general section. Then for any section $\sigma \in H^0(X, \mathscr{O}_X(-K_X))$ such that*

$$\sigma|_S \equiv \sigma_S \mod \Omega_S^2 \tag{5.1}$$

(*see* (4.2)) *the divisor $D := \operatorname{div}(\sigma)$ is a normal surface with only Du Val singularities. Furthermore, the configuration of $\Delta(D, C)$ is as described in Theorem 5.1.*

Below we outline the proof of Theorem 5.2 following [2], § 2. We treat the possibilities (IC), (IIB), (k3A), (kAD) and (k2A$_2$) case by case.

5.2. Case (IC). By [6], (A.3), we have the following identification at P:

$$(X, C) = (\mathbb{C}^3_{y_1,y_2,y_4}, \{y_1^{m-2} - y_2^2 = y_4 = 0\})/\boldsymbol{\mu}_m(2, m-2, 1).$$

A general divisor $S \in |-2K_X|$ is given by $y_1 = \xi(y_2, y_4)$, where $\xi \in (y_2, y_4)^2$ is such that $\operatorname{wt}(\xi) \equiv 2 \mod m$. Thus we have

$$S \simeq \mathbb{C}^2_{y_2,y_4}/\boldsymbol{\mu}_m(m-2, 1), \qquad \boldsymbol{\omega}_S = (\mathscr{O}_{S^\sharp,P^\sharp}\, \mathrm{d}y_2 \wedge \mathrm{d}y_4)^{\boldsymbol{\mu}_m}, \tag{5.2}$$

$$\boldsymbol{\omega}_S \otimes \mathbb{C}_P = \mathbb{C} \cdot y_2^{(m-1)/2}\, \mathrm{d}y_2 \wedge \mathrm{d}y_4 \oplus \mathbb{C} \cdot y_4\, \mathrm{d}y_2 \wedge \mathrm{d}y_4. \tag{5.3}$$

Furthermore,

$$\mathrm{gr}^0_C\, \boldsymbol{\omega}^* = (P^\sharp) = \left(-1 + \frac{m+1}{2} \cdot 2P^\sharp\right) \simeq \mathscr{O}_C(-1), \tag{5.4}$$

where

$$\Omega^{-1} := (\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_4)^{-1}$$

is an ℓ-free ℓ-basis at P. Hence $H^0(C, \mathrm{gr}^0_C\, \boldsymbol{\omega}^*) = 0$ and

$$H^0(X, \mathscr{O}_X(-K_X)) = H^0(X, \mathscr{I}_C \mathbin{\widetilde{\otimes}} \mathscr{O}_X(-K_X)), \tag{5.5}$$

where $\mathscr{I}_C$ is the defining ideal of C in X. Furthermore, by [2], (2.10.4),

$$\mathrm{gr}^1_C\, \boldsymbol{\omega}^* = (5P^\sharp) \mathbin{\widetilde{\oplus}} (0), \tag{5.6}$$

where the $\boldsymbol{\mu}_m$-semi-invariants

$$(y_1^{m-2} - y_2^2) \cdot \Omega^{-1} \quad \text{and} \quad y_4 \cdot \Omega^{-1} \tag{5.7}$$

form an ℓ-free ℓ-basis at P. Therefore,

$$\mathrm{gr}^1_C\, \boldsymbol{\omega}^* \simeq \begin{cases} \mathscr{O}_C(-1) \oplus \mathscr{O}_C & \text{if } m \geqslant 9, \\ \mathscr{O}_C \oplus \mathscr{O}_C & \text{if } m = 7, \\ \mathscr{O}_C(1) \oplus \mathscr{O}_C & \text{if } m = 5. \end{cases}$$

We have natural homomorphisms

$$\delta \colon H^0(X, \mathscr{O}_X(-K_X)) \to \mathrm{gr}^1_C\, \boldsymbol{\omega}^* \to (\mathrm{gr}^1_C\, \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp}.$$

Since $(y_1 - \xi) \cdot (\mathrm{gr}^1_C\, \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp} = 0$, the map δ factors as

$$\delta \colon H^0(X, \mathscr{O}_X(-K_X)) \to \boldsymbol{\omega}_S \to (\mathrm{gr}^1_C\, \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp}.$$

As in [11], (3.1.1), we see that

$$\Omega^2_S \subset (\mathfrak{m}_{S,P} \cdot y_4 + \mathfrak{m}_{S,P} \cdot y_2^{(m-1)/2})\, \mathrm{d}y_2 \wedge \mathrm{d}y_4 = \mathfrak{m}_{S,P} \cdot \boldsymbol{\omega}_S,$$

because for arbitrary elements ϕ_1 and ϕ_2 of the set of generators

$$\{y_2^m, y_4^m, y_2 y_4^2, y_2^{(m-1)/2} y_4\}$$

of the ring $\mathscr{O}_{S^\sharp}^{\boldsymbol{\mu}_m}$ we have

$$\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \in \big((y_2, y_4)y_4 + (y_2, y_4)y_2^{(m-1)/2}\big)\, \mathrm{d}y_2 \wedge \mathrm{d}y_4.$$

Thus δ factors further as follows:

$$\delta\colon H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \boldsymbol{\omega}_S/\Omega_S^2 \twoheadrightarrow \boldsymbol{\omega}_S \otimes \mathbb{C}_P \to (\mathrm{gr}_C^1\, \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp},$$

where the last map is a surjection if $m = 5$ and the image is generated by $y_4\Omega^{-1}$ if $m \geqslant 7$ (see (5.3) and (5.7)). If $m \geqslant 7$, this implies that the coefficient of $y_4\Omega^{-1}$ in σ_S is nonzero. If $m = 5$, then the coefficients of $y_4\Omega^{-1}$ and $(y_1^{m-2} - y_2^2)\Omega^{-1}$ in σ_S are independent and the image $\overline{\sigma}$ of σ in $\mathrm{gr}_C^1\, \boldsymbol{\omega}^*$ is not contained in $\mathscr{O}_C(1)$. Hence $\overline{\sigma}$ is nowhere vanishing and so the singular locus of D does not meet $C \setminus \{P\}$. Then we can again take σ_S so that it contains the term $y_4\Omega^{-1}$. Therefore, $D \in |-K_X|$ can be given by the equation $y_4 + \cdots = 0$. Then Computation 2.10.5 in [2] shows that D is Du Val at P and its graph is as given for type (IC) in Theorem 5.1.

5.3. Case (IIB). Then by [6], (A.3), the germ (X, C) at P can be given as follows

$$(X, C) = \big(\{\phi = 0\} \subset \mathbb{C}^4_{y_1,\dots,y_4},\ \{y_1^2 - y_2^3 = y_3 = y_4 = 0\}\big)/\boldsymbol{\mu}_4(3, 2, 1, 1),$$
$$\phi = y_1^2 - y_2^3 + \psi, \qquad \mathrm{wt}(\psi) \equiv 2 \mod 4, \qquad \psi(0, 0, y_3, y_4) \notin (y_3, y_4)^3.$$

A general divisor $S \in |-2K_X|$ is given by $y_2 = \xi(y_1, y_3, y_4)$ with $\xi \in (y_1, y_3, y_4)^2$ such that $\mathrm{wt}(\xi) \equiv 2 \mod 4$. Thus S is the quotient by $\boldsymbol{\mu}_4(3, 1, 1)$ of the hypersurface $\phi(y_1, \xi, y_3, y_4) = 0$ in $\mathbb{C}^3_{y_1,y_3,y_4}$. We have

$$\boldsymbol{\omega}_S = \left(\mathscr{O}_{S^\sharp, P^\sharp}\, \frac{\mathrm{d}y_3 \wedge \mathrm{d}y_4}{y_1 + \cdots}\right)^{\boldsymbol{\mu}_4},$$
$$\boldsymbol{\omega}_S \otimes \mathbb{C}_P = \mathbb{C} \cdot y_3\, \frac{\mathrm{d}y_3 \wedge \mathrm{d}y_4}{y_1 + \cdots} \oplus \mathbb{C} \cdot y_4\, \frac{\mathrm{d}y_3 \wedge \mathrm{d}y_4}{y_1 + \cdots}. \tag{5.8}$$

Furthermore,

$$\mathrm{gr}_C^0\, \boldsymbol{\omega}^* = (P^\sharp) = (-1 + 3P^\sharp + 2P^\sharp) \simeq \mathscr{O}_C(-1), \tag{5.9}$$

where

$$\Omega^{-1} := \left(\frac{\mathrm{d}y_2 \wedge \mathrm{d}y_3 \wedge \mathrm{d}y_4}{\partial\phi/\partial y_1}\right)^{-1}$$

is an ℓ-free ℓ-basis at P. Hence $H^0(C, \mathrm{gr}_C^0\, \boldsymbol{\omega}^*) = 0$ and

$$H^0(X, \mathscr{O}_X(-K_X)) = H^0(X, \mathscr{I}_C \mathbin{\widetilde{\otimes}} \mathscr{O}_X(-K_X)), \tag{5.10}$$

where $\mathscr{I}_C$ is the defining ideal of C in X. Furthermore, by [2], (2.11),

$$\mathrm{gr}_C^1\, \boldsymbol{\omega}^* = (0) \mathbin{\widetilde{\oplus}} (1) \simeq \mathscr{O}_C \oplus \mathscr{O}_C(1), \tag{5.11}$$

where the $\boldsymbol{\mu}_m$-invariants

$$y_3 \cdot \Omega^{-1}, \quad y_4 \cdot \Omega^{-1} \tag{5.12}$$

form an ℓ-free ℓ-basis at P. As in case (IC), we have natural homomorphisms

$$\delta\colon H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \boldsymbol{\omega}_S \otimes \mathbb{C}_P \to (\mathrm{gr}^1_C\,\boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp},$$

where the last homomorphism is an isomorphism (see (5.8) and (5.12)). Thus the coefficients of $y_3\Omega^{-1}$ and $y_4\Omega^{-1}$ in σ_S are independent, and so Computation 2.11.2 in [2] shows that D is Du Val at P, and the image $\overline{\sigma}$ of σ in $\mathrm{gr}^1_C\,\boldsymbol{\omega}^*$ is not contained in $\mathscr{O}_C(1)$. Hence $\overline{\sigma}$ is nowhere vanishing and D is smooth outside P. Hence the graph $\Delta(D,C)$ is as given for type (IIB) in Theorem 5.1.

5.4. Case (k3A). The configuration of singular points on (X,C) is the following: a type (IA) point P of odd index $m \geqslant 3$, a type (IA) point Q of index 2 and a type (III) point R. According to [6], (A.3), and [2], (2.12), the local structure of the points is given by

$$\begin{aligned}
(X,C,P) &= (\mathbb{C}^3_{y_1,y_2,y_3}, (y_1\text{-axis}), 0)/\boldsymbol{\mu}_m\Bigl(1, \frac{m+1}{2}, -1\Bigr),\\
(X,C,Q) &= (\mathbb{C}^3_{z_1,z_2,z_3}, (z_1\text{-axis}), 0)/\boldsymbol{\mu}_2(1,1,1),\\
(X,C,R) &= \bigl(\{\gamma(w_1,w_2,w_3,w_4)=0\}, (w_1\text{-axis}), 0\bigr),
\end{aligned}$$

where $\gamma \equiv w_1w_3 \mod (w_2,w_3,w_4)^2$.

For a general divisor $S \in |-2K_X|$ we have $S \cap C = \{P\}$ and S is given by $y_1 = \xi(y_2,y_3)$, where $\xi \in (y_2,y_3)^2$ is such that $\mathrm{wt}(\xi) \equiv 1 \mod m$. Thus,

$$S \simeq \mathbb{C}^2_{y_2,y_3}/\boldsymbol{\mu}_m\Bigl(\frac{m+1}{2}, -1\Bigr), \qquad \boldsymbol{\omega}_S = (\mathscr{O}_{S^\sharp,P^\sharp}\,\mathrm{d}y_2 \wedge \mathrm{d}y_3)^{\boldsymbol{\mu}_m}, \tag{5.13}$$

$$\boldsymbol{\omega}_S \otimes \mathbb{C}_P = \mathbb{C}\cdot y_2\,\mathrm{d}y_2 \wedge \mathrm{d}y_3 \oplus \mathbb{C}\cdot y_3^{(m-1)/2}\,\mathrm{d}y_2 \wedge \mathrm{d}y_3. \tag{5.14}$$

By the proof of Lemma (2.12.2) in [2] we have

$$\mathrm{gr}^0_C\,\boldsymbol{\omega}^* = \Bigl(-1 + \frac{m+1}{2}P^\sharp + Q^\sharp\Bigr) \simeq \mathscr{O}_C(-1), \tag{5.15}$$

where an ℓ-free ℓ-basis at P, Q and R, respectively, can be written as follows:

$$\Omega_P^{-1} := (\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3)^{-1}, \qquad \Omega_Q^{-1} := (\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3)^{-1},$$
$$\Omega_R^{-1} := \Bigl(\frac{\mathrm{d}w_2 \wedge \mathrm{d}w_3 \wedge \mathrm{d}w_4}{\partial\gamma/\partial w_1}\Bigr)^{-1}.$$

Hence $H^0(C, \mathrm{gr}^0_C\,\boldsymbol{\omega}^*) = 0$ and

$$H^0(X, \mathscr{O}_X(-K_X)) = H^0(X, \mathscr{I}_C \mathbin{\widetilde{\otimes}} \mathscr{O}_X(-K_X)), \tag{5.16}$$

where $\mathscr{I}_C$ is the defining ideal of C in X. Furthermore, as in [2], (2.12.4), we can further arrange that

$$\mathrm{gr}^1_C\,\boldsymbol{\omega}^* = (0) \mathbin{\widetilde{\oplus}} \Bigl(-1 + \frac{m+3}{2}P^\sharp\Bigr), \tag{5.17}$$

where $y_2 \cdot \Omega_P^{-1}$, $z_2 \cdot \Omega_Q^{-1}$ and $w_2 \cdot \Omega_R^{-1}$ form ℓ-free ℓ-bases for (0) at P, Q and R, respectively, and $y_3 \cdot \Omega_P^{-1}$, $z_3 \cdot \Omega_Q^{-1}$, $w_4 \cdot \Omega_R^{-1}$ form such a basis for $(-1+(m+3)/2P^\sharp)$. Moreover,

$$\gamma \equiv w_1 w_3 + c_1 w_4^2 + c_2 w_4 w_2 + c_3 w_2^2 \mod (w_3, w_2^2, w_2 w_4, w_4^2) \cdot \mathscr{I}_C$$

for some $c_1, c_2, c_3 \in \mathbb{C}$ such that $c_1 \neq 0$ if $m \geqslant 5$ (see [2], (2.12.6), and [8], Remark 2) and $(c_1, c_2, c_3) \neq 0$ if $m = 3$ (see [2], (2.12.7), and [8], Remark 2).

As in [11], (3.1.1), we see that

$$\Omega_S^2 \subset (\mathfrak{m}_{S,P} \cdot y_2 + \mathfrak{m}_{S,P} \cdot y_3^{(m-1)/2}) \, \mathrm{d}y_2 \wedge \mathrm{d}y_3 = \mathfrak{m}_{S,P} \cdot \boldsymbol{\omega}_S,$$

because for arbitrary elements ϕ_1 and ϕ_2 of the set of generators

$$\{y_2^m, y_3^m, y_2^2 y_3, y_2 y_3^{(m+1)/2}\}$$

of the ring $\mathscr{O}_{S^\sharp}^{\boldsymbol{\mu}_m}$ we have

$$\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \in \bigl((y_2, y_3) y_2 + (y_2, y_3) y_3^{(m-1)/2}\bigr) \, \mathrm{d}y_2 \wedge \mathrm{d}y_3.$$

Thus the image of the homomorphism

$$\delta \colon H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \boldsymbol{\omega}_S \otimes \mathbb{C}_P \to (\mathrm{gr}_C^1 \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp},$$

is equal to $(\mathrm{gr}_C^1 \boldsymbol{\omega}^*)^\sharp \otimes \mathbb{C}_{P^\sharp}$ if $m = 3$, and $\mathbb{C} \cdot y_2 \Omega_P^{-1}$ if $m \geqslant 5$.

If $m \geqslant 5$, this implies that the coefficient of $y_2 \Omega_P^{-1}$ in the image $\overline{\sigma}$ of σ in $\mathrm{gr}_C^1 \boldsymbol{\omega}^*$ is nonzero and hence nowhere vanishing. If $m = 3$, then the coefficients of $y_2 \Omega_P^{-1}$ and $y_3 \Omega_P^{-1}$ are independent and hence $\overline{\sigma}$ is a general global section of $\mathrm{gr}_C^1 \boldsymbol{\omega}^* \simeq \mathscr{O}_C \oplus \mathscr{O}_C$. Then the proof of Lemma (2.12.5) in [2] shows that D is Du Val and that its graph is as given in type (k3A) in Theorem 5.1.

Lemma 5.3. *In the notation of* § 5.4 *there exists a deformation* $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$ *of* (X, C) *which is trivial outside* R *such that for* $\lambda \neq 0$ *the germ* (X_λ, C_λ) *has a cyclic quotient singularity at* Q, *and is of type* (kAD), *case* (5.22), *if* $m \geqslant 5$ *and type* (k2A$_2$), *case* (5.22), *if* $m = 3$.

Proof. Let (X_λ, C_λ) be the twisted extension [6], (1b.8.1), of the germ

$$(X_\lambda, R) = \{\gamma - \lambda w_2 = 0\} \supset (C_\lambda, R) = (w_1\text{-axis})$$

by $u = (w_2, w_4)$. Then in $\mathrm{gr}_{C_\lambda}^1 \mathscr{O}$ we have $w_1 w_3 = \lambda w_2$ for $\lambda \neq 0$. Since $\mathrm{gr}_C^1 \boldsymbol{\omega}^* = \mathscr{O}_C \cdot w_2 \Omega_R^{-1} \oplus \mathscr{O}_C \cdot w_4 \Omega_R^{-1}$ at R, we have

$$\mathrm{gr}_{C_\lambda}^1 \boldsymbol{\omega}^* = \mathscr{O}_{C_\lambda} \cdot w_3 \Omega_R^{-1} \oplus \mathscr{O}_{C_\lambda} \cdot w_4 \Omega_R^{-1}$$

at R, where $w_3 \Omega_R^{-1} = (\lambda w_1)^{-1} w_2 \Omega_R^{-1}$. Thus

$$\mathrm{gr}_{C_\lambda}^1 \boldsymbol{\omega}^* = (R) \, \widetilde{\oplus} \left(-1 + \frac{m+3}{2} P^\sharp\right). \tag{5.18}$$

For $\lambda \neq 0$ the germ (X_λ, C_λ) is either of type (kAD) or (k2A$_2$). Comparing (5.18) with (5.31) in the first case, and in view of § 5.7 in the second, we see that (X_λ, C_λ) is (kAD) if $m \geqslant 5$, and (k2A$_2$) if $m = 3$. The lemma is proved.

5.5. Case (kAD). The configuration of singular points on (X, C) is the following: a type (IA) point P of odd index $m \geqslant 3$ and a type (IA) point Q of index 2. According to [6], (A.3), [2], (2.13), and [8] we can write

$$
\begin{aligned}
(X, C, P) &= (\mathbb{C}^3_{y_1,y_2,y_3}, (y_1\text{-axis}), 0)/\boldsymbol{\mu}_m\biggl(1, \frac{m+1}{2}, -1\biggr),\\
(X, C, Q) &= \bigl(\{\beta = 0\} \subset \mathbb{C}^4_{z_1,\dots,z_4}, (z_1\text{-axis}), 0\bigr)/\boldsymbol{\mu}_2(1,1,1,0),
\end{aligned}
$$

where $\beta = \beta(z_1, \dots, z_4)$ is a semi-invariant with $\operatorname{wt}(\beta) \equiv 0 \mod 2$.

For a general divisor $S \in |-2K_X|$ we have $S \cap C = \{P\}$ and S is given by $y_1 = \xi(y_2, y_3)$ with $\xi \in (y_2, y_3)^2$ such that $\operatorname{wt}(\xi) \equiv 1 \mod m$. Thus

$$
\begin{gathered}
S \simeq \mathbb{C}^2_{y_2,y_3}/\boldsymbol{\mu}_m\biggl(\frac{m+1}{2}, -1\biggr), \qquad \boldsymbol{\omega}_S = (\mathscr{O}_{S^\sharp, P^\sharp}\, \mathrm{d}y_2 \wedge \mathrm{d}y_3)^{\boldsymbol{\mu}_m},\\
\boldsymbol{\omega}_S \otimes \mathbb{C}_P = \mathbb{C} \cdot y_2\, \mathrm{d}y_2 \wedge \mathrm{d}y_3 \oplus \mathbb{C} \cdot y_3^{(m-1)/2}\, \mathrm{d}y_2 \wedge \mathrm{d}y_3.
\end{gathered} \tag{5.19}
$$

Then

$$
\operatorname{gr}^0_C \boldsymbol{\omega}^* = \biggl(-1 + \frac{m+1}{2}P^\sharp + Q^\sharp\biggr) \simeq \mathscr{O}_C(-1), \tag{5.20}
$$

where an ℓ-free ℓ-basis at P and Q, respectively, can be written as follows:

$$
\Omega_P^{-1} = (\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3)^{-1} \quad \text{and} \quad \Omega_Q^{-1} = \biggl(\frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{\partial\beta/\partial z_4}\biggr)^{-1}.
$$

Hence $H^0(C, \operatorname{gr}^0_C \boldsymbol{\omega}^*) = 0$ and

$$
H^0(X, \mathscr{O}_X(-K_X)) = H^0(X, \mathscr{I}_C \mathbin{\widetilde{\otimes}} \mathscr{O}_X(-K_X)), \tag{5.21}
$$

where $\mathscr{I}_C$ is the defining ideal of C in X.

As in [2], Lemma (2.13.3), we distinguish two subcases:

$$
\ell(Q) \leqslant 1, \qquad i_Q(1) = 1, \quad \operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1); \tag{5.22}
$$

$$
\ell(Q) = 2, \qquad i_Q(1) = 2, \quad \operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-1). \tag{5.23}
$$

5.6. Subcase (5.23). This is treated similarly to § 5.4. Since $\ell(Q) = 2$, we have

$$
\beta \equiv z_1^2 z_4 \mod (z_2, z_3, z_4)^2.
$$

As in [2], Lemma (2.13.4), we can arrange that

$$
\operatorname{gr}^1_C \boldsymbol{\omega}^* = (0) \mathbin{\widetilde{\oplus}} \biggl(-1 + \frac{m+3}{2}P^\sharp\biggr), \tag{5.24}
$$

where

$$
(y_2 \cdot \Omega_P^{-1}, z_2 \cdot \Omega_Q^{-1}) \quad \text{and} \quad (y_3 \cdot \Omega_P^{-1}, z_3 \cdot \Omega_Q^{-1}) \tag{5.25}
$$

form an ℓ-free ℓ-basis at P and Q for (0) and $(-1 + (m+3)/2P^\sharp)$, respectively, and

$$
\beta \equiv z_1^2 z_4 + c_1 z_3^2 + c_2 z_2 z_3 + c_3 z_2^2 \mod (z_4, z_3^2, z_2 z_3, z_2^2)(z_2, z_3, z_4)
$$

for some $c_1, c_2, c_3 \in \mathbb{C}$ such that $(c_1, c_2, c_3) \neq 0$ if $m = 3$ by the classification of 3-fold terminal singularities (see [15], Theorem (6.1)), and $c_1 \neq 0$ if $m \geqslant 5$ (see [2], Lemma (2.12.6), and [8], Remark 2). The rest of the argument is the same as § 5.4 (for type (k3A)), except that we use [2], Lemma (2.13.5), instead of [2], Lemma (2.12.6).

Remark 5.4. We note that the lowest power of the $\boldsymbol{\mu}_2$-invariant variable z_4 that appears in β (that is, the axial multiplicity for (X, P)) remains the same for the defining equation of $D^\sharp$ under the elimination of variable of wt $\equiv 1 \mod 2$. Thus the graph $\Delta(D, C)$ is as given for type (kAD) in Theorem 5.1.

Lemma 5.5. *In the situation of* § 5.5 *with* (5.23), *let* (X_λ, C_λ) *be the twisted extension of the germ*

$$(X_\lambda, Q) = \{\beta - \lambda z_4 = 0\}/\boldsymbol{\mu}_2 \supset (C_\lambda, Q) = (z_1\textit{-axis})/\boldsymbol{\mu}_2$$

by $u = (z_1 z_2, z_1 z_3)$ (*see* [6], *Definition* (1b.8.1)). *Then for* $\lambda \neq 0$ *the germ* (X_λ, C_λ) *is of type* (k3A).

Proof. When $0 < |\lambda| \ll 1$, a small neighbourhood $X_\lambda \ni Q$ has two singular points on C_λ: a cyclic quotient at Q and a Gorenstein point at $(\sqrt{\lambda}, 0, 0, 0)$. The lemma is proved.

5.7. Subcase (5.22). Note that in this case $m \geqslant 5$ (see [2], Lemma (2.13.10), and [8]). Since $\ell(Q) \leqslant 1$, we have

$$\beta \equiv z_4 \mod (z_2^2, z_3, z_4)(z_2, z_3, z_4)$$
$$\bigl(\beta \equiv (z_1 z_3 + z_2^2) \mod (z_2^2, z_3, z_4)(z_2, z_3, z_4), \text{ respectively}\bigr).$$

By [2], Lemma (2.13.10), we have

$$\begin{gathered} \operatorname{gr}^1_C \mathscr{O} = \left(\frac{m-1}{2} P^\sharp + Q^\sharp\right) \mathbin{\widetilde{\oplus}} (-1 + P^\sharp + Q^\sharp) \\ \left(\operatorname{gr}^1_C \mathscr{O} = \left(\frac{m-1}{2} P^\sharp\right) \mathbin{\widetilde{\oplus}} (-1 + P^\sharp + Q^\sharp), \text{ respectively}\right). \end{gathered} \tag{5.26}$$

Tensoring these with (5.20) we obtain

$$\begin{gathered} \operatorname{gr}^1_C \boldsymbol{\omega}^* = (1) \mathbin{\widetilde{\oplus}} \left(-1 + \frac{m+3}{2} P^\sharp\right) \\ \left(\operatorname{gr}^1_C \boldsymbol{\omega}^* = (Q^\sharp) \mathbin{\widetilde{\oplus}} \left(-1 + \frac{m+3}{2} P^\sharp\right), \text{ respectively}\right), \end{gathered} \tag{5.27}$$

where $(y_2 \Omega_P^{-1}, z_3 \Omega_Q^{-1})$ $((y_2 \Omega_P^{-1}, z_4 \Omega_Q^{-1})$, respectively) is the ℓ-free ℓ-basis for the first ℓ-summand of $\operatorname{gr}^1_C \boldsymbol{\omega}^*$ and $(y_3 \Omega_P^{-1}, z_2 \Omega_Q^{-1})$ for the second. Take the ideal $\mathscr{J} \subset \mathscr{I}$ as in [2], Lemmas (2.13.10) and (2.13.11). Thus

$$(\mathscr{I}/\mathscr{J}) \mathbin{\widetilde{\otimes}} \boldsymbol{\omega}^* = \left(-1 + \frac{m+3}{2} P^\sharp\right), \qquad H^0(X, \mathscr{O}_X(-K_X)) = H^0(\mathrm{F}^2(\boldsymbol{\omega}^*, \mathscr{J})),$$
$$\mathscr{J}^\sharp = (y_3^2, y_2) \ \text{ at } P \quad \text{and} \quad \mathscr{J}^\sharp = (z_2^2, z_3, z_4) \ \text{ at } Q.$$

Then by Lemma (2.13.11) in [2] we have

$$\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}) = (0) \mathbin{\widetilde{\oplus}} \left(-1 + \frac{m+5}{2}P^\sharp + Q^\sharp\right),$$

where

$$y_2 \cdot \Omega_P^{-1}, \quad y_3^2 \cdot \Omega_P^{-1}, \quad z_3 \cdot \Omega_Q^{-1}, \quad z_2^2 \cdot \Omega_Q^{-1} \quad (z_4 \cdot \Omega_Q^{-1}, \text{ respectively}) \tag{5.28}$$

form an ℓ-free ℓ-basis at P and Q. Thus we investigate

$$H^0(\mathscr{O}_X(-K_X)) = H^0(F^2(\boldsymbol{\omega}^*, \mathscr{J})) \to \operatorname{gr}^2_C(\boldsymbol{\omega}^*, \mathscr{J})$$

via the induced homomorphism

$$H^0(\mathscr{O}_X(-K_X)) \to \operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}) \to (\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}))^\sharp \otimes \mathbb{C}_{P^\sharp},$$

which, since $(y_1 - \xi) \cdot (\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}))^\sharp \otimes \mathbb{C}_{P^\sharp} = 0$, factors as

$$H^0(\mathscr{O}_X(-K_X)) \to \boldsymbol{\omega}_S \to (\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}))^\sharp \otimes \mathbb{C}_{P^\sharp},$$

and further to

$$\delta\colon H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \boldsymbol{\omega}_S \otimes \mathbb{C}_P \to (\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}))^\sharp \otimes \mathbb{C}_{P^\sharp},$$

since $\Omega_X^2 \subset \mathfrak{m}_{S,P} \cdot \boldsymbol{\omega}_S$ as in the (k3A) case.

The image of δ is generated by $y_2 \cdot \Omega_P^{-1}$ if $m > 5$, and by $y_2 \cdot \Omega_P^{-1}$ and $y_3^2 \cdot \Omega_P^{-1}$ if $m = 5$ (see (5.19) and (5.28)). Hence if σ_S is chosen to be general, the image $\overline{\sigma}$ of σ in $\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J})$ globally generates the direct summand $\mathscr{O}_C$ if $m > 5$ and a general global section of $\operatorname{gr}^2(\boldsymbol{\omega}^*, \mathscr{J}) \simeq \mathscr{O}_C \oplus \mathscr{O}_C$ if $m = 5$. Hence

$$\overline{\sigma} \equiv \begin{cases} (\lambda_P y_2 + \mu_P y_3^2)\Omega_P^{-1} & \text{at } P, \\ (\lambda_Q z_3 + \mu_Q z_2^2)\Omega_Q^{-1} \ ((\lambda_Q z_3 + \mu_Q z_4)\Omega_Q^{-1}, \text{ respectively}) & \text{at } Q, \end{cases}$$

where

$$\begin{aligned} m \geqslant 7 \quad &\Longrightarrow \quad \lambda_P(P)\lambda_Q(Q) \neq 0, \\ m = 5 \quad &\Longrightarrow \quad \lambda_P(P) \text{ and } \mu_P(P) \text{ are independent,} \\ &\qquad\quad \text{and } \lambda_Q(Q) \text{ and } \mu_Q(Q) \text{ are independent.} \end{aligned}$$

These mean that the corresponding $D \in |-K_X|$ is smooth outside P and Q and D is Du Val at P and Q by Computation (2.13.6) in [2]. See Remark 5.4 for further details.

Lemma 5.6. *In the situation of* § 5.5 *with* (5.22), *let* (X_λ, C_λ) *be the twisted extension* (*see* [6], *Definition* (1b.8.1)) *of the germ* $(X_\lambda, P) = (X, P) \supset (C_\lambda, P) = (C, P)$ *by* $u = (y_1^{(m-1)/2} y_2 + \lambda y_1 y_3, y_1, y_3)$. *Then for* $\lambda \neq 0$ *the germ* (X_λ, C_λ) *is of type* (k2A$_2$).

Proof. It is clear that (X_λ, C_λ) is of type (k2A$_2$) or (kAD), subcase (5.22), because $\ell(Q) \leqslant 1$. In either case, there is only one nonzero section s (up to constant multiplication) of $\operatorname{gr}^1_C \mathscr{O}$ (cf. (5.31)). Since $s = y_1^{(m-1)/2} y_2 \in \operatorname{gr}^1_C \mathscr{O}$ at P (up to a constant), we have an extension

$$s_\lambda = y_1^{(m-1)/2} y_2 + \lambda y_1 y_3 = y_1 (y_1^{(m-3)/2} y_2 + \lambda y_3) \in \operatorname{gr}^1_{C_\lambda} \mathscr{O}$$

on (C_λ, P), which generates $(P^\sharp) \subset \operatorname{gr}^1_{C_\lambda} \mathscr{O}$. In view of (5.26) the germ (X_λ, C_λ) is of type (k2A$_2$) by § 5.7. The lemma is proved.

Lemma 5.7. *In the situation of* § 5.5 *with* (5.22) *and* $\ell(Q) = 1$, *let* (X_λ, C_λ) *be the twisted extension* (*see* [6], *Definition* (1b.8.1)) *of the germ*

$$(X_\lambda, Q) = \{\beta - \lambda z_4 = 0\}/\boldsymbol{\mu}_2 \supset (C_\lambda, Q) = (z_1\textit{-axis})/\boldsymbol{\mu}_2$$

by $u = (z_1 z_2, z_4)$. *Then for* $\lambda \neq 0$ *the germ* (X_λ, C_λ) *is of type* (kAD) *and* $\ell(Q) = 0$.

In fact, a global section s of $\operatorname{gr}^1_C \mathscr{O}$ extends to the section $s_\lambda = y_4$ of $\operatorname{gr}^1_{C_\lambda} \mathscr{O}$ at Q, and s_λ vanishes at $P^\sharp$ to order $(m-1)/2 > 1$ by § 5.7. Thus (X_λ, C_λ) is of type (kAD).

5.8. Case (k2A$_2$). This case comes from Lemmas (2.13.1) and (2.13.9) in [2]. The configuration of singular points on (X, C) is the following: a type (IA) point P of odd index $m \geqslant 3$ and a type (IA) point Q of index 2. According to [6], (A.3), [2], (2.13), and [8] we can write

$$(X, C, P) = \bigl(\{\alpha = 0\} \subset \mathbb{C}^4_{y_1,\dots,y_4}, (y_1\text{-axis}), 0\bigr)/\boldsymbol{\mu}_m(1, a, -1, 0),$$
$$(X, C, Q) = \bigl(\{\beta = 0\} \subset \mathbb{C}^4_{z_1,\dots,z_4}, (z_1\text{-axis}), 0\bigr)/\boldsymbol{\mu}_2(1, 1, 1, 0),$$

where a is an integer prime to m such that $m/2 < a < m$, and α and β are invariants with

$$\alpha = y_1 y_3 - \alpha_1(y_2, y_3, y_4), \qquad \alpha_1 \in (y_2, y_3)^2 + (y_4),$$
$$\beta = z_1 z_3 - \beta_1(z_2, z_3, z_4), \qquad \beta_1 \in (z_2, z_3)^2 + (z_4).$$

Then

$$\operatorname{gr}^0_C \boldsymbol{\omega}^* = (-1 + aP^\sharp + Q^\sharp) \simeq \mathscr{O}_C(-1), \tag{5.29}$$

where an ℓ-free ℓ-basis at P and Q, respectively, can be written as follows:

$$\Omega_P^{-1} = \left(\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3}{\partial\alpha/\partial y_4}\right)^{-1} \quad \text{and} \quad \Omega_Q^{-1} = \left(\frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{\partial\beta/\partial z_4}\right)^{-1}.$$

Hence $H^0(C, \operatorname{gr}^0_C \boldsymbol{\omega}^*) = 0$ and

$$H^0(X, \mathscr{O}_X(-K_X)) = H^0(X, \mathscr{I}_C \mathbin{\widetilde{\otimes}} \mathscr{O}_X(-K_X)), \tag{5.30}$$

where $\mathscr{I}_C$ is the defining ideal of C in X.

As in the argument in [2], Theorem (2.13.8) and Lemma (2.13.9), we have

$$\operatorname{gr}^1_C \mathscr{O} = \mathscr{L} \mathbin{\widetilde{\oplus}} \operatorname{gr}^0_C \boldsymbol{\omega} \quad \text{and} \quad \operatorname{gr}^1_C \boldsymbol{\omega}^* = \mathscr{L} \mathbin{\widetilde{\otimes}} (\operatorname{gr}^0_C \boldsymbol{\omega}^*) \mathbin{\widetilde{\oplus}} (0), \tag{5.31}$$

where $\mathscr{L}$ is an ℓ-invertible sheaf such that $\mathscr{L} = (P^\sharp + Q^\sharp)$ $((P^\sharp), (Q^\sharp), (0))$ if $y_4 \in \alpha$ and $z_4 \in \beta$ ($y_4 \in \alpha$ and $z_4 \notin \beta$, $y_4 \notin \alpha$ and $z_4 \in \beta$, $y_4 \notin \alpha$ and $z_4 \notin \beta$, respectively). We also see that $y_3\Omega_P^{-1}$ ($y_4\Omega_P^{-1}$) and $y_2\Omega_P^{-1}$ form an ℓ-free ℓ-basis for $\mathrm{gr}^1_C\,\boldsymbol{\omega}^*$ at P if $y_4 \in \alpha$ ($y_4 \notin \alpha$, respectively).

For a general divisor $S \in |-2K_X|$ we have $S \cap C = \{P\}$ and S in X is given by

$$\gamma := y_1^{2a-m} + y_2^2 + y_3^{2m-2a} + \cdots = 0, \tag{5.32}$$

with $\mathrm{wt}(\gamma) \equiv 2a \mod m$. Let Ω be a generator of the dualizing sheaf $\boldsymbol{\omega}_{S^\sharp}$ of $S^\sharp$ at $P^\sharp$. Then

$$\boldsymbol{\omega}_S = (\mathscr{O}_{S^\sharp,P^\sharp}\Omega)^{\boldsymbol{\mu}_m}, \qquad \mathrm{wt}(\Omega) \equiv -a \mod m \tag{5.33}$$

and

$$\boldsymbol{\omega}_S \otimes \mathbb{C}_P = \mathbb{C}\cdot y_2\Omega \oplus \mathbb{C}\cdot y_3^{m-a}\Omega. \tag{5.34}$$

Lemma 5.8. *The induced map*

$$\Omega_S^2 \to \boldsymbol{\omega}_S \otimes \mathbb{C}_P \tag{5.35}$$

is zero, where Ω_S^2 is the sheaf of holomorphic 2-forms on S.

Proof. We have

$$\Omega = \pm\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{\varDelta_{3,4}} = \cdots = \pm\frac{\mathrm{d}y_3 \wedge \mathrm{d}y_4}{\varDelta_{1,2}}, \quad \text{where } \varDelta_{i,j} := \begin{vmatrix} \dfrac{\partial\alpha}{\partial y_i} & \dfrac{\partial\alpha}{\partial y_j} \\ \dfrac{\partial\gamma}{\partial y_i} & \dfrac{\partial\gamma}{\partial y_j}\end{vmatrix}. \tag{5.36}$$

Note that $\mathrm{wt}(\varDelta_{i,j}) \equiv 2a - \mathrm{wt}(y_i) - \mathrm{wt}(y_j)$ and $\mathrm{wt}(\Omega) \equiv -a \mod m$. Since $\boldsymbol{\omega}_S = (\mathscr{O}_{S^\sharp}\Omega)^{\boldsymbol{\mu}_m}$, it is sufficient to show that for any $\phi_1, \phi_2 \in \mathbb{C}\{y_1, \dots, y_4\}^{\boldsymbol{\mu}_m}$ the inclusion

$$\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \in \mathfrak{m}_{S,0}\cdot(\mathscr{O}_{S^\sharp}\Omega)^{\boldsymbol{\mu}_m} \tag{5.37}$$

holds. By (5.36) the form $\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2$ is a linear combination of the following:

$$\frac{\partial(\phi_1,\phi_2)}{\partial y_i\,\partial y_j}\,\mathrm{d}y_i \wedge \mathrm{d}y_j = \frac{\partial(\phi_1,\phi_2)}{\partial y_i\,\partial y_j}\varDelta_{k,l}\Omega, \qquad \{i,j,k,l\} = \{1,\dots,4\}.$$

Set

$$\Xi[i,j,k,l] := \frac{\partial\phi_1}{\partial y_i}\cdot\frac{\partial\phi_2}{\partial y_j}\cdot\frac{\partial\alpha}{\partial y_k}\cdot\frac{\partial\gamma}{\partial y_l}, \qquad \{i,j,k,l\} = \{1,\dots,4\}.$$

Since $\partial(\phi_1,\phi_2)/(\partial y_i\,\partial y_j)\varDelta_{k,l}$ are linear combinations of $\Xi[i,j,k,l]$, it is sufficient to show that the following holds:

$$\Xi[i,j,k,l]\big|_{S^\sharp} \in (\mathfrak{m}_{S^\sharp})_{\mathrm{wt}=0}\cdot(\mathscr{O}_{S^\sharp})_{\mathrm{wt}=a}. \tag{5.38}$$

First we note that (5.38) holds if $\{1,3\} \subset \{i,j,k\}$. Suppose, for example, that $i = 1$ and $j = 3$. Then

$$\frac{\partial\phi_1}{\partial y_1}, \frac{\partial\phi_2}{\partial y_3} \in \mathfrak{m}_{S^\sharp}, \qquad \mathrm{wt}\biggl(\frac{\partial\phi_1}{\partial y_1}\cdot\frac{\partial\phi_2}{\partial y_3}\biggr) \equiv 0 \quad \text{and} \quad \mathrm{wt}\biggl(\frac{\partial\alpha}{\partial y_k}\cdot\frac{\partial\gamma}{\partial y_l}\biggr) \equiv a$$

(because $\{\mathrm{wt}(y_k), \mathrm{wt}(y_l)\} = \{0, a\}$). Thus,

$$\frac{\partial\phi_1}{\partial y_1}\cdot\frac{\partial\phi_2}{\partial y_3} \in (\mathfrak{m}_{S^\sharp})_{\mathrm{wt}=0} \quad\text{and}\quad \frac{\partial\alpha}{\partial y_k}\cdot\frac{\partial\gamma}{\partial y_l} \in (\mathscr{O}_{S^\sharp})_{\mathrm{wt}=a}.$$

Therefore, $l = 3$ or 1.

Let $l = 3$. Then $\mathrm{wt}(\partial\gamma/\partial y_3) \equiv 2a+1$ and

$$\mathrm{wt}\Big(\frac{\partial\phi_1}{\partial y_i}\Big), \mathrm{wt}\Big(\frac{\partial\phi_2}{\partial y_j}\Big), \mathrm{wt}\Big(\frac{\partial\alpha}{\partial y_k}\Big) \equiv -1, -a, 0$$

up to permutations of i, j and k. We claim that any product Π of three monomials of weights -1, $-a$ and $2a+1$ belongs to $(\mathfrak{m}_{S^\sharp})_{\mathrm{wt}=0}\cdot(\mathscr{O}_{S^\sharp})_{\mathrm{wt}=a}$. This is obvious if Π is divisible by y_4, y_1y_3 or y_2. So it is enough to consider the case when Π is a power of y_1 or y_3. In the former case Π is divisible by $y_1^{m-1}\cdot y_1^{m-a}\cdot y_1^{2a+1-m} = y_1^m\cdot y_1^a$, and in the latter Π is divisible by $y_3\cdot y_3^a\cdot y_3^{2m-2a-1} = y_3^m\cdot y_3^{m-a}$, which settles the claim.

Finally, let $l = 1$. Similarly to the previous case, we show that any product Π of monomials of weights $-a$, 1 and $2a-1$ belongs to $(\mathfrak{m}_{S^\sharp})_{\mathrm{wt}=0}\cdot(\mathscr{O}_{S^\sharp})_{\mathrm{wt}=a}$. Again we can assume that Π is a power of y_1 or y_3. In the latter case, Π is divisible by $y_3^a\cdot y_3^{m-1}\cdot y_3^{2m-2a+1} = y_3^{3m-a} = y_3^m\cdot y_3^{2m-a}$. Similarly, in the former case Π is divisible by y_1^a. By (5.32), the monomial y_1^{2a-m} belongs to $(y_2, y_3)\mathfrak{m}_{S^\sharp}$, and we have $y_1^a \in (\mathfrak{m}_{S^\sharp})_{\mathrm{wt}=0}\cdot(\mathscr{O}_{S^\sharp})_{\mathrm{wt}=a}$ as $a > 2a-m$. This concludes the proof of Lemma 5.8.

By Lemma 5.8 the homomorphism δ is factored as in other cases:

$$\delta\colon H^0(\mathscr{O}_X(-K_X)) \twoheadrightarrow \boldsymbol{\omega}_S\otimes\mathbb{C}_P \to (\mathrm{gr}^1_C\,\boldsymbol{\omega}^*)^\sharp\otimes\mathbb{C}_{P^\sharp}. \tag{5.39}$$

Thus we see that δ is surjective if and only if $a = m-1$ and $y_4 \in \alpha$. If δ is not surjective, then $y_2\Omega_P^{-1}$ generates its image, which is the second summand (0) of $\mathrm{gr}^1_C\,\boldsymbol{\omega}^*$, and hence $\overline{\sigma}$ is a nowhere vanishing section. The remainder of the proof is the same as [2], Lemma (2.13.9). This concludes the proof of Theorem 5.2.

Proposition 5.9. *Let $(X, \overline{C})$ be an extremal curve germ whose central fibre $\overline{C}$ is reducible. Suppose that $\overline{C}$ contains a component C of type* (k2A$_2$) *and another component C' of type* (k1A) *meeting at a point P of index $m > 2$. Assume further that $(X, \overline{C})$ satisfies condition $(*)$ in Theorem 1.3. Then a general member $D \in |-K_X|$ is Du Val in a neighbourhood of $C \cup C'$.*

Proof. The (k2A$_2$) case of Theorem 5.2, considered in § 5.8, shows that a general member D $(\supset C)$ of $|-K_{\overline{X}}|$ is Du Val in a neighbourhood of C. If $D \not\supset C'$, then $D\cap C' = \{P\}$ because otherwise D contains a Gorenstein point P' of X and so $D\cdot C' > 1$, which contradicts $D\cdot C' = -K_X\cdot C' < 1$ by [6], (2.3.1), and [9], (3.1.1). Thus we can assume that $D \supset C'$. We use the notation in § 5.8. In view of (IA) and (IA$^\vee$) in [6], (A.3), the fact that D defined by $y_2 + \cdots = 0$ contains C' means that (X, C') is of type (IA), $C'^\sharp$ is smooth at $P^\sharp$, and either y_1 or y_3 is a coordinate of $C'^\sharp$.

Lemma 5.10. *The variable y_3 is a coordinate of $C'^\sharp$, and hence $C'^\sharp$ can be taken to be the y_3-axis modulo a $\boldsymbol{\mu}_m$-equivariant change of coordinates.*

Proof. Assume that y_1 is a coordinate of $C'^\sharp$. Then $\mathscr{I}^\sharp_{C'}$ is generated by

$$y_1^a\gamma_2+\delta_2,\quad y_1^{m-1}\gamma_3+\delta_3,\quad y_1^m\gamma_4+\delta_4,$$

with $\gamma_i\in\mathscr{O}_X$ and $\delta_i\in(y_2,y_3,y_4)^2\mathscr{O}^\sharp_X$. Thus,

$$\mathscr{I}^\sharp_C+\mathscr{I}^\sharp_{C'}\subset(y_2,y_3,y_4,y_1^a)$$

and $\boldsymbol{\omega}^\sharp_X\otimes\mathscr{O}^\sharp_X/(\mathscr{I}^\sharp_C+\mathscr{I}^\sharp_{C'})$ contains a nonzero $\boldsymbol{\mu}_m$-invariant element $y_1^{m-a}\Omega_P$, since $m-a<a$. In view of the exact sequence

$$0\to\operatorname{gr}^0_{C\cup C'}\boldsymbol{\omega}\to\operatorname{gr}^0_C\boldsymbol{\omega}\oplus\operatorname{gr}^0_{C'}\boldsymbol{\omega}\to(\boldsymbol{\omega}^\sharp_X\otimes\mathscr{O}^\sharp_X/(\mathscr{I}^\sharp_C+\mathscr{I}^\sharp_{C'}))^{\boldsymbol{\mu}_m}\to 0,$$

we see that $H^1(\operatorname{gr}^0_{C\cup C'}\boldsymbol{\omega})\neq 0$. This implies that $(\overline{X},C\cup C')$ is a conic bundle germ and $C\cup C'$ is a whole fibre of the conic bundle (see [9], Corollary (4.4.1)). However, $(C+C'\cdot D)<2$, and this is impossible. The lemma is proved.

From now on we assume that $C'^\sharp$ is the y_3-axis. Hence y_2, y_1 (or y_2, y_4) form an ℓ-free ℓ-basis of $\operatorname{gr}^1_{C'}\mathscr{O}$, and $y_2\Omega_P^{-1}$, $y_1\Omega_P^{-1}$ (or $y_2\Omega_P^{-1}$, $y_4\Omega_P^{-1}$) form an ℓ-free ℓ-basis of $\operatorname{gr}^1_{C'}\boldsymbol{\omega}^*$ at P. Furthermore, we see that $\operatorname{gr}^1_{C'}(\boldsymbol{\omega}^*)$ has a global section $\overline{\sigma}=(y_2+\cdots)\Omega_P^{-1}$ induced by the section σ defining D. We also note that $\operatorname{gr}^0_{C'}\boldsymbol{\omega}^*=((m-a)P^\sharp)$ since the weight wt' for C' is $\mathrm{wt}'\equiv-\mathrm{wt}\mod m$. According to the (k2A$_2$) case of Theorem 5.2 considered in § 5.8, the divisor D is defined at P by

$$y_2\Psi_1+y_3^{m-a}\Psi_2=0,$$

where the Ψ_i are invariant functions by (5.33) and (5.34). We have the surjections

$$H^0(\mathscr{O}_X(-K_X))\twoheadrightarrow\boldsymbol{\omega}_S/\Omega^2_S\twoheadrightarrow\boldsymbol{\omega}_S\otimes\mathbb{C}_P$$

by (4.1) and (4.2) for the first case and by Lemma 5.8 for the second. In particular, $\Psi_2(P)\neq 0$. Since $C'=(y_3\text{-axis})/\boldsymbol{\mu}_m$, we have $D\not\supset C'$. This contradicts our assumption. Thus, we have shown that a general elephant D of $(\overline{X},\overline{C})$ is Du Val in a neighbourhood of $C\cup C'$. Proposition 5.9 is proved.

§ 6. Proof of the main theorem

Notation 6.1. Let $(\overline{X},\overline{C})$ be an extremal curve germ with reducible central fibre $\overline{C}$ such that on each irreducible component C_i of $\overline{C}$ there exists at most one point of index >2. Let $\{P_a\}_{a\in A}$ be the collection of such points. For each C_i without points of index >2, choose one general point of C_i. Let $\{P_b\}_{b\in B}$ be the collection of such points. For each $i\in A\cup B$, let $S_i\in|-2K_{(X,P_i)}|$ be a general element on the germ (X,P_i), and set $S=\sum_{i\in A\cup B}S_i$. Then S extends to an element $|-2K_X|$ by Theorem (7.3) in [6].

Proof of the Main Theorem 1.3. Take a general element $\sigma_i\in\mathscr{O}_{S_i}(-K_X)$, and

$$\sigma_S:=\sum_i\sigma_i\in\mathscr{O}_S(-K_X).$$

By Proposition 4.2, (i) or (ii), the section $\sigma_S \mod \Omega_S^2$ lifts to

$$s \in H^0(X, \mathscr{O}_X(-K_X)).$$

Let $C_{\mathrm{main}} \subset \overline{C}$ be the union of the irreducible components of type (IC), (IIB), (kAD), (k3A) and ($\mathrm{k2A_2}$).

By Theorem 5.2 the divisor $D := \{s = 0\} \supset C_{\mathrm{main}}$ is Du Val in a neighbourhood of C_{main} and, for each irreducible component C of C_{main}, the graph $\Delta(D, C)$ is as described in Theorem 5.1 by Theorem 5.2. If $C_{\mathrm{main}} = \varnothing$, then we are done by Propositions 3.1 and 3.2. So we assume $C_{\mathrm{main}} \neq \varnothing$ and each irreducible component $C \subset \overline{C}$ intersects C_{main} (because singular points of $\overline{C}$ are non-Gorenstein on $\overline{X}$, see [6], Corollary (1.15), and [9], Lemma (4.4.2)). Then D is normal and C_{main} is connected. If $C \not\subset D$, then

$$C \cap D \subset C_{\mathrm{main}} \cap D,$$

and D is Du Val in a neighbourhood of C as well, because $C \cdot D < 1$ (see [6], (0.4.11.1)) and D is a Cartier divisor outside C_{main} (see [6], Corollary (1.15)).

Suppose C_{main} contains a component of one of the types (IC), (IIB), (kAD) or (k3A) and let $C \subset D$. Let $\upsilon\colon D \to D_0$ be the contraction of C_{main} and let $P_0 := \upsilon(C_{\mathrm{main}})$. Then the point $P_0 \in D_0$ is Du Val of type D or E (using the fact that υ is crepant and Theorem 5.1). Apply Lemma 2.3 to D_0. We conclude that the surface D_0 has only Du Val singularities and the same is true for D.

If C meets C_{main} at an index 2 point, then $D \not\supset C$ by Lemma 4.3. The case where C_{main} consists of curves of type ($\mathrm{k2A_2}$) and C intersects C_{main} at a point P of index $m > 2$ is treated in Proposition 5.9. Theorem 1.3 is proved.

Proof of Corollary 1.4. If $C_i \subset C$ is of type (IIB), then it contains a point P of type cAx/4, and all other components $C_j \subset C$ passing through P are of types (IIA) or ($\mathrm{II}^\vee$). But according to [6], Theorems (6.4) and (9.4), P is the only non-Gorenstein point on C_j. Since X is not Gorenstein at any intersection point $C_i \cap C_j$ by [6], Corollary (1.15), and [9], Lemma (4.4.2), all the components $C_k \subset C$ must pass through P. This proves (i).

From now on we can assume that $C = C_i \cup C_j$. We can also assume that C_j is not of type ($\mathrm{k2A_{n,m}}$), $n, m \geqslant 3$ (otherwise there is nothing to prove). Thus (X, C) satisfies condition $(*)$ in Theorem 1.3. Let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. Consider a general member $D \in |-K_X|$ and the Stein factorization

$$f_D\colon D \supset C \to f' D_Z \ni o_Z \to f(D) \ni o. \tag{6.1}$$

By Theorem 1.3 the surface D has only Du Val singularities. The contraction $f'\colon D \to D_Z$ is crepant, and the point $D_Z \ni o_Z$ is Du Val. Now we note that the germs (D, C_i) and $(D,\ C_j)$ are as described by Theorem 5.1. Thus the whole configuration of $\Delta(D, C)$ is one of the Dynkin diagrams A, D or E. In particular, $\Delta(D, C)$ has no vertices of valency $\geqslant 4$ and at most one vertex, say v, of valency 3. On the other hand, by Theorem 5.2 the configuration of $\Delta(D, C)$ is obtained by 'gluing' the configurations of $\Delta(D, C_i)$ described in Theorem 5.1 along one connected component of the white subgraph. Since the whole configuration of $\Delta(D, C)$ is Du Val, at most one component of C is of type (IC), (IIB), (kAD) or (k3A).

For (ii) it is enough to note that all the singularities along C_i are of type cA and so the extremal germs cD/m, cAx/2, cE/2, (IIA), and ($\mathrm{II}^\vee$) do not occur as a component of (X,C). A similar argument is applied in (iii) but in this case the singularities cD/2 and cAx/2 are allowed, see [8]. It remains to prove (iv).

6.1. Let T be a point of C_j. It is easy to observe that a twisted extension $(X_{j,\lambda}, C_{j,\lambda})$ (see [6], (1b.8.1)) of the germ $(X_{j,\lambda}, T) \supset (C_{j,\lambda}, T)$ by u in a neighbourhood of $C_{j,\lambda}$ can naturally contain a neighbourhood of C_i if

(a) $T \notin C_i$ or

(b) $(X_{j,\lambda}, T) \supset (C_{j,\lambda}, T)$ is a trivial deformation, that is, $X_{j,\lambda} = X_j$ and $C_{j,\lambda} = C_j$.

We can make successive deformations of (X,C) in a neighbourhood of C_j, which are trivial on a neighbourhood of C_i, in such a way that (X, C_j) deforms as follows:

1) from (kAD) of case (5.23) to (k3A), as treated in Lemma 5.5;
2) from (k3A) to (k2A_2) or (kAD) of case (5.22), as treated in Lemma 5.3;
3) from (kAD) of case (5.22) to (k2A_2), as treated in Lemma 5.6;

and ultimately to (X, C_j) of type (k2A_2). Indeed, we use (a) when T is the type (III) point $\notin C_i$ for deformation 2); we apply (b) when T is the index m point for deformation 3).

6.2. For deformation 1), we need to take the index 2 point Q with $\ell(Q) = 2$ as T. Suppose $Q \in C_i$, in which case the divisor D cannot be Du Val because $\Delta(D, C_j)$ of type $\mathrm{D_k}$ with $k \geqslant 8$ and $\Delta(D, C_i)$ of type $\mathrm{A_q}$ with $q \geqslant 4$ are connected at the index 2 point Q. So $Q \notin C_i$ and (a) applies, and we are left with the case when C_j is of type (k2A_2). It remains to eliminate the case where both components of C are of type (k2A_2). This follows from Lemma 6.4 below. The corollary is proved.

Lemma 6.2. *Let (X,C) be an extremal curve germ such that any component $C_i \subset C$ is of type* (k2A). *Assume that a general member $D \in |-K_X|$ is Du Val. Then a general hyperplane section $H \in |\mathscr{O}_X|_C$ passing through C has only cyclic quotient singularities and the pair (H,C) is log canonical and purely log terminal outside* $\mathrm{Sing}(C)$.

Remark 6.3. If in the conditions of Lemma 6.2 the germ (X,C) is a $\mathbb{Q}$-conic bundle, then its base surface is smooth. Indeed, if the base surface is singular, then by Theorem (1.3) in [10] each component $C_i \subset C$ must be locally imprimitive.

Proof of Lemma 6.2. Let $f\colon (X,C) \to (Z,o)$ be the corresponding contraction. Note that $D \supset C$. Consider the Stein factorization (6.1). It is easy to see that the configuration $\Delta(D,C)$ is a linear chain. Therefore, $D_Z \ni o_Z$ is a (Du Val) singularity of type A. Then the arguments in the proof of Proposition 2.6 in [12] work and show that the pair $(X, D+H)$ is log canonical. Since $D \supset C$, the pair (X,H) is purely log terminal by Bertini's theorem. Hence H is normal and by the inversion of adjunction the pair (H,C) is log canonical. Moreover, $C = D \cap H$ and $K_H + C$ is a Cartier divisor on H. By the classification of two-dimensional log canonical pairs (see [3], Theorem 4.15) the singularities of H are cyclic quotients and the pair (H,C) is purely log terminal outside $\mathrm{Sing}(C)$. The lemma is proved.

Lemma 6.4. *Let (X,C) be an extremal curve germ, where C is reducible and has exactly two components. Then both components cannot be of type* (k2A_2).

Proof. Assume the contrary. The computation below is very similar to that in [7], Proposition 2.6. Let $C = C_1 \cup C_2$, let $C_1 \cap C_2 = \{P_0\}$, and let $P_i \in C_i$ for $i = 1, 2$ be the non-Gorenstein point other than P_0. Let $H \in |\mathscr{O}_X|_C$ be a general hyperplane section passing through C. According to Lemma 6.2 the surface H has only cyclic quotient singularities. Consider the minimal resolution $\mu\colon \widetilde{H} \to H$ and write

$$K_{\widetilde{H}} = \mu^* K_H - \Theta, \tag{6.2}$$

where Θ is an effective $\mathbb{Q}$-divisor with support in the exceptional locus (the codiscrepancy divisor). Let $\widetilde{C}_i$ be the proper transform of C_i. Then $K_{\widetilde{H}} \cdot \widetilde{C}_i < K_H \cdot C_i < 0$. Hence $\widetilde{C}_i$ is a (-1)-curve on $\widetilde{H}$. Moreover,

$$\Theta \cdot \widetilde{C}_i = 1 + K_H \cdot C_i < 1.$$

Since H is a Cartier divisor on X such that (X, H) is purely log terminal, the singularities of H are of type T (see [5]). Hence

$$(H \ni P_i) \simeq (\mathbb{C}/\boldsymbol{\mu}_{m_i^2 p_i}(1, m_i p_i a_i - 1) \ni 0)$$

for some positive m_i, p_i and a_i such that $a_i < m_i$ and $\gcd(m_i, a_i) = 1$. Here m_i is the index of P_i. Write $\Theta = \Theta_0 + \Theta_1 + \Theta_2$ so that $\operatorname{Supp}(\Theta_i) = \mu^{-1}(P_i)$. Computations with weighted blowups (see [2], (10.1)–(10.3)) show that the coefficients of Θ_i in the ends of the chain $\operatorname{Supp}(\Theta_i)$ are equal to $(m_i - a_i)/m_i$ and a_i/m_i. Since $\widetilde{C}_1$ and $\widetilde{C}_2$ meet different ends of the chain $\operatorname{Supp}(\Theta_0)$, up to permutating P_1 and P_2 and changing the generators of $\boldsymbol{\mu}_{m_i^2 p_i}$ we have

$$\Theta_1 \cdot \widetilde{C}_1 = \frac{a_1}{m_1}, \qquad \Theta_0 \cdot \widetilde{C}_1 = \frac{m_0 - a_0}{m_0}, \qquad \Theta_0 \cdot \widetilde{C}_2 = \frac{a_0}{m_0}, \qquad \Theta_2 \cdot \widetilde{C}_2 = \frac{m_2 - a_2}{m_2}.$$

Set

$$\delta_1 := a_0 m_1 - a_1 m_0 \quad \text{and} \quad \delta_2 := a_2 m_0 - a_0 m_2. \tag{6.3}$$

Then by (6.2)

$$-K_H \cdot C_1 = \frac{a_0}{m_0} - \frac{a_1}{m_1} = \frac{\delta_1}{m_0 m_1} > 0, \qquad -K_H \cdot C_2 = \frac{a_2}{m_2} - \frac{a_0}{m_0} = \frac{\delta_2}{m_0 m_2} > 0.$$

Further, put

$$\varDelta_i := m_0^2 p_0 + m_i^2 p_i - m_0 p_0 m_i p_i \delta_i, \qquad i = 1, 2. \tag{6.4}$$

Then

$$C_i^2 = \frac{-\varDelta_i}{m_0^2 p_0 m_i^2 p_i}, \qquad C_1 \cdot C_2 = \frac{1}{m_0^2 p_0}.$$

Since the configuration is contractible, we have $\varDelta_i > 0$ and

$$\varDelta_1 \varDelta_2 - m_1^2 p_1 m_2^2 p_2 \geqslant 0. \tag{6.5}$$

Assume that $m_0 > 2$. Since C_i is of type (k2A$_2$), $m_1 = m_2 = 2$. Then $a_1 = a_2 = 1$ and (6.3) implies

$$\delta_1 = 2a_0 - m_0 > 0 \quad \text{and} \quad \delta_2 = m_0 - 2a_0 > 0,$$

which is impossible. Hence $m_0 = 2$. Then $a_0 = 1$ and (6.4) can be written as follows

$$\Delta_i = m_i^2 p_i - 2p_0(m_i p_i \delta_i - 2) > 0, \qquad i = 1, 2,$$

where $m_i p_i \delta_i - 2 > 0$. Then (6.5) reads

$$2p_0(m_1p_1\delta_1 - 2)(m_2p_2\delta_2 - 2) \geqslant (m_1p_1\delta_1 - 2)m_2^2p_2 + (m_2p_2\delta_2 - 2)m_1^2p_1.$$

Combining these inequalities we obtain

$$\frac{m_1^2p_1}{m_1p_1\delta_1 - 2} > 2p_0 \geqslant \frac{m_2^2p_2}{m_2p_2\delta_2 - 2} + \frac{m_1^2p_1}{m_1p_1\delta_1 - 2} > \frac{m_1^2p_1}{m_1p_1\delta_1 - 2}.$$

The contradiction proves the lemma.

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