Alexander Petrovich Veselov


View the article online for updates and enhancements.
Alexander Petrovich Veselov  
(on his 60th birthday)

May 4, 2015 was the 60th birthday of Alexander Petrovich Veselov, an excellent mathematician, a brilliant teacher, and a remarkable colleague and friend. Veselov’s scientific interests are unusually broad. Besides his contributions to the theory and development of integrable systems, the main direction of his research activities, he has also made important discoveries in geometry, differential equations and spectral theory, representation theory, and the theory of special and symmetric functions.

Veselov spent his childhood in the village of Volchikhovo near Udomlya, a town in Kalinin (Tver) Oblast. The development of his mathematical talent was encouraged by his mother, a schoolteacher. He had to walk several kilometers to school, and did his homework by the dim light of a kerosene lamp. The beauty of these places is reflected in Isaac Levitan’s famous landscapes March and Above Eternal Peace.

A decisive step in Veselov’s mathematical formation was his enrollment in 1970 in the physics-mathematics Boarding School no. 18 at Moscow State University (now the Advanced Education and Science Centre, also known as Kolmogorov’s Boarding School), as a result of his success in the All-Union Mathematical Olympiad in Simferopol. Andrey Kolmogorov was then actively engaged in the school life, and this had a strong influence on Veselov. Subsequently, Veselov himself taught in this boarding school and organized meetings of school students with leading mathematicians.

The journal Kvant for school students was also founded in 1970. Veselov became an active reader of this journal, and later a contributing author [6]–[8].

In 1972 Veselov enrolled in the Faculty of Mechanics and Mathematics of Moscow State University. Having been interested in geometry since his school days, he first chose A.T. Fomenko as his scientific advisor, but then moved to S.P. Novikov, whose influence was decisive in the development of his mathematical tastes. After finishing his postgraduate studies in 1981, Veselov defended his Ph.D. thesis “Geometry of Hamiltonian systems connected with non-linear partial differential equations”. He became a member of the Landau Institute of Theoretical Physics in

AMS 2010 Mathematics Subject Classification. Primary 01A70.

© 2016 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
Chernogolovka. It can rightly be said that this time was a period in which the institute flourished. The research being done there and the contact with leading theoretical physicists (such as A. A. Belavin, V. A. Fateev, V. A. Kazakov, A. B. Migdal, the brothers Zamolodchikov, and others) significantly influenced Veselov’s future research interests.

In 1984 he transferred to the Department of Higher Geometry and Topology in the Faculty of Mechanics and Mathematics at Moscow State University. After defending his D.Sc. thesis “Geometry of integrable systems with discrete and continuous time” in 1991, he became a professor at the university and held this position until 1995.

His direct work with students revealed his wonderful teaching talents. Veselov gave new breath to the traditional courses of analytic geometry and linear algebra. His unusually brilliant and interesting lectures attracted many students and postgraduates to his special courses. He organized a very successful research seminar, whose meetings often lasted until late in the evening. Among his students in this period were V. M. Goncharenko, A. A. Oblomkov, A. V. Penskoi, M. V. Feigin, and O. A. Chalykh. Yu. Yu. Berest also became an active participant of the seminar at that time.

In addition to his research and teaching duties at the university, Veselov worked with pupils in his own physics-mathematics School no. 18. He taught there for more than ten years, from 1978 to 1988. Besides teaching general courses in geometry and analysis, he supervised a mathematical study group and often accompanied the school team on the traditional festival of young mathematicians in Batumi.

From 1995 to the present Veselov has been a professor at Loughborough University (Great Britain). He has contributed much to the transformation of the mathematical department into an internationally famous research centre in the field of geometry and integrable systems. There are many widely renowned members of the department, including representatives of the Moscow mathematical school such as A. V. Bolsinov, A. I. Neishtadt, and E. V. Ferapontov. Veselov’s love of beautiful mathematics, his impeccable taste, and his encyclopedic knowledge of the subject attract both experts and beginners in mathematics.

On Veselov’s initiative, the London Mathematical Society launched a series of summer mathematical schools for British undergraduates. The first of them took place at Loughborough in 2015. The seminar “Geometry and Mathematical Physics”, supervised by Veselov in Loughborough for more than 20 years, is widely appraised in Great Britain for its distinguished scientific level along with an informal and friendly style of discussions. Seminar meetings sometimes last for more than two hours, after which the scientific discussions are moved to a pub. Another very successful project of Veselov is the “Math Reviews” seminar. The members and postgraduates meet weekly during lunch and discuss various topics in general mathematics. The mathematical “Christmas Challenge” for Loughborough students, which has become a tradition, successfully continues the best traditions of Russian mathematical olympiads, a rarity in Great Britain. Finally, the annual “Integrable Day”, organized by Veselov, has become a genuine visiting card of Loughborough. Every year experts from the Universities of Bath, Cambridge,
Glasgow, London, Kent, Leeds, Oxford, York, and other places eagerly look forward to this event.

Veselov has educated a considerable number of postgraduates in Loughborough, including V. M. Goncharenko (matrix solitons and monodromy-free Schrödinger operators), M.-P. Grosset (elliptic Bernoulli polynomials and the theory of Lamé operators), G. Kemp (magnetic Dirac monopoles and their discrete analogues), M. V. Feigin (quantum integrable systems), A. Hemery (theory of Whittaker–Hill operators and trivial monodromy), W. Haese-Hill (spectral properties of integrable Schrödinger operators with singular potentials), and V. Schreiber (special configurations of vectors in geometry and integrable systems).

It is remarkable that, though very busy, Veselov finds the time and energy for another lifelong fiery passion: school mathematics. For many years, helped only by his postgraduate students, he held weekly mathematical study groups for Loughborough school students, including the now popular mathematical combats. In 2012–2015 he was a member of the Education Committee of the London Mathematical Society.

In speaking of Veselov’s work and life in Loughborough, we should also mention his wife Larisa Evgen’evna and the unusual hospitality of their home towards visitors and friends. Many of us harbour warm memories of the apple trees in their remarkable garden, the piano and guitar evening parties, and their devoted friend Jim, a spaniel.

To describe Veselov’s main results and scientific achievements we should start with his works from the late 1970s. A central theme in mathematical physics of that time was the theory of finite-gap operators developed by Novikov’s school. Veselov actively participated in the programme of development of the finite-gap method of integration.

One of his first papers [1] (1979) was devoted to computing the kernel of the Hamiltonian operator for the Gelfand–Dikii hierarchy and constructing the Hamiltonian formalism for the equations of commutativity of ordinary linear differential operators. Finite-gap integration of equations in one space variable hinges essentially on the property that stationary submanifolds for linear combinations of higher flows of the corresponding hierarchy, in the case of operators of coprime orders, form finite-dimensional invariant subspaces determined by the condition of commutativity of the operators that appear in the Lax pair of the equation and admit a natural algebro-geometric description. One naturally wants to compute the restriction of the Hamiltonian formalism to this submanifold. The Lagrangian (and hence also the Hamiltonian) formalism of the restriction was studied by I.M. Gelfand and L. A. Dikii, but the presence of a non-trivial Hamiltonian operator in the initial system makes it necessary to compute its kernel and verify certain non-degeneracy conditions. Veselov calculated the kernel of this Hamiltonian operator by showing that it is a gradient, which enabled him to obtain a standard variational problem using Lagrange multipliers. He also showed that the non-degeneracy conditions for the Hamiltonian operator always hold in the case of operators of coprime orders and thereby completed the solution of the problem stated above.

The Hamiltonian formalism of the restrictions of soliton systems to stationary solutions of higher equations was originally described in terms of field variables.
However, Veselov and Novikov [2], [3] showed that this formalism takes its most transparent form in terms of ‘spectral’ variables (the spectral curve and a divisor). This gives rise to the so-called ‘analytic’ brackets, the theory of which was developed in [2] and [3]. In [2] this formalism enabled the authors to solve the classical problem of computing the action-angle variables for the Kovalevskaya top, which is traditionally regarded as the most important integrable system of the 19th century.

In 1983 Veselov and Novikov solved the problem of distinguishing those algebrogeometric data that generate two-dimensional Schrödinger operators with zero magnetic field and real potential. The idea that the scattering problem at one level of energy could be used in a natural way for two-space-dimensional soliton equations was proposed in a 1976 paper by B.A. Dubrovin, I.M. Krichever, and Novikov (two-dimensional Schrödinger operators that are ‘finite-gap at one energy’) which made essential use of S.V. Manakov’s formalism of $L$-$A$-$B$-triples. This naturally raised the question of characterizing the spectral data (the complex Fermi curve at the given energy level and a divisor on this curve) that correspond to the important reduction of an identically zero magnetic field (or to potential operators). This problem turned out to be unexpectedly difficult. Its solution was given in [4] in terms of the so-called Cherednik differential and a holomorphic involution with two fixed points (the additional condition of reality of the electric field means that the spectral data are invariant under an anti-holomorphic involution which commutes with the holomorphic one). It is known that the Prym variety of a curve with a holomorphic involution is principally polarized in a natural way (and hence one can construct $\Theta$-functions on the Prymian of the curve) only when the number of fixed points is either 0 or 2. Veselov and Novikov showed in [5] that the Riemann $\Theta$-functions in the formula for the electric field are reduced to $\Theta$-functions on the Prymian in the potential case. Moreover, in [5] they constructed a hierarchy of integrable systems with two spatial variables, which is now known as the Veselov–Novikov hierarchy.

These papers strongly influenced the further development of the theory of integrable systems, exactly soluble linear operators, and connections of the soliton theory with algebraic geometry. In one direction, the scattering theory at a fixed energy was constructed for localized potentials (R.G. Novikov, P.G. Grinevich, Manakov, S.P. Novikov). This reduction and its analogues for discrete systems were then extensively used in the characterization of Prymians of curves (the Riemann–Schottky problem) using methods in the theory of integrable systems (I.A. Taimanov, a complete solution by I.M. Krichever and S. Grushevskii).

Veselov also obtained several fundamental results in the theory of integrable dynamical systems with discrete time and of two-dimensional discrete equations. In this area he originated new directions of research which are still being extensively developed by mathematicians worldwide.

One such direction is connected with the study of Lagrangian systems with discrete time. In [9] Veselov established the integrability of the stationary problem for an anisotropic discrete Heisenberg chain: he found a generating function for the first integrals, described a connection with finite-gap difference Schrödinger operators, and constructed solutions in terms of theta functions. Besides these concrete results, the following simple observation turned out to be conceptually
important: the shift map on the vertices of the chain admits a natural interpretation as a dynamical system. The only previously known example of such an integrable map was billiards on an ellipsoid. Veselov soon came up with a series of analogous examples, which included discretizations of the Euler top and the Neumann and Jacobi systems in dimension $n$. He developed a discrete version of the Lagrangian and Hamiltonian formalism for such maps and established a discrete analogue of the Liouville–Arnold theorem [11], [13]. The integrable systems of classical mechanics arise from these discrete systems in the continuous limit. Conversely, one can regard the discrete system as a special numerical scheme for solving the continuous system. This scheme possesses remarkable properties because of its integrable nature.

Together with J. Moser, Veselov also discovered a general algebraic mechanism underlying most of the interesting examples: representation of the map as a result of factorization of an appropriate matrix polynomial in the spectral parameter followed by a permutation of the factors. Such a transformation preserves the spectrum and thus gives rise to integrals of the motion along with algebro-geometric formulae for the solution that linearize the dynamics on the Jacobian of the spectral curve.

This scheme is known as the Moser–Veselov algorithm and serves as a natural discrete analogue of the Lax representation. It was proposed and studied in [15] and [19] and generalized in [43] to the case of maps with values in the infinite-dimensional group of symplectic diffeomorphisms of the plane. The last two papers were written with Moser, whose collaboration deeply influenced Veselov’s research activities. In [61] Veselov told about his acquaintance with Moser and the circle of problems they discussed.

Another important direction of Veselov’s work in those years was a study of general polynomial and rational maps and correspondences in $\mathbb{C}^n$. Unlike the symplectic case, here it is difficult to give a universal definition of integrability. In [10] he proposed an approach based on the existence of commuting maps. He also discussed the problem of classifying such maps in $\mathbb{C}^n$, constructed an important family of these maps connected with Lie algebras, and investigated their dynamical properties. The results in [10] are non-trivial extensions of results obtained in the beginning of the 20th century for $n = 1$ by G. Julia, P. Fatou, and J.F. Ritt. On the other hand, the problem mentioned is a natural discrete analogue of the theory of commuting differential operators and generalized symmetries for partial differential equations. These theories were extensively developed in mathematical physics after the discovery of the inverse scattering method. Veselov also used the approach with commuting maps to classify polynomial maps in the Cremona group (see [12], where the important connection between integrability of a map and the polynomial growth of the degrees of its iterates was first observed).

We point out a specific feature of the discrete case: multivalued maps or correspondences. They arise naturally and are very important, for example, in Buchstaber’s representation theory of multivalued groups [31]. But what should be meant by a solution of such a correspondence? Multivaluedness plays no great role for differential equations, since distinct branches of the solution can be separated for continuity reasons. But the very concept of a branch loses meaning for discrete equations, and the solution has to be understood as the whole set of quantities related by the given correspondence. The structure of this set is described
by a graph, which can be of various types, from a regular lattice to an $n$-ary tree. Veselov undoubtedly has priority in the analysis of this phenomenon. In [16] and [20] he showed that a test for the integrability of a symplectic correspondence is provided by the confluence of images under the iterates, so that their number grows polynomially. For general algebraic correspondences, this growth is exponential, but the presence of a commuting correspondence restricts the growth even in this case. For single-valued maps, an analogous numerical characteristic is given, for example, by the Arnold complexity of the dynamical system (in the planar case this is the intersection number of a curve with images of another curve).

Results in the study of integrable maps along with conjectures and open problems in this area were described by Veselov in his remarkable surveys [17] and [18], which still give food for thought for experts and young mathematicians alike. After 1992 his interests moved towards discrete analogues of partial differential equations.

Two-dimensional discrete and semidiscrete equations arise in modern mathematical physics both as independent models and as auxiliary tools for studying continuous equations. An instructive example is the theory of Bäcklund transformations, which can be interpreted as correspondences on the infinite-dimensional space of strings. A commutativity condition (discovered by L. Bianchi) is still central in this set-up. The paper [21] by Veselov and A.B. Shabat is devoted to dressing chains, which are differential-difference equations that describe the iterates of the Darboux transform for one-dimensional Schrödinger operators and that commute with the isospectral flow determined by the KdV equation. They constructed a Hamiltonian theory of such a chain and showed that finite-gap Schrödinger operators can be completely described in terms of its periodic solutions. They also made the important observation that a certain weakening of the periodicity condition distinguishes a new class of exactly soluble potentials: generalizations of a harmonic oscillator with linearly growing discrete spectrum which are expressed in terms of the Painlevé transcendents and their higher analogues. These results laid the foundations for extensive further research and generalizations in the theory of integrable equations, the spectral theory of Schrödinger operators, and the theory of Painlevé-type equations. Veselov returned to this theme repeatedly: see [30] and [38] for the multiloop and matrix versions of dressing chains, and [60] for applications to the theory of periodic continued fractions.

Yang–Baxter maps are a related class of equations whose theory has benefited from Veselov’s significant contributions. Together with his co-authors he obtained a number of important results on the dynamical properties of these maps, their Hamiltonian structure, the structure of Lax matrices, and connections with other types of integrable equations such as the quad-equation and the matrix KdV equation ([47]–[49], [51], [55], [56], [65]).

Another important contribution of Veselov and his school is the multidimensional generalization of the finite-gap theory. S.P. Novikov showed in 1974 that one-dimensional Schrödinger operators $L = -\partial^2 + u(x)$ with periodic potentials $u(x)$ admitting a commuting operator $M = \partial^{2m+1} + \cdots$ possess a remarkable spectral property: their spectrum has finitely many gaps. An important example is given by the Lamé operator $L = -\partial^2 + m(m+1)\wp(x, \tau)$, where $\wp$ is the Weierstrass elliptic function.
J. L. Burchnall and T. Chaundy noted as early as the 1920s that if $L$ is a finite-gap operator, then $L$ and $M$ are algebraically dependent: they satisfy a polynomial equation of the form $M^2 = f(L)$, where $f = -t^{2m+1} + \cdots \in \mathbb{C}[t]$. This equation determines a hyperelliptic curve $X_L$, which is called the spectral curve of $L$. Moreover, the stationary Schrödinger equation $L\psi = \lambda \psi$ is explicitly soluble since it can be replaced by the system $L\psi = \lambda \psi$, $M\psi = \mu \psi$ $(\lambda, \mu) \in X_L$), which has rank 1, and therefore can be reduced to a linear homogeneous first-order differential equation. In other words, for general $\lambda$ we obtain a basis of solutions of $L\psi = \lambda \psi$, formed by eigenfunctions of the maximal commutative subalgebra generated by $L$ and $M$ in the algebra of differential operators. This subalgebra is isomorphic to the algebra $O(X_L)$ of regular functions on the spectral curve.\(^1\)

A generalization of the finite-gap theory to the multidimensional case was first undertaken by Krichever and then developed by Veselov and Chalykh in [14]. By definition, an algebraically integrable Schrödinger operator in several variables is an operator of the form

$$L = -\Delta + u(x_1, \ldots, x_n),$$

where $\Delta$ is the Laplacian in $\mathbb{R}^n$ and $u$ is a function on $\mathbb{R}^n$ such that $L$ can be included in a supercomplete algebra of differential operators, that is, a (maximal) commutative subalgebra $A$ of the algebra of differential operators on $\mathbb{R}^n$ such that the eigenvalue problem $M\psi = \lambda(M)\psi$, $M \in A$, has a one-dimensional space of solutions for every character $\lambda \in \text{Spec } A$ (called the spectral variety of the algebra $A$).

Veselov and Chalykh conjectured that algebraic integrability holds for the elliptic Calogero–Moser operator $L$ with potential determined from an arbitrary root system by the formula

$$u = \sum_{\alpha} m_\alpha(m_\alpha + 1)(\alpha, \alpha)\wp(\alpha(x), \tau),$$

where $\alpha$ runs over the positive part of the root system, and the $\mathbb{Z}_+\text{-valued function } m_\alpha = m(\alpha)$ is invariant under the Weyl group. In the case of the root system $A_{n-1}$ the Calogero–Moser system is a system of $n$ pairwise interacting particles on the line. The potentials (1) and their rational and trigonometric degenerations (obtained by replacing $\wp(z, \tau)$ by $z^{-2}$ and $\sin^{-2}z$, respectively, which involves non-crystallographic Coxeter groups $W$ in the rational case) were first introduced in the theory of integrable systems by M. A. Ol’shanetskii and A. M. Perelomov (1977) with arbitrary complex $m_\alpha$. Moreover, Veselov and Chalykh have proved the conjectures for rational and trigonometric degenerations.\(^2\) The proof is based on the important technique of multidimensional Baker–Akhiezer functions developed in these papers. A generalization of these results to the case of finite reflection

---

\(^1\)A complete theory of commutative subalgebras of the algebra of differential operators in one variable was developed by Krichever in the late 1970s.

\(^2\)In the elliptic case this conjecture was proved for type $A$ by A. Braverman, P.I. Etingof, and D. Gatesgory in 1997. They used a result due to G. Felder and A. Varchenko, who computed the spectral variety of this system. In 2002 Oblomkov, Chalykh, and Etingof proved the conjecture in the general form using a result of I. Cherednik (1994) on the ordinary quantum integrability of this system. Note that in the one-dimensional case this reduces to the finite-gap property of the Lamé operator.
groups was obtained in [23]. Later, Veselov and co-authors found new interesting examples, including the deformed Calogero–Moser systems (see below).

An interesting development of these papers took place in the subsequent 25 years. First of all, the rings $A$ of quantum integrals of the Calogero–Moser operators $L$ as well as their spectral varieties turned out to be very interesting in the rational and trigonometric cases. Namely, the algebra $A = Q_m$ of quantum integrals of the rational system turned out to be the algebra of $m$-quasi-invariant polynomials of the finite Coxeter group $W$. By definition, this is the subalgebra of all polynomials $f$ on the geometric representation $\mathbb{R}^n$ of $W$ such that $f(x) - f(s_\alpha x)$ is divisible by $\alpha(x)^{2m_\alpha + 1}$ for all roots $\alpha$ ($s_\alpha$ is the reflection corresponding to $\alpha$).

In [44] Veselov and Feigin proposed studying the algebraic properties of $Q_m$. In particular, they conjectured that it is Cohen–Macaulay and, moreover, Gorenstein. They proved this in the case of rank 2 and a constant function $m_\alpha$. In [46] Veselov and Felder later calculated the Hilbert series of $Q_m$. Since it is Stanley-palindromic, the Gorenstein property would follow whenever the Cohen–Macaulay property is known. Subsequently, V. Ginzburg and Etingof and then Ginzburg, Etingof, and Berest proved that $Q_m$ is Cohen–Macaulay (and hence Gorenstein) using the representation theory of the Cherednik rational algebra. It was also shown in the last paper that the ring of differential operators on $Q_m$ is Morita-equivalent to the ring of differential operators on an affine space (as had been conjectured by Berest). Later D. Ben-Zvi and T. Nevins showed that it is a general property of manifolds with a bijective smooth normalization (in particular, the spectral variety $\text{Spec } Q_m$ has this property). Berest and Chalykh extended the theory of quasi-invariants to complex reflection groups and established its connections with the representation theory of Cherednik rational algebras for such groups. Finally, Chalykh generalized the theory of quasi-invariants and Baker–Akhiezer functions to the $q$-deformed case and used this generalization to obtain a new proof of Macdonald’s conjectures. Thus, the theory of quasi-invariants stimulated the development of several directions in representation theory, combinatorics, and algebraic geometry.

In 1994 Buchstaber, Veselov, and Felder [27] defined elliptic Dunkl operators and proposed using them in proving the quantum integrability of the elliptic Calogero–Moser system. A proof based on a similar approach was given by Cherednik at nearly the same time.

Veselov’s broad scientific interests and remarkable intuition manifested themselves in [32]. An essential tool in [27] was a functional equation whose complete analytic solution was found there. The elliptic functions involved in this solution play an important role in the theory of Hirzebruch genera of manifolds. Veselov suggested looking for topological applications of the results in [27]. As a result, he and Buchstaber [32] discovered a connection between the classical transformations of elliptic functions and the Adams–Novikov operators in complex cobordism theory. A consequence of this result was a surprising relation between the signature of a stably complex manifold and the signatures of its submanifolds which are dual to the Pontryagin classes in cobordisms of the manifold. This direction was further developed by K. E. Feldman and others.
The approach in [27] to the integrability of the quantum elliptic Calogero–Moser system was realized and generalized to crystallographic complex reflection groups in [66], where there are new examples of elliptic quantum integrable systems.

At the same time Veselov, Feigin, and Chalykh found new examples of quantum integrable Schrödinger operators: deformed Calogero–Moser systems [34], [35]. These systems (which are purely quantum and have no integrable classical analogues) possess a rich structure, including their connections with the theory of classical Lie superalgebras (developed by Veselov and A.N. Sergeev) and with representations of Cherednik algebras (established by Feigin).

For special values of the parameters some of these systems admit additional quantum integrals, and the algebra of quantum integrals can be supercomplete, that is, can admit an eigenfunction which is a multidimensional analogue of the stationary rational Baker–Akhiezer function of the form \( \psi(k, x) = e^{(k,x)}Q(k, x) \), where \( Q \) is a rational function. The theory of such algebraically integrable operators was developed in [39] and [45]. It was further investigated in later works by Sergeev and Veselov and by E. Rains and P. Etingof and is presently the subject of extensive studies.

Algebraically integrable multidimensional Schrödinger operators turned out to be closely related to Huygens’ principle. The question of Huygens’ principle was stated in exact mathematical terms by J. Hadamard in his Lectures on the Cauchy problem (1923). Physically, this phenomenon means that a wave propagating from an initial localized perturbation does not leave a ‘tail’ behind its fastest front. According to Hadamard, this property is expressed mathematically by the vanishing of the fundamental solution of a hyperbolic wave operator inside the propagation domain. Hadamard’s original question “Which wave operators satisfy Huygens’ principle?” appeared to be a difficult mathematical problem. It was thought for a long time that the classical d’Alembertian \( \Box_{n+1} = \partial^2_t - \partial^2_{x_1} - \cdots - \partial^2_{x_n} \) in odd-dimensional spaces \( n \geq 3 \) is the only Huygens operator (modulo trivial transformations). This conjecture (sometimes referred to as Hadamard’s conjecture) was partially supported by a theorem of M. Mathisson (1939), who proved that for \( n = 3 \) the operator \( \Box_{3+1} \) is indeed the only Huygens operator among all second-order hyperbolic operators with constant principal symbol. Nevertheless, ‘Hadamard’s conjecture’ does not hold in higher dimensions: the first counterexamples for \( n = 5 \) were constructed by K. Stellmacher in 1953, and in his later joint papers with J. Lagnese (1960s) he was able to describe all Huygens operators of the form \( \mathcal{L} = \Box_{n+1} + V(x) \), where \( V(x) \) is an algebraic potential depending on one variable (say, \( x = x_1 \)). It is remarkable that the Stellmacher–Lagnese potentials coincide precisely with the Adler–Moser potentials which determine rational solutions of the well-known KdV equation. This coincidence indicated a close relation between Huygens’ principle and integrability, and suggested that methods and results in the theory of integrable systems could be applied to the study of Hadamard’s problem. During the last 20 years Veselov and his collaborators have confirmed this intuitive suggestion and strengthened and extended the connection with integrable systems.

The first new results on Hadamard’s problem, inspired by the connection with integrable systems, appeared in joint papers with Berest (see [24] and [25]) which described a new class of Huygens operators \( \mathcal{L} = \Box_{n+1} + V \) with potentials
$V = V(x_1, \ldots, x_n)$ depending on arbitrarily many variables. Namely, $V$ is a rational degeneration of the elliptic Calogero–Moser potential (1). The main theorem in [24] and [25] states that for each pair $(W, m)$ as above, the wave operator $\mathcal{L}$ is Huygens if $n$ is odd and $n \geq 3 + 2 \sum m_\alpha$. In the case of rank 1 this result covers Stellmacher’s examples. The proofs in [24] and [25] are based on a remarkable connection between the Baker–Akhiezer function for the Schrödinger operator $L = -\Delta + V$ and the fundamental solution of the corresponding wave operator $\mathcal{L}$. Roughly speaking, the Baker–Akhiezer function and the fundamental solution are constructed in a universal way from the same ingredients: a sequence of functions $\{U_\nu(x, \xi): \nu \geq 0\}$ called Hadamard coefficients. The finiteness of this sequence (that is, the vanishing of $U_\nu$ for $\nu \gg 0$) guarantees the algebraic integrability of the Schrödinger operator on the one hand, and Huygens’ principle for the corresponding wave operator on the other. Connections of this type have played a key role in all the subsequent studies of Hadamard’s problem using the methods of integrable systems and representation theory. For example, this approach was used in [22] and [33] to prove Huygens’ principle for modified wave equations on certain symmetric spaces of non-zero curvature.

It was conjectured in [25] that the Coxeter potentials (along with the one-dimensional Stellmacher–Lagnese potentials) give a complete solution of Hadamard’s problem in the class of wave operators of the form $\mathcal{L} = \Box_{n+1} + V$. But this turned out not to be the case. In joint papers with Feigin and Chalykh ([34], [35], [39]) Veselov discovered surprising new examples of algebraically integrable Schrödinger operators connected with non-Coxeter configurations of hyperplanes. The corresponding potentials come from (very special) deformations of the Coxeter root systems of type $A$ and $B$, and in the case of integer multiplicities they give new examples of Huygens operators. The investigations of Veselov, Feigin, and Chalykh motivated Berest and I.M. Lutsenko (1997) to find another family of non-Coxeter examples of two-dimensional potentials depending on continuous parameters. These examples may be regarded as deformations of the Coxeter dihedral system by means of KdV soliton flows acting in the angle variable. In all the cases the Huygens potentials are rational functions with singularities along a special divisor of hyperplanes in $\mathbb{C}^n$. This situation was axiomatized in [39] by introducing the notion of a locus configuration and the corresponding Baker–Akhiezer function. A locus configuration is an arrangement of hyperplanes in $\mathbb{C}^n$ satisfying a certain explicit overdetermined system of algebraic equations which are called locus equations. If the locus equations admit a solution (that is, the locus configuration is non-empty), then there exist a related Baker–Akhiezer function and an explicit potential $V$ (with singularities along the locus configuration) such that the wave operator $\mathcal{L} = \Box_{n+1} + V$ satisfies Huygens’ principle. Based on earlier joint papers with Berest ([36] and [41]), Veselov and his co-authors were able to prove the converse in [39]. Namely, when all the Hadamard coefficients are rational functions, the potential $V$ is Huygens if and only if it corresponds to some locus configuration. Thus, the original analytic problem of finding and classifying Huygens operators was reduced to an algebraic problem of finding and classifying locus configurations. Due to the efforts of Veselov and his co-authors, locus configurations have been carefully studied in the last years, and new interesting connections with other areas
of mathematics have been found (see, for example, [50]). The relation between
Huygens’ principle and integrability was the theme of Veselov’s talk at the 2nd
European Mathematical Congress in Budapest (1996), a brief survey of which is
given in [37]. A more detailed survey covering the classical work on Huygens’
principle can be found in [26].

A number of Veselov’s papers (mainly joint with Felder) concern the theory of
Coxeter groups. Veselov’s contribution is based on combining the classical geo-
metric and topological principles of this theory with modern mathematical physics.
He clarified and used the deep connections between the geometry and topology
of complements to reflection hyperplanes of Coxeter groups, the theory of quan-
tum integrable Calogero–Moser systems, and the Knizhnik–Zamolodchikov equa-
tion. In the $A_n$-case, the Calogero–Moser system is a system of $n + 1$ particles
with pairwise interactions on the line, and the Knizhnik–Zamolodchikov equation
is the equation for correlation functions of a two-dimensional conformal field theory,
the Wess–Zumino–Novikov–Witten model with the group $U(n + 1)$. Both ingre-
dients extend to crystallographic Coxeter groups, and in the rational case to all
Coxeter groups.

It was shown in [28] and [29] that the shift operator introduced by E. M. Opdam
and G. Heckman, which connects eigenfunctions of the Schrödinger operator
with different coupling parameters, has a very natural interpretation in terms of
the Matsuo–Cherednik isomorphism between these eigenfunctions and solutions
of the Knizhnik–Zamolodchikov equation. In [46] the authors used a version of
the Matsuo–Cherednik isomorphism in the degenerate limit of zero spectral
parameter to calculate the Poincaré polynomial of the space of $m$-harmonic
polynomials for an arbitrary isotypic component of the Coxeter group action. These
polynomials generalize the harmonic polynomials from the theory of Coxeter groups
and are polynomial eigenfunctions of the quantum integrals of Calogero–Moser
systems. They were introduced in [44]. Motivated by the Matsuo–Cherednik
correspondence, Veselov and Felder found a complete set of polynomial solutions
of the Knizhnik–Zamolodchikov equations and a formula for the Baker–Akhiezer
function of Calogero–Moser systems with integer parameters in terms of iterated
residues [57], [62].

In [52] Veselov and Felder introduced the geometric notion of a special involution
in a Coxeter group $G$. This notion is central in the description of the Coxeter group
action on the cohomology of the complement to the complexified configuration of
reflection hyperplanes. It is well known that the dimension of the cohomology space
is equal to the order of the Coxeter group $G$, but it is perhaps surprising that, as
shown by G. I. Lehrer and L. Solomon, this representation is in general not isomor-
phic to the regular representation $\rho$. A universal description of this representation
for all Coxeter groups was obtained in [52] in the form of a sum $\sum_a (2 \operatorname{Ind}_1^2 1 - \rho)$
over the conjugacy classes of the special involutions of representations induced from
the trivial representation of the subgroups generated by involutions. In particular,
it was shown that the sign representation never occurs. The authors also showed
that the cohomology Poincaré polynomial of the corresponding Artin braid group
has the form $P(t) = \sum_a t^{d(a)}$, where the sum is taken over the classes of special
involutions and $d(a)$ is the dimension of the eigenspace with eigenvalue $(-1)$ in the
geometric realization. The authors explained this by giving a conjectural cohomology basis labelled by special involutions. They proved this conjecture in all the cases but $E_7$, $E_8$, $F_4$, $H_3$, and $H_4$.

In [67] Veselov together with Felder and his postgraduate student L. Aguirre considered a universal version of the Gaudin integrable spin chain. In this situation, the Gaudin system is an Abelian Lie subalgebra of maximal dimension $n - 1$ in the space $V_n$ spanned by the generators of the Kohno–Drinfeld Lie algebra $t_n$. In a natural way the set of Gaudin systems forms a submanifold in the Grassmannian of $(n - 1)$-dimensional planes in $V_n$. It was proved in [67] that this submanifold is smooth and isomorphic to the Deligne–Mumford moduli space $\mathcal{M}_{0,n+1}$ of stable rational curves with $n + 1$ marked points. Moreover, a geometric description was given for the tautological bundle whose fibres are Gaudin systems. In the recent paper [73] these results were extended to general Coxeter configurations and the corresponding De Concini–Procesi compactifications of the projectivized complements to Coxeter configurations of hyperplanes.

In a series of joint papers with Sergeev, Veselov developed a new approach to the study of deformed integrable Calogero–Moser–Sutherland systems. Their fruitful collaboration began in 2001 when they (being former fellow students at Kolmogorov’s Boarding School and former neighbours in the hostel of the Faculty of Mechanics and Mathematics at Moscow State University) met at the Newton Institute in Cambridge during the scientific programme “Symmetric Functions 2001: Survey of Developments and Perspectives”.

Sergeev’s fresh results on Jack superpolynomials suggested possible deep connections between the theory of Lie superalgebras and the theory of deformed Calogero–Moser systems. Their subsequent joint research resulted in the paper [50], which confirmed this conjecture. More precisely, a deformed Calogero–Moser–Sutherland operator was constructed for each simple classical Lie superalgebra with its associated generalized root system (introduced previously by V. Serganova). The main objects for further studies were also introduced there: admissible deformations of generalized root systems and the corresponding deformed second-order operator, generalized algebras of invariants, and their difference analogues. The case of the generalized root system $A(n, m)$ was considered in more detail in [53] (the differential version) and [63] (the corresponding difference version). The paper [54], which appeared as a byproduct, was devoted to the investigation of Jack polynomials for special values of the parameter in connection with the classical Sylvester–Cayley question of describing polynomials with prescribed multiplicities of roots.

These papers showed that there are deep connections between Lie superalgebras and deformed quantum integrable systems, connections which can be used to develop both directions. In particular, the study of infinite-dimensional analogues of deformed integrable systems made it possible to explain the integrability of the deformed Macdonald–Ruijsenaars difference operator and the deformed differential operator of type $BC$, and also to obtain explicit formulae for their eigenfunctions [63], [64]. These eigenfunctions depend on several parameters. A natural question is how to interpret specializations of eigenfunctions from the point of view of representation theory. In the case of classical Jack polynomials it is known that they specialize either to characters of irreducible finite-dimensional representations
of Lie algebras or to spherical functions associated with finite-dimensional representations, and the corresponding second-order operators are interpreted as the radial parts of the Laplace operators.

The situation for deformed Calogero–Moser–Sutherland operators is still not completely clear, because simple Lie superalgebras can possess reducible indecomposable representations. Partial results in this direction were obtained in [68] and [69]. It was proved in [68] that for deformed operators of type $BC$ there is a natural specialization of eigenfunctions to Euler characters. The paper [69], which was published in *Annals of Mathematics*, contains a description of the representation rings of the classical Lie superalgebras as invariants of a certain groupoid arising naturally from the point of view of deformed quantum integrable systems.

The connection with the theory of Lie superalgebras made it possible to construct a natural generalization of classical symmetric Jack functions and to describe the action of the algebra of integrals in generalized eigenspaces for certain values of the parameters [70], [72].

In [71] the infinite-dimensional analogues of Dunkl operators were used to find Lax pairs for deformed Calogero–Moser–Sutherland operators corresponding to the classical series of generalized root systems. This led to a simpler proof of integrability of these operators.

Veselov also made a significant contribution to the theory of Frobenius manifolds. In 1999 he noted that generalized root systems enable one to obtain new solutions of the Witten–Dijkgraaf–Verlinde–Verlinde equations (the associativity equations), and he introduced the notion of a $\vee$-system [40]. This is a set of vectors satisfying natural linear-algebraic conditions which are equivalent to the associativity equations for the corresponding prepotential with logarithmic singularities along the configuration of hyperplanes corresponding to the given $\vee$-system. In the case of root systems, such prepotentials are Dubrovin-dual to the polynomial Frobenius structure on the space of orbits of the Coxeter group. They also arise in the Seiberg–Witten theory. In [42] Veselov and Chalykh found $n$-parametric $\vee$-systems that deformed the classical root systems $A_n$ and $B_n$. In [58] Veselov and Feigin found a natural interpretation of such solutions in terms of Strachan’s Frobenius structures on the discriminants of Coxeter groups, thus extending the Dubrovin duality to discriminant strata, which led to new classes of $\vee$-systems. In [58] and [59] Veselov and Feigin showed that the class of $\vee$-systems is closed under the natural operations of taking subsystems and restrictions, and in the recent work [74] they investigated the subclass of harmonic $\vee$-systems and considered its connections with the theory of Saito-free configurations of hyperplanes. A complete description of $\vee$-systems is still an interesting open question.

Veselov presently continues his active and eventful mathematical life, participating in many scientific activities in Great Britain, Russia, and other countries. He is a member of the editorial boards or editorial councils of the journals *Funktsional’nyi Analiz i ego Prilozheniya*, *Journal of Integrable Systems, Regulyarnaya i Khaoticheskaya Dinamika*, and *Journal of Nonlinear Mathematical Physics.*

---

3Translated as *Functional Analysis and its Applications.*

4Translated as *Regular and Chaotic Dynamics.*
We wish Alexander Petrovich Veselov robust health and many more active years.


Cited papers of A. P. Veselov


“Periodic continued fractions and hyperelliptic curves”, J. Lond. Math. Soc. (2) 77:3 (2008), 593–606. (with M.-P. Grosset)


Translated by A. DOMRIN