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RATE OF CONVERGENCE IN THE "CIRCLE FORMATION" PROBLEM

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Abstract. We consider transformations obtained from local homogeneous rules of motion of polygons in the plane for which regular polygons are stationary (the so-called "circle formation" problem). These transformations are studied for initial states that are close to regular polygons. A method of determination of the rate of convergence to regular polygons is presented; on this basis we obtain estimates of the highest rate of convergence.

Figures: 1. Bibliography: 4 items.

Introduction

The "circle formation" problem was formulated in [7]. This problem deals with the motion of polygons in the plane in accordance with local homogeneous rules of motion of their vertices that are invariant with respect to motions of the plane. A rule of motion is specified by a vector function $f(r_{-k}, \cdots, r_0, \cdots, r_k)$ that depends on $2k+1$ vectors $r_{-k}, \cdots, r_0, \cdots, r_k$ and is invariant with respect to motions of the plane (see (1.3)–(1.4)). It is possible to consider either the case of discrete time, when the position of a polygon with vertices $A_1, \cdots, A_n$ at the instant $t+1$ is determined on the basis of the position of the polygon at the instant t as follows:

$$A_i(t+1) = f(A_{i-k}(t), \ldots, A_i(t), \ldots, A_{i+k}(t)), \; i = 1, \ldots, n, \qquad (*)$$

or the case of continuous time, when the following system of differential equations is studied:

$$\frac{d}{dt} A_i(t) = f(A_{i-k}(t), \ldots, A_i(t), \ldots, A_{i+k}(t)), \; i = 1, \ldots, n. \qquad (**)$$

Both these case are analyzed in almost the same way (though for continuous time we have some minor simplifications). For definiteness, in the Introduction we shall consider the case of continuous time only.

Thus the motion of a vertex will depend on its position and the position of a small number (k from the left and k from the right) of neighboring vertices (the local property),

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the rules of motion are the same for all the vertices (the homogeneity property), and the motion of a vertex relative to its neighbors depends only on its position with respect to $2k$ neighboring vertices (invariance of the rules with respect to motions of the plane).

In [1], various examples were considered of rules of vertex motion. In all these examples the vertex A_i moves to a point A_i' that is equidistant from the neighboring vertices A_{i-1} and A_{i+1}. The point A_i' is taken differently for different rules. It can be a point whose distance to the vertices A_{i-1} and A_{i+1} is equal to a prescribed number d, or a point such that the angle $\angle A_{i-1}A_i'A_{i+1}$ is equal to $\pi - 2\pi/n$, where n is the number of vertices of the polygon, or such that this angle is equal to the average of the angles at the vertices A_{i-1}, A_i and A_{i+1}, or such that the cosines of these angles are averaged, or a point A_i' that lies on a circle of given radius R that passes through the vertices A_{i-1} and A_{i+1}, or a point that halves the segment connecting the points equidistant from the vertices A_{i-1} and A_{i+1} and lying on two circles, one of which passes through the vertices A_{i-2}, A_{i-1} and A_{i+1}, and the other through the vertices A_{i-1}, A_{i+1} and A_{i+2} (for more details see §5). In [1], computer simulation results are presented for these rules and their linear combinations. For all the initial states used in the simulation, with appropriate choice of the parameters, convergence took place to a regular polygon, i.e., "circle formation."

A polygon with n vertices $r_1, \cdots, r_n$ can be represented by a vector $r = (r_1, \cdots, r_n)$ of $2n$-dimensional space R^{2n}. Then the rules of motion (**) will specify in R^{2n} a dynamical system T_t (just as (*) specifies in R^{2n} a transformation T). The question arises of the behavior of the trajectories of this dynamical system, and also of the form of the overall pattern. It is evident that the answer will be strongly dependent on the function f. Of special interest are functions f such that for $t \longrightarrow \infty$ almost all the trajectories converge to points corresponding to regular polygons. With such rules of motion, the polygons tend more and more to regular polygons for almost all initial positions, i.e. we have "circle formation."

In studying a dynamical system, we must first of all find fixed points in R^{2n}, i.e. stationary polygons. In general, the form of such polygons for some function f can be most diverse; moreover, such polygons may not exist at all. However, it follows from the homogeneity of the rules and their invariance with respect to motions that there exist natural models of stationary polygons, namely starlike regular polygons, i.e. polygons (possibly self-intersecting) in which all the sides are the same and all the angles are the same. It is evident that even if such polygons are not stationary, they will in any case go over into self-similar polygons.

A starlike regular polygon is determined by the length of its sides and the magnitude of the angle between neighboring sides. This angle can be equal to $\pi - 2\pi l/n$, $l = 1, 2, \cdots, [n/2]$; the vertices can be numbered either clockwise or counter-clockwise. For $l = 1$ the polygon will be regular. For an $l > 1$ that is not a divisor of n the polygon will be self-intersecting. If l is a divisor of n, the polygon will be an n-polygon with n/l distinct vertices forming a regular n/l-polygon, and at each of these vectices we have l merging vertices of an n-polygon. The set of points of P^{2n} that are starlike regular polygons with vertex angles equal to $\pi - 2\pi l/n$, $l = 1, \cdots, [n/2]$, will be denoted by V_l^4 if the vertices are numbered clockwise, and by V_{n-l}^4 if they are numbered counter-clockwise. It is easy to see that all these sets are four-dimensional subspaces of R^{2n}. It is evident that $V_l^4 = \mathbf{U}_{d>0} V_l^3(d)$, $i = 1, \cdots, n-1$, where $V_l^3(d)$ denotes a three-dimensional manifold of points of $V_l^4(d)$ that are polygons with a side length d.

For many functions f it is easy to prove that all stationary polygons are starlike regular polygons. If the function f is invariant not only with respect to motions, but also to stretching (for more details, see §1) we shall say that the rule is dimensionless and assume that the set $\mathfrak{N}$ of fixed points of the dynamical system consists of a number of subspaces V_l^4 (in all the available examples we have either $\mathfrak{N} = V_1^4$, or $\mathfrak{N} = \mathbf{U}_1^{n-1} V_l^4$, or $\mathfrak{N} = \mathbf{U}_1^{[(n-1)/2]} V^4$). If the rule of motion is not dimensionless, we shall assume that $\mathfrak{N}$ consists of a number of manifolds $V_l^3(d_l)$. For definiteness we shall henceforth consider dimensionless rules of motion.

For all l such that $V_l^4 \subset \mathfrak{N}$ let us consider a stable manifold Γ_l^+ and an unstable manifold Γ_l^- (i.e. sets of points ρ of R^{2n} such that the trajectory $T_t\rho$ converges to V_l^4 for $t \longrightarrow +\infty$ in the case of Γ_l^+, and for $t \longrightarrow -\infty$ in the case of Γ_l^-). It is of interest to study the mutual position of these manifolds, and obtain Smale's diagram [2].

First of all it is necessary to find the dimension of these manifolds. For this purpose it is useful to study a linear system of differential equations that is a linearization of the system (**) in the vicinity of the points of V_l^4, and find its eigenvalues. As a result of the invariance of dimensionless rules with respect to motion and stretching, four eigenvalues of the linearized system must be equal to zero (for dimensionless rules, three eigenvalues must always be equal to zero). Suppose that the real parts of the other eigenvalues are nonzero (precisely this is the general case); thus let the number of eigenvalues with positive real part be equal to ρ_l^-, and with negative real part to ρ_l^+, so that $\rho_l^- + \rho_l^+ = 2n - 4$. (It follows from the invariance of the rules with respect to motion and stretching that the eigenvalues, and hence also the numbers ρ_l^- and ρ_l^+, depend only on l, and not on the neighborhood of which point of V_l^4 the linear system is studied.) It is easy to see [3] that $\dim \Gamma_l^\pm = \rho_l^\pm + 4$.

Thus it is important to be able to find the eigenvalues of the corresponding linear systems. This can be done as follows (§§3 and 4). In $R^{2n} = R^2 \times \cdots \times R^2$ let us consider the following coordinate system: The lth component r_l of the vector $r = (r_1, \cdots, r_n) \in R^{2n}$ will be written in a coordinate system in the plane, obtained from a fixed coordinate system by rotation by an angle $2\pi l/n$, $l = 1, \cdots, n-1$. It easily follows from the homogeneity of the rules that the $2n \times 2n$ matrix A of any such linear system will consist of n^2 matrices B_{ij} (of dimension 2×2), and that all the matrices standing in the same diagonal (more precisely, for which the difference between the numbers i and j modulo n is the same) are the same; if $i - j \equiv l$ (modulo n), then

$$B_{ij}=B_l=\begin{pmatrix} b_l^{11} & b_l^{12} \\ b_l^{21} & b_l^{22} \end{pmatrix}.^{1)}$$

It hence easily follows that the matrix A commutes with the matrix $\mathfrak{A}$ in which for $l \neq 1$ we have

$$B_l=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } B_1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.^{2)}$$

The matrix $\mathfrak{A}$ has n distinct eigenvalues, to each of which there corresponds a two-dimensional (complex) proper subspace. From the commutativity with $\mathfrak{A}$ it follows that these two-dimensional subspaces will be invariant for all the matrices for which the matrices B_{ij} depend only on the difference $i-j$ (modulo n). Therefore the problem of finding the eigenvalues of the matrix A reduces to finding the eigenvalues of matrices acting in these two-dimensional subspaces, i.e. to the solving of quadratic equations.

Of particular interest is the case that all the eigenvalues, apart from four, have negative real parts, i.e. $\rho_l^+ = 2n-4$ and $\rho_l^- = 0$. In this case (in any event for points ρ that are close to V_l^4) we have $T_t\rho \longrightarrow V_l^4$ for $t \to \infty$. If this occurs for $l=1$, then the polygons that are close to regular ones will become even closer, i.e. we are solving the "circle formation" ("rounding") problem. The quantity $\mathrm{Re}\lambda_{\max}$, where $\lambda_{\max}$ is the eigenvalue with a negative real part that is maximal, characterizes the rate of convergence to a regular polygon (see [3]); the more this quantity differs from zero, the faster will be the convergence. In §5 we obtain for the examples under consideration an asymptotic formula for the rate of convergence for $n \to \infty$. This asymptotic expression is of great interest, since the values of n under consideration are fairly large (of the order of 20 or 100), and it can be assumed that for such n the asymptotic formulas will yield values close to the true values. The obtained rates of convergence are in good agreement with the computer simulation results [1]; in some cases the asymptotic expression for the rate of convergence was predicted on the basis of the simulation results. (Let us only note that for polygons that are very unlike regular polygons the rate of convergence is often higher. At first a fast process of "crude rounding" is taking place, and then, when the system is close to linear, we have a slow process of "precise rounding" [1]).

In §§6–8 we obtain asymptotic estimates of the rate of convergence. In §6 an asymptotic expression of this rate is obtained for a fixed function f for $n \to \infty$. In §7 we obtain an estimate of the maximum value of the rate of convergence for a fixed (though very large) n. In §8 we study in more detail the case $k=1$, when the dependence on each of the neighbors is the same.

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1) Let us note that in the case of a transformation T the matrix of the corresponding linear transformation will have the same property.

2) This commutativity is an algebraic manifestation of the geometrical fact that the dynamical system T_t is invariant with respect to cyclic replacment of indices

$$(r_1,\dots,r_n) \to (r_i,\dots,r_n,r_1,\dots,r_{i-1}).$$

Let us formulate an hypothesis with regard to the location of the manifolds Γ_l^+ and Γ_l^-. This will be done for the case that the cosines of the angles are averaged, although it seems to us that this hypothesis (perhaps with some modifications) is of a general character. For this rule we have $N = \bigcup_1^{[(n-1)/2]} V_l^4$. It is possible to prove (see §5) that $\rho_l^- = 2(l-1)$. In particular, the points of attraction will be only points belonging to V_1^4 (i.e. regular polygons). It is natural to assume (although this has not been proved at all) that $T_t\, \rho \to V_1^4$ for $t \to \infty$ for all points $\rho \in R^{2n} \setminus \bigcup_{l \geq 2} \Gamma_l^+$.

Let us assume that this hypothesis is true. It would seem that in this case for $t \to \infty$ all the trajectories of the manifold Γ_l^- must converge to V_1^4. It would hence follow that for $s \neq l$ the subspaces V_s^4 do not intersect with Γ_l^-.

But this is not the case. Indeed, let us consider a subspace W_d (d being a divisor of the number n) consisting of polygons with n/d distinct vertices at each of which d vertices are merging, i.e. $W_d = \{r = (r_1, \cdots; r_m): r_i = r_{i+n/d},\ i = 1, \cdots, n\}$. It is evident that all these subspaces are invariant with respect to T_t. In fact, in a subspace W_d (of dimension $2n/d$) there acts the same dynamical system as in T_t, with the only difference that n has been replaced by n/d. Next, $V_{ld}^4 \subset W_d$, $l = 1, \cdots, n/d - 1$. It is evident that for a dynamical system bounded on W_d, the quantity V_d^4 plays the same role as V_1^4 for the entire dynamical system T_t. Hence if all the manifolds Γ_l^- contain V_1^4, the manifolds Γ_{ld}^-, $l = 1, 2, \cdots$, will contain the subspace V_d^4.

Therefore it is justifiable to assume that Γ_s^- will contain V_d^4 if and only if both n and s are divisible by d, and that Γ_s^+ will contain V_j^4 if and only if n and j are divisible by s. However, we have not been able to prove this.

The "circle formation" problem has arisen in connection with the biological process of morphogenesis [4]. It seems, however, that the "circle formation" problem yields also an interesting example of a dynamical system (or transformation); it is therefore useful to study it thoroughly (the same applies also to the "rectification" problem [4], and in general to problems with local homogeneous rules).

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§1. Transformation rule

Let us consider a vector function $f(r_{-k}, \cdots, r_0, \cdots, r_k)$ that depends on $2k+1$ vectors $r_{-k}, \cdots, r_0, \cdots, r_k$ (all the vectors are assumed to belong to the plane R^2). Such a function f specifies the rules of motion of the vertices of closed polygons. The time can be either discrete or continuous. In the case of discrete time we assign a rule of transformation of polygons. Thus the position of the vertices of the n-polygon $A_1 \cdots A_n$ at the instant $t+1$ will depend on the position of the vertices at the instant t as follows:

$$A_i(t+1) = f(A_{i-k}(t), \ \ldots, \ A_{i+k}(t)).^{3)} \tag{1.1}$$

3) The subscript i is defined modulo n; in the following this will not be especially mentioned.

For continuous time the transformation rule (1.1) is replaced by a dynamical system described by the following system of differential equations:

$$\frac{d}{dt} A_i(t) = f(A_{i-k}(t), \ \ldots, \ A_{i+k}(t)). \tag{1.2}$$

The formulas (1.1) and (1.2) specify the rules of motion of the vertices of any polygon. These rules are homogeneous, i.e. they are the same for all the vertices; they are also local, since the motion of a vertex depends only on the positions of this vertex and its nearest neighbors (k from the right and k from the left). For the motion of a vertex with respect to its neighbors to depend only on the mutual position of the vertex and its neighbors, without depending on its position in the plane, the function f will be everywhere assumed to be invariant with respect to motions (without reflections) in the following sense: For any vectors $r_{-k}, \cdots, r_k$, ρ, and any rotation A we have

$$f(r_{-k} + \rho, \ \ldots, \ r_k + \rho) = f(r_{-k}, \ \ldots, \ r_k) + \rho, \tag{1.3}$$

$$f(Ar_{-k}, \ \ldots, \ Ar_k) = Af(r_{-k}, \ \ldots, \ r_k). \tag{1.4}$$

The problem under consideration and the function f are said to be *dimensionless* if A in (1.4) can be a similarity transformation (without reflections), and *symmetrical* if A is a rotation and reflection. For a symmetrical problem the motion of a vertex will depend equally on the right and left neighbors.

In §5 we present examples of various functions f. By taking all their possible linear combinations, we obtain a sufficently large set of various rules of motion of polygons. In all the examples under consideration the function f is symmetrical.

For each discrete-time problem (with a function f) it is possible to find a continuous-time problem with a function $f(r_{-k} \cdots, r_0, \cdots, r_k) - r_0$, and by considering a family of discrete-time problems with functions $(1 - \sigma)r_0 + \sigma f(r_{-k}, \cdots, r_0, \cdots, r_k)$, we can see that for $\sigma \to 0$ such problems approximate the corresponding continuous-time problem.

In the following we shall consider only the case of the transformation (1.1). The system (1.2) can be studied in the same way, though sometimes this involves minor simplifications.

The totality of vectors $r_{-k}, \cdots, r_k$, i.e. a $(2k + 1)$-sectional plane broken line, can be represented as a point of $(4k + 2)$-dimensional space $R^{4k+2} = R^2 \times \cdots \times R^2$. Therefore the function f is defined on R^{4k+2}, taking its values in R^2, i.e. f: $R^{4k+2} \to R^2$. Here it is possible that the function f is defined not on the entire space R^{4k+2}. In all the available examples the function f is defined if $r_1 \neq r_{-1}$. It is often sufficient (for example, in studying the asymptotic behavior for $n \to \infty$) to require that the function f be defined in a neighborhood of points of R^{4k+2} representing broken lines whose vertices lie on the same straight line at equal distances from one another; any vertex lies between neighboring vertices, so that the angles between neighboring sections are equal to π, and not 0.[4)]

4) Here and in the following all the angles are in a counter-clockwise direction.

To each ordered set of vectors $r_1, \cdots, r_n$, i.e. to each plane closed n-polygon (in which some of the vertices may coincide) it is possible to assign in a natural manner a vector of the $2n$-dimensional space $R^{2n} = R^2 \times \cdots \times R^2$; conversely, any vector of the space $R^{2n} = R^2 \times \cdots \times R^2$ can be represented by an n-polygon. Then the rules of motion of the vertices expressed by formula (1.1) will define a transformation T acting in R^{2n}.[5] This transformation acts not on the entire space R^{2n}, but on those parts for which the right-hand side in (1.1) is defined. Let us denote by $\mathfrak{M}$ the set of points of R^{2n} for which the right-hand side in (1.1) is not defined. In all the examples under consideration we have

$$\mathfrak{M} = \bigcup_{i=1}^{n} \mathfrak{M}_i, \text{ where } \mathfrak{M}_i = \{r = (r_1, \ldots, r_n) : r_{i-1} = r_{i+1}\};$$

on $\mathfrak{M}_i$ the rule of motion of the ith vertex is not defined. In the open manifold $R^{2n} \setminus \mathfrak{M}$ the transformation T is everywhere defined and continuous. In general the transformation T cannot be extended in a continuous manner to the entire space R^{2n}. Let us also note that T is not necessarily a one-to-one mapping (yet if T is close to a dynamical system in the above sense, then it will be a one-to-one mapping).

In the space R^{2n} there acts a group $\mathfrak{G}$ induced by the group of motions (without reflections) of the plane R^2 (the elements of this group are likewise called motions). In fact, to any motion g of the plane R^2 there corresponds a transformation G by the following rule:

$$G(r_1, \ldots, r_n) = (gr_1, \ldots, gr_n). \tag{1.5}$$

From (1.3) and (1.4) it follows that the action of the group $\mathfrak{G}$ commutes with the transformation T. If g in (1.5) can represent not only the motion of the plane, but any similarity transformation (without reflection), we shall denote the resulting group by $\tilde{\mathfrak{G}}$;[6] it is evident that for a dimensionless problem the actions of this group commute with T. The groups $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ fiber the space R^{2n} into orbits, all of which are three-dimensional for the group $\mathfrak{G}$ and four-dimensional for the group $\tilde{\mathfrak{G}}$, apart from a single two-dimensional orbit consisting of points of the form $(r, r, \cdots, r)$.

Since $\mathfrak{G}$ and T commute, it follows that T carries orbits into orbits. Hence it can be assumed that the transformation T acts on a $(2n-3)$-dimensional (open) manifold $M^{2n-3} = (R^{2n} \setminus \mathfrak{M})/\mathfrak{G}$. Such a point of view is often useful. For example, in R^{2n} the set of fixed points for T is necessarily a union of $\mathfrak{G}$-orbits; hence there cannot be any isolated fixed points. On the other hand, the fixed points in M^{2n-3} are in general isolated points. In the same way, for a dimensionless problem it is useful to represent the transformation T acting on a $(2n-4)$-dimensional manifold $M^{2n-4} = (R^{2n} \setminus \mathfrak{M})/\tilde{\mathfrak{G}}$.[7]

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5) Similarly, formula (1.2) will specify in R^{2n} a dynamical system T_t with continuous time.

6) In a similar way we shall define the action of the groups $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ in the space R^{4k+2}.

7) The homogeneity of the rules causes T to commute with a group of transformations $\mathbf{Z}_n$ consisting of cyclic permutations γ_j: $(r_1, \cdots, r_n) \to (r_{1+j}, \cdots, r_n, \cdots, r_j)$, $j = 0, 1, \cdots, n-1$.

Our task is to study the transformation T. In [1], numerical results were obtained for particular cases, and several theorems were proved. In the present paper this problem is studied from a more general point of view. Unfortunately we were unable to obtain any global results. This paper is devoted to a study of the transformation T in the vicinity of fixed points.

§2. Fixed points

Thus we must find at first the fixed points, i.e. stationary polygons. By virtue of the homogeneity of the rules it is natural to seek the stationary polygons among the "locally regular" polygons, in which all the angles and all the sides are equal. Such polygons will be called *quasiregular* polygons. From the homogeneity of the rules and formulas (1.3)–(1.4) it is evident that even if such polygons are not stationary, they are in any case transformed into similar polygons.[8] Therefore it is important to describe all quasiregular polygons.

This can be easily done (see [1]). The vertices of any such polygonal line lie on the same circle as the vertices of a regular polygon (we shall assume that in this polygon the vertices are numbered in a clockwise direction), though their numeration is not the usual one, i.e., between vertices that are neighbors in a polygonal line, we have in a regular polygon the $(s-1)$th vertex, where $s = 1, 2, \cdots, n-1, n$. Such polygonal lines are called quasiregular polygons of order s. For $s = 1, \cdots, n-1$, they form in R^{2n} a four-dimensional manifold (subspace) $V_s^4 = \bigcup_{0 \le d < \infty} V_s^3(d)$, where $V_s^3(d)$ is a three-dimensional (for $d > 0$) manifold of quasiregular polygons of order s with a side length equal to d. Each manifold $V_s^3(d)$ $(s = 1, 2, \cdots, n-1, n;\ 0 \le d < \infty)$ is an orbit of the group $\mathfrak{G}$, and for $d > 0$ all of them are nonintersecting. The manifolds $V_s^3(0)$, as well as V_n^4, represent the same two-dimensional subspace V^2 formed by points $(r, r, \cdots, r)$; as noted above, they constitute a singular orbit.

For $s = 1$ and $s = n-1$, quasiregular polygons are regular; for $s = 1$ the vertices are numbered clockwise, and for $s = n-1$ they are numbered counter-clockwise (in the following, only quasiregular polygons of first order will be called regular polygons). For $s = n$, all the vertices merge. If s is divisible by n, a quasiregular polygon will form a regular n/s-polygon traversed s times, i.e. s vertices of an n-polygon will merge at each vertex of this n/s-polygon. If $1 < s < n-1$ and $(s, n) = 1$, the polygon will be a starlike regular (self-intersecting) polygon. For $(s, n) = d$, d vertices of the polygon will merge into one vertex of a quasiregular polygon of order s/d that has n/d vertices; for $d < s < n-d$ this polygon with n/d vertices will be a starlike regular (self-intersecting) polygon. Quasiregular polygons of order s and order $n-s$ are

8) Hence in the case of a dimensionless problem a point of M^{2n-4} that is a quasiregular polygon will be a fixed point; but in R^{2n} such a point is not necessarily fixed, although it goes over into a point of its $\widetilde{\mathfrak{G}}$-orbit. In this connection let us note that to each fixed point of R^{2n} there corresponds a fixed point of M^{2n-3} (or M^{2n-4}). The converse may not be true. But if on a $\mathfrak{G}$-orbit ($\widetilde{\mathfrak{G}}$-orbit), corresponding to a fixed point of M^{2n-3} (M^{2n-4}) we have at least one fixed point, then all the points of this orbit will be fixed.

mirror symmetrical. The vertex angles of a quasiregular polygon of order s are equal to $\pi - 2\pi s/n$ for $s < n/2$, and to $\pi + 2\pi(n-s)/n$ for $s > n/2$. Let us note that the subspaces V^2 and $V^4_{n/2}$ lie in the intersection of the $\mathfrak{M}_i$.

We shall be interested in the case that only quasiregular polygons can be stationary. It is assumed that in the manifold M^{2n-3} (or M^{2n-4} in the case of a dimensionless problem) the fixed points are isolated points. Thus in the case of a dimensionless problem it is assumed that the set $\mathfrak{N}$ of fixed points of the transformation T consists of a number of subspaces V^4_s. One of them must be the subspace V^4_1, i.e. all the regular polygons are stationary. In the general case it will be assumed that $\mathfrak{N}$ consists of a finite number of manifolds $V^3_s(d_s)$, one of which is the manifold $V^3_1(d)$ for some positive d.

Let us note that the function f can be such that the set $\mathfrak{N}$ has the above-described form for all n, or for only some n, for example, one value of n, or even for no value of n at all.

In Table 1 of §5 we shall list the form of the sets $\mathfrak{N}$ for the examples under consideration.

§3. Linearization

For studying the transformation T in the neighborhood of a fixed point $r \in \mathfrak{N} \subset R^{2n}\setminus\mathfrak{M}$, it is necessary to consider a linear transformation A_r that is a linearization of T in the neighborhood of the point r, i.e.

$$A_r\rho = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(T(r+\varepsilon\rho) - r\right), \quad \rho \in R^{2n}. \tag{3.1}$$

It is assumed that this limit exists; for this we must require that the function f be continuously differentiable. In all the available examples the function f is differentiable (even infinitely many times) at all the points, apart from a finite number of manifolds of codimension 1.

For studying the behavior of the transformation T in the neighborhood of the point r, we must first of all find the eigenvalues of A_r. It is evident that if r^1 and r^2 belong to the same $\mathfrak{G}$-orbit (or $\tilde{\mathfrak{G}}$-orbit in the case of a dimensionless problem), the transformations A_{r^1} and A_{r^2} will be conjugate, and hence they will have the same eigenvalues. Since by virtue of the assumptions made in §2 the set $\mathfrak{N}$ consists of finitely many orbits, we must calculate the eigenvalues of finitely many operators. Next, it is evident that the tangents to $\mathfrak{N}$ (i.e. to the orbit) of a vector are invariant with respect to A_r. Therefore several eigenvalues, i.e. four for the dimensionless problem and three in the general case, must be equal to unity.

Let $r^0 \in \mathfrak{N}$ and let V_1 and V ($V_1 \subset V$) be sufficiently small neighborhoods of the point r^0 (such that V intersects with only one connectivity component of the set $\mathfrak{N}$). As was noted above, all the operators A_r, $r \in \mathfrak{N} \cap V$, have the same eigenvalues. Let us denote by Λ the set of all eigenvalues, apart from those whose corresponding eigenvectors are tangent to $\mathfrak{N}$ [9] (as we noted above, the eigenvalues that do not belong to Λ must be equal to unity). Let $\lambda_{\max}$ be the maximum eigenvalue of this set. Next let us denote by Γ a set of points $\rho \in V_1$ such that $T^t\rho \in V$ for all integers $t \geq 0$, and by $\Gamma^+ \subset \Gamma$ a set of points $\rho \in V_1$ such that $T^t\rho \in V$ for all $t \geq 0$, and $T^t\rho \to \mathfrak{N}$ for $t \to \infty$ (so that for all the points of Γ^+ the trajectories originating at them will tend to $\mathfrak{N} \cap V$, whereas all the trajectories beginning in $V_1 \setminus \Gamma$ will originate at V).

After these remarks we evidently obtain (see [3])

Theorem 3.1. 1) *Let s_1 be the number of eigenvalues in Λ that are larger than* 1, *and s_2 the number of eigenvalues smaller than* 1. *Then Γ^+ will contain an open subset of a manifold of dimension $s_2 + 3$ ($s_2 + 4$ in the case of a dimensionless problem), whereas Γ is contained in a manifold of dimension $2n - s_1$. If $s_1 + s_2 = 2n - 3$ ($2n-4$ in the case of a dimensionless problem),*[10] *then $\Gamma^+ = \Gamma$, and the trajectories $T^t\rho$ for all $\rho \in V$ apart from an open subset of a manifold of dimension $2n - s_1$ will originate at V. In particular, if $s_1 = 0$, all the trajectories $T^t\rho$, $\rho \in V_1$, will tend to $\mathfrak{N} \cap V$ when $t \to \infty$.*

2) *Let $\lambda_{\max} < 1$ (i.e. $s_1 = 0$). Then the quantity $\lambda_{\max}$ will specify the rate of convergence of the trajectories $T^t\rho$ to $\mathfrak{N}$, i.e. for any positive ϵ there exists a constant c dependent on ρ and ϵ, but not on t, such that the distance between the vector $T^t\rho$ and $\mathfrak{N}$ does not exceed $c(|\lambda_{\max}| + \epsilon)^t$. If the maximum eigenvalue is simple, it is possible to set ϵ equal to zero, and for almost all $\rho \in V$ the distance between $T^t\rho$ and $\mathfrak{N}$ will be asymptotically (for $t \to \infty$) proportional to $|\lambda_{\max}|^t$.*

Remark. A similar theorem holds also for a dynamical system, with the only difference that instead of equating $|\lambda_{\max}|$ to one, we must equate $\mathrm{Re}\lambda_{\max}$ to zero; here $\lambda_{\max}$ is an eigenvalue belonging to Λ that has maximal real part; we must also replace $|\lambda_{\max}|^t$ by $\exp\{t \cdot \mathrm{Re}\lambda_{\max}\}$. Let us also note that in going over from a transformation to the corresponding dynamical system (as described in §1), the eigenvalue λ is replaced by $\lambda - 1$. It is likewise evident that in the transformations approximating a dynamical system (with functions $(1 - \sigma)r_0 + \sigma f$), the quantity λ is replaced by $1 - \sigma + \sigma\lambda$; hence if $\mathrm{Re}\lambda < 1$, then in the case of a sufficiently small σ the corresponding eigenvalue will be smaller than 1 (even if $|\lambda| \geq 1$); only the eigenvalues λ with $\mathrm{Re}\lambda > 1$ do not permit convergence for any positive σ.

It is evident from this theorem that the quantity $\tau = \tau(f) = -1/\ln|\lambda_{\max}|$ is very characteristic, i.e., if $|\lambda_{\max}| < 1$, then τ will be the time during which $\mathfrak{N}$ is approximated roughly e times; hence it is natural in our case to say that τ is the "rounding time." If $|\lambda_{\max}| \sim 1$, then $\tau \sim (1 - |\lambda_{\max}|)^{-1}$.

Thus it is important to be able to find the eigenvalues of the transformation A_r. For this purpose let us find the matrix A of this transformation. Let $r^0 \in V_s^3(d) \subset V_s^4$, V denoting a neighborhood of the point r^0. In the region V we shall use the following

9) Thus Λ is the set of all eigenvalues of the corresponding linear operator acting in a space tangent to the manifold M^{2n-3} or M^{2n-4}.

10) So that the absolute values of all the eigenvalues belonging to Λ are distinct from 1.

system of coordinates. To a point $r^0 \in \mathfrak{R}$ there corresponds a quasiregular polygon of order s with vertices $r_1^0, \cdots, r_n^0$ that has its center at the point O. In the plane let us consider the following n systems of coordinates: In the ith system of coordinates the origin coincides with the point r_i^0, the first unit vector $\mathfrak{h}_{i,1}$ is perpendicular to the ray Or_i^0, and the second unit vector $\mathfrak{h}_{i,2}$ is directed along this ray, the angle between the first and the second unit vector being equal to $\pi/2$ (and not $3\pi/2$; see footnote 4). Thus the ith coordinate system is rotated with respect to the first system by an angle $2\pi(i-1)/n$. Suppose that to a vector $r \in V$ there corresponds in the plane a polygon with vertices $r_1, \cdots, r_n$. Let us denote the coordinates of the vector r_i in the ith coordinate system by α_i and β_i. Then we can take as the coordinates of the vector r the numbers $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$. Thus we have described the coordinate system in V. The origin of coordinates coincides with the point r^0. The unit vectors of this system will be denoted by $h_{i,1}$ and $h_{i,2}$, $i = 1, \cdots, n$.

Let us find the form of the matrix A in this coordinate system. Let $A = \|B_{ij}\|$, i, $j = 1, \cdots, n$, where

$$B_{ij} = \begin{pmatrix} b_{ij}^{11} & b_{ij}^{12} \\ b_{ij}^{21} & b_{ij}^{22} \end{pmatrix}$$

is a 2×2 matrix. It easily follows from the homogeneity of the rules that all the matrices B_{ij} for which $i - j \equiv l$ (modulo n) are equal:

$$B_{ij} = B_l = \begin{pmatrix} b_l^{11} & b_l^{12} \\ b_l^{21} & b_l^{22} \end{pmatrix}.$$

Matrices possessing these properties will be called *2-circulants* (since a matrix $B = \|b_{ij}\|$, $i, j = 1, \cdots, n$, in which $b_{ij} = b_l$ for $i - j \equiv l$ (modulo n), is called a circulant).[11] Thus it follows from the homogeneity of the rules that the matrix A is a 2-circulant.

Since the rules are local, it follows that

$$b_l^{\nu\mu} = 0 \quad \text{for} \quad |l| > k, \quad \nu, \mu = 1, 2. \tag{3.2}$$

Next, the vectors tangent to $V_s^3(d)$ (V_s^4 in the case of a dimensionless problem) are invariant with respect to A_r. It is evident that the vectors tangent to $V_s^3(d)$ will be all possible linear combinations of vectors ζ_0, ζ_s and $\hat{\zeta}_s$, whereas the vectors tangent to V_s^4 will be linear combinations of vectors ζ_0, $\hat{\zeta}_0$, ζ_s and $\hat{\zeta}_s$, where

$$\zeta_0 = (1, 1, \ldots, 1, 0, 0, \ldots, 0),$$

$$\hat{\zeta}_0 = (0, 0, \ldots, 0, 1, 1, \ldots, 1),$$

$$\zeta_s = \left(1, \cos\frac{2\pi s}{n}, \ldots, \cos\frac{2\pi(n-1)s}{n}, 0, \sin\frac{2\pi s}{n}, \ldots, \sin\frac{2\pi(n-1)s}{n}\right),$$

$$\hat{\zeta}_s = \left(0, -\sin\frac{2\pi s}{n}, \ldots, -\sin\frac{2\pi(n-1)s}{n}, 1, \cos\frac{2\pi s}{n}, \ldots, \cos\frac{2\pi(n-1)s}{n}\right)$$

11) A matrix $\|B_{ij}\|$, $i, j = 1, \cdots, n$, where B_{ij} are $s \times s$ matrices, such that B_{ij} depends only on the difference $i - j$ modulo n, will be called an s-circulant.

(to the vectors ζ_s and $\hat{\zeta}_s$ there correspond polygons obtained from r^0 by parallel shift, to the vector ζ_0 there correspond polygons obtained by rotation, and to the vector $\hat{\zeta}_0$ there correspond polygons obtained by stretching). This yields the following conditions:

$$\sum_{l=-k}^{k} b_l^{11} = 1, \qquad \sum_{l=-k}^{k} b_l^{21} = 0, \tag{3.3}$$

$$\sum_{l=-k}^{k} b_l^{11}\cos\frac{2\pi ls}{n} + \sum_{l=-k}^{k} b_l^{12}\sin\frac{2\pi ls}{n} = 1, \quad \sum_{l=-k}^{k} b_l^{21}\cos\frac{2\pi ls}{n} + \sum_{l=-k}^{k} b_l^{22}\sin\frac{2\pi ls}{n} = 0,$$

$$-\sum_{l=-k}^{k} b_l^{21}\sin\frac{2\pi ls}{n} + \sum_{l=-k}^{k} b_l^{22}\cos\frac{2\pi ls}{n} = 1, \quad -\sum_{l=-k}^{k} b_l^{11}\sin\frac{2\pi ls}{n} + \sum_{l=-k}^{k} b_l^{12}\cos\frac{2\pi ls}{n} = 0; \tag{3.4}$$

for a dimensionless problem we have the additional conditions

$$\sum_{l=-k}^{k} b_l^{22} = 1, \qquad \sum_{l=-k}^{k} b_l^{12} = 0. \tag{3.5}$$

It is easy to see that in the case of a symmetrical problem we have

$$b_{-l}^{11} = b_l^{11}, \quad b_{-l}^{22} = b_l^{22}, \quad b_{-l}^{12} = -b_l^{12}, \quad b_{-l}^{21} = b_l^{21}. \tag{3.6}$$

It is easy to prove also the converse, i.e., if a matrix A is a 2-circulant of order $2n$ and it satisfies the conditions (3.2)–(3.4), then there exists a function $f(r_{-k}, \cdots, r_k)$ that satisfies the conditions (1.3)–(1.4) and such that a quasiregular n-polygon of order s will be stationary and the matrix of a linearized transformation in the neighborhood of this polygon will be the matrix A. If, moreover, the condition (3.5) is also satisfied, then the function f will be dimensionless, whereas if (3.5) holds it will be symmetrical.

With the aid of (3.1) it is possible to express the coefficients of the matrix A in terms of the function f. After simple calculations we obtain

$$b_l^{\nu\mu} = (\mathfrak{F}\mathfrak{h}_{l,\mu}, \mathfrak{h}_{l,\nu}), \qquad \nu, \mu = 1, 2, \quad l = -k, \ldots, k, \tag{3.7}$$

where $\mathfrak{F} = (\partial f/\partial r_l)|_{r_{-k}^0, \cdots, r_k^0}$ is a matrix (of second order, since f and r_l are vectors in R^2) constructed from the first-order partial derivatives of the function f with respect to the lth vector variable, taken for $r_j = r_j^0$, $j = -k, \cdots, k$ where (as mentioned above) the points $r_{-k}^0, \cdots, r_{-k}^0$ form $2k+1$ successive vertices of a regular polygon of order s, and $\mathfrak{h}_{l,1}$ and $\mathfrak{h}_{l,2}$ are the unit vectors of the above-contructed lth coordinate system in the plane. Let us recall that $f(r_{-k}^0, \cdots, r_k^0) = r_0^0$.

§4. Calculation of eigenvalues

In this section we shall describe a method of calculation of the eigenvalues of a 2-circulant.[12)]

12) Theorem 4.1 is well known to experts. Unfortunately the author was unable to find a reference. In view of the importance and simplicity of this theorem, we shall present its proof in full.

Theorem 4.1. *The eigenvalues of the 2-circulant A are $\lambda_{l,1}$ and $\lambda_{l,2}$, $l = 0, 1, \cdots, n-1$, where $\lambda_{l,1}$ and $\lambda_{l,2}$ satisfy the following second-degree equation:*

$$F_l(\lambda) = \lambda^2 - (\beta_l^{11} + \beta_l^{22})\lambda + \beta_l^{11}\beta_l^{22} - \beta_l^{12}\beta_l^{21} = 0, \tag{4.1}$$

where

$$\beta_l^{\nu\mu} = \sum_{j=0}^{n-1} b_j^{\nu\mu} e^{-i\frac{2\pi jl}{n}}, \quad \nu, \mu = 1, 2.$$

Remark. In the case of an arbitrary s-circulant, the eigenvalues will be the eigenvalues of the matrices $\|\beta_l^{\nu\mu}\|$, ν, $\mu = 1, \cdots, s$, where

$$\beta_l^{\nu\mu} = \sum_{j=0}^{n-1} b_j^{\nu\mu} e^{-i\frac{2\pi jl}{n}} \quad (l = 0, 1, \ldots, n-1).$$

Proof of Theorem 4.1. Let us consider a complex space C^{2n} and take a basis in it. The matrix A can be regarded as the matrix of a linear operator acting in this space. In the proof of the theorem we shall mainly rely on

Lemma 4.1. *For a 2-circulant A the two-dimensional subspaces π_l, $l = 0, 1, \cdots, n-1$, spanned over the vectors*

$$\eta_l = (1, e^{i\frac{2\pi l}{n}}, \ldots, e^{i\frac{2\pi(n-1)l}{n}}, 0, 0, \ldots, 0),$$
$$\hat{\eta}_l = (0, 0, \ldots, 0, 1, e^{i\frac{2\pi l}{n}}, \ldots, e^{i\frac{2\pi(n-1)l}{n}}),$$

will be invariant with respect to A.

Proof. Indeed, we have

$$A\eta_l = A\left(\sum_{j=1}^{n} e^{i\frac{2\pi(j-1)l}{n}} h_{j,1}\right) = \sum_{j=1}^{n} e^{i\frac{2\pi(j-1)l}{n}} \left(\sum_{m=0}^{n-1} b_m^{11} h_{j+m,1} + \sum_{m=0}^{n-1} b_m^{21} h_{j+m,2}\right)$$
$$= \sum_{v=1}^{n}\left(\sum_{m=0}^{n-1} b_m^{11} e^{-i\frac{2\pi ml}{n}}\right) e^{i\frac{2\pi(v-1)l}{n}} h_{v,1} + \sum_{v=1}^{n}\left(\sum_{m=0}^{n-1} b_m^{21} e^{-i\frac{2\pi ml}{n}}\right) e^{i\frac{2\pi(v-1)l}{n}} h_{v,2}$$
$$= \beta_l^{11}\eta_l + \beta_l^{21}\hat{\eta}_l.$$

By performing similar calculations for $A\hat{\eta}_l$, we obtain

$$A\eta_l = \beta_l^{11}\eta_l + \beta_l^{21}\hat{\eta}_l, \quad A\hat{\eta}_l = \beta_l^{12}\eta_l + \beta_l^{22}\hat{\eta}_l, \tag{4.2}$$

whence $A\pi_l \subset \pi_l$. This completes the proof of the lemma.

After this lemma the proof of the theorem is obvious. Indeed, the matrix of the operator A, bounded in an invariant subspace π_l, will be in accordance with (4.2) as follows:

$$\begin{pmatrix} \beta_l^{11} & \beta_l^{12} \\ \beta_l^{21} & \beta_l^{22} \end{pmatrix},$$

and therefore the characteristic equation of an operator A, bounded in a subspace π_l, will be (4.1). Hence follows the validity of the thoerem.

It is easy to find the eigenvectors $\xi_{l,1}$ and $\xi_{l,2}$ corresponding to the eigenvalues $\lambda_{l,1}$ and $\lambda_{l,2}$. In fact, it follows from (4.2) that as eigenvectors we can take, for example, the vectors

$$\xi_{l,\nu} = \beta_l^{12}\eta_l + (\beta_l^{11} - \lambda_{l,\nu})\,\hat{\eta}_l, \quad \nu = 1, 2. \tag{4.3}$$

Since

$$e^{i\frac{2\pi l}{n}} = \overline{e^{i\frac{2\pi(n-l)}{n}}}$$

and the $b_l^{\nu\mu}$ are real numbers, it follows that the coefficients, and hence also the roots of the lth and $(n-l)$th equation (4.1) are conjugate; for definiteness, let

$$\lambda_{n-l,1} = \overline{\lambda_{l,1}}, \quad \lambda_{n-l,2} = \overline{\lambda_{l,2}}, \quad l = 0, 1, \ldots, n-1. \tag{4.4}$$

It hence follows from (4.3) that

$$\xi_{n-l,1} = \overline{\xi_{l,1}}, \quad \xi_{n-l,2} = \overline{\xi_{l,2}}, \quad l = 0, 1, \ldots, n-1.$$

From these relations (and from (4.3)) follows the invariance of A with respect to the two-dimensional real subspace $\Pi_{l,\nu} \subset R^{2n}$, spanned over the vectors

$$\frac{1}{2}(\xi_{l,\nu} + \xi_{n-l,\nu}) = \frac{1}{2}(\xi_{l,\nu} + \overline{\xi_{l,\nu}}) = \mathrm{Re}\,(\beta_l^{12}\eta_l) + \mathrm{Re}\,((\beta_l^{11} - \lambda_{l,\nu})\,\hat{\eta}_l),$$

$$\frac{1}{2i}(\xi_{l,\nu} - \xi_{n-l,\nu}) = \frac{1}{2i}(\xi_{l,\nu} - \overline{\xi_{l,\nu}}) = \mathrm{Im}\,(\beta_l^{12}\eta_l) + \mathrm{Im}\,((\beta_l^{11} - \lambda_{l,\nu})\,\hat{\eta}_l).$$

If $\lambda_{l,\nu}$ is real, it will be a double eigenvalue and $\Pi_{l,\nu}$ will be the corresponding proper subspace (of dimension 2). If $\lambda_{l,\nu}$ is complex the operator A will act in $\Pi_{l,\nu}$ as an elliptical rotation.

By virtue of (3.2), $\beta_l^{\nu\mu}$ will assume the simpler form

$$\beta_l^{\nu\mu} = \sum_{j=-k}^{k} b_j^{\nu\mu} e^{-i\frac{2\pi jl}{n}}, \quad \nu, \mu = 1, 2, \; l = 0, 1, \ldots, n-1. \tag{4.5}$$

In the case of a symmetrical problem it evidently follows from (3.6) that β_l^{11} and β_l^{22} are real, whereas β_l^{12} and β_l^{21} are purely imaginary. Therefore the equation (4.1) will have real coefficients. Hence the eigenvalues $\lambda_{l,1}$ and $\lambda_{l,2}$ are either both real or both complex conjugate. Using (4.4), we find that each eigenvalue is double, i.e. in the first case we have $\lambda_{l,1} = \lambda_{n-l,1}$ and $\lambda_{l,2} = \lambda_{n-l,2}$, whereas in the second case we have $\lambda_{l,1} = \lambda_{n-l,2}$ and $\lambda_{l,2} = \lambda_{n-l,1}$.

It is often necessary to find eigenvalues that are close to unity. They are easy to calculate if we know the values of the characteristic polynomial and its derivative for $\lambda = 1$. In fact, if $F(\lambda) = \lambda^2 + p\lambda + q$ is a characteristic polynomial with roots λ_1 and λ_2, with

$$F(1)=\Phi,\quad \left.\frac{dF(\lambda)}{d\lambda}\right|_{\lambda=1}=\Omega,$$

then (as can be easily seen)

$$\lambda_{1,2}=1-\frac{\Omega}{2}\pm\sqrt{\frac{\Omega^2}{4}-\Phi}. \tag{4.6}$$

This formula is often useful. For example, knowing Φ and Ω we can find how many

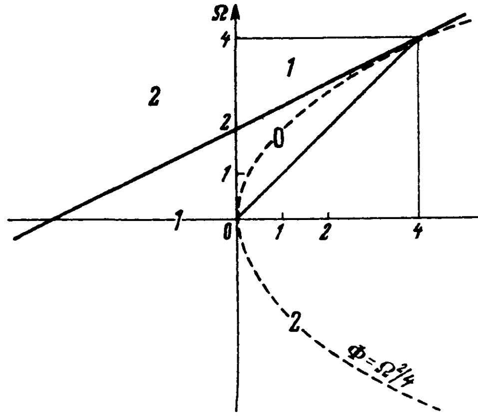

Figure 1

roots have an absolute value larger than unity. For real Φ and Ω we have plotted the results in Figure 1. Below we shall also use the following result, easily obtained from (4.6). Let Φ and Ω depend on n, and for $n\to\infty$ let $\Phi\sim c_1 n^{-\kappa_1}$, $\Phi\sim c_2 n^{-\kappa_2}$, $c_1\neq 0$, $c_2\neq 0$, $\kappa_1>0$, $\kappa_2>0$. Let $|\lambda_1|\leq|\lambda_2|$.[13] Hence

$$1-|\lambda_2|<cn^{-\varkappa},\text{where } \varkappa=\max(\varkappa_2,\varkappa_1-\varkappa_2),\quad c>0. \tag{4.7}$$

Next let us write

$$\frac{2\pi l}{n}=\varphi,\quad F_l(1)=\Phi_s(\varphi)=\Phi(\varphi),\quad \left.\frac{dF_l(\lambda)}{d\lambda}\right|_{\lambda=1}=\Omega_s(\varphi)=\Omega(\varphi).$$

Using (4.6), we can find the roots $\lambda_{l,1}$ and $\lambda_{l,2}$, $l=0,1,\cdots,n-1$. It follows from (4.5) that Φ and Ω are trigonometric polynomials of ϕ, the first of degree $2k$ and the second of degree k. Let

$$\Phi(\varphi)=\sum_{j=-2k}^{2k} f_j e^{ij\varphi},\ \Omega(\varphi)=\sum_{j=-k}^{k}\omega_j e^{ij\varphi},\ f_j=f_j(s),\ \omega_j=\omega_j(s).$$

It follows from (4.1) and (4.5) that

$$f_j=\sum_{r=-k}^{k}(\tilde{b}_r^{11}\tilde{b}_{j-r}^{22}-b_r^{12}b_{j-r}^{21}),\quad j=-2k,\ \ldots,\ 2k, \tag{4.8}$$

$$\omega_j=-\tilde{b}_j^{11}-\tilde{b}_j^{22},\quad j=-k,\ \ldots,k, \tag{4.9}$$

where

13) In the following this will be assumed throughout.

$$\tilde{b}_r^{11} = \begin{cases} b_r^{11}, & r \neq 0, \\ b_0^{11} - 1, & r = 0, \end{cases} \qquad \tilde{b}_r^{22} = \begin{cases} b_r^{22} & , r \neq 0, \\ b_0^{22} - 1 & , r = 0. \end{cases}$$

Hence we can see that f_j and ω_j are real coefficients. It follows from (3.6) that in the case of a symmetrical problem we have $f_j = f_{-j}$ and $\omega_j = \omega_{-j}$; therefore Φ and Ω will be real even functions.

The polynomials Φ and Ω cannot be arbitrary. In fact, f_j and ω_j must be real and such that the equations (4.8) and (4.9) have real solutions $b_l^{\nu\mu}$, $\nu, \mu = 1, 2$, $l = -k, \cdots, k$, satisfying the conditions (3.3) and (3.4). For a dimensionless problem the solutions must additionally satisfy the condition (3.5), and for a symmetrical problem, the condition (3.6). It follows from (3.3) and (3.4) that $F_0(1) = F_s(1) = 0$, i.e.

$$\Phi_s(0) = 0, \tag{4.10}$$

$$\Phi_s\left(\frac{2\pi s}{n}\right) = 0, \tag{4.11}$$

and from (3.5) it follows that

$$\left.\frac{dF_0(\lambda)}{d\lambda}\right|_{\lambda=1} = 0,$$

i.e.

$$\Omega_s(0) = 0. \tag{4.12}$$

§5. Examples

Now let us apply our results to the rules of motion considered in [1]. There rules were obtained from geometrical considerations, i.e. angles, or their cosines, or curvature radii, were averaged, or it was tried (by vertex motion) to set a prescribed value of the side length, or angle, or curvature radius. In all the subsequent examples the point $f = f(r_{-k}, \cdots, r_k)$ will lie on a straight line perpendicular to the vector $r_1 - r_{-1}$ that passes through the point $\frac{1}{2}(r_1 + r_{-1})$ and such that the distance between the point f and the points r_{-1} and r_1 is the same:

$$|f - r_1| = |f - r_{-1}|. \tag{5.1}$$

The following functions f_i were considered in [1] (with the use of (5.1) the definitions given below determine these functions uniquely):

1) (rule of setting a prescribed side length) $f = f_1(r_{-1}, r_0, r_1)$ is determined from the condition

$$|f - r_{-1}| = |f - r_1| = \min\left\{d, \frac{1}{2}|r_1 - r_{-1}|\right\}, \quad 0 \leqslant \angle r_{-1} f_1 r_1 \leqslant \pi;$$

2) (rule of setting a prescribed angle) $f = f_{2,m}(r_{-1}, r_0, r_1)$ is determined from the condition

$$\angle r_{-1} f_{2,m} r_1 = \pi - \frac{2\pi}{m};$$

3) (rule of averaging of angles) $f = f_3(r_{-2}, r_{-1}, r_0, r_1, r_2)$ is determined from the condition

$$\angle r_{-1}f_3r_1=\gamma \cdot \angle r_{-1}r_0r_1 + \frac{1-\gamma}{2}(\angle r_{-2}r_{-1}r_0 + \angle r_0r_1r_2);$$

4) (rule of averaging of cosines of angles) $f = f_4(r_{-2}, r_{-1}, r_0, r_1, r_2)$ is determined from the conditions

$$\cos\angle r_{-1}f_4r_1 = \gamma \cdot \cos\angle r_{-1}r_0r_1 + \frac{1-\gamma}{2}(\cos\angle r_{-2}r_{-1}r_0 + \cos\angle r_0r_1r_2)$$

and $0 \leq \angle r_{-1}f_4r_1 \leq \pi$;

5) (rule of averaging of curvature radii)

$$f = f_5(r_{-2}, r_{-1}, r_0, r_1, r_2) = \frac{1}{2}(f_5^+(r_{-2}, r_{-1}, r_0, r_1, r_2) + f_5^-(r_{-2}, r_{-1}, r_0, r_1, r_2)),$$

where the point $f_5^{\pm}$ lies on a circle passing through the points r_{-1}, r_1 and $r_{\pm 2}$;

6) (rule of setting a prescribed curvature radius) $f = f_6(r_{-1}, r_0, r_1)$ lies on a circle of radius $\max\{R, \frac{1}{2}|r_1 - r_{-1}|\}$, with $0 \leq \angle r_{-1}f_6r_1 \leq \pi$.

By taking the weighted mean of these functions and of the function $f_0 = r_0$, i.e. by setting

$$f = (1 - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6)r_0 + \sigma_1f_1 + \sigma_2f_{2,m} + \sigma_3f_3 + \sigma_4f_4 + \sigma_5f_5 + \sigma_6f_6, \tag{5.2}$$

we can obtain a suffiently large collection of various rules of motion of polygons.

In [1] the results of a computer simulation for the functions f_1, $f_{2,n}$, f_4, f_5 and their linear combinations were presented. The simulation showed that in the case of sufficiently small σ_1, σ_2, σ_4 and σ_5 we have convergence to a regular polygon for a wide class of initial states. For the function $(1 - \sigma)r_0 + \sigma f_6$ it was shown in [1] with the aid of simple geometrical considerations that for any σ there may be no convergence to a regular polygon even if the initial states are arbitrarily close to a regular polygon.

For the functions under consideration it is easy to find fixed points on the basis of geometrical considerations. With the aid of elementary (though cumbersome) calculations it is possible to find the coefficients $b_l^{\nu\mu}$. These results are listed in Table 1. Since all the functions under consideration are symmetrical, it follows from (3.6) that it suffices to calculate b_l^{11} and b_l^{22} for $l \geq 0$, and b_l^{12} and b_l^{21} for $l \geq 1$. In the table it is assumed that $r^0 \in V_s^4$. Let us note that the same numbers stand in the columns for b_l^{11} and b_l^{12}. This is a consequence of (5.1); more precisely, it follows from (5.1) that $b_l^{12} = b_l^{11}\mathrm{tg}(\pi s/n)$. It also follows from (5.1) that $b_l^{11} = b_l^{12} = 0$ for $l > 1$. For a function f of the form (5.2) the coefficients $b_l^{\nu\mu}$ are linear combinations of the coefficients $b_l^{\nu\mu}$ in Table 1.

For a function f of the form (5.2) and $\sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 \neq 0$, all the fixed points will be quasiregular polygons, and for $\sigma_1 = 0$ and $\sigma_6 = 0$ the problem will be dimensionless. As we noted above, any function of the form (5.2) is symmetrical. Let us note that if n is not divisible by m, then for $f = (1 - \sigma_2)r_0 + \sigma_2f_{2,m}$ there will be no

Table 1

Sets $\mathfrak{N}$ and coefficients $b_1^{\nu\mu}$ for functions f_i

	k	$\mathfrak{N}$	b_0^{11}	b_0^{22}	b_1^{11}	b_1^{12}	b_1^{21}	b_1^{22}	b_2^{11}	b_2^{12}	b_2^{21}	b_2^{22}
$f_0=r_0$	0	R^{2n}	1	1								
f_1	1	$\mathfrak{N}(d)$	0	0	$\frac{1}{2}$	$\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	$-\frac{1}{2}\operatorname{ctg}\frac{\pi s}{n}$	$-\frac{1}{2}$				
$f_{2,m}$	1	$V_s^4,\ s=\frac{n}{m}$	0	0	$\frac{1}{2}$	$\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	$-\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	$\frac{1}{2}$				
f_3	2	$\bigcup\limits_{\substack{s=1\\ s\neq n/2}}^{n-1} V_s^4$	0	$1-3\gamma$	$\frac{1}{2}$	$\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	$-\gamma\operatorname{tg}\frac{\pi s}{n}$	2γ	0	0	$\frac{1}{2}\gamma\operatorname{tg}\frac{\pi s}{n}$	$-\frac{1}{2}\gamma$
f_4	2	$\bigcup\limits_{\substack{s=1\\ s\neq n/2}}^{\left[\frac{n-1}{2}\right]} V_s^4$	Same as for f_3									
f_5	2	$\bigcup\limits_{\substack{s=1\\ s\neq n/2}}^{n-1} V_4^s$	0	0	$\frac{1}{2}$	$\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	0	$\frac{1}{2}\left(1+\frac{1}{1+2\cos\frac{2\pi s}{n}}\right)$	0	0	0	$-\frac{1}{2\left(1+2\cos\frac{2\pi s}{n}\right)}$
f_6	1	$\bigcup\limits_{\substack{s=1\\ s\neq n/2}}^{\left[\frac{n-1}{2}\right]} V_s^3\left(2R\sin\frac{\pi s}{n}\right)$	0	0	$\frac{1}{2}$	$\frac{1}{2}\operatorname{tg}\frac{\pi s}{n}$	0	$\frac{1}{2\cos\frac{2\pi s}{n}}$				

($\mathfrak{N}(d)$ signifies a set of equilateral convex polygons with a side length d and clockwise numeration (i.e. with angles not exceeding π)).

fixed points in $R^{2n}\backslash\mathfrak{M}$ (in this case fixed points are those of the form $(r, \cdots, r)$ for which the function $f_{2,m}$ is not defined).

Knowing the coefficients $b_l^{\nu\mu}$, we can find the functions $\Phi(\phi)$ and $\Omega(\phi)$. After calculations, we obtain for a function f of the form (5.2) the formulas

$$\Phi_s(\varphi) = (\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6)\cdot\left\{\sigma_2\cdot 4\left(\sin\frac{\varphi}{2}\right)^4\left(1-\left(\frac{\operatorname{tg}\frac{\pi s}{n}}{\operatorname{tg}\frac{\varphi}{2}}\right)^2\right)\right.$$
$$+(\sigma_3+\sigma_4)\cdot 16\gamma\left(\sin\frac{\varphi}{2}\right)^6\left(1-\left(\frac{\operatorname{tg}\frac{\pi s}{n}}{\operatorname{tg}\frac{\varphi}{2}}\right)^2\right)+\sigma_5\cdot\frac{16\left(\sin\frac{\varphi}{2}\right)^6}{1+2\cos\frac{2\pi s}{n}}\left(1-\left(\frac{\sin\frac{\pi s}{n}}{\sin\frac{\varphi}{2}}\right)^2\right)$$
$$\left.+\sigma_6\cdot\frac{4\left(\sin\frac{\varphi}{2}\right)^4}{\cos\frac{2\pi s}{n}}\left(1-\left(\frac{\sin\frac{\pi s}{n}}{\sin\frac{\varphi}{2}}\right)^2\right)\right\}, \tag{5.3}$$

$$\Omega_s(\varphi) = 2\sigma_1+4\sigma_2\left(\sin\frac{\varphi}{2}\right)^2+2(\sigma_3+\sigma_4)\left(\sin\frac{\varphi}{2}\right)^2\left(1+4\gamma\left(\sin\frac{\varphi}{2}\right)^2\right)$$
$$+2\sigma_5\left(\sin\frac{\varphi}{2}\right)^2\left(1+\frac{\left(\sin\frac{\varphi}{2}\right)^2-\left(\sin\frac{\pi s}{n}\right)^2}{1+2\cos\frac{2\pi s}{n}}\right)$$
$$+\sigma_6\left(\left(\sin\frac{\varphi}{2}\right)^2+\frac{\left(\sin\frac{\varphi}{2}\right)^2-\left(\sin\frac{\pi s}{n}\right)^2}{\cos\frac{2\pi s}{n}}\right). \tag{5.4}$$

Using (5.3)–(5.4) and (4.6), we can calculate the eigenvalues.

Let $\sigma_6 > 0$. It then follows from (5.4) that $\Omega_s(0) < 0$. From Figure 1 (§4) it is easy to see that the absolute value of one of the roots $\lambda_{0,1}$ and $\lambda_{0,2}$ is larger than 1. This is true for all s. It hence follows from Theorem 3.1 that for $t \to \infty$ there is no convergence of T_ρ^t to $\mathfrak{R}$. The same applies to the case that $\sigma_6 < 0$ and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0$, since in this case we have $\Phi_s(\phi) < 0$ for $\phi = 2\pi l/n$, $s < l < n - s$. Therefore we shall henceforth assume that $\sigma_6 = 0$.

Let $s = 1$ (and hence $m = n$), i.e. we study polygons that are close to regular. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 be nonnegative. It follows from (5.3) and (5.4) that in this case $\Phi > 0$ and $\Omega > 0$ for all $k \neq 0, 1, n-1$. From this result and from Figure 1 it follows that if σ_1, σ_2, σ_3, σ_4 and σ_5 are sufficiently small, then all the eigenvalues in Λ will be smaller than unity (if $\sigma_2+\sigma_3+\sigma_4+\sigma_5 > 0$), i.e. the polygons close to regular polygons will converge to regular polygons. It is possible to find the rate of convergence and the rounding time τ. In fact, since σ_1, σ_2, σ_3, σ_4 and σ_5 are sufficiently small, all the eigenvalues will be real and positive; it is easy to see that in this case $\lambda_{\max} = \lambda_{2,2} = \lambda_{n-2,2}$. By studying the asymptotic behavior of Φ and Ω for $s = 1$, $l = 2$, $n \to \infty$, and using (4.6), we obtain after some calculations[14] (let us recall that λ_{max} is positive, so that $|\lambda_{max}| = \mathrm{Re}\lambda_{max} = \lambda_{max}$):

$$1-\lambda_{\max}=\begin{cases}\dfrac{3\sigma_2(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5)}{2\sigma_1}\cdot\left(\dfrac{2\pi}{n}\right)^4(1+O(n^{-2})), & \text{if}\quad \sigma_1>0,\sigma_2>0,\\ \dfrac{9(4\gamma(\sigma_3+\sigma_4)+\sigma_5)(\sigma_1+\sigma_3+\sigma_4+\sigma_5)}{2\sigma_1}\left(\dfrac{2\pi}{n}\right)^6(1+O(n^{-2})), & \\ \qquad\text{if}\quad \sigma_1>0,\ \sigma_2=0,\ \sigma_3+\sigma_4+\sigma_5>0, & \\ (2\sigma_2+\sigma_3+\sigma_4+\sigma_5)\left(1-\sqrt{1-\dfrac{2\sigma_2+\sigma_3+\sigma_4+\sigma_5}{3\sigma_2(\sigma_2+\sigma_3+\sigma_4+\sigma_5)}}\cdot\left(\dfrac{2\pi}{n}\right)^2\right. & \\ \qquad\times(1+O(n^{-2})), \quad \text{if}\quad \sigma_1=0,\ \sigma_2>0, & \\ (6\gamma(\sigma_3+\sigma_4)+2\sigma_5)\left(\dfrac{2\pi}{n}\right)^4(1+O(n^{-2})), & \text{if}\quad \sigma_1=\sigma_2=0,\sigma_3+\sigma_4+\sigma_5>0.\end{cases} \tag{5.5}$$

(Let us note that for $\sigma_1 = 0$ and $\sigma_2 > 0$ the radicand must be smaller than 1 and larger than ¼.)

In particular (see also [1]), by setting $\sigma_1 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$ and $\sigma_2 = \cos(2\pi s/n)$ in (5.2), we obtain

$$1-|\lambda_{\max}|=\left(\frac{2\pi}{n}\right)^2(1+O(n^{-2})). \tag{5.6}$$

In the case of arbitrary s, if $\sigma_5 = \sigma_6 = 0$ and σ_1, σ_2, σ_3 and σ_4 are positive and sufficiently small, we easily find from (5.3)–(5.4) and Figure 1 that there exist $\min\{2(s-1), 2(n-s-1)\}$ eigenvalues larger than unity. This result was mentioned in the Introduction.

§6. Estimation of rate of convergence in the case of a fixed function

In this and the subsequent sections we shall consider the action of a transformation in a small neighborhood of the point $r^0 \in \mathfrak{N}$. Let $r^0 \in V_s^4$. As it follows from Theorem 3.1, if $|\lambda_{max}| < 1$, then all the trajectories originating in this neighborhood will converge to $\mathfrak{N} \cap V_s^4$ and the quantity $1 - |\lambda_{max}|$ will determine the rate of convergence. In particular, if $s = 1$, then $1 - |\lambda_{max}|$ will determine the rate of convergence to a regular polygon. Therefore it is of interest to estimate how large this quantity can be.

Suppose we are given a function $f(r_{-k}, \cdots, r_k)$ such that regular n-polygons will be stationary for all sufficiently large n. In this section we shall find an asymptotic expression for the rate of convergence to a regular n-polygon for $n \to \infty$.

A broken line with vertices $r_{-k}, \cdots, r_k$ is said to be (d, ϕ)-regular if each section of this line has a length d and each angle is equal to ϕ. The set of all (d, ϕ)-regular polygonal lines forms a three-dimensional manifold $\mathcal{P}^3_{d,\phi} \subset R^{4k+2}$. In particular,

14) These results are mentioned in part in [1].

the manifold $\mathcal{P}^3_{d,0}$ consists of polygonal lines all whose vertices lie on the same straight line, the distance between each vertex and the next being equal to d.

Theorem 6.1. *Suppose that the function* $f(r_{-k}, \cdots, r_k)$ *is five times continuously differentiable in a neighborhood of the manifold* $\mathcal{P}^3_{d,0} \subset R^{4k+2}$, $d > 0$, *and let* $V^3_s(d) \in \mathfrak{N}$ *for all sufficiently large* n. *Then*

$$1 - |\lambda_{\max}| < Dn^{-4}, \tag{6.1}$$

where D *is a constant that depends on* f *and* s, *but not on* n.

Proof. The theorem will be proved on the assumption that the function f is analytic in a neighborhood of the manifold $\mathcal{P}^3_{d,0}$. However, it follows from the proof that it suffices to require fivefold continuous differentiability.

For each n it is possible to calculate the matrix A of the linear transformation A_r, $r \in V^3_s(d)$. The elements of this matrix depend on n. Let us write $\theta = 2\pi s/n$. Since f is an analytic function, it follows from (3.7) that the coefficients $b^{\nu\mu}_l$, ν, $\mu = 1, 2$, and $l = -k, \cdots, k$, are likewise analytically dependent on θ, and they can be expanded in a power series in θ: $b^{\nu\mu}_l(\theta) = \Sigma^k_{j=-k} b^{\nu\mu}_{l,j}\theta^j$.

For $\theta = 2\pi s/n$ the coefficients $b^{\nu\mu}_l(\theta)$ satisfy the conditions (3.3) and (3.4); it follows from the analyticity that these conditions hold for all θ. Next, from (1.3)–(1.4) it is easy to obtain

$$b^{\nu\mu}_l(\theta) = b^{\nu\mu}_{-l}(-\theta), \qquad \nu, \mu = 1, 2, \ l = -k, \ \dots, \ k. \tag{6.2}$$

If the points $r_{-k}, \cdots, r_k$ form a (d, θ)-regular broken line, $\theta = 2\pi s/n$, then $f(r_{-k}, \cdots, r_0, \cdots, r_k) = r_0$; it follows from the analyticity that this is true not only for $\theta = 2\pi s/n$, but for all θ. Hence we find with the aid of (3.4) that the coefficients $b^{\nu\mu}_l(\theta)$ satisfy also the following conditions:

$$\sum_{l=-k}^{k} l b^{11}_l(\theta) = \frac{1}{2} \operatorname{ctg} \frac{\theta}{2} \sum_{l=-k}^{k} b^{12}_l(\theta), \quad \sum_{l=-k}^{k} l b^{21}_l(\theta) = \frac{1}{2} \operatorname{ctg} \frac{\theta}{2} \left(\sum_{l=-k}^{k} b^{22}_l(\theta) - 1 \right). \tag{6.3}$$

Let us write $B^{\nu\mu}_{j,m} = \Sigma^k_{l=-k} b^{\nu\mu}_{l,j} l^m$. It follows from (6.2) that

$$B^{\nu\mu}_{j,m} = 0 \quad \text{for} \quad j + m \quad \text{odd}. \tag{6.4}$$

From (3.3), (3.4) and (6.3) we obtain the following equations:

$$\begin{gathered} B^{11}_{0,0} = 1, B^{11}_{j,0} = 0 \ \text{ for } \ j > 0, \quad B^{21}_{j,0} = 0 \ \text{ for } \ j \geqslant 0, \\ B^{12}_{0,0} = 0, \quad B^{22}_{0,0} = 1, \quad B^{12}_{1,1} = \frac{1}{2} B^{11}_{0,2}, \quad B^{22}_{1,1} = \frac{1}{2} B^{21}_{0,2}, \\ B^{12}_{0,2} = 0, \quad B^{22}_{0,2} = 0, \quad B^{11}_{1,1} = B^{12}_{2,0}, \quad B^{21}_{1,1} = B^{22}_{2,0}. \end{gathered} \tag{6.5}$$

Let us prove that

$$1 - |\lambda_{2s,2}| < Dn^{-4}, \tag{6.6}$$

where the constant D depends only on f and s, but not on n. For this purpose we shall expand $\Phi_s(4\pi s/n)$ and $\Omega_s(4\pi s/n)$ in power series in $1/n$. It is evident that the coefficients of these series are polynomials in $B^{\nu\mu}_{j,m}$. It easily follows from (6.4) that only the coefficients of even powers of $1/n$ are nonzero. After simple calculations we obtain from (6.5) the formulas

$$\Phi_s\left(\frac{4\pi s}{n}\right) = O(n^{-6}) \text{ and } \Omega_s\left(\frac{4\pi s}{n}\right) = O(n^{-2}).$$ [15]

From these relations and from Theorem 3.1 and formula (4.7) we obtain (6.6). From (6.6) we obtain (6.1), which completes the proof of the theorem.

In the case $s = 1$ we shall study convergence to regular polygons. Theorem 6.1 asserts in this case that for $n \to \infty$ the rounding time τ increases not slower than n^4. Let us note that in general $\tau \sim cn^4$.

It follows from Theorem 6.1 that although there exists a sequence of functions (for example, the functions $f_{2,n}$ of §5) for which $\tau \sim cn^2$, there does not exist any single such function f. It is of interest to ascertain how the rounding time can vary for $n \to \infty$ in the case of a sequence of functions f_n that depend on n. We shall say that this sequence is smooth if the coefficients $b_l^{\nu\mu}$, as functions of $\theta = 2\pi s/n$, are four times continuously differentiable with respect to θ in a neighborhood of the point $\theta = 0$. We have

Theorem 6.2. *If $f_n(r_{-k}, \cdots, r_k)$ is a smooth sequence of functions and if $V_s^3(d) \in \mathfrak{N}$, $d > 0$, for sufficiently large n, then*

$$1 - |\lambda_{\max}| < Dn^{-2},$$

where D is a constant that depends on the sequence f_n and on s, but not on n.

Theorem 6.2 can be proved in the same way as Theorem 6.1, with the only difference that in general the condition (6.3) does not hold.

§7. Maximum rate of convergence

By virtue of Theorem 6.2 we can hope that the maximum value of the rate of convergence for fixed n will decrease as n^{-2} with increasing n. This result does not follow from Theorem 6.2, since the sequence of functions f_n that maximize the rate of convergence for fixed n may not be smoothly dependent on n. In this section we shall calculate by another method the maximum rate of convergence, thus obtaining a number of bounds for the constant D in Theorem 6.2 (not given in §6).

Let us denote the maximum value of the rate of convergence for fixed n by $\mathfrak{L}_k(n)$. As we noted in §3,

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15) This assertion apparently contradicts formula (5.4); as a matter of fact, the function f_1 of §5 does not satisfy the conditions of Theorem 6.1, since it is not differentiable at the points of the manifold $\mathfrak{P}^3_{d,0}$.

$$\mathscr{L}_k(n) = \max_{A \in M_{k,n}} (1 - |\lambda_{\max}|), \tag{7.1}$$

where $M_{k,n}$ denotes a set of matrices of order $2n$ that are 2-circulants and satisfy the conditions (3.2)–(3.4). In the same way it is possible to find the highest rate of convergence in the case of a dimensionless problem, a symmetrical problem, and a dimensionless symmetrical problem. The corresponding quantities will be denoted by $\mathfrak{L}^{\mathrm{D}}_{k,n}$, $\mathfrak{L}^{\mathrm{S}}_{k,n}$ and $\mathfrak{L}^{\mathrm{DS}}_{k,n}$; for finding them, we must consider in (7.1) instead of $M_{k,n}$ the sets $M^{\mathrm{D}}_{k,n}$, $M^{\mathrm{S}}_{k,n}$ and $M^{\mathrm{DS}}_{k,n}$, which denote the sets of 2-circulants of order $2n$ that satisfy respectively the conditions (3.2)–(3.5), the conditions (3.2)–(3.4) and (3.6), and the conditions (3.2)–(3.6). In this section we shall obtain asymptotic estimates for these quantities for $n \to \infty$. Thus we shall prove

Theorem 7.1. *We have the following inequalities:*

$$k\left(\frac{2\pi}{n}\right)^2\left(1 - \frac{D}{n^2}\right) \leqslant \mathscr{L}_k(n) \leqslant 2\sqrt{D_{3,2k}}\left(\frac{2\pi}{n}\right)^{3/2}\left(1 + \frac{D}{n}\right), \tag{7.2}$$

$$k\left(\frac{2\pi}{n}\right)^2\left(1 - \frac{D}{n^2}\right) \leqslant \mathscr{L}^{\mathrm{D}}_k(n) \leqslant 4D_{2,k}\left(\frac{2\pi}{n}\right)^2\left(1 + \frac{D}{n}\right), \tag{7.3}$$

$$k\left(\frac{2\pi}{n}\right)^2\left(1 - \frac{D}{n^2}\right) \leqslant \mathscr{L}^{\mathrm{S}}_k(n) \leqslant \sqrt{2D_{4,2k}}\left(\frac{2\pi}{n}\right)^2\left(1 + \frac{D}{n}\right), \tag{7.4}$$

where $D_{s,l} = 2\sum_{j=1}^{l} j^s$ *and* D *is a constant that depends on* k, *but not on* n.

Proof. First we shall prove the left-hand sides of (6.2)–(6.4). Let

$$\tilde{f} = (1 - \sigma_n) r_0 + \sigma_n f_{2,n}, \quad \sigma_n = \cos\frac{2\pi}{n},$$

and let A be the matrix of the corresponding linear transformation. It is evident that A^k belongs to $M_{k,n}$, as well as to $M^{\mathrm{D}}_{k,n}$ and $M^{\mathrm{S}}_{k,n}$ (it is the linear transformation matrix for a function $\tilde{f}^{(k)}$ that is a k-fold superposition of the function $\tilde{f}$). But the eigenvalues of A^k are the kth powers of the eigenvalues of A. It hence follows from (5.6) that for A^k we have

$$1 - |\lambda_{\max}| = k\left(\frac{2\pi}{n}\right)^2 (1 + O(n^{-2})).^{16)}$$

From this formula we can obtain the left-hand sides of (7.2)–(7.4).

For proving the right-hand sides of (7.2)–(7.4), we shall estimate the eigenvalues $\lambda_{2,1}$ and $\lambda_{2,2}$. Let us recall that $|\lambda_{2,1}| \le |\lambda_{2,2}|$.

In the following we shall need some lemmas relating to the quadratic trinomial (with complex coefficients) $F(\lambda) = \lambda^2 + p\lambda + q$, whose roots λ_1 and λ_2, $|\lambda_1| \le |\lambda_2|$, do not exceed unity in absolute value (it is appropriate to compare these lemmas with Figure 1). Let $\alpha = F(1)$ and $\beta = F'(1)$.

16) It is evident that the rounding time decreases precisely k times; the functions $\tilde{f}$ and $\tilde{f}^{(k)}$ yield the same rule of transformation of polygons; only the unit of time for $\tilde{f}^{(k)}$ is equal to k units of time for $\tilde{f}$.

Lemma 7.1. $|\alpha| \le 4$, $|\beta| \le 4$.

Proof. We have

$$|\alpha| = |1+p+q| = |1-\lambda_1-\lambda_2+\lambda_1\lambda_2| \leqslant 1+1+1+1=4,$$
$$|\beta| = |2+p| = |2-\lambda_1-\lambda_2| \leqslant 2+1+1=4.$$

This proves the lemma.

Lemma 7.2. $|\lambda_2| \ge 1 - \frac{1}{2}\mathrm{Re}\,\beta$.

Proof. It is easy to see from (4.6) that at least one of the roots λ_1 and λ_2 is not smaller in absolute value than $|1-\beta/2|$. But

$$|1-\beta/2| \geqslant \mathrm{Re}(1-\beta/2) = 1 - \frac{1}{2}\mathrm{Re}\,\beta.$$

This proves the lemma

It follows from the lemma that

$$|\lambda_2| \geqslant 1 - \frac{1}{2}|\beta|. \tag{7.5}$$

Lemma 7.3.

$$|\lambda_2| \geqslant 1 - \sqrt{|\alpha|}. \tag{7.6}$$

Proof. We can have the following two cases:

1) $|\beta^2/4| \le |\alpha|$. Then $\frac{1}{2}|\beta| \le \sqrt{|\alpha|}$. Hence (7.5) yields (7.6).

2) $|\beta^2/4| > |\alpha|$. Then

$$\sqrt{\beta^2/4-\alpha} = \frac{\beta}{2}\sqrt{1-\varepsilon} = \frac{\beta}{2}\left(1-\sum_{j=1}^{\infty} c_j\varepsilon^j\right), \quad \varepsilon = \frac{4\alpha}{\beta^2}.$$

It is easy to see that

$$c_j \geqslant 0, \quad \sum_{j=1}^{\infty} c_j = 1. \tag{7.7}$$

Since $|2\sqrt{\alpha}/\beta| < 1$, we obtain from (4.6) and (7.7) the formula

$$|\lambda_2| = \left|1 - \frac{\beta}{2} + \frac{\beta}{2}\left(1 - \sum_{j=1}^{\infty} c_j\left(\frac{4\alpha}{\beta^2}\right)^j\right)\right| = \left|1 - \sqrt{\alpha}\left(\sum_{j=1}^{\infty} c_j\left(\frac{2\sqrt{\alpha}}{\beta}\right)^{2j-1}\right)\right|$$
$$\geqslant 1 - \sqrt{|\alpha|}\sum_{j=1}^{\infty} c_j = 1 - \sqrt{|\alpha|}.$$

This completes the proof of the lemma.

Let us note that if $\alpha = \beta^2/4$, then $\lambda_1 = \lambda_2 = 1 - \sqrt{\alpha}$.

Now let us estimate $|\lambda_{2,2}|$, assuming that $|\lambda_{\max}| \le 1$. It follows from Lemma 7.1 that

$$|\Phi(\varphi)| \leqslant 4, \quad |\Omega(\varphi)| \leqslant 4 \quad \text{for} \quad \varphi = 2\pi l/n, \ l = 0, 1, \ldots, n-1. \tag{7.8}$$

Lemma 7.4. *Suppose that the function* $\Phi(\phi) = \Sigma_{j=-2k}^{2k} f_j e^{ij\phi}$ *satisfies the condition* (7.8), *and that* $M = \max_{0 \leq \phi \leq 2\pi} |\Phi(\phi)|$. *Then there exists a constant* C *that depends on* k, *but not on* n, *and such that*

$$M < 4\left(1 + \frac{C}{n}\right). \tag{7.9}$$

Proof. Let

$$M_1 = \max_{0 \leqslant \varphi \leqslant 2\pi} |\Phi'(\varphi)|, \quad \Phi(\tilde{\varphi}) = M, \frac{2\pi l}{n} \leqslant \tilde{\varphi} < \frac{2\pi(l+1)}{n}.$$

By virtue of (7.8) we have

$$M - 4 \leqslant |\Phi(\tilde{\varphi})| - \left|\Phi\left(\frac{2\pi l}{n}\right)\right| \leqslant \left|\Phi(\tilde{\varphi}) - \Phi\left(\frac{2\pi l}{n}\right)\right|$$

$$= \left|\int_{\frac{2\pi l}{n}}^{\tilde{\varphi}} \Phi'(\varphi)\, d\varphi\right| \leqslant M_1\left(\tilde{\varphi} - \frac{2\pi l}{n}\right) < M_1 \cdot \frac{2\pi}{n},$$

i.e.

$$M_1 \geqslant \frac{n}{2\pi}(M - 4). \tag{7.10}$$

Next, from the inversion formula

$$f_j = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\varphi) e^{-ij\varphi}\, d\varphi$$

we find that

$$|f_j| \leqslant M, \quad j = -2k, \ldots, 2k. \tag{7.11}$$

Hence

$$|\Phi'(\varphi)| = \left|\sum_{j=-2k}^{2k} i f_j e^{ij\varphi}\right| \leqslant M \cdot 2k(k+1),$$

i.e.

$$M_1 \leqslant 2k(k+1)M. \tag{7.12}$$

From (7.10) and (7.12) follows (7.9). This completes the proof of the lemma.

Let us note that this lemma is applicable not only to the function $\Phi(\phi)$, but also to the function $\Omega(\phi)$. In this case

$$|\omega_j| \leqslant M, \quad j = -k, \ldots, k. \tag{7.13}$$

Let us prove the right-hand side of (7.2). Let us expand $\Phi(\phi) = \Sigma_{j=-2k}^{2k} f_j e^{ij\phi}$ in a series in ϕ:

$$\Phi(\varphi)=\left\{\sum_j f_j-\frac{\varphi^2}{2}\sum_j j^2 f_j+\frac{\varphi^4}{24}\sum_j j^4 f_j+O(\varphi^6)\right\}$$
$$+i\left\{\varphi\sum_j jf_j-\frac{\varphi^3}{6}\sum_j j^3 f_j+O(\varphi^5)\right\}. \tag{7.14}$$

By using (4.10) and (4.11) (for $s = 1$), we obtain

$$\sum_{j=-2k}^{2k} f_j=0,\quad \sum_{j=-2k}^{2k} j^2 f_j=\frac{1}{12}\left(\frac{2\pi}{n}\right)^2\sum_{j=-2k}^{2k} j^4 f_j+O(n^{-4}),$$
$$\sum_{j=-2k}^{2k} jf_j=\frac{1}{6}\left(\frac{2\pi}{n}\right)^2\sum_{j=-2k}^{2k} j^3 f_j+O(n^{-4}).$$

From this formula and (7.14) it follows that

$$\Phi\left(\frac{4\pi}{n}\right)=\left\{\frac{1}{2}\left(\frac{2\pi}{n}\right)^4\sum_{j=-2k}^{2k} j^4 f_j+O(n^{-6})\right\}+i\left\{\left(\frac{2\pi}{n}\right)^3\sum_{j=-2k}^{2k} j^3 f_j+O(n^{-5})\right\}. \tag{7.15}$$

From the latter and (7.11) and (7.9), we obtain

$$\left|\Phi\left(\frac{4\pi}{n}\right)\right|=\left(\frac{2\pi}{n}\right)^3\left|\sum_{j=-2k}^{2k} j^3 f_j\right|+O(n^{-4})$$
$$\leqslant\left(\frac{2\pi}{n}\right)^3 D_{3,2k}\cdot M+O(n^{-4})\leqslant 4D_{3,2k}\left(\frac{2\pi}{n}\right)^3+O(n^{-4}).$$

Using Lemma 7.3, we find that

$$|\lambda_{2,2}|\geqslant 1-\sqrt{4D_{3,2k}\left(\frac{2\pi}{n}\right)^3+O(n^{-4})},$$

which proves formula (7.2).

Let us prove the right-hand side of (7.3). Let us expand $\Omega(\phi)=\Sigma_{j=-k}^{k}\omega_j e^{ij\phi}$ in a series in n:

$$\Omega(\varphi)=\left\{\sum_j \omega_j-\frac{\varphi^2}{2}\sum_j j^2\omega_j+O(\varphi^4)\right\}+i\left\{\varphi\sum_j j\omega_j+O(\varphi^3)\right\}.$$

It follows from (4.12) that $\Sigma_{j=-k}^{k}\omega_j=0$, whence we obtain with the aid of (7.9) and (7.13) the formula

$$\left|\operatorname{Re}\Omega\left(\frac{4\pi}{n}\right)\right|\leqslant 2\left(\frac{2\pi}{n}\right)^2\left|\sum_{j=-k}^{k} j^2\omega_j\right|+O(n^{-4})$$
$$\leqslant 2MD_{2,k}\left(\frac{2\pi}{n}\right)^2+O(n^{-4})\leqslant 8\,D_{2,k}\left(\frac{2\pi}{n}\right)^2+O(n^{-3}).$$

With the use of Lemma 7.2 we find that

$$|\lambda_{2,2}| \geqslant 1 - 4D_{2,k}\left(\frac{2\pi}{n}\right)^2 + O(n^{-3}),$$

whence follows (7.3).

Let us prove the right-hand side of (7.4). In this case, as was noted in §4, the function $\Phi(\phi)$ (just as $\Omega(\phi)$) will be real. Instead of (7.15) we hence obtain

$$\Phi\left(\frac{4\pi}{n}\right) = \frac{1}{2}\left(\frac{2\pi}{n}\right)^4 \sum_{j=-2k}^{2k} j^4 f_j + O(n^{-6}),$$

whence

$$\left|\Phi\left(\frac{4\pi}{n}\right)\right| \leqslant 2\left(\frac{2\pi}{n}\right)^4 D_{4,2k} + O(n^{-6}).$$

Using Lemma 7.3, we obtain (7.4).

Thus we have completed the proof of Theorem 7.1.

Let us note that in (7.3) and (7.4) the quantity n occurs in the same power in both the right and the left sides of the inequalities; only the constants (which depend on k) in front of n are different. In the case of (7.2) the situation is worse, i.e. the right-hand side has the order $n^{-3/2}$, and the left-hand side the order n^{-2}. It would be desirable to elucidate the true asymptotic behavior of $\mathfrak{L}_k(n)$.

In the formulas (7.3) and (7.4) it would be desirable to sharpen the dependence of the constant in front of $(2\pi/n)^2$ on k. It is easy to see that $4D_{2,k} \sim 8k^3/3$ and $\sqrt{2D_{4,2k}} \sim 16k^{2.5}/\sqrt{10}$ for $k \to \infty$. It is true (in any case for (7.4)) that for $k \to \infty$ this constant is asymptotically proportional to k^2.

From (7.2)–(7.4) we can obtain estimates for the rounding time τ. For example, in the case of a symmetrical problem the minimum value for τ is asymptotically proportional to n^2 for $n \to \infty$.

§8. The case of a symmetrical problem, $k = 1$

Of greater interest is undoubtedly the case of small k, for example, $k = 1$ or 2, and a symmetrical problem. (Let us note that all the available examples are of this type.) In this section we obtain a more exact estimate of the quantity $\mathfrak{L}_1^S(n)$, i.e. the maximum rate of convergence in the case of a symmetrical problem with $k = 1$. In this case formula (7.4) goes over into the following inequality:

$$\left(\frac{2\pi}{n}\right)^2\left(1 - \frac{D}{n^2}\right) < \mathscr{L}_1^S(n) < \sqrt{34}\left(\frac{2\pi}{n}\right)^2\left(1 + \frac{D}{n}\right). \tag{8.1}$$

In (8.1) it is very easy to reduce the constant $\sqrt{34}$ to $\sqrt{3}$. In this section we shall prove

Theorem 8.1. *For $n \to \infty$,*

$$\mathscr{L}_1^S(n) \sim \left(\frac{2\pi}{n}\right)^2.$$

Proof. Since $k = 1$, we obtain $\Phi(\phi) = f_0 + f_1 e^{i\phi} + f_{-1}e^{-i\phi} + f_2 e^{i2\phi} + f_{-2}e^{-i2\phi}$, and since the problem is "symmetrical" we have $f_{-1} = f_1$ and $f_{-2} = f_2$. By virtue of (4.10) we have $f_0 = -2f_1 - 2f_2$; hence

$$\Phi(\varphi) = -4f_1\left(\sin\frac{\varphi}{2}\right)^2 - 4f_2(\sin\varphi)^2.$$

From (4.11) it easily follows that

$$f_1 = -4f_2\left(\cos\frac{\pi}{n}\right)^2 \tag{8.2}$$

whence

$$\Phi(\varphi) = 16f_2\left(\sin\frac{\varphi}{2}\right)^4\left(1 - \left(\frac{\sin\frac{\pi}{n}}{\sin\frac{\varphi}{2}}\right)^2\right). \tag{8.3}$$

Since the eigenvalues are not larger than unity in absolute value, we obtain (by setting $l = n/2$) from (7.8) and (8.3) the formula

$$0 \leqslant f_2 \leqslant \frac{1}{4\left(\cos\frac{\pi}{n}\right)^2}. \tag{8.4}$$

Next, since we are considering a "symmetrical problem," it follows that $\omega_{-1} = \omega_1$ and $\Omega(\phi) = \omega_0 + 2\omega_1\cos\phi$. By setting $\Omega(0) = \omega_0 + 2\omega_1 = \omega$, we easily find that

$$\Omega(\varphi) = \omega - 4\,\omega_1\left(\sin\frac{\varphi}{2}\right)^2. \tag{8.5}$$

From this formula and (7.8) it follows that

$$0 \leqslant \omega \leqslant 4. \tag{8.6}$$

For brevity let us write $b_1^{11} = \alpha_1$ and $b_1^{22} = \alpha_2$. By virtue of (3.3) we have

$$1 - b_0^{11} = 2\alpha_1, \tag{8.7}$$

and from (4.9) it follows that

$$1 - b_0^{22} = \omega + 2\alpha_2. \tag{8.8}$$

Next, from (4.8) and (8.7)–(8.8) we have

$$f_1 = -4\alpha_1\alpha_2 - \alpha_1\omega, \quad f_2 = \alpha_1\alpha_2 - b_1^{12}b_1^{21}.$$

Using (8.2), we hence obtain

$$b_1^{12}b_1^{21} = -\alpha_1\alpha_2\left(\operatorname{tg}\frac{\pi}{n}\right)^2 - \alpha_1\omega\,\frac{1}{4\left(\cos\frac{\pi}{n}\right)^2},$$

whence

$$f_2 = \alpha_1\alpha_2 \frac{1}{\left(\cos\frac{\pi}{n}\right)^2} + \alpha_1\omega \frac{1}{4\left(\cos\frac{\pi}{n}\right)^2}. \tag{8.9}$$

From (4.9) it follows that $\omega_1 = -\alpha_1 - \alpha_2$. From (8.9) we hence obtain

$$\alpha_1^2 - \alpha_1 \frac{\omega - 4\omega_1}{4} + f_2\left(\cos\frac{\pi}{n}\right)^2 = 0.$$

The root of this quadratic (in α_1) equation must be real. Hence

$$\frac{1}{16}(\omega - 4\omega_1)^2 - 4f_2\left(\cos\frac{\pi}{n}\right)^2 \geqslant 0. \tag{8.10}$$

Now let us write $\phi = 4\pi/n$. From (8.3) and (8.5) we obtain

$$\Phi\left(\frac{4\pi}{n}\right) \sim 12 f_2\left(\sin\frac{2\pi}{n}\right)^4,\ \Omega\left(\frac{4\pi}{n}\right) = \omega - 4\omega_1\left(\sin\frac{2\pi}{n}\right)^2$$

$$= (c - 4\omega_1)\left(\sin\frac{2\pi}{n}\right)^2,\ c = \frac{\omega}{\left(\sin\frac{2\pi}{n}\right)^2}, \tag{8.11}$$

and by virtue of (8.6) we have

$$0 < \omega < c. \tag{8.12}$$

By using Theorem 4.1 and formula (4.6), we obtain from (8.11)

$$\lambda_2 = 1 - \left\{\frac{c - 4\omega_1}{2} \pm \sqrt{\frac{(c - 4\omega_1)^2}{4} - 12f_2}\right\}\left(\sin\frac{2\pi}{n}\right)^2 + O(n^{-4}). \tag{8.13}$$

But by (8.4), (8.10) and (8.12) we have for sufficiently large n

$$\frac{(c - 4\omega_1)^2}{4} - 12f_2 \geqslant 4\left(\frac{(\omega - 4\omega_1)^2}{16} - 4f_2\left(\cos\frac{\pi}{n}\right)^2\right) \geqslant 0.$$

Therefore the eigenvalues $\lambda_{2,1}$ and $\lambda_{2,2}$ will be real. It hence follows from (8.13) that

$$1 - |\lambda_{\max}| \leqslant 1 - \lambda_{2,2} = \left\{\frac{c - 4\omega_1}{2} - \sqrt{\frac{(c - 4\omega_1)^2}{4} - 12f_2}\right\}\left(\sin\frac{2\pi}{n}\right)^2 + O(n^{-4})$$

$$= \frac{c - 4\omega_1}{2}(1 - \sqrt{1 - \varepsilon})\left(\sin\frac{2\pi}{n}\right)^2 + O(n^{-4}) = \frac{\sqrt{12f_2}}{\sqrt{\varepsilon}}(1 - \sqrt{1 - \varepsilon})\left(\sin\frac{2\pi}{n}\right)^2 + O(n^{-4}), \tag{8.14}$$

where $\epsilon = 48f_2/(c - 4\omega_1)^2$.

From (8.10) and (8.12) we obtain

$$\varepsilon \leqslant \frac{3}{4\left(\cos\frac{\pi}{n}\right)^2}. \tag{8.15}$$

It is easy to see that the function $(1 - \sqrt{1-\epsilon})/\sqrt{\epsilon}$ increases with ϵ. It hence follows from (8.4) and (8.14)–(8.15) that

$$1 - |\lambda_{\max}| < 2\left(1 - \sqrt{1 - \frac{3}{4\left(\cos\frac{\pi}{n}\right)^2}}\right)\left(\sin\frac{2\pi}{n}\right)^2 + O(n^{-4}) = \left(\frac{2\pi}{n}\right)^2 + O(n^{-4}).$$

From this formula and from the left-hand side of (8.1) follows the theorem.

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