Self-adjoint commuting differential operators of rank two

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Self-adjoint commuting differential operators of rank two

A. E. Mironov

Abstract. This is a survey of results on self-adjoint commuting ordinary differential operators of rank two. In particular, the action of automorphisms of the first Weyl algebra on the set of commuting differential operators with polynomial coefficients is discussed, as well as the problem of constructing algebro-geometric solutions of rank $l > 1$ of soliton equations.

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Keywords: commuting differential operators of rank two, self-adjoint operators, Weyl algebra.

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1. Introduction

The purpose of this paper is to give a survey of recent results about commuting ordinary differential operators of rank $l > 1$.

The first steps towards a classification of commuting ordinary differential operators of rank one were made by Burchnall and Chaundy in the 1920s (see [1]–[3]). In particular, they proved the familiar lemma on the existence of a polynomial that annihilates two commuting operators. The classification problem was solved completely by Krichever in the late 1970s [4], [5] (see the papers by Drinfeld [6] and Mumford [7] for algebraic aspects of this classification).
The coefficients of the operators of rank one as well as their common eigenfunctions are expressed in terms of the theta function of the Jacobian variety of the spectral curve (see [4]).

The case of operators of rank \( l > 1 \) is very complicated. Their common eigenfunctions have not been found explicitly. In 1979 Krichever and Novikov [8], [9] used the method of deformation of Tyurin’s parameters to find operators of rank two in the case of an elliptic spectral curve. Later Mokhov used the same method to find operators of rank three, also corresponding to elliptic spectral curves [10], [11]. For a long time, no examples of commuting operators of rank more than one were known in the case of spectral curves of genus \( g > 1 \). The first examples of this kind were found in [12]–[14] for \( l = 2 \) and \( g = 2, 3, 4 \) and in [15] for \( l = 3 \) and \( g = 2 \).

Formally self-adjoint operators of rank two and orders \( 4g + 2 \) were studied in [16]–[30] in the case of hyperelliptic spectral curves of genus \( g \). In particular, these papers contained the following results:

- examples of operators of rank two for \( g > 1 \);
- with the help of coordinate changes and automorphisms of the first Weyl algebra, examples of operators of arbitrary rank \( l > 1 \) for \( g > 1 \) were constructed from operators of rank two;
- a connection was established between certain special functions (eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials, Bessel functions, Heun functions) and the Baker–Akhiezer function of rank two;
- for the action of the automorphisms of the first Weyl algebra on the set of commuting operators with polynomial coefficients for a fixed spectral curve, the set of orbits was shown to be infinite for a certain class of hyperelliptic spectral curves (in particular, for all elliptic curves and generic curves of genus two).

We shall give a survey of these results.

There are many interesting unsolved problems in the theory of commuting differential operators of rank \( l > 1 \). We mention one of them. Commuting differential operators have applications to soliton equations. Soliton equations arise from conditions for the compatibility of auxiliary linear problems. Virtually all the exact algebro-geometric solutions of soliton equations which were found in the last decades are solutions of rank one. This means that the auxiliary linear operators commute with operators of rank one. At the same time, Krichever and Novikov [9] (see also [31]) showed that soliton equations may have solutions of rank \( l > 1 \). After [9] and [31] there has been no essential progress in the construction of such solutions for spectral curves of genus \( g > 1 \). This difficult and interesting problem remains open, and we discuss it for a special evolution system in \( \S \, 3.3.5 \).

The paper is organized as follows.

In \( \S \, 2 \) we consider operators of rank one. In \( \S \, 2.2 \) we recall some spectral properties of finite-gap one-dimensional Schrödinger operators (which will be contrasted in \( \S \, 3.3.1 \) with the spectral properties of fourth-order self-adjoint operators of rank two) and consider the finite-gap Lamé and Treibich–Verdier operators as examples.

Self-adjoint operators of rank two are considered in \( \S \, 3 \), with \( \S \, 3.2 \) devoted to the case of elliptic spectral curves. Here we discuss the Krichever–Novikov operators with polynomial coefficients and rank-two solutions of the Kadomtsev–Petviashvili equation. In \( \S \, 3.3 \) we consider operators in the case of hyperelliptic spectral curves.
2. Operators of rank one

2.1. The Baker–Akhiezer function. The condition of commutation of ordinary differential operators

\[ L_n = \sum_{k=0}^{n} u_k(x) \partial_x^k, \quad L_m = \sum_{j=0}^{m} v_j(x) \partial_x^j \]

is equivalent to a complicated system of non-linear differential equations in the coefficients. By a change of variables and a suitable conjugation, one can ensure that \( u_n = 1 \) and \( u_{n-1} = 0 \). Then we can assume that \( v_m = 1 \) and \( v_{m-1} = \text{const} \). The first results on the commutation equations are due to Wallenberg [32]. He integrated the equations for \( n = 2 \) and \( m = 3, 5 \).

In 1905, Schur [33] proved the following lemma.

**Lemma 1.** If \( L_n L_m = L_m L_n \) and \( L_n L_k = L_k L_n \), where \( n \geq 1 \), then \( L_m L_k = L_k L_m \).

This lemma means that the set of operators commuting with a given operator forms a ring. To prove this lemma, Schur used formal pseudodifferential operators of the form

\[ u_n(x) \partial_x^n + \cdots + u_0(x) + u_{-1}(x) \partial_x^{-1} + u_{-2}(x) \partial_x^{-2} + \cdots. \]

Such operators form an associative ring. Their multiplication is defined by the natural requirement

\[ \partial_x^n \circ \partial_x^{-m} = \partial_x^{-m} \circ \partial_x^n = \partial_x^{n-m} \]

along with the Leibniz rule

\[ \partial_x^n \circ f = \sum_{j \geq 0} \binom{n}{j} (\partial_x^j f) \circ \partial_x^{n-j}, \]

where \( \binom{n}{j} = \frac{n!}{j!(n-j)!} \) for \( 0 \leq j \leq n \) and \( \binom{n}{j} = 0 \) for \( j > n \). It is easily verified that elements of the form

\[ S = 1 + s_1(x) \partial_x^{-1} + s_2(x) \partial_x^{-2} + \cdots \]

are invertible. For every differential operator \( L_n \) of order \( n \) there exists an operator \( S \) of this form such that \( S^{-1} \circ L_n \circ S = \partial_x^n \). The coefficients of \( S \) are found successively from the equation

\[ L_n \circ S = S \circ \partial_x^n. \]

Consider the two pseudodifferential operators

\[ \widetilde{L}_m = S^{-1} L_m S, \quad \widetilde{L}_k = S^{-1} L_k S. \]

It follows from the hypothesis of the lemma that

\[ \partial_x^n \widetilde{L}_m = \widetilde{L}_m \partial_x^n, \quad \partial_x^n \widetilde{L}_k = \widetilde{L}_k \partial_x^n. \]
We see from direct calculations that if a pseudodifferential operator commutes with \( \partial_x^n \), then its coefficients are constant. Therefore, \( \hat{L}_m \) and \( \hat{L}_k \) commute. It follows that \( L_m \) and \( L_k \) commute.

The following lemma was proved by Burchnall and Chaundy \cite{1} in 1924.

**Lemma 2.** If \( L_n L_m = L_m L_n \), then there exists a non-zero polynomial \( R(z, w) \) such that \( R(L_n, L_m) = 0 \).

The set of zeros of \( R \) parametrizes the joint eigenvalues of \( L_n \) and \( L_m \). Namely, if \( \psi \) is a common eigenfunction of them, that is,

\[
L_n \psi = z \psi, \quad L_m \psi = w \psi,
\]

then \( z \) and \( w \) satisfy the equation \( R(z, w) = 0 \). The *spectral curve* of the operators \( L_n \) and \( L_m \) is defined as the smooth compactification of the complex algebraic curve in \( \mathbb{C}^2 \) given by the equation \( R(z, w) = 0 \).

Suppose that \( L_n \) and \( L_m \) are commuting differential operators of orders \( n \) and \( m \). The integer

\[
l = \dim \{ \psi : L_n \psi = z \psi, \ L_m \psi = w \psi, \ P = (z, w) \in \Gamma \}
\]

is a divisor of \( n \) and \( m \) for a generic point \( P \in \Gamma \). The *rank* of the operators \( L_n \) and \( L_m \) is defined as the greatest common divisor of the orders of all the operators in the maximal commutative ring containing \( L_n \) and \( L_m \). In this section we consider the case of operators of rank one. Hence we may assume that \( (n, m) = 1 \). The eigenfunctions of operators of rank one are the Baker–Akhiezer functions found by Krichever \cite{4}. They are constructed from the following *spectral data*:

\[
\{ \Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_g \},
\]

where \( \Gamma \) is a Riemann surface of genus \( g \), \( q \in \Gamma \) is a distinguished point, \( k^{-1} \) is a local parameter near \( q \), and \( \gamma_1 + \cdots + \gamma_g \) is a non-special divisor. There exists a unique function \( \psi(x, P) \), \( P \in \Gamma \), satisfying the following conditions:

1) near \( q \) the function \( \psi \) has the form

\[
\psi = e^{xk} \left( 1 + \frac{\xi_1(x)}{k} + \frac{\xi_2(x)}{k^2} + \cdots \right);
\]

2) \( \psi \) is meromorphic on \( \Gamma \setminus \{q\} \), with simple poles at \( \gamma_1, \ldots, \gamma_g \).

The commutative ring of differential operators corresponding to the spectral data is isomorphic to the ring of meromorphic functions on \( \Gamma \) with a pole at \( q \). Let \( f(P) \) be a meromorphic function on \( \Gamma \) with a unique pole in a neighbourhood of \( q \) which has order \( n \). Then there is a differential operator

\[
L(f) = \partial_x^n + u_{n-1}(x)\partial_x^{n-1} + \cdots + u_0(x)
\]

such that

\[
L(f)\psi - f(P)\psi = e^{kx} O \left( \frac{1}{k} \right).
\]

The coefficients \( u_i(x) \) are found from \( \xi_i(x) \). The right-hand side of the last equality vanishes, since otherwise we can add it to the Baker–Akhiezer function and obtain
a new function satisfying the conditions 1) and 2). Thus, by uniqueness we have
\[ L(f)\psi = f(P)\psi. \] For another meromorphic function \( g(P) \) with a pole at \( q \) we similarly have \( L(g)\psi = g(P)\psi \). The operators \( L(f) \) and \( L(g) \) commute.

**Example 1.** Let \( \Gamma \) be an elliptic curve:
\[ \Gamma = \mathbb{C}/\{2m\omega_1 + 2n\omega_2, n, m \in \mathbb{Z}\}. \]
The Baker–Akhiezer function corresponding to the spectral data \( \{\Gamma, 0, -z, -\gamma\} \) is given by
\[ \psi(x, z) = e^{-x\zeta(z)} \frac{\sigma(z + x + \gamma)\sigma(\gamma)}{\sigma(z + \gamma)\sigma(x + \gamma)}. \]
We have
\[ (\partial_x^2 - 2\wp(x + \gamma))\psi(x, z) = \wp(z)\psi(x, z). \]
Here \( \zeta(z) \), \( \sigma(z) \), and \( \wp(z) \) are Weierstrass elliptic functions. The operator \( \partial_x^2 - 2\wp(x + \gamma) \) commutes with the operator
\[ \partial_x^3 - 3\wp(x + \gamma)\partial_x - \frac{3}{2}\wp'(x + \gamma). \]

In the general case the Baker–Akhiezer function is expressed in terms of the theta function of the Jacobian variety of the spectral curve \( \Gamma \) [4]. We choose a basis of cycles \( a_i, b_i, 1 \leq i \leq g \), on \( \Gamma \) with the following intersection numbers:
\[ a_i \circ a_j = 0, \quad b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}. \]
Let \( \omega_1, \ldots, \omega_g \) be the basis of Abelian differentials normalized by the condition \( \int_{a_i} \omega_j = \delta_{ij} \). The theta function of the Jacobian variety \( J(\Gamma) = \mathbb{C}^g/\{\mathbb{Z}^g + \Omega\mathbb{Z}^g\} \) is given by the series
\[ \theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i \langle \Omega n, n \rangle + 2\pi i \langle n, z \rangle), \quad z \in \mathbb{C}^g, \]
where \( \langle n, z \rangle = n_1 z_1 + \cdots + n_g z_g \) and the entries of the matrix \( \Omega \) are
\[ \Omega_{ij} = \Omega_{ji} = \int_{b_i} \omega_j. \]
The theta function satisfies
\[ \theta(z + \Omega m + n) = \exp(-\pi i \langle \Omega m, m \rangle - 2\pi i \langle m, z \rangle)\theta(z), \quad m, n \in \mathbb{Z}^g. \]
Let \( \omega \) be a meromorphic 1-form on \( \Gamma \) with a unique pole of order two at \( q \). We normalize \( \omega \) by the condition \( \int_{a_i} \omega = 0 \). Let \( U \) be the vector of \( b \)-periods of \( \omega \):
\[ U = \left( \int_{b_1} \omega, \ldots, \int_{b_g} \omega \right). \]
We write \( A(P): \Gamma \to J(\Gamma) \) for the Abel map

\[
A(P) = \left( \int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g \right),
\]

where \( P_0 \) is a fixed point. The function

\[
\varphi(x, P) = \frac{\theta(A(P) - A(\gamma_1) - \cdots - A(\gamma_g) - K_\Gamma + xU)}{\theta(A(P) - A(\gamma_1) - \cdots - A(\gamma_g) - K_\Gamma)} \exp\left(2\pi i x \int_{P_0}^P \omega \right),
\]

where \( K_\Gamma \) is the vector of Riemann constants, is well defined on \( \Gamma \). The function \( \varphi \) has poles at \( \gamma_1, \ldots, \gamma_g \), and near \( q \) it has the form

\[
\varphi = e^{kx} \left( \xi_0(x) + \frac{\xi_1(x)}{k} + \cdots \right).
\]

After the normalization \( \psi(x, P) = \varphi(x, P)/\xi_0(x) \) we obtain the Baker–Akhiezer function.

### 2.2. Finite-gap Schrödinger operators.

Consider the first-order equation that annihilates the Baker–Akhiezer function \( \psi \):

\[
(\partial_x - \chi(x, P))\psi(x, P) = 0, \quad \chi(x, P) = \frac{\psi_x(x, P)}{\psi(x, P)}.
\]

The function \( \chi \) is meromorphic on \( \Gamma \) with simple poles \( P_1(x), \ldots, P_g(x) \) and \( q \). Near \( q \) it has the form

\[
\chi(x, P) = k + O(1).
\]

Let \( k - \gamma_i(x) \) be a local parameter near \( P_i(x) \). Then

\[
\chi(x, P) = \frac{-\gamma'_i(x)}{k - \gamma_i(x)} + O(1).
\]

The operator \( \partial_x - \chi \) is a right divisor of the operator \( L(f) - f(P) \):

\[
L(f) - f(P) = \tilde{L}(\partial_x - \chi),
\]

where \( \tilde{L} \) is a differential operator whose order is smaller by one than the order of \( L(f) \). The coefficients of \( \tilde{L} \) depend on \( P \).

Let us find a decomposition into factors of the form (1) in the case when \( L(f) \) is a one-dimensional Schrödinger operator. Novikov [34] proved that if a periodic Schrödinger operator

\[
L_2 = -\partial_x^2 + u(x)
\]

commutes with \( L_{2g+1} \), then there are finitely many gaps in the spectrum of \( L_2 \). The converse was proved by Dubrovin [35]. In this case \( L_2 \) is said to be a finite-gap operator. The spectral curve of the operators \( L_2 \) and \( L_{2g+1} \) is a hyperelliptic curve

\[
w^2 = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0,
\]
and $q$ coincides with the point at infinity. The potential $u(x)$ of a finite-gap Schrödinger operator was found by Its and Matveev [36] in terms of the theta function of the Jacobian variety of the hyperelliptic spectral curve:

$$u(x) = -2\partial_x^2 \log \theta(xV_1 + V_2) + \text{const}, \quad V_1, V_2 \in \mathbb{C}^g.$$ 

The operator $L_2 - z$ is factorized as follows (see, for example, [37]):

$$L_2 - z = -(\partial_x + i\chi_0)(\partial_x - i\chi_0), \quad \chi_0 = -\frac{iQ_x}{2Q} + \frac{w}{Q},$$

where $Q = z^g + \alpha_{g-1}(x)z^{g-1} + \cdots + \alpha_0(x)$. The function $Q$ satisfies the equation

$$4w^2 = 4(z - u(x))Q^2 - Q_x^2 + 2QQ_{xx}. \quad (2)$$

Differentiating (2) with respect to $x$, we obtain

$$Q_{xxx} + 4(z - u(x))Q_x - 2u_x(x)Q = 0. \quad (3)$$

Now we look at examples of finite-gap operators.

**Example 2.** One example of a finite-gap operator is given by the Lamé operator

$$-\partial_x^2 + g(g + 1)\wp(x),$$

where $\wp(x)$ satisfies the equation

$$((\wp'(x))^2 = 4\wp^3(x) - 2\wp(x) - g_3.$$ 

The function $Q$ for the Lamé operator turns out to be a polynomial of degree $g$ in $\wp(x)$ (see [17]):

$$Q = A_g\wp^g(x) + A_{g-1}(z)\wp^{g-1}(x) + \cdots + A_0(z).$$

Substituting this ansatz into (3), we obtain

$$A_s = \frac{(s + 1)(8A_{s+1}z - A_{s+2}g_2(s + 2)(2s + 3) - 2A_{s+3}g_3(s + 2)(s + 3))}{4(2s + 1)(g^2 + g - s(s + 1))},$$

where $A_g$ is a constant, and $A_k = 0$ for $k > g$.

To find an equation of the spectral curve, we substitute $Q$ in (2), put $x = x_0$, where $x_0$ is a zero of $\wp(x)$, and use the equalities

$$((\wp'(x_0))^2 = -g_3 \quad \text{and} \quad \wp''(x_0) = -\frac{g_2}{2}.$$ 

This yields an explicit equation for the spectral curve of the Lamé operator.

**Theorem 1** [17]. *The spectral curve of the Lamé operator is given by the equation*

$$w^2 = \frac{1}{4} \left(4A_0^2z - A_0(4A_2g_3 + A_1g_2) + A_1^2g_3\right).$$
Another approach to finding the spectral curve of the Lamé operator was proposed in [38]. The problem of an explicit description of the spectral curve of the Lamé operator has been studied in many papers (see, for example, [39]).

**Example 3.** Consider the Treibich–Verdier operator

\[-\partial_x^2 + n_0(n_0 + 1)\varphi(x) + \sum_{j=1}^2 n_j(n_j + 1)\varphi(x + \omega_j), \quad \omega_3 = \omega_1 + \omega_2, \quad n_0, n_1, n_2 \in \mathbb{N}.

The following seems to be the simplest way of proving that the Treibich–Verdier operator is finite-gap. We restate Theorem 1.1 in the paper [40] by Gesztesy, Unterkoffler, and Weikard as follows.

**Theorem 2** [40]. The Schrödinger operator \(-\partial_x^2 + u(x)\) with an elliptic potential \(u(x)\) is finite-gap if and only if the Laurent decomposition of \(u(x)\) near each pole \(x_0 \in \mathbb{C}\) has the form

\[u(x) = \frac{n(n+1)}{(x-x_0)^2} + \alpha_0 + \alpha_2(x-x_0)^2 + \cdots + \alpha_{2n}(x-x_0)^{2n} + O((x-x_0)^{2n+1}).\]

**Corollary 1.** The Treibich–Verdier operator is finite-gap.

**Remark 1.** A new proof of Theorem 2 was recently obtained by Oganesyan. The equation (2) plays a key role in this proof.

### 3. Operators of rank two

#### 3.1. The method of deformation of Tyurin’s parameters.

In the case of operators of rank \(l > 1\) the common eigenfunctions (the Baker–Akhiezer functions of rank \(l > 1\)) correspond to the spectral data (see [5])

\[
\{\Gamma, q, k^{-1}, \gamma, \alpha, v(x)\},
\]

where \(\Gamma\) is a Riemann surface of genus \(g\), \(q\) is a fixed point on \(\Gamma\), \(k^{-1}\) is a local parameter near \(q\),

\[v(x) = (v_1(x), \ldots, v_{l-1}(x)),\]

\(v_j(x)\) is a smooth function, \(\gamma = \gamma_1 + \cdots + \gamma_{lg}\) is a divisor on \(\Gamma\), and \(\alpha\) is a tuple of vectors

\[\alpha_1, \ldots, \alpha_{lg}, \quad \alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,l-1}).\]

The pair \((\gamma, \alpha)\) is called Tyurin’s parameters. Tyurin’s parameters determine a stable holomorphic vector bundle of rank \(l\) and degree \(lg\) on \(\Gamma\) along with holomorphic sections \(\eta_1, \ldots, \eta_l\). The points \(\gamma_1, \ldots, \gamma_{lg}\) are the points of linear dependence of these sections:

\[\eta_l(\gamma_k) = \sum_{i=1}^{l-1} \alpha_{j,i} \eta_j(\gamma_k).\]

The Baker–Akhiezer vector-valued function \(\psi = (\psi_1, \ldots, \psi_l)\) is determined by the following conditions.
1) Near \( q \), \( \psi \) has the form
\[
\psi(x, P) = \left( \sum_{s=0}^{\infty} \xi_s(x)k^{-s} \right) \Phi(x, k),
\]
where \( \xi_0 = (1, 0, \ldots, 0) \), \( \xi_i(x) = (\xi^1_i(x), \ldots, \xi^l_i(x)) \), and the matrix \( \Phi \) satisfies
\[
\frac{d\Phi}{dx} = A\Phi, \quad A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k + v_1 & v_2 & v_3 & \ldots & v_{l-1} & 0
\end{pmatrix}.
\]

2) The components of \( \psi \) are meromorphic functions on \( \Gamma \setminus \{q\} \) with simple poles at \( \gamma_1, \ldots, \gamma_l \) such that
\[
\text{Res}_{\gamma_i} \psi_j = \alpha_{i,j} \text{Res}_{\gamma_i} \psi_l, \quad 1 \leq i \leq lg, \quad 1 \leq j \leq l - 1.
\]

For every rational function \( f(P) \) on \( \Gamma \) with a unique pole of order \( n \) at \( q \), there exists a linear differential operator \( L(f) \) of order \( ln \) such that \( L(f)\psi(x, P) = f(P)\psi(x, P) \). For any pair \( f(P), g(P) \) of such functions, the operators \( L(f) \) and \( L(g) \) commute.

The main difficulty in the construction of operators of rank \( l > 1 \) is that the Baker–Akhiezer function cannot be found explicitly. Nevertheless, the operators themselves can be found using the following method of deformation of Tyurin’s parameters (the Krichever–Novikov method).

The common eigenfunctions of commuting differential operators of rank \( l \) satisfy a linear differential equation of order \( l \):
\[
\psi^{(l)}(x, P) = \chi_0(x, P)\psi(x, P) + \cdots + \chi_{l-1}(x, P)\psi^{(l-1)}(x, P).
\]  
\text{(4)}

The coefficients \( \chi_i \) are rational functions on \( \Gamma \) (see [5]) with simple poles \( P_1(x), \ldots, P_{lg}(x) \in \Gamma \). Near \( q \) they have the form
\[
\chi_0(x, P) = k + g_0(x) + O(k^{-1}), \quad \chi_{l-1}(x, P) = O(k^{-1}),
\]
\[
\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad 0 < j < l - 1.
\]

Let \( k - \gamma_i(x) \) be a local parameter near \( P_i(x) \). Then
\[
\chi_j = \frac{c_{i,j}(x)}{k - \gamma_i(x)} + d_{i,j}(x) + O(k - \gamma_i(x)).
\]

The functions \( c_{i,j}(x), d_{i,j}(x) \) satisfy the following equations (see [5]):
\[
c_{i,1}(x) = -\gamma_i'(x),
\]  
\text{(5)}
\[
d_{i,0}(x) = \alpha_{i,0}(x)\alpha_{i,1}(x) + \alpha_{i,0}(x)d_{i,1}(x) - \alpha_{i,1}'(x),
\]  
\text{(6)}
\[
d_{i,j}(x) = \alpha_{i,j}(x)\alpha_{i,j-2}(x) - \alpha_{i,j-1}(x)
\quad + \alpha_{i,j}(x)d_{i,j-1}(x) - \alpha_{i,j}'(x), \quad j \geq 1,
\]  
\text{(7)}

where \( \alpha_{i,j}(x) = c_{i,j}(x)/c_{i,l-1}(x), \ 0 \leq j \leq l - 1, \ 1 \leq i \leq lg. \)
3.2. Elliptic spectral curves.

3.2.1. The Krichever–Novikov operators. To find \( \chi_i \), one must solve the equations (5)–(7). Using \( \chi_i \), one can find the coefficients of the commuting operators. Krichever and Novikov [8], [9] solved these equations and found these operators for \( g = 1 \) and \( l = 2 \).

Theorem 3 [8], [9]. The fourth-order operator has the form

\[
L_{KN} = (\partial_x^2 + u)^2 + 2c_x(\varphi(\gamma_2) - \varphi(\gamma_1))\partial_x + (c_x(\varphi(\gamma_2) - \varphi(\gamma_1)))_x - \varphi(\gamma_2) - \varphi(\gamma_1),
\]

where \( \gamma_1(x) = \gamma_0 + c(x), \gamma_2(x) = \gamma_0 - c(x) \);

\[
u(x) = -\frac{1}{4c_x^2} + \frac{1}{4} c_{xx}^2 + 2\Phi(\gamma_1,\gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0 + c,\gamma_0 - c) - \Phi^2(\gamma_1,\gamma_2)),
\]

\[
\Phi(\gamma_1,\gamma_2) = \zeta(\gamma_2 - \gamma_1) + (1 - \zeta(\gamma_1) - \zeta(\gamma_2)),
\]

\( \zeta(z), \varphi(z) \) are Weierstrass functions, \( c(x) \) is an arbitrary smooth function, and \( \gamma_0 \) is a constant.

The operator \( \tilde{L}_{KN} \) that commutes with \( L_{KN} \) can be found from the identity

\[
\tilde{L}_{KN}^2 = 4L_{KN}^3 - g_2L_{KN} - g_3.
\]

The operators \( L_{KN} \) and \( \tilde{L}_{KN} \) have been studied by many authors (see [41]–[49]). Dixmier [50] constructed the first example of a commutative subalgebra of the first Weyl algebra \( \mathcal{A}_1 = \mathbb{C}[x][\partial_x] \), namely, the following operators with polynomial coefficients commute:

\[
L_D = (\partial_x^2 + x^3 + h)^2 + 2x, \tag{8}
\]

\[
\tilde{L}_D = (\partial_x^2 + x^3 + h)^3 + \frac{3}{2}(x(\partial_x^2 + x^3 + h) + (\partial_x^2 + x^3 + h)x). \tag{9}
\]

These operators are operators of rank two with the spectral curve

\[
w^2 = z^3 - h,
\]

whence the operator \( L_D \) coincides with \( L_{KN} \) for some \( c(x) \). The natural question arises as to how \( L_D \) can be obtained from \( L_{KN} \). The answer is given in the following theorem of Grinevich [42].

Theorem 4. The operator \( L_{KN} \) corresponding to a curve

\[
w^2 = 4z^3 - g_2z - g_3
\]

has rational coefficients if and only if

\[
c(x) = \int_{q(x)}^\infty \frac{dt}{\sqrt{4t^3 - g_2t - g_3}},
\]

where \( q(x) \) is a rational function. If \( \gamma_0 = 0 \) and \( q(x) = x \), then the operator \( L_{KN} \) coincides with \( L_D \).

Grinevich and Novikov [43] found conditions for \( L_{KN} \) to be self-adjoint.

Theorem 5. The operator \( L_{KN} \) is self-adjoint if and only if \( \varphi(\gamma_1) = \varphi(\gamma_2) \).

Spectral properties of periodic operators of rank \( l > 1 \) with spectral curve of any genus were studied by Novikov in [51].
3.2.2. Operators with polynomial coefficients. One can complement Grinevich’s theorem (Theorem 4). It was shown in [52] that among the Krichever–Novikov operators there are operators with polynomial coefficients of arbitrarily high degree.

**Theorem 6** [52]. For an arbitrary spectral curve $\Gamma$ given by an equation

$$w^2 = F(z) = z^3 + c_2 z^2 + c_1 z + c_0, \quad c_i \in \mathbb{C}, \quad (10)$$

an arbitrary point $(z_0, w_0) \in \Gamma$ with $w_0 \neq 0$, and an arbitrary integer $n \geq 1$, there exist polynomials

$$R = \delta_{2n+2} x^{2n+2} + \cdots + \delta_0, \quad \delta_{2n+2} \neq 0,$$

$$P = \beta_n x^n + \cdots + \beta_0, \quad \beta_n \neq 0,$$

such that the operator

$$L_4 = (\partial_x^2 + R)^2 + (4w_0 P_x) \partial_x + \partial_x (4w_0 P_x)$$

$$- 16F(z_0) P^2 + 4F'(z_0) P - \frac{1}{2} F''(z_0) + z_0,$$

commutes with a sixth-order operator $L_6$ with polynomial coefficients. The spectral curve of the operators $L_4$ and $L_6$ is $\Gamma$.

We sketch the proof of this theorem. The coefficients of $L_{_{KN}}$ are expressed in terms of the functional parameter $c(x)$ by means of Weierstrass functions (see Theorem 3). This parametrization is not very convenient for investigating operators with polynomial coefficients. The proof of Theorem 6 uses another parametrization.

**Lemma 3.** Let $\Gamma$ be the curve given by an equation (10). Then the operator

$$L_4 = (\partial_x^2 + R)^2 + (4w_0 P_x) \partial_x + \partial_x (4w_0 P_x)$$

$$- 16F(z_0) P^2 + 4F'(z_0) P - \frac{1}{2} F''(z_0) + z_0,$$

where $(z_0, w_0) \in \Gamma$, $P(x) = 1/(W(x) + 2z_0 + c_2)$,

$$R(x) = \frac{1}{4P_x^2} (2P(x) - 16F(z_0) P^4(x) + 8F'(z_0) P^3(x)$$

$$- 2F''(z_0) P^2(x) + P_{xx}^2 - 2P_x P_{xxx}), \quad (13)$$

and $W(x)$ is an arbitrary smooth function, commutes with some sixth-order operator $L_6$. The spectral curve of the operators $L_4, L_6$ is $\Gamma$.

The coefficients of the operator (12) can be parametrized by the functional parameter $W(x)$ in the following way. Consider the formally self-adjoint commuting operators $\tilde{L}_4, \tilde{L}_6$ of orders four and six with the spectral curve $\Gamma$. As will be explained in § 3.3.1, $\tilde{L}_4$ can be written in the form

$$\tilde{L}_4 = (\partial_x^2 + V(x))^2 + W(x),$$

where $V(x)$ is expressed in terms of $W(x)$ by

$$V(x) = -\frac{16F(-c_2 + W)/2 + W_{xx}^2 - 2W_x W_{xxx}}{4W_x^2}.$$
The operator $\tilde{L}_4 - z_0$ is factorized as follows:

$$\tilde{L}_4 - z_0 = (\partial_x^2 + V(x))^2 + W(x) - z_0 = (\partial_x^2 + \chi_1 \partial_x + \chi_2)(\partial_x^2 - \chi_1 \partial_x - \chi_0),$$  \hspace{1cm} (15)

where

$$\chi_0 = -\frac{Q_{xx}}{2Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q_x}{Q}, \quad \chi_2 = \frac{3Q_{xx}}{2Q} + \frac{w - Q_x^2}{Q} + V,$$

and $Q = z_0 + (W + c_2)/2$. Furthermore,

$$\tilde{L}_6 - w_0 = l_4(\partial_x^2 - \chi_1 \partial_x - \chi_0),$$  \hspace{1cm} (16)

where $l_4$ is an operator of fourth order.

By the Latham–Previato theorem [46], non-self-adjoint Krichever–Novikov operators can be obtained from the formally self-adjoint operators $\tilde{L}_4 - z_0$ and $\tilde{L}_6 - w_0$ by a Darboux transformation, that is, by interchanging the factors in (15) and (16):

$$L_4 = (\partial_x^2 - \chi_1 \partial_x - \chi_0)(\partial_x^2 + \chi_1 \partial_x + \chi_2) + z_0,$$

$$L_6 = (\partial_x^2 - \chi_1 \partial_x - \chi_0)l_4 + w_0.$$

One can show by direct calculations that $L_4$ takes the form (12). In what follows we assume that $w_0 \neq 0$. We easily see that the operator (12) has polynomial coefficients if and only if $P(x)$ and $R(x)$ are polynomials (see [52]).

It follows from (13) that

$$4P_x^2 R = 2P - 16F(z_0)P^4 + 8F'(z_0)P^3 - 2F''(z_0)P^2 + P_{xx}^2 - 2P_xP_{xxx}.$$  \hspace{1cm} (17)

It turns out that for each $n > 0$ there exist polynomials $P(x)$ and $R(x)$ of the form (11) which satisfy this equation. Differentiating both sides of (17) with respect to $x$ and then dividing by $P_x$, we get that

$$32F(z_0)P^3 + 4RP_{xx} + 2R_xP_x - 12F'(z_0)P^2 + 2F''(z_0)P + P_{xxxx} - 1 = 0.$$  \hspace{1cm} (18)

The equation (17) is equivalent to the system consisting of (18) and the equation for the constant term in (17):

$$2\alpha_0^2 \beta_1^2 = \beta_0 - 8F(z_0)\beta_0^4 + 4F'(z_0)\beta_0^3 - F''(z_0)\beta_0^2 + 2\beta_2^3 - 6\beta_1 \beta_3.$$  

The equation (18) is equivalent to a system of $3n + 1$ equations with $3n + 4$ variables $\alpha_i$, $\beta_j$. Note that all the equations are of degree three and the set of their solutions consists of the points in $\mathbb{C}^{3n+4}$ (with coordinates $\alpha_i$, $\beta_j$) that lie in the intersection of the $3n + 1$ cubics given by these equations. In [52] it was shown using methods of algebraic geometry that this intersection is non-empty.

3.2.3. The Kadomtsev–Petviashvili equation. Krichever and Novikov [9] found rank-two solutions of the Kadomtsev–Petviashvili (KP) equation in the case of an elliptic spectral curve. In this subsection we recall how to construct these solutions. Instead of the parametrization of the coefficients of $L_{KN}$ in terms of the
Weierstrass elliptic functions indicated in Theorem 3, we shall use the parametrization in (12), (13). We remark that the Krichever–Novikov solutions were studied in [46].

The Kadomtsev–Petviashvili equation

$$\frac{3}{4} U_{yy} = \partial_x \left( U_t + \frac{3}{2} U U_x - \frac{1}{4} U_{xxx} \right)$$

is equivalent to the following condition of commutation of operators:

$$[\partial_y - M, \partial_t - A] = 0,$$

where

$$M = \partial_x^2 - U(x, y, t), \quad A = \partial_x^2 - \frac{3}{2} U \partial_x + S(x, y, t),$$

$$S_x = -\frac{3}{4} U_y - \frac{3}{4} U_{xx}, \quad S_y = -U_t - \frac{3}{4} U_{xy} + \frac{U_{xxx}}{4} - \frac{3}{2} U U_x.$$

A solution $U$ is said to be algebrao-geometric of rank $l$ if, in addition, the operators $\partial_y - M, \partial_t - A$ commute with commuting ordinary differential operators of rank $l$ whose coefficients depend on $x, y, t$:

$$[L_n, \partial_y - M] = [L_n, \partial_t - A] = 0,$$

where $L_n = \sum_{j=0}^{n} u_j(x, y, t) \partial_x^j$ is an operator of rank $l$.

Krichever found a solution of KP of rank one:

$$U = 2 \partial_x^2 \log \theta(V_1 x + V_2 y + V_3 t + V_4, \Omega) + \text{const}, \quad (19)$$

where $\theta(z, \Omega)$ is the theta function of the Jacobian variety of the spectral curve, and $\Omega$ is the matrix of periods of normalized Abelian differentials. Novikov conjectured that (19) gives a solution of KP if and only if $\Omega$ is the matrix of periods of Abelian differentials for some Riemann surface (solution of the Riemann–Schottky problem). This conjecture was proved by Shiota [53].

Suppose that the spectral curve $\Gamma$ is given by an equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

The Krichever–Novikov operator (12) can be written in the form

$$L_{KN} = (\partial_x^2 - U)^2 + f_1 \partial_x + \partial_x f_1 + f_0,$$

$$U = -V - \frac{2(c_2 + W + 2z_0)W_{xx} - 2W_x^2}{(c_2 + W + 2z_0)^2},$$

where $V$ is expressed in terms of $W$ by (14).

In what follows we assume that $z_0 = z_0(y)$ and $W = W(x, t)$. We substitute the operator $L_{KN}$ in

$$[L_{KN}, \partial_y - M] = [L_{KN}, \partial_t - A] = [\partial_y - M, \partial_t - A] = 0.$$
It follows that $U$ is a solution of KP if and only if $W$ satisfies the Krichever–Novikov equation

$$W_t = \frac{48F_1(-(c_2 + W)/2) - W^2_{xx} + 2W_xW_{xxx}}{8W_x}$$

and the function $z_0(y)$ satisfies the ordinary differential equation

$$(z'_0(y))^2 = 4F_1(z_0(y)).$$

**Theorem 7.** The algebro-geometric solutions of rank two corresponding to an elliptic spectral curve (the Krichever–Novikov solutions) have the form

$$U(x, y, t) = -V(x, t) - \frac{2(c_2 + W(x, t) + 2z_0(y))W_{xx}(x, t) - 2W^2_x(x, t)}{(c_2 + W(x, t) + 2z_0(y))^2},$$

$$V(x, t) = \frac{-16F(-c_2 + W(x, t))/2 + W^2_{xx}(x, t) - 2W_x(x, t)W_{xxx}(x, t)}{4W^2_x(x, t)},$$

where $W(x, t)$ satisfies the Krichever–Novikov equation (20) and $z_0(y)$ satisfies (21).

Thus, the rank-two algebro-geometric solutions of KP in the case of an elliptic spectral curve correspond to a surprising separation of variables in KP. The formula (22) produces a solution of KP from a solution of the (1 + 1)-Krichever–Novikov equation (20) and a solution of the ordinary differential equation (21).

### 3.3. Hyperelliptic spectral curves.

#### 3.3.1. Self-adjoint operators.**

Consider a pair $L_4$, $L_{4g+2}$ of commuting differential operators of rank two with a hyperelliptic spectral curve $\Gamma$ of genus $g$:

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0.$$  

The operators $L_4$ and $L_{4g+2}$ satisfy the equation $(L_{4g+2})^2 = F_g(L_4)$. The curve $\Gamma$ has a holomorphic involution

$$\sigma: \Gamma \to \Gamma, \quad \sigma(z, w) = (z, -w).$$

We recall that the common eigenfunctions of $L_4$ and $L_{4g+2}$ satisfy the following second-order equation (see (4)):

$$\psi''(x, P) = \chi_1(x, P)\psi'(x, P) + \chi_0(x, P)\psi(x, P), \quad P = (z, w) \in \Gamma.$$  

The coefficients $\chi_0(x, P)$, $\chi_1(x, P)$ are rational functions on $\Gamma$ with $2g$ simple poles depending on $x$, and $\chi_0$ also has an additional pole at infinity. These functions satisfy (5)–(7). To find the operators $L_4$ and $L_{4g+2}$, it suffices to find $\chi_0$ and $\chi_1$.

If the function $\chi_1$ is invariant under the involution $\sigma$, then the operator $L_4$ is self-adjoint. Near $q = \infty$ we have decompositions

$$\chi_0 = \frac{1}{k} + a_0(x) + a_1(x)k + O(k^2), \quad \chi_1 = b_1(x)k + b_2(x)k^2 + O(k^3),$$

where $k = 1/\sqrt{z}$ is a local parameter near $q$. The coefficients of $L_4$ are expressed in terms of the coefficients $a_j(x)$ and $b_i(x)$ by the following formulae (see [16]):

$$L_4 = \partial_x^4 + f_2(x)\partial_x^2 + f_1(x)\partial_x + f_0(x),$$

$$f_0 = a_0^2 - 2a_1 - 2b'_1 - a''_0, \quad f_1 = -2(b_1 + a'_0), \quad f_2 = -2a_0.$$
If $\chi_1$ is invariant under $\sigma$, then $b_1 = 0$ and in this case

$$L_4 = (\partial_x^2 - a_0(x))^2 - 2a_1(x).$$

Novikov conjectured that the converse also holds: if $b_1 = 0$, then $b_{2j+1}$ is equal to zero for all $j \geq 1$. This conjecture was proved in [16].

**Theorem 8** [16]. The operator $L_4$ is formally self-adjoint if and only if

$$\chi_1(x, P) = \chi_1(x, \sigma(P)).$$

For $g = 1$ Theorem 8 is equivalent to Theorem 5, which was proved by Grinevich and Novikov [43].

Suppose that the operator $L_4$ is formally self-adjoint. Then it can be written as

$$L_4 = (\partial_x^2 + V(x))^2 + W(x).$$

**Theorem 9** [16]. If $L_4$ is self-adjoint, then

$$\chi_0 = -\frac{1}{2} \frac{Q_{xx}}{Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q_x}{Q},$$

where

$$Q = z^g + \alpha_{g-1}(x)z^{g-1} + \cdots + \alpha_0(x).$$

The function $Q$ satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_x Q_{xxx}$$

$$+ 2Q(2V_x Q_x + 4V Q_{xx} + Q_{xxxx}).$$

(25)

Differentiating both sides of (25) with respect to $x$ and dividing by $Q$, we obtain a linear (skew-symmetric) equation for $Q$.

**Corollary 2** [16]. The function $Q$ satisfies the linear equation

$$\partial_x^5 Q + 4V Q_{xxx} + 2Q_x(2z - 2W + V_{xx}) + 6V_x Q_{xx} - 2Q W_x = 0.$$  (26)

The potentials $V$ and $W$ have the form

$$V = \frac{(Q'')^2 - 2Q'(Q^{(3)}) - 4F_g(z)}{4(Q')^2} \bigg|_{z=\gamma_j},$$

$$W = -2(\gamma_1 + \cdots + \gamma_g) - c_2,$$

(27)

where the $\gamma_j(x)$ are the roots of the equation $Q = 0$.

For $g = 1$, (25) implies the formula (14) expressing the potential $V(x)$ in terms of $W(x)$.

Theorem 9 reveals an analogy between the spectral theories of a finite-gap Schrödinger operator $L_2 = -\partial_x^2 + u(x)$ and a self-adjoint operator $L_4$ commuting with $L_{4g+2}$. Indeed, (25) and (26) are analogues of (2) and (3), and (27) is an analogue of the trace formula for the potential $u(x)$. 

3.3.2. Examples. Theorem 9 enables us to construct explicit examples.

**Example 4** [16]. The operator

\[ L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g + 1)\alpha_3 x, \quad \alpha_3 \neq 0, \]

commutes with a differential operator \( L_{4g+2}^\sharp \) of order \( 4g + 2 \). Here

\[
Q = \delta_g x^q + \cdots + \delta_1 x + \delta_0, \quad \delta_i = \delta_i(z),
\]

\[
\delta_s = \frac{s + 1}{\alpha_3(g - s)(s + g + 1)(2s + 1)} \left( 2(\alpha_2(s + 1)^2 + z)\delta_{s+1}
+ \alpha_1(s + 2)(2s + 3)\delta_{s+2} + 2\alpha_0(s + 2)(s + 3)\delta_{s+3}
+ \frac{1}{2}(s + 2)(s + 3)(s + 4)(s + 5)\delta_{s+5} \right),
\]

where \( 0 \leq s < g - 1 \), \( \delta_g \) is a constant, and \( \delta_s = 0 \) for \( s > g \).

If \( g = 1 \), \( \alpha_1 = \alpha_2 = 0 \), and \( \alpha_3 = 1 \), then the operators \( L_4^\sharp \) and \( L_{4g+2}^\sharp \) coincide with the Dixmier operators (8) and (9).

**Example 5** [17]. Let \( \mathcal{P} \) be a solution of the equation

\[
(\mathcal{P}'(x))^2 = g_2 \mathcal{P}^2(x) + g_1 \mathcal{P}(x) + g_0, \quad g_2 \neq 0.
\]

Then the operator

\[ L_4 = (\partial_x^2 + \alpha_1 \mathcal{P}(x) + \alpha_0)^2 + \alpha_1 g_2 (g + 1) \mathcal{P}(x), \quad \alpha_1 \neq 0, \]

commutes with an operator \( L_{4g+2} \) of order \( 4g + 2 \).

For \( g_0 = 1, g_1 = 0, \) and \( g_2 = -1 \) we obtain the following operator with periodic coefficients:

\[ L_4 = (\partial_x^2 + \alpha_1 \cos x + \alpha_0)^2 - \alpha_1 g(g + 1) \cos x, \quad \alpha_1 \neq 0. \]

For \( g_0 = -1, g_1 = 0, \) and \( g_2 = 1 \) we obtain the operator

\[ L_4^\flat = (\partial_x^2 + \alpha_1 \cosh x + \alpha_0)^2 + \alpha_1 g(g + 1) \cosh x, \quad \alpha_1 \neq 0, \]

which will be used in what follows.

**Example 6** [17]. Let \( \wp(x) \) be a solution of the equation

\[
(\wp'(x))^2 = 4\wp^3(x) - g_1 \wp^2(x) - g_2 \wp(x) - g_3.
\]

Then the operator

\[ L_4 = (\partial_x^2 + \alpha_1 \wp(x) + \alpha_0)^2 + s_1 \wp(x) + s_2 \wp^2(x), \]

where

\[
\alpha_1 = \frac{1}{4} - 2g^2 - 2g, \quad s_1 = \frac{1}{4} g(g + 1)(16\alpha_0 - 5g_1), \quad s_2 = -4g(g + 2)(g^2 - 1)
\]

and \( \alpha_0 \) is an arbitrary constant, commutes with an operator of order \( 4g + 2 \).
Let us consider further examples of fourth-order operators commuting with operators of order $4g + 2$.

**Example 7** [21].
\[
L_4 = \left( \frac{1}{4} \alpha^2 \cos^2 x + \alpha \cos x + \frac{(2g + 1)^2 - \alpha^2}{4} \right)^2 \\
- g(g + 1)(\alpha^2 \cos^2 x + 2\alpha \cos x), \quad \alpha \neq 0.
\]

**Example 8** [27].
\[
L_4 = \left( \partial_x^2 + \alpha_1 x^6 + \alpha_2 x^2 \right)^2 + 16\alpha_1 g(g + 1)x^4.
\]

**Example 9** [27].
\[
L_4 = \left( \partial_x^2 + \alpha_1 x^4 + \alpha_2 x^2 + \alpha_3 \right)^2 + 4\alpha_1 g(g + 1)x^2.
\]

It was proved in [19] and [21] that the maximal commutative rings containing the operators of orders $4$ and $4g + 2$ in Examples 4, 5 (in the case (29)), and 7 do not contain operators of odd orders.

Oganesyan [25] investigated the special case of self-adjoint fourth-order operators of the form
\[
L_4 = \partial_x^4 + W(x).
\]

It was proved in [25] that if this operator commutes with an operator of order $4g + 2$, then $W(x)$ can have poles only of order 4, with the following Laurent series expansion in a neighbourhood of any pole $a \in \mathbb{C}$:
\[
W(x) = \frac{n(4n + 1)(4n + 3)(4n + 4)}{(x - a)^4} + \sum_{j=0}^{\infty} d_j(x - a)^j, \quad n \in \mathbb{N},
\]
where $d_{4k-l} = 0$ for $k = 1, \ldots, n$ and $l = 1, 2, 3$, and $d_{4r-1} = d_{4r-3} = 0$ for $r = n + 1, \ldots, g$. Moreover, if $\psi(x)$ is a solution of the equation
\[
(\partial_x^4 + W(x))\psi(x) = z\psi(x),
\]
then the singularity of $\psi(x)$ at the point $a \in \mathbb{C}$ is of the type $x^{\sigma_{1,2}^\pm} g(x)$, where $g(x)$ is analytic at $a$, $r = 1, 2$, and
\[
\sigma_{1,2}^\pm = \frac{1}{2} \left( 1 - 4n \pm \sqrt{1 - 16n - 16n^2} \right),
\]
\[
\sigma_{2}^\pm = \frac{1}{2} \left( 5 + 4n \pm \sqrt{1 - 16n - 16n^2} \right).
\]

An example of such an operator commuting with $L_{4g+2}$ was also constructed in [25].

**Example 10** [25]. The operator
\[
L_4 = \partial_x^4 + g(4g + 1)(4g + 3)(4g + 4)\varphi^2(x),
\]
where $\varphi(x)$ satisfies the equation
\[
(\varphi'(x))^2 = 4(\varphi(x))^3 - g_2 \varphi(x),
\]
commutes with an operator of order $4g + 2$. 
3.3.3. Automorphisms of the first Weyl algebra. Let $L_n, L_m \in \mathcal{A}$ be commuting operators with polynomial coefficients: $L_n L_m = L_m L_n$. Suppose also that $A, B \in \mathcal{A}$ satisfy the identity $[A, B] = 1$. Since $[\partial_x, x] = 1$, by replacing $x$ by $B$ and $\partial_x$ by $A$ in $L_n$ and $L_m$ we obtain new operators in $\mathcal{A}$ which again commute. We note that the map $\varphi: \mathcal{A} \to \mathcal{A}$ given by the formulae

$$\varphi(\partial_x) = A_1 \quad \text{and} \quad \varphi(x) = B$$

is an endomorphism of the algebra $\mathcal{A}$. Dixmier proved that the automorphism group $\text{Aut}(\mathcal{A})$ is generated by the following automorphisms:

$$\varphi_1(x) = \alpha x + \beta \partial_x, \quad \varphi_1(\partial_x) = \gamma x + \delta \partial_x, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \alpha \delta - \beta \gamma = 1,$$

$$\varphi_2(x) = x + M_1(\partial_x), \quad \varphi_2(\partial_x) = \partial_x,$$

$$\varphi_3(x) = x, \quad \varphi_3(\partial_x) = \partial_x + M_2(x),$$

where $M_1$ and $M_2$ are arbitrary polynomials (see [50]), that is, any automorphism is a composite of automorphisms of the form $\varphi_j$.

Mokhov [22], [23] applied automorphisms in $\text{Aut}(\mathcal{A})$ to operators of rank two in order to find examples of operators of rank $l > 1$.

Example 11 [22]. Applying the automorphism

$$\partial_x \mapsto -x + \partial_x^k, \quad x \mapsto \partial_x$$

to the operators in Example 4 with $\alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 0$, we obtain commuting operators $L_{4k}^i$ and $L_{4kg+2k}^i$ of orders $4k$ and $4kg+2k$ and rank $l = 2k, k > 1$, where

$$L_{4k}^i = (\partial_x^{2k} - 2x \partial_x^k - k \partial_x^{k-1} + \partial_x^3 + x^2 + \alpha_0)^2 + g(g+1) \partial_x.$$ 

Applying the automorphism

$$\partial_x \mapsto -\partial_x, \quad x \mapsto -x + \partial_x^k$$

to the same operators, we obtain operators $L_{6k}^i$ and $L_{6kg+3k}^i$ of rank $l = 3k, k > 1$, and orders $6k$ and $6kg + 3k$, where

$$L_{6k}^i = (\partial_x^{3k} - 3x \partial_x^{2k} - 3k \partial_x^{2k-1} + 3x^2 \partial_x^k + 3k x \partial_x^{k-1} + k(k-1) \partial_x^{k-2} + \partial_x^3 - x^2 + \alpha_0) - g(g+1)x + g(g+1) \partial_x^k.$$ 

Example 12 [23]. Consider the operator (29), which commutes with an operator of order $4g + 2$. Mokhov observed that after a change of variables

$$x = \ln(y + \sqrt{y^2 - 1})^l, \quad l = \pm 1, \pm 2, \ldots,$$

the coefficients of these operators will be polynomial in $y$, and

$$L_4^i = ((1 - y^2) \partial_y^2 - 3y \partial_y + a T_l(y) + b)^2 - ar^2 g(g+1) T_l(y), \quad a \neq 0,$$

where $T_l(y)$ is the Chebyshev polynomial of degree $|l|$. We recall that

$$T_0(y) = 1, \quad T_1(y) = y, \quad T_l(y) = 2y T_{l-1}(y) - T_{l-2}(y), \quad T_{-l}(y) = T_l(y).$$
Acting on these operators by the automorphism
\[ \partial_y \mapsto -y, \quad y \mapsto \partial_y, \]
we obtain operators of rank \( l \) and orders \( 2l \) and \( (2g + 1)l \).

It is an interesting problem to describe the orbits of the action of Aut(\( \mathcal{A}_1 \)) (the group of automorphisms of the first Weyl algebra) on the set of commuting operators with a fixed spectral curve, or more generally, on the set of solutions of the equation
\[ R(X, Y) = \sum_{i,j=0}^{N} c_{ij} X^i Y^j, \quad X, Y \in \mathcal{A}_1, \quad c_{ij} \in \mathbb{C}. \]

A description of all orbits enables us to compare the set End(\( \mathcal{A}_1 \)) of endomorphisms with Aut(\( \mathcal{A}_1 \)). We recall Dixmier’s conjecture (see [50]) that End(\( \mathcal{A}_1 \)) = Aut(\( \mathcal{A}_1 \)), or in other words, for any solution of the equation
\[ [A, B] = 1, \quad A, B \in \mathcal{A}_1, \]
the operators \( A \) and \( B \) are obtained from \( \partial_x \) and \( x \) by compositions of the automorphisms \( \varphi_j \). This is related to the Jacobian conjecture. Namely, it was shown in [54] that Dixmier’s general conjecture for the \( n \)-dimensional Weyl algebra \( \mathcal{A}_n \) is stably equivalent to the Jacobian conjecture.

Berest conjectured the following. If the genus of the Riemann surface corresponding to the equation \( R = 0 \) with general \( c_{ij} \in \mathbb{C} \) is equal to \( g = 1 \), then the set of orbits is infinite. But if \( g > 1 \), then there are only finitely many orbits.

If the set of orbits is finite for at least one equation, then it follows that End(\( \mathcal{A}_1 \)) = Aut(\( \mathcal{A}_1 \)).

The following theorem was proved in [29].

**Theorem 10.** The set of orbits of the action of Aut(\( \mathcal{A}_1 \)) on the set of solutions of an arbitrary equation
\[ Y^2 = X^3 + c_2 X^2 + c_1 X + c_0, \quad X, Y \in \mathcal{A}_1, \quad c_j \in \mathbb{C}, \]
is infinite.

This theorem follows from the next theorem (a version of Theorem 6) and Lemma 4.

**Theorem 11** [29]. For an arbitrary integer \( m > 0 \) and an arbitrary spectral curve \( \Gamma \) given by an equation \( w^2 = z^3 + c_2 z^2 + c_1 z + c_0 \), there are polynomials
\[ V_m(x) = \alpha_{m+2} x^{m+2} + \cdots + \alpha_0, \quad W_m(x) = \beta_m x^m + \cdots + \beta \]
with coefficients \( \alpha_{m+2} \neq 0, \beta_m \neq 0 \) such that the operator
\[ L_{4,m} = (\partial_x^2 + V_m(x))^2 + W_m(x) \]
commutes with a sixth-order operator \( L_{6,m} \) and the spectral curve of the operators \( L_{4,m}, L_{6,m} \) coincides with \( \Gamma \).
Lemma 4 [29]. Consider the following family of fourth-order operators with polynomial coefficients:

\[ L(r) = (a(x)\partial_x^2 + b(x)\partial_x + c_r(x))^2 + d_r(x), \quad r \in \mathbb{N}, \]

where \( a(x) \) and \( b(x) \) are polynomials of a fixed degree such that

\[ \deg a(x) > \deg b(x), \quad \deg c_r(x) = r, \quad r \geq \deg d_r(x). \]

If \( r > \deg a(x) + 8 \), then

\[ \varphi(L(r)) \neq L(r_1) \]

for \( r \neq r_1 \) and an arbitrary \( \varphi \in \text{Aut}(\mathcal{A}_1) \). Here it is assumed that \( \deg b(x) = -\infty \) if \( b(x) = 0 \).

Thus, \( L_{4,n} \) and \( L_{4,m} \) lie in different orbits for \( n \neq m \).

Let

\[ w^2 = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_1z + c_0 \tag{30} \]

be the equation of the spectral curve for the commuting operators \( L^b_4 \) and \( L^b_{4g+2} \) in Example 5.

**Theorem 12** [29]. The set of orbits of the action of \( \text{Aut}(\mathcal{A}_1) \) on the set of solutions of the equation

\[ Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_1X + c_0, \quad X, Y \in \mathcal{A}_1, \]

is infinite.

It was shown in Example 12 that the operators \( L^b_4 \) and \( L^b_{4g+2} \) commute for all \( l \) and have the same spectral curve (30). Hence by Lemma 4, these pairs of operators with different \( l \) lie in different orbits.

The set of orbits is also infinite in the case of generic hyperelliptic curves of genus two.

**Theorem 13** [55]. The operator

\[ L^b_4 = ((\alpha_1x^2 + 1)\partial_x^2 + (\alpha_2x + \alpha_3)\partial_x + \alpha_4x + \alpha_5)^2 + \alpha_1\alpha_4(g+1)x + \alpha_6 \]

commutes with the operator \( L^b_{10} \) for \( g = 2 \) and all \( \alpha_i \in \mathbb{C} \).

The pair \( L^b_4, L^b_{10} \) is a solution of the equation

\[ Y^2 = X^5 + c_4X^4 + c_3X^3 + c_2X^2 + c_1X + c_0, \quad X, Y \in \mathcal{A}_1, \quad c_i \in \mathbb{C}, \tag{31} \]

where the numbers \( c_i \) depend polynomially on the \( \alpha_i \). The set of orbits of the action of \( \text{Aut}(\mathcal{A}_1) \) on the set of solutions of (31) with generic \( c_i \) is infinite.

We have verified that \( L^b_4 \) commutes with \( L^b_6 \) for \( g = 1 \) and with \( L^b_{14} \) for \( g = 3 \). It would be interesting to prove that \( L^b_4 \) commutes with an operator of order \( 4g+2 \).

To do this, one should develop the methods in [16] in the case of non-self-adjoint operators.
3.3.4. Schrödinger operators with polynomial potentials. As already mentioned, the Baker–Akhiezer function has not been found explicitly in the general case \( l > 1 \). In this subsection we indicate a connection between the eigenfunctions of certain commuting ordinary differential operators of rank two with polynomial coefficients and the eigenfunctions of one-dimensional Schrödinger operators with potential of degree three or four. We note that Schrödinger operators with polynomial potentials arise in many questions of mathematical physics (see, for example, [56]–[58]).

Recall that the operator

\[
 L_4^I = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \quad \alpha_3 \neq 0,
\]

commutes with an operator of order \( 4g + 2 \) (see Example 4). The following theorem describes a connection between certain eigenfunctions \( \psi(x) \) of this operator,

\[
 ((\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x)\psi(x) = z\psi(x),
\]

and eigenfunctions of one-dimensional Schrödinger operators with cubic potentials.

**Theorem 14** [28]. Let \( \varphi(x) \) be a solution of the equation

\[
 (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)\varphi(x) = 0.
\]

Then the following assertions hold.

1) For \( g = 2 \) the function \( \psi(x) = (6\alpha_3 x + z + 4\alpha_2)\varphi(x) \) satisfies (32), where the eigenvalue \( z \) is a solution of the equation

\[
 z^2 + 4\alpha_2 z + 12\alpha_1 \alpha_3 = 0. \tag{33}
\]

2) For \( g = 4 \) the function

\[
 \psi(x) = (280\alpha_3^2 x^2 + 20\alpha_3 (z + 16\alpha_2)x + z^2 + 20\alpha_2 z + 64\alpha_2^2 + 168\alpha_1 \alpha_3)\varphi(x)
\]

satisfies (32), where the eigenvalue \( z \) is a solution of the equation

\[
 z^3 + 20\alpha_2 z^2 + 16(4\alpha_2^2 + 13\alpha_1 \alpha_3)z + 320\alpha_3(7\alpha_0 \alpha_3 + 2\alpha_1 \alpha_2) = 0. \tag{34}
\]

Theorem 14 means that the Schrödinger operator

\[
 L_2 = \partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0
\]

commutes with some operators of rank two modulo \( L_2 \).

**Corollary 3** [28]. For \( g = 2, 4 \) the operators

\[
 p^{-1}L_4^g p, \quad p^{-1}L_{4g+2}^g p, \quad L_2
\]

form a commutative ring modulo \( L_2 \), that is,

\[
 [p^{-1}L_4^g p, L_2] = B_1 L_2, \quad [p^{-1}L_{4g+2}^g p, L_2] = B_2 L_2, \tag{35}
\]

where \( B_1 \) and \( B_2 \) are some operators. Here for \( g = 2 \)

\[
 p(x) = 6\alpha_3 x + z + 4\alpha_2
\]

and \( z \) satisfies (33), while for \( g = 4 \)

\[
 p(x) = 280\alpha_3^2 x^2 + 20\alpha_3 (z + 16\alpha_2)x + z^2 + 20\alpha_2 z + 64\alpha_2^2 + 168\alpha_1 \alpha_3
\]

and \( z \) satisfies (34).
One can verify that the operator
\[ L_4 = (\partial_x^2 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + 2g(g + 1)x(\alpha_3 + 2\alpha_4 x), \] (36)
where
\[ \alpha_3^2 - 4\alpha_2 \alpha_3 \alpha_4 + 8\alpha_1 \alpha_4^2 = 0, \] (37)
commutes with an operator of order \(4g + 2\) for small \(g\). It would be interesting to prove that they commute for all \(g\). The operator in Example 9 is a particular case of (36). The next theorem describes a connection between certain eigenfunctions \(\psi(x)\) of the operator (36),
\[ L_4 \psi(x) = z \psi(x), \] (38)
and eigenfunctions of one-dimensional Schrödinger operators with potentials of degree four.

**Theorem 15** [28]. Let \(\varphi(x)\) be a solution of the equation
\[ (\partial_x^2 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0) \varphi = 0, \]
where the \(\alpha_j\) satisfy (37). Then the following assertions hold.
1) For \(g = 1\) the function
\[ \psi(x) = (4\alpha_4 x + \alpha_3) \varphi(x) \]
satisfies (38), where \(z = \alpha_3^2 / \alpha_4 - 4\alpha_2\).
2) For \(g = 2\) the function
\[ \psi(x) = (24\alpha_4^2 x^2 + 12\alpha_3 \alpha_4 x - 3\alpha_3^2 + \alpha_4 (z + 16\alpha_2)) \varphi(x) \]
satisfies (38), where the eigenvalue \(z\) is a solution of the equation
\[ z^2 - \left( \frac{3\alpha_3^2}{\alpha_4} - 16\alpha_2 \right) z + 24\alpha_1 \alpha_3 + 192\alpha_0 \alpha_4 = 0. \]

For Theorem 15 there is also an analogue of Corollary 3.

To complete this subsection, we show that some eigenfunctions of operators of rank two can be expressed in terms of a Bessel function. It was proved in [18] that the operator
\[ L_4^\vee = (\partial_x^2 + \alpha_1 e^{x} + \alpha_0)^2 + \alpha_1 g(g + 1)e^{x} \]
commutes with an operator of order \(4g + 2\). We write \(\varphi(x)\) for a solution of the equation
\[ \left( \partial_x^2 + \alpha_1 e^{x} + \alpha_0 + \frac{1}{4}(g + \varepsilon)^2 \right) \varphi = 0. \]

**Theorem 16** [28]. 1) For \(\varepsilon = 0\) the function \(\psi(x) = e^{g \varepsilon / 2} \varphi(x)\) satisfies the equation
\[ L_4^\vee \psi(x) = -\frac{1}{4} g^2 (4\alpha_0 + g^2) \psi(x). \]
2) For \(\varepsilon = 1\) the function \(\psi(x) = e^{-(g+1) \varepsilon / 2} \varphi(x)\) satisfies the equation
\[ L_4^\vee \psi(x) = -\frac{1}{4} (g + 1)^2 (4\alpha_0 + (g + 1)^2) \psi(x). \]
Remark 2. The solutions of the equation \((\partial_x^2 + \alpha_1 e^x + \alpha_0)\varphi = 0\) are expressed in terms of a Bessel function. Namely, the change of variables by the formula \(x = \log(y^2/(4\alpha_1))\) reduces this equation to the Bessel equation \((y^2 \partial_y^2 + y \partial_y + (y^2 + 4\alpha_0))\varphi = 0\).

A connection between some eigenfunctions of operators of rank two and Heun’s functions was indicated in [26].

3.3.5. Evolution equations. The equation
\[[L_4, \partial_{t_n} - A_{2n+1}] = 0,\] (39)
where \(L_4 = (\partial_x^2 + V(x, t_n))^2 + W(x, t_n)\) and \(A_{2n+1}\) is a skew-symmetric operator of order \(2n + 1\), determines a system of evolution equations for \(V(x, t_n)\) and \(W(x, t_n)\). For \(n = 1\) the operator \(A_3\) has the form
\[A_3 = \partial_x^3 + \frac{3}{2} V(x, t_1) \partial_x + \frac{3}{4} V_x(x, t_1),\]
and the equation (39) takes the form
\[V_{t_1} = \frac{1}{4}(6V V_x + 6W_x + V_{xxx}), \quad W_{t_1} = \frac{1}{2}(-3VW_x - W_{xxx}).\] (40)

For \(n = 2\) we have
\[A_5 = \partial_x^5 + \frac{5}{2} V \partial_x^3 + \frac{15}{4} V_x \partial_x^2 + \frac{5}{8}(3V^2 + 2W + 5V_{xx}) \partial_x \]
\[+ \frac{5}{16}(6VV_x + 2W_x + 3W_{xxx}),\]
\[V_{t_2} = \frac{1}{16}(30V^2V_x + 20V_x(W + V_{xx}) + 10V(2V_x + V_{xxx}) + 10W_{xxx} + \partial_x^5V),\]
\[W_{t_2} = \frac{1}{8}(-5V^2W_x + 5W_x(2W - V_{xx}) - 10V_xW_{xx} - 10W_{xxx} - 2\partial_x^5W).\] (41)

Following [9], we say that a solution \(V, W\) of (39) is a solution of rank two if \(L_4\) commutes with \(L_{4g+2}\) for all \(t_n\) and together they form a pair of operators of rank two. Solutions of rank one (that is, when \(L_4\) commutes with an operator of odd order) were found by Drinfeld and Sokolov [59].

By Theorem 9, the operator \(L_4\) is associated with a function
\[Q = z^g + \alpha_{g-1}(x, t_n)z^{g-1} + \cdots + \alpha_0(x, t_n).\]
It was proved in [20] that \(Q\) satisfies the following evolution equations for \(n = 1, 2\):
\[Q_{t_1} = \frac{1}{2}(-3VQ_x - Q_{xxx}),\] (43)
\[Q_{t_2} = \frac{1}{8}(-4QW_x + 2V_xQ_{xx} + Q_x(8z - 5V^2 + 2W - V_{xx}) - 2VQ_{xxx}).\] (44)

These equations determine symmetries of the equation (25).
There exists an analogue of (43) in the case of finite-gap solutions of the Korteweg–de Vries equation

\[ u_t = \frac{1}{4}(6uu_x + u_{xxx}), \quad u = u(x,t) \]

(see [37]). The Korteweg–de Vries equation has the following Lax representation:

\[
\begin{bmatrix}
\partial_x^2 + u, \partial_t - \left( \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x \right)
\end{bmatrix} = 0.
\]

Finite-gap solutions are distinguished by the condition

\[
[\partial_x^2 + u, L_{2g+1}] = 0,
\]

where \( L_{2g+1} \) is a differential operator of order \( 2g + 1 \) with coefficients depending on \( t \). The polynomial \( Q \) associated with a finite-gap Schrödinger operator (see § 2.2) satisfies the equation

\[
Q_t = \frac{1}{2}(2z + u)Q_x - \frac{1}{2}Qu_x
\]

(see [37]). This equation is analogous to (43). It determines symmetries of the equation (2).

Suppose that the spectral curve is given by the equation

\[
w^2 = F_1(z) = z^3 + c_2z^2 + c_1z + c_0.
\]

Then, as mentioned in § 3.2.2, the function \( V(x,t_n) \) is expressed in terms of \( W(x,t_n) \) by

\[
V(x,t_n) = \frac{-16F_1((-c_2 - W(x,t_n))/2) + W_{xx}^2 - 2W_xW_{xxx}}{4W_x^2}.
\]

For \( n = 1 \) the system (40) and also (43) reduce to the Krichever–Novikov equation

\[
W_{t_1} = \frac{48F_1((-c_2 - W)/2) - 3W_{xx}^2 + 2W_xW_{xxx}}{8W_x}.
\]

Similarly, for \( n = 2 \) the system (41), (42) and also the equation (44) reduce to the second equation in the Krichever–Novikov hierarchy:

\[
W_{t_2} = \frac{1}{128W_x^3}(-1280F_1^2(\gamma) - 16(4c_2 + 2W - F_{1xx}(\gamma))W_x^4 - 45W_{xx}^4
\]

\[
+ 100W_xW_{xx}W_{xxx} + 160F_1(\gamma)(5W_{xx}^2 - 2W_xW_{xxx})
\]

\[
+ 20W_x^2(-W_{xx}^2 + 2W_{xx}(4F_{1x}(\gamma) - \partial_x^4W)) + 8W_x^3\partial_x^3W),
\]

where \( \gamma = -(c_2 + W)/2 \).

Self-similar rank-two solutions of these equations take the following form.
Theorem 17 [20]. The systems (40) and (41), (42) have solutions

\[ V(x, t_1) = -10\wp(at_1 + x) - \frac{2a}{21}, \]
\[ W(x, t_1) = -40\wp^2(at_1 + x) - \frac{20}{21}(8a - 7g_1)\wp(at_1 + x) \]

and

\[ V(x, t_2) = -8\wp(at_2 + x) + \frac{2g_1}{3}, \]
\[ W(x, t_2) = 24\wp^2(at_2 + x) - 4g_1\wp(at_2 + x) + \frac{4a}{5} \]

respectively, where \( a \) is a constant and \( \wp(x) \) is the Weierstrass elliptic function satisfying the equation

\[(\wp'(x))^2 = 4\wp^3(x) - g_1\wp^2(x) - g_2\wp(x) - g_3.\]

It would be very interesting to obtain an analogue of the Krichever–Novikov equation from (43) for \( g \geq 2 \).

The solutions \( V(x, t_1) \) and \( W(x, t_1) \) in Theorem 17 and (22) give explicit rank-two solutions of the Kadomtsev–Petviashvili equation.

Bibliography


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