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Some unsolved problems in the theory of differential equations and mathematical physics [1]

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Problems suggested by V.I. Arnol'd.

1. What is the largest number of parts into which the zeros of an n-th degree polynomial spherical function subdivide a sphere?

[The well-known theorem of Courant gives (for a two-dimensional sphere) an upper bound of the form $n^2/2+O(n)$, while the examples of Karpushkin give a lower bound of the form $n^2/4+O(n)$.]

What is the largest number of maxima of such a function?

2. Find the number of components of the space of non-degenerate homogeneous equations $\dot{x} = P(x)$, where $x \in \mathbb{R}^n$, the components of P being homogeneous second-degree polynomials having no common zeros apart from the origin.

[The geometric problem (for $n = 4$) reduces to the study of deformations of foursomes of quadrics (ellipsoids) in projective space. The quadrics can be degenerate, they can even vanish, but they are not allowed to have a point common to all of them. The question is, how many foursomes that cannot be deformed by homotopy into each other are there (in the case $n = 3$, when we are dealing with sets of three ellipses, the answer is 2; ellipses of the first threesome do not intersect, while for the other one each ellipse separates the two points of intersection of the other two ellipses.]

3. How many limit cycles can arise by perturbing an integrable polynomial system of degree n by a small perturbation with a polynomial of degree n?

[The question reduces to studying the number of zeros of the integral $I(h) = \oint \frac{P\,dx + Q\,dy}{M}$ along the contours $H = h$ of the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ with integrating factor M, where X, Y, P, Q are polynomials of degree n. This question is unresolved even in the case $n = 2$ and even for

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[1] The problems below were suggested and discussed during a session of the second international conference of graduates of the Faculty of Mechanics and Mathematics of the Moscow State University. The theme of the conference was "Differential equations and their applications".

$M = 1$, when H is a polynomial. In the case when $M = 1$, and when H, P, Q are polynomials of fixed degree, there is a uniform upper bound for the number of zeros (due to Varchenko and Khovanskii), but it is ineffective.]

4. The sequence of meandrous numbers 1, 1, 2, 3, 8, 14, 42, 81, ... is defined as follows. Let an infinite river flowing from south-west to east cross an infinite highway that goes directly from west to east, under n bridges numbered by $1, \ldots, n$ from west to east along the highway. The order in which the bridges are encountered by the river defines a meandrous permutation of the numbers $1, \ldots, n$. The meandrous number M_n is the number of meandrous permutations on n elements.

[Meandrous numbers have remarkable properties, for example, M_n is odd if and only if n is a power of two (Lando).] Find the asymptotics of M_n as $n \to \infty$. [It is known that $c4^n < M_n < C16^n$, $c, C = \text{const}$.]

5. Is it true that the minimum of Hausdorff dimensions of minimal attractors of the Navier-Stokes equation (on, say, the two-dimensional torus) grows as the Reynolds number increases?

[Even the existence of minimal attractors of any dimension that grows with the Reynolds number has not been proved; only upper bounds of the dimension of all attractors by a power of the Reynolds number are known (these are the results of Il'yashenko, Vishik, and Babin).]

Problems suggested by M.I. Vishik.

1. a) Let us consider a system of reaction-diffusion equations

$$(1) \qquad \partial_t u - \Delta u - f(x, u, \lambda) - g(x) \equiv A(u, \lambda), \quad u = (u^1, \ldots, u^m),$$
$$f = (f^1, \ldots, f^m), \quad |\lambda| \leqslant \lambda_0, \quad x \in \Omega \Subset \mathbb{R}^n,$$

$$(2) \qquad u\,|_{t=0} = u_0(x), \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0,$$

containing a small parameter λ. Under certain conditions on f and g (see, for example, [1]), to the problem (1), (2) there corresponds a semigroup $\{S_t(\lambda),\ t \geqslant 0\}$ $(S_t(\lambda)u_0 = u(t, \lambda)$, where $u(t, \lambda)$ is the solution of (1), (2)), which acts in the space $E = (H_1(\Omega))^m$, $S_t(\lambda): E \to E$ for all $t \geqslant 0$. If for $\lambda = 0$ $f(x, u, 0) = \nabla_u F(x, u)$, then, if certain conditions are satisfied, the principal term $\tilde{u}_0(t)$ of the stabilized asymptotics of $u(t, \lambda)$ in λ $(u(t, \lambda) \in E$ for all t and $\lambda)$, uniformly in t and u_0, $\|u_0\|_E \leqslant R$, is constructed in [1]. The function $\tilde{u}_0(t) = \tilde{u}_0(t, \lambda)$ is piecewise continuous in t, is dependent on λ, and its continuous pieces satisfy the limiting system of equations for $\lambda = 0$:

$$(3) \qquad \partial_t \tilde{u}_0(t) = A(\tilde{u}_0(t), 0),$$
$$\tilde{u}_0\,|_{t=0} = u_0 = u_0(x).$$

All the continuous pieces of $\tilde{u}_0(t)$, except for the first one, belong to a finite-parameter family of solutions of (3). Moreover, we have the estimate

$$(4) \qquad \sup_{0 \leqslant t < \infty} \|u(t, \lambda) - \tilde{u}_0(t)\|_E \leqslant C|\lambda|^q, \quad q > 0, \quad C = C(R).$$

The problem consists in finding the next term of the stabilized asymptotics of $u(t, \lambda)$, that is, in constructing a vector function $\widetilde{u}_1(t) \in E$, piecewise continuous in t, which except for its first continuous piece belongs to a finite-parameter family of curves and satisfies the estimate

$$(5) \qquad \sup_{t \geqslant 0} \| u(t, \lambda) - \widetilde{u}_0(t) - \widetilde{u}_1(t) \| \leqslant C_1 |\lambda|^{q_1}, q_1 > q.$$

b) Consider a hyperbolic equation with dissipation that contains a small parameter λ multiplying the term $\partial_t^2 u$:

$$(6) \qquad \lambda \partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) - g(x), \; u|_{\partial\Omega} = 0, \; \gamma > 0,$$

$$(7) \qquad u|_{t=0} = u_0, \; \partial_t u|_{t=0} = p_0,$$

Under certain conditions on $f(u)$ and $g(x)$ the principal term $\widetilde{u}_0(t)$ of the stabilized asymptotics of the solution $u(t, \lambda)$ in λ, which satisfies an estimate of the form (4), where $E = H_1$, was constructed in [1]. The function $\widetilde{u}_0(t)$ is piecewise continuous in t and at its points of continuity satisfies the limiting parabolic equation

$$(8) \qquad \gamma \partial_t \widetilde{u}_0(t) = \Delta \widetilde{u}_0(t) - f(\widetilde{u}_0(t)) - g, \; \widetilde{u}_0|_{t=0} = u|_{t=0} = 0.$$

All the continuous pieces of $\widetilde{u}_0(t)$, except for the first one, belong to a finite-parameter family of solutions of (8).

The problem consists in finding the next term $\widetilde{u}_1(t) \in E$ of the stabilized asymptotics of $u(t, \lambda)$ such that an estimate of the form (5) is satisfied.

c) An analogous question consists in finding the second term of the stabilized asymptotics of trajectories $u(t, \lambda)$ of the semigroups $\{S_t(\lambda)\}$ that depend on a small parameter λ. Here it is assumed that the semigroup $\{S_t(\lambda)\}$, $|\lambda| \leqslant \lambda_0$, satisfies conditions such as the ones satisfied by the semigroups corresponding to the problems (1), (2) and (6), (7).

2. Find a lower bound for the Hausdorff dimension of the attractor $\mathfrak{A}$ of the two-dimensional Navier-Stokes system of equations for large values of Reynolds number Re. Determine the exponent q for which we have the estimate

$$(9) \qquad C(\mathrm{Re})^q \leqslant \dim_H \mathfrak{A}$$

($\dim_H \mathfrak{A}$ is the Hausdorff dimension of $\mathfrak{A}$).

Remark. In [2] it is shown that for periodic Kolmogorov flows with periods $2\pi/\alpha$ in x_1 (α is a small parameter) and 2π in x_2 we can take $q = 1$ in (9). Apparently, this estimate can be improved.

3. Let there be given a system of ordinary differential equations

$$(10) \qquad \partial_t \widetilde{u} = f(\widetilde{u}), \widetilde{u} = (\widetilde{u}^1, \ldots, \widetilde{u}^m), f = (f^1, \ldots, f^m),$$

$f_u \geqslant -CI$ (I is the identity matrix). Moreover, additional conditions may be imposed on f. It is assumed that the system (10) has a compact maximal attractor. Let $u\,(t,\ x,\ \varepsilon)$ be the solution of the following partial differential boundary-value problem:

$$\text{(11)} \qquad \partial_t u = \varepsilon \Delta u + f\,(u),\ \varepsilon > 0,\ x \in T^n\ (T^n \text{ is a torus})$$

$$\text{(12)} \qquad u\,|_{t=0} = u_0\,(x).$$

The problem consists in finding the principal term $\widetilde{u}$ of the stabilized asymptotics in ε of the solution $u\,(t,\ x,\ \varepsilon)$ uniformly in t, $0 \leqslant t \leqslant +\infty$, and in $u_0(x)$, $\|u_0(x)\|_C \leqslant R$. Here for $\widetilde{u}$ we have the estimate

$$\sup_{0 \leqslant t < +\infty} |u\,(t, x, \varepsilon) - \widetilde{u}\,(t, x, \varepsilon)| \leqslant C\varepsilon^q, q > 0, C = C\,(R).$$

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Problems suggested by Yu.S. Il'yashenko.

The problems suggested below are closely related to Hilbert's 16th problem. This problem admits a number of formulations of various strengths.

A formulation that is intermediate in strength is as follows. Prove that for every n there is a number N such that a polynomial vector field of degree n in the real plane has no more than N limit cycles. At the beginning of the century, when the problem was being posed, polynomials of fixed degree were the most natural finite-parameter family of vector fields. Nowadays "typical" finite-parameter families are popular.

1. **The Hilbert-Arnol'd problem.**

Prove that for a typical finite-parameter family of smooth, that is C^∞, vector fields on the two-dimensional sphere there is a number N such that the equations of the family have no more than N limit cycles. It is assumed that the basis of the family is compact. This problem is close to the one posed by Arnol'd in [1]; hence its name. The condition of being typical is essential, since an individual C^∞-vector field may have countably many limit cycles. We note that smooth functions encountered in typical finite-parameter families behave in many respects as analytic functions.

The two following problems are auxiliary problems for the foregoing one.

2. Bifurcation of elementary complex cycles.
A critical point of a vector field in the plane is called *elementary* if at least one eigenvalue of the linearization of the field at that point is not zero. A complex cycle (a polygon of separatrices) is called *elementary* if all its critical points are elementary.

Let us consider an elementary complex cycle of a smooth vector field in the plane. We shall assume that its monodromy transformation has a Dulac series that is not identically zero. It has to be proved that under bifurcation of this vector field in an arbitrary finite-parameter smooth family, the number of limit cycles arising does not exceed some constant that depends on the vector field being deformed and not on the deformation itself.

Remark. The case in which the assumption above does not hold, that is, if the correction to the Dulac series is zero, defines a set of vector fields of codimension infinity; equations from this set will not be encountered in typical finite-parameter families.

3. Resolution of singularities in families.
The classical Bendixson-Dumortier theorem states that an isolated critical point of an analytic vector field, or a critical point of finite multiplicity of a smooth vector field, can be resolved into finitely many elementary ones by finitely many σ-processes. Is an analogous theorem for local families of vector fields true? To be more specific, let us consider a family of differential equations $\dot{x} = v(x, \varepsilon)$ in a neighbourhood of the point (0, 0) of the product of the two-dimensional phase space and a finite-dimensional parameter space B. Do there exist a manifold M and a direction field α on M such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{H} & U \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{B} & \xrightarrow{h} & B \end{array}$$

commutes and such that the following conditions are satisfied: $\tilde{B}$ is an analytic manifold of the same dimension as B, h is an analytic but not necessarily bijective mapping; π is the projection $(x, \varepsilon) \mapsto \varepsilon$; M is an analytic manifold of dimension $\dim B+2$; $\tilde{\pi}$ is an analytic mapping, the fibers of which are two-dimensional manifolds; the directions of the field α are mapped by the analytic mapping H into the direction field generated by the vector field $(v, 0)$ on U; the field α is tangent to the fibers of the mapping $\tilde{\pi}$; the restriction of the field α to each fiber in a neighbourhood of each point is generated by a vector field having only elementary critical points.

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Problems suggested by A.S. Kalashnikov.

1. Let us set $\Omega_T = \{(x, t) \mid x \in \mathbb{R}^N,\ 0 < t < T\}$, where $0 < T \leqslant +\infty$ and let us consider on Ω_T the Cauchy problem for the system

$$(1) \qquad \frac{\partial u_i}{\partial t} = a_i \Delta_x (u_i^{m_i}) - b_i u_1^{p_i} u_2^{q_i} \qquad (i = 1, 2)$$

with initial conditions

$$(2) \qquad u_i (x, +0) = u_{0i} (x) \ (i = 1, 2).$$

Here

$$(3) \qquad a_i > 0,\ m_i \geqslant 1,\ p_i > 0,\ q_i > 0\ (i = 1, 2),\ \ b_1 > 0,\ b_2 \geqslant 0$$

are constants; $u_{0i} \in L^\infty (\mathbb{R}^N)$, $u_{0i} (x) \geqslant 0$ $(i = 1, 2)$ for almost all (abbreviated as a.a.) $x \in \mathbb{R}^N$. A generalized solution of (1), (2) (abbreviated as g.s. of (1), (2)) in Ω_T is taken to be a vector-valued function $(u_1, u_2) : \Omega_T \to (\overline{\mathbb{R}}_+)^2$ which belongs to $(L^\infty (\Omega_T))^2$ and satisfies (1) in $\mathscr{D}' (\Omega_T)$ and (2) in $\mathscr{D}' (\mathbb{R}^N)$. The existence of a g.s. of (1), (2) follows from the results of [1].

The following questions have not been exhaustively studied: A) is the g.s. of (1), (2) in Ω_∞ unique? B) how regular is it?

2. Let (u_1, u_2) be a g.s. of (1), (2) in Ω_∞ under the assumptions (3) and under the additional restrictions

$$(4) \qquad m_1 = m_2 = 1,\ a_1 = a_2,\ b_2 > 0,\ p_1 < 1,\ q_2 < 1,\ q_1 \geqslant q_2;$$

$$(5) \qquad \operatorname{ess\,inf}_{\mathbb{R}^N} \{(1 - p_1 + p_2)^{-1}\, b_2\, [u_{01} (x)]^{1-p_1+p_2} - (1 + q_1 - q_2)^{-1}\, b_1\, [u_{02} (x)]^{1+q_1-q_2}\} > 0.$$

Then: a) $\operatorname{ess\,inf}_{\Omega_\infty} u_1 (x, t) > 0$; b) there is a T such that $u_2(x, t) = 0$ for a.a. $(x, t) \in \mathbb{R}^N \times [T, +\infty)$. This is proved in [1]. Examples constructed there show that if (5) is violated, assertions a) and b) are invalid.

Question: can restriction (4) be weakened, in particular, can one allow $m_i > 1$, $m_1 \neq m_2$, $a_1 \neq a_2$?

3. Let (u_1, u_2) be a g.s. of (1), (2) in Ω_∞ under the assumptions (3) and under the additional restrictions:

$$(6) \qquad p_1 \geqslant 1,\ b_2 \geqslant 0,\ m_i > 1,\ u_{0i} (x) \text{ are of compact support } (i = 1, 2),\ u_{01} (x) \not\equiv 0.$$

Then by well-known results about finite rate of propagation of perturbations, the functions $u_i(x, t)$ are of compact support in x for a.a. $t \geqslant 0$ $(i = 1, 2)$. It is proved in [1] that if (6) and the inequality

$$(7) \qquad q_1 > m_2 - 1 + 2/N$$

hold, the support of $u_1(x, t)$ has a non-empty intersection with the set $\{(x, t) \in \Omega_\infty \mid |x| > L\}$ for any $L > 0$; if on the other hand

$$q_1 < m_2 - 1 + 2/N \tag{8}$$

and

$$m_1 > m_2, \ b_2 = 0, \ N = 1, \tag{9}$$

then the function $u_1(x, t)$ is localized in space, that is, $u_1(x, t) = 0$ for a.a. (x, t) such that $|x| \geqslant L_0$, $0 \leqslant t < +\infty$, for some $L_0 \in (0, +\infty)$.

Questions. A) What is the asymptotic behaviour of the boundary of the support of $u_1(x, t)$ as $t \to +\infty$ in the case when (7) holds? B) Is the function $u_1(x, t)$ localized in space when (8) holds and (9) is violated?

4. Let us now change the assumptions (3) in the following way:

$$b_1 < 0, \ p_1 > 1, \ a_i > 0, \ m_i \geqslant 1, \ q_i \geqslant 0 \ (i = 1, 2), \ b_2 \geqslant 0, \ p_2 \geqslant 0. \tag{10}$$

In view of the first two of the inequalities (10), the problem (1), (2) in Ω_T is soluble in general only for sufficiently small $T > 0$. In [2] it is proved that there is a g.s. of (1), (2) in Ω_∞ if together with (10) we have the inequality

$$|b_1|(p_1 - 1)(\text{ess sup } u_{01}(x))^{p_1-1}(\text{ess sup } u_{02}(x))^{1+q_1-q_2} < \\ < b_2(1 + q_1 - q_2)(\text{ess inf } u_{01}(x))^{p_2}; \tag{11}$$

if (11) is violated this statement is no longer valid.

Question: For what T can we guarantee the existence of g.s. of (1), (2) in Ω_T if (11) is violated?

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Problems suggested by V.A. Kondrat'ev.

Let us consider the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x, t) \frac{\partial u}{\partial x_j}, \tag{1}$$

where $x = (x_1, \dots, x_n)$, $(x, t) \in Q = \{(x, t) : |x - x_0| \leqslant \alpha, \ t_0 < t < t_0 + \beta\}$, $\alpha = \text{const} > 0$, $\beta = \text{const} > 0$. It is assumed that $a_{ij}(x, t)$ are bounded measurable functions in Q and

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqslant \lambda \sum_{i=1}^{n} \xi_i^2, \quad \lambda = \text{const} > 0.$$

A (generalized) solution of equation (1) is a function $u(x, t) \in L_2(Q)$ such that $\frac{\partial u}{\partial x_i} \in L_2(Q)$, $i \leqslant n$,

$$-\int_Q u \frac{\partial \psi}{\partial t}\, dx\, dt + \int_Q \Big[\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \psi}{\partial x_i} \Big]\, dx\, dt = 0,$$

for any $\psi(x, t)$ such that $\psi \in L_2(Q)$, $\frac{\partial \psi}{\partial x_i} \in L_2(Q)$, $\psi|_{|x-x_0|=\alpha} = 0$, $\psi|_{t=t_0+\beta} = 0$.

1. Under what conditions on $a_{ij}(x, t)$ is it true that

$$\frac{\partial u}{\partial t} \in L_2(Q)?$$

[This statement is easily proved if

$$|a_{ij}(x, t+h) - a_{ij}(x, t)| \leqslant Lh^{\gamma}$$

for $i, j = 1, \ldots, n$, $\gamma = \text{const} > 1/2$.]

Can the restriction $\gamma > 1/2$ be weakened?

2. It is not hard to prove that $u(x, t) \in H^{0, 1/2}(Q)$, that is,

$$\int_{|x-x_0|\leqslant\alpha} \int_{\substack{t_0<t<t_0+\beta \\ t_0<\tau<t_0+\beta}} \frac{|u(x, t) - u(x, \tau)|^2}{|t-\tau|^2}\, dx\, dt\, d\tau < \infty.$$

Is it possible to prove that $u(x, t) \in H^{0, s}$ for any $s > 1/2$?

Problems suggested by S.N. Kruzhkov.

1. Let $u_0(x) \in L_\infty(\mathbb{R}^1)$, $u(t, x) \in L_\infty(\Pi_T)$, where $\Pi_T = (0, T] \times \mathbb{R}^1$, while in Π_T we have $u_t + (u^2/2)_x = 0$, $(u^2/2)_t + (u^3/3)_x \leqslant 0$, and $u(t, x) \to u_0(x)$ as $t \to +0$ in the sense of generalized functions.

Consider the question of uniqueness of the function $u(t, x)$ for a given fixed function $u_0(x)$.

2. Let $u = (u^1, u^2) \in \mathbb{R}^2$, $\varphi(u) = (\varphi^1(u), \varphi^2(u))$, where $\varphi^1(u)$, $\varphi^2(u)$ are smooth functions on $\mathbb{R}^2$, and such that the Jacobian matrix $\varphi'(u)$ has real and distinct eigenvalues. Let us consider the linear hyperbolic system

$$\varphi^1_{u^1} F_{u^1} + \varphi^2_{u^1} F_{u^2} = G_{u^1},$$
$$\varphi^1_{u^2} F_{u^1} + \varphi^2_{u^2} F_{u^2} = G_{u^2}$$

(for short: $\varphi'(u)F_u = G_u$) with respect to scalar Lipschitz continuous functions $F(u)$ and $G(u)$ on $\mathbb{R}^2$ that satisfy this system of equations almost everywhere.

Under what conditions on $\varphi(u)$ does there exist a family of solutions $F(u, k)$ and $G(u, k)$ depending on a parameter $k = (k^1, k^2) \in \mathbb{R}^2$ that have the following properties: 1) $F(u, k) = F(k, u)$, $G(u, k) = G(k, u)$; 2) $F(u, k) \geqslant 0$ and $F(u, k) = 0 \Leftrightarrow u = k$; 3) on any compact set $K \subset \mathbb{R}^2 \times \mathbb{R}^2$ we have the inequality $|G(u, k)| \leqslant C_k F(u, k)$, $C_k = \text{const}$; 4) the function $F(u, k)$ is convex in u and in k? The study of this question is of interest for the theory of quasi-linear hyperbolic systems, both in the global and in the local setting.

3. Let the vector function $\varphi(u)$ satisfy the conditons of the previous problem, with the components of the function $u_0(x) = (u_0^1(x), u_0^2(x))$ belonging to $C_0^\infty(\mathbb{R}^1)$. Let us consider the Cauchy problem for the parabolic system $u_t^\varepsilon + (\varphi(u^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon$, $\varepsilon = \mathbf{const} > 0$, with initial condition $u^\varepsilon(0, x) = u_0(x)$.

Do there exist $\varphi(u)$ and $u_0(x)$ such that the corresponding solutions $u^\varepsilon(t, x)$ are uniformly bounded as $\varepsilon \to +0$, while their variation in x on some fixed interval is not bounded uniformly in ε?

4. Construct a theory of the Cauchy problem and of the main boundary-value problems for the parabolic equation $u_t = u_{xx} + \operatorname{sign} u_x$.

5. Describe the possible singularities of the generalized solution of the Cauchy problem for the KdV equation $u_t + uu_x = u_{xxx}$ with initial condition $u(0, x) = u_0(x)$, where $u_0(x)$ belongs only to $L_2(\mathbb{R}^1)$.

Problems suggested by E.M. Landis.

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial}{\partial x_i \partial x_j}$ be a uniformly elliptic operator in Ω with measurable bounded coefficients. Let f be a function continuous on $\partial\Omega$. A function $v(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is called an upper (lower) function for the Dirichlet problem if $Lv \leqslant 0$ ($Lv \geqslant 0$) in Ω and $v|_{\partial\Omega} \geqslant f$ ($v|_{\partial\Omega} \leqslant f$). Let us set $u^+(x) = \inf v(x)$, where the infimum is taken over all upper functions. Similarly, let $u^-(x) = \sup v(x)$, where the supremum is taken over all lower functions.

Questions: a) Is it true that u^+ (u^-) satisfies locally a Hölder condition?

b) Let $x_0 \in \partial\Omega$ be a regular point of the boundary of the domain (see [1]). Is it true that $u^\pm(x) \to f(x_0)$ as $x \to x_0$ ($x \in \Omega$)?

c) Let $a_{ij}^h(x)$ be a homogenization of the coefficient a_{ij}, $L^h = \sum_{i,j=1}^n a_{ij}^h(x) \frac{\partial^2}{\partial x_i \partial x_j}$, assume that all the points of $\partial\Omega$ are e-regular (Ω can be taken to be a ball), and let $u^{(h)}$ be the solution of the Dirichlet problem $L^h u^{(h)} = 0$, $u^{(h)}|_{\partial\Omega} = f$. From a theorem of Krylov and Safonov [2], [3] and from the e-regularity condition it follows that there is a sequence $\{u^{(h_k)}\}$, $h_k \to 0$, such that $u^{(h_k)} \rightrightarrows u^*$.

Is it true that $u^- \leqslant u^* \leqslant u^+$?
d) Is it true that $u^- \equiv u^+$?

2. Let C_ρ be the semi-infinite cylinder $C_\rho = \{x \in \mathbb{R}^n, |x'| < \rho, 0 < x_n < \infty\}$, $\rho > 0$, $x' = (x_1, \ldots, x_{n-1})$. Let $u(x)$ be a solution of the equation $\Delta u = uf(|u|)$ in C_ρ, where $f(t)$, $t > 0$, is a positive monotone increasing function. Let us set $M(t) = \sup_{|x_n|=t} |u(x)|$. If

$$(1) \qquad \int_0^1 \frac{dt}{t\sqrt{f(t)}} < \infty,$$

then there is in C_ρ a solution $u(x) \neq 0$ that is zero for sufficiently large x_n (see [4]). If $\Delta u = 0$ or if u is a solution of the equation $\Delta u = uf(|u|)$ with f bounded, then the limiting rate of decrease of $M(x_n)$ as $x_n \to \infty$ for which $u \not\equiv 0$ is of the order

$$(2) \qquad \exp(-A \exp x_n)$$

($A > 0$ depends on ρ). It can be shown that this limiting rate of decrease of a non-zero solution stays the same, at least up to $f(t) \leqslant |\ln t|^{2-\delta}$, $\delta > 0$ (only A changes). But somewhere in the interval $|\ln t|^{2-\delta}$, $|\ln t|^2$ this limiting rate of decrease of a non-zero solution starts to change, since there are functions $f(t) \leqslant C|\ln t|^2$ such that the solution can decrease at a rate $(-\exp(\exp \ldots \exp t))$, while for $f(t) \geqslant |\ln t|^{2+\delta}$ we naturally have (1).

It is required to: a) find the sharp boundary of the growth rate of $f(t)$, $t \to +0$, for which the limiting rate of decrease of a non-zero solution as $x_n \to \infty$ has the form (2);

b) find the dependence between the growth rate of the function $\varphi(\varepsilon) = \int_\varepsilon^1 \frac{dt}{t\sqrt{f(t)}}$ as $\varepsilon \to 0$ and the admissible rate of decrease of $M(x_n)$ as $x_n \to \infty$.

3. Let $\Omega \subset \mathbb{R}^n$, $n > 2$, be a bounded domain and let the origin O lie in $\partial\Omega$. Let $\partial\Omega$ be contained in the cone $\{x \in \mathbb{R}^n : |x'| \leqslant a|x_n|, x_n \geqslant 0\}$ in a neighbourhood of O; here $a > 0$, $x' = (x_1, \ldots, x_{n-1})$, and apart from the point O the hyperplane $x_n = 0$ belongs to Ω in a neighbourhood of O. Let f be a function continuous on $\partial\Omega$, and let u_f be the generalized (in the sense of Wiener) solution of the Dirichlet problem $\Delta u_f = 0$, $u_f|_{\partial\Omega} = f$. Let us denote by $v(x')$ the restriction of u_f to the hyperplane $x_n = 0$ (with the origin removed). Let us set $E_k = \{x \in \mathbb{R}^n : 2^{-(k+1)} < |x| < 2^{-k}, x \notin \Omega\}$. It is known (see [5]) that if m is a non-negative integer,

$$(3) \qquad \sum_{k=1}^{n} \operatorname{cap} E_k \cdot 2^{k(n-2)+m+\alpha} < \infty,$$

then $v(x')$ can be extended to the point O so that $v \in C^{m,\alpha}$ in a neighbourhood of O. Condition (3) is sharp.

It is required to change condition (3) so that v: a) belongs to a given Gevrey class, b) is an analytic function, in such a way that the established conditions are sharp.

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Problems suggested by V.M. Millionshchikov.

1. For every integer $k > 2$ find out whether the zero solution of the equation $x^{(k)} + (\cos t + \sin \sqrt{2}t)\, x = 0$ is stable. (For $k = 2$ we have instability. This was proved by Filippov [1].)

2. Does there exist an $\alpha \in \mathbb{R}$ for which the equation

$$\ddot{x} + (\cos t + \alpha \sin \sqrt{2}t)\, x = 0;$$

is a) irreducible, b) not almost reducible, c) irregular? (For the concepts of reducible, almost reducible, and regular linear systems of differential equations see [2], col. 633, 547-548, 551-552.)

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Problems suggested by O.A. Oleinik.

1. Let us consider the stationary linear system of elasticity theory

(1) $$\frac{\partial}{\partial x_i} a_{kh}^{ij}(x) \frac{\partial u_j}{\partial x_h} = f_k, k = 1, \dots, n,$$

where

$$u = (u_1, \dots, u_n),\ x = (x_1, \dots, x_n),\ f = (f_1, \dots, f_n),$$
$$a_{kh}^{ij}(x) = a_{ih}^{kj}(x) = a_{hk}^{ji}(x),$$
$$\lambda_1 |\eta|^2 \leqslant a_{ih}^{kj} \eta_i^k \eta_h^j \leqslant \lambda_2 |\eta|^2, \eta_i^k = \eta_k^i, |\eta|^2 = \eta_i^k \eta_i^k.$$

Here and in what follows we assume summation on repeated indices from 1 to n. For the system (1), let us consider the boundary-value problem in the layer $\Omega = \{x : 0 < x_n < 1\}$ with the boundary conditions

(2) $$\sigma_k(u) \equiv a_{kh}^{ij}(x) \frac{\partial u_j}{\partial x_h} \nu_i = 0, \quad k = 1, \dots, n,$$

for $x_n = 0$ and $x_n = 1$, where $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward normal vector to $\partial\Omega$. It can be shown (see [1], [2]) that for $f = 0$ the solutions of the problem (1), (2) can only be vector-valued functions $u = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector, if it is assumed that

$$\mathcal{D}(u, \Omega) = \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j}\right)^2 dx < \infty.$$

It is of interest to consider the class of solutions of (1), (2) such that the energy integral

$$E(u, \Omega) = \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 dx < \infty.$$

Can one claim that in this class of solutions the problem (1), (2) in Ω for $f = 0$ has only solutions of the form $u = Ax + B$? For what unbounded domains is this not true?

2. In [3], [4] it is shown that the generalized solution of the Dirichlet problem for the biharmonic equation

(3) $$\Delta\Delta u = f$$

in a domain $\Omega \subset \mathbb{R}^2$ with boundary conditions

(4) $$u = 0,\ \operatorname{grad} u = 0 \ \text{ on } \partial\Omega,$$

which belongs to the Sobolev space $W_2^2(\Omega)$, satisfies in a neighbourhood of a point O of the boundary $\partial\Omega$ of the domain Ω (we take O to be the origin) the estimates

(5) $$|u(x)| \leqslant C_1 |x|^{1+\delta(\omega)},\ C_1 = \text{const},$$

where $\delta(\omega)$ is the solution of the transcendental equation

$$\sin^2(\omega\delta) = \delta^2 \sin^2 \omega;$$

the constant ω is determined by the geometric structure of the domain Ω in a neighbourhood of the point O; the length of any arc belonging to the intersection $\Omega \cap \{x: |x| = t\}$ does not exceed ωt. The estimate (5) is sharp in the indicated class of domains. This is proved [3], [4] for $1.24\pi \leqslant \omega \leqslant 2\pi$. (The estimate (5) is sharp in the sense that it will not hold if we substitute $C_1 |x|^{1+\delta(\omega)+\varepsilon}$, $\varepsilon = \text{const} > 0$.) Some estimates of the form (5) were obtained in [5] for any ω.

It is of interest to obtain sharp estimates in the case $0 < \omega < 1.24\pi$ also.

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Problems suggested by A.F. Filippov.

Let $F(t, x)$ be a compact set in $\mathbb{R}^n$ that depends continuously on t, x. Let all the solutions of the differential inclusion $\dot{x} \in F(t, x)$ with initial condition $x(t_0) = x_0$ exist for $t_0 \leqslant t \leqslant t^*$, let M be the integral funnel of the point (t_0, x_0), that is, the set of points t, x lying on the graphs of such solutions, let $A(t_1)$ be the section of this funnel by the plane $t = t_1$, where $t_0 \leqslant t_1 \leqslant t^*$, and let M^+ and M^- be the parts of the funnel M lying in the half-spaces $t \geqslant t_1$ and $t \leqslant t_1$, respectively. The *upper tangent cone* to a set B at a point x is the set

$$T(B, x) = \overline{\lim_{h \to +0}} \frac{1}{h} (B - x)$$

(the upper topological limit is being taken). It is known that for $x_1 \in A(t_1)$ $T(M^+, (t_1, x_1))$ is a set (in the tangent space t, x) whose section by the plane $\tau = t - t_1 \geqslant 0$ is described by the formula $T(A(t_1), x_1) + (t - t_1)F(t_1, x_1)$.

1. Describe the set $T(M^-, (t_1, x_1))$.

2. Under what conditions (without the assumption of convexity of $F(t, x)$ and $A(t)$) does the upper tangent cone to $A(t_1)$ at a point $x_1 \in \partial A(t_1)$ coincide with the lower cone, which is defined in the same way, but with $\underline{\lim}$ instead of $\overline{\lim}$?

Problems suggested by M.A. Shubin.

1. Colin de Verdière [1] proved that any set of numbers $0 = \lambda_1 < \lambda_2 < \ldots$ $\ldots < \lambda_N$ can be the set of the first N eigenvalues of a planar membrane with a free end, that is, the collection of the first N eigenvalues of the operator $(-\Delta)$ with Neumann boundary conditions in some planar domain.

Let us pose a similar question for a membrane with clamped ends: what set of positive numbers $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ can be the set of the first N eigenvalues of the operator $(-\Delta)$ in some planar domain with Dirichlet boundary conditions?

A number of results show that the answer here has to be quite complicated. For example, we have the inequalities $\lambda_{j+1} \leqslant 3\lambda_j$ for all $j = 1, 2, \ldots,$ (see [2]) $\lambda_3 + \lambda_2 \leqslant 6\lambda_1$ (see [2]), $\lambda_3 + \lambda_2 \leqslant (3 + \sqrt{7})\,\lambda_1$ (see [3]), $\lambda_2/\lambda_1 \leqslant 2.6578$ (see [4]), $\lambda_3 \leqslant \lambda_1 + \lambda_2 + \sqrt{\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2}$ (see [5] and also the survey paper [6]). Let us note that a homothety of the domain allows us to multiply each of the numbers $\lambda_1, \ldots, \lambda_N$ by the same positive factor.

In [2] it is conjectured that $\max \lambda_{n+1}/\lambda_n$ for all membranes and for all n is attained for the disk and for $n = 1$ (for the disk $\lambda_2/\lambda_1 = 2.53873\ldots$). The proof of this conjecture would have provided us with some information about the answer to the question formulated above. Let us note however that it has not yet been proved even that $\max \lambda_2/\lambda_1$ is attained on the disk.

Another interesting particular case of the general question is: what is $\max \lambda_3/\lambda_2$ over all planar membranes? [From the conjecture of [2] stated above it follows that this maximum is obtained for a pair of two non-intersecting identical disks (for a disk $\lambda_2 = \lambda_3$, while for a pair of identical disks $\lambda_1 = \lambda_2$, and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$ coincides with the eigenvalue λ_2 of the disk). Let us note that among polygons $\max \lambda_3/\lambda_2$ is obtained for a rectangle with side ratio of $3:8$ (and equals 1.75).]

2. Let X be a compact Riemannian manifold, and let M be its universal covering with the induced metric. Let us consider the heat equation on p-forms on M, and let $\mathscr{E}_p = \mathscr{E}_p\,(t, x, y)$ be its fundamental solution (the Schwartz kernel of the operator $\exp(-t\Delta_p)$, where $\Delta_p = d\delta + \delta d$ is the Laplace operator on exterior p-forms). Let $L_p = L_p(x, y)$ be the kernel of the orthogonal projection operator onto harmonic square integrable p-forms on M,

that is, forms that are stationary solutions of the heat equation. Then we can expect that the kernel $R_p(t, x, y) = \mathscr{E}_p(t, x, y) - L_p(x, y)$ decays as $t \to +\infty$.

Conjecture: there is a number $\varepsilon_p > 0$ such that $R_p(t, x, y) = O(t^{-\varepsilon_p})$ uniformly in x, y. In [7], [8] a somewhat weaker conjecture is made, and it is noted that the indicated estimate makes it possible to define the von Neumann torsion of Ray–Singer type for multiply-connected manifolds. Moreover, the sharp upper bound $\bar{\varepsilon}_p$ of the possible numbers ε_p does not depend on the metric on X and is an invariant of a multiply-connected smooth manifold X.

Can the invariants $\bar{\varepsilon}_p$ be expressed in terms of known differential topological invariants?

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