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On the determination of minimal global attractors for the Navier-Stokes and other partial differential equations

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CONTENTS

§1. Introduction

In this paper I want to clarify and present in a systematic way a number of results in a field which, in the context of partial differential equations (PDE's), has been developing rapidly since the seventies, and which appears to be very fruitful and full of promise. This field, a global stability theory for PDE's, qualifies as a new direction in the study of initial boundary-value problems for PDE's. We shall deal with the problem of finding minimal sets which in time attract some part of the phase space X. We shall mainly concentrate on determining the minimal set attracting any bounded subset B of X. Let us clarify this statement.

Initial boundary-value problems for PDE's for which these problems are to be considered must be uniquely soluble for all $t \in \mathbf{R}^+ \equiv [0, \infty)$ in some complete metric space X, which I shall call the phase space. In this paper I shall restrict myself to the case when there is no explicit dependence on time, and when, therefore, the solution operators V_t, $t \in \mathbf{R}^+$, form a semigroup $V_t: X \to X$, $t \in \mathbf{R}^+$. The choice of phase-space for any given problem (that is, for a given PDE together with a boundary condition) is not unique. It is natural to try to choose the largest possible X. The limit to the size of X is set by requiring a uniqueness theory to hold, so that the dynamics are deterministic. It frequently happens that if a uniqueness theorem can be proved, one can also prove continuity of the semigroup $V_t: X \to X$, $t \in \mathbf{R}^+$, that is, continuity of $V_t(v)$ in $(t, v) \in \mathbf{R}^+ \times X$. However, many papers and even monographs on the subject omit this proof, partly since it is not essential for their purposes. In these works, instead of proving continuity of $V_t(v)$ in (t, v) the authors at most prove "weak" continuity of $V_t(v)$ in $t \in \mathbf{R}^+$ for a fixed $v \in X$, and continuity of $V_t(v)$ in v for almost all t in $\mathbf{R}^+$. For some of the questions we consider here, continuity in t is not needed. These questions can in fact be resolved positively in the case of discrete semigroups V^n $(n = 0, 1, \ldots)$. On the other hand, the continuity of the operators V_t for all $t \geqslant 0$ (or of the operator V in the discrete case) is necessary. From the point of view of physics and geometry, it is unnatural to withdraw the requirement that the semitrajectory $\gamma^+(v) = \{V_t(v),\ t \in \mathbf{R}^+\}$ be continuous in t. The same is true about the continuity of $V_t(v)$ in $(t, v) \in \mathbf{R}^+ \times X$. In all thoroughly investigated cases of initial boundary-value problems for PDE's, continuity of $V_t(v)$ in $(t, v) \in \mathbf{R}^+ \times X$ holds. Let us denote various bounded subsets of X by B, and the family of all such sets by $\mathscr{B}$.

The statement that the set B_0 attracts the set B has the following interpretation: for any $\varepsilon > 0$ there exists a number $t_1(\varepsilon, B)$ such that $V_t(B) \subset O_\varepsilon(B_0)$ for all $t \geqslant t_1(\varepsilon, B)$, where $O_\varepsilon(B_0)$ is the union of all balls of radius ε with centres at points of B_0.

We call the set B_0 *absorbing* if for all $B \subset \mathscr{B}$ there exists a $t_1(B)$ such that $V_t(B) \subset B_0$ for all $t \geqslant t_1(B)$.

We define the *minimal* (or *true*) *global B-attractor of the problem* (more precisely, of the corresponding semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$) to be the smallest closed non-empty set that attracts all bounded sets $B \subset X$, and we denote it by $\mathfrak{M}$.

We define the *minimal global attractor of the problem* (of the semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$) to be the smallest non-empty closed set that attracts all points of X, and we denote it by $\hat{\mathfrak{M}}$.

It is clear that $\hat{\mathfrak{M}} \subset \mathfrak{M}$, and it is important to note that $\hat{\mathfrak{M}}$ can be much smaller than $\mathfrak{M}$. It turns out that in many interesting problems of natural sciences $\mathfrak{M}$ is a "minuscule" subset of X.

My paper [1], which was published in February 1972, concerns the problem of finding the set $\mathfrak{M}$ for PDE's and describing the dynamics V_t on $\mathfrak{M}$. In this paper the set $\mathfrak{M}$ was found for a certain class (class 1) of semigroups, which encompasses a number of problems of the flow of viscous incompressible fluids, as well as quasilinear parabolic equations and systems of parabolic type for which there exists a bounded absorbing set. This class of semigroups is discussed in §2.

Reflection on the problem of describing turbulent (that is, complex) fluid flow in the framework of Navier-Stokes (N.-S.) equations gave me a hint of the construction by which such a set $\mathfrak{M}$ can be found. My thoughts on this matter are to be found in the beginning of §2. To describe this construction of the set $\mathfrak{M}$ here, I shall first review a number of concepts and facts from the theory of ordinary differential equations (ODE's) and from the theory of semigroups $V_t : X \to X$, $t \in \mathbf{R}^+$, acting on a locally compact space X. To every semitrajectory $\gamma^+(v) \equiv \{V_t(v),\ t \in \mathbf{R}^+\}$ there corresponds the set

$$\text{(1.1)} \qquad \omega(v) \equiv \bigcap_{t\geqslant 0} [\gamma_t^+(v)]_X = \bigcap_{t\geqslant T} [\gamma_t^+(v)]_X \quad \text{for all } T>0,$$

where $[\cdot]_X$ stands for closure in X, and $\gamma_t^+(v) \equiv \{V_\tau(v),\ \tau \in [t, \infty)\}$. The set in (1.1) is called the ω-limit set of $\gamma^+(v)$, (or, what amounts to the same thing, of v). For any $B \subset \mathscr{B}$ the ω-limit set is defined as follows:

$$\text{(1.2)} \qquad \omega(B) \equiv \bigcap_{t\geqslant 0} [\gamma_t^+(B)]_X = \bigcap_{t\geqslant T} [\gamma_t^+(B)]_X \quad \text{for all } T>0,$$

where $\gamma_t^+(B) \equiv \bigcup_{v\in B} \gamma_t^+(v)$.

If $\gamma^+(B) \subset \mathscr{B}$, then $\omega(B)$ is a non-empty compact set that attracts B. It is an invariant set, that is,

$$V_t(\omega(B)) = \omega(B) \quad \text{for all } t \in \mathbf{R}^+.$$

If there exists a set $B_0 \subset \mathscr{B}$ such that $V_t(B_0) \subset B_0$ for all $t \in \mathbf{R}^+$, and if B_0 absorbs all $B \subset \mathscr{B}$, then the set $\omega(B_0)$ is the set $\mathfrak{M}$ for the system of equations (or, alternatively, for the semigroup V_t, $t \in \mathbf{R}^+$, in X). In other words, $\omega(B_0)$ is the minimal global B-attractor for the system. Moreover,

$$\text{(1.3)} \qquad \mathfrak{M} = \omega(B_0) = \bigcap_{t\geqslant 0} [V_t(B_0)]_X = \bigcap_{t\geqslant T} [V_t(B_0)]_X \quad \text{for all } T>0.$$

I shall not quote in the Introduction less restrictive conditions under which a system of equations admits an attractor $\mathfrak{M}$. The statements listed above are not hard to prove, and the same is true about many other important general principles. It was much harder to discover and formulate them. With the help of these notions a number of interesting results in the theory of ODE's have been obtained.

The idea of extending these concepts to cover the case of PDE's had been considered, I think, more than once, especially by dynamical systems experts. However, its realization is from the very beginning hindered by serious obstacles. The reason for this lies in the fact that the phase spaces X for PDE's (equipped with a boundary condition) are not locally compact. Therefore all the statements above, starting with the claim that $\omega(v)$ and $\omega(B)$ are non-empty, could in fact be false. Apparently, this consideration hampered the use of ω-limit sets in PDE's, and created the impression that this direction of enquiry was promising. However, on closer inspection, this impression was found to be wrong, and the work of the last fifteen years has completely dispelled all doubts. The first instance in which this approach bore unexpected fruit was that of initial boundary-value problems for the N.-S. equations [1]. For these in the case $n = 2$, that is, in the case of two-dimensional problems with any of the main boundary conditions, it was found that the set $\mathfrak{M}$ exists, and that it is compact and finite-dimensional in some sense. As an example of the conclusions reached in the beginning of [1], I quote the following theorem.

***Theorem* 1.1.** *Let the semigroup $V_t : X \to X$, $t \in \mathbb{R}^+$, in a complete metric space X be such that the operators V_t are completely continuous*[1] *for all $t > 0$. Then, if $\gamma^+(B) \subset \mathscr{B}$, $\omega(B)$ is a non-empty invariant compact set that attracts B. If there exists a set $B_0 \subset \mathscr{B}$ such that $V_t(B_0) \subset B_0$ for all $t \in \mathbb{R}^+$, and if B_0 absorbs all the sets $B \subset \mathscr{B}$, then $\omega(B_0)$ is $\mathfrak{M}$, that is, the minimal global B-attractor of the system (of the semigroup). It is invariant and compact. If X is connected, so is $\mathfrak{M}$.*

Under the conditions of this theorem $\omega(B_0)$ is found from (1.3).

All the assumptions of Theorem 1.1 are satisfied in the case of two-dimensional N.-S. equations. In [1] the space $\overset{0}{J}(\Omega)$, which is the largest of all admissible Hilbert spaces, serves as the phase space X under the boundary condition $v|_{\partial\Omega} = 0$. The definition of this space, and the definition of the spaces D^s, $s \geqslant 0$, are recalled in §2; the space D^{s_2} is compactly embedded in D^{s_1} if $s_1 < s_2$; $D^0 = \overset{0}{J}(\Omega)$. Any of the spaces D^s, $s \geqslant 0$, could be chosen to play the role of X as long as $\partial\Omega$ and f are sufficiently smooth. I shall not formulate here the relevant smoothness conditions on $\partial\Omega$ and f. In this case the set $\mathfrak{M}$ is a bounded and closed subset of D^{s_1}, $s_1 \geqslant s$, where s_1 is determined by $\partial\Omega$ and f only.

For the three-dimensional N.-S. equations with the same boundary condition, the space $\overset{0}{J}(\Omega)$ is too big: it appears that in this space there is no uniqueness, that is, the operators V_t: $\overset{0}{J}(\Omega) \to \overset{0}{J}(\Omega)$, $t \in \mathbb{R}^+$, are not single-values (on this, see [2], Ch. VI, §7). On the other hand, in the spaces D^s, $s > 1/2$, a uniqueness theorem does hold, but the existence of solution operators $V_t : D^s \to D^s$ for arbitrary (not small) Reynolds numbers has been established only for "short" times. The length of the interval $[0, t(v_0)]$ of existence of the solution $V_t(v_0)$, $v_0 \in D^s$, depends on $\|v_0\|_{D^s}$; moreover, $t(v_0) < \infty$ and $t(v_0) \to 0$ as $\|v_0\|_{D^s} \to \infty$. No interesting bounded subsets B of D^s, $s > 1/2$, for which the sets $\gamma^+(B)$ are defined and bounded in D^s have been found. However, sometimes one should follow the practice of historians, which the "General history, as revised by the Satyriconians" describes thus: "All that pertains to ancient times and of which we know nothing is called the prehistoric period. Scientists, although they have no information about this period (for if they did, they would have had to call it a historic period), divide it nonetheless into three epochs: 1)..., 2)..., 3)..." In this vein, we quote the following result.

***Corollary* 1.1.** *If for the three-dimensional N.-S. equations there exists a set B in D^s, $s > 1/2$, for which $\gamma^+(B)$ is bounded in D^s, then one can choose as the phase space X the set $[\gamma^+(B)]_{D^s}$ equipped with the metric induced by the norm of D^s.*

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[1] That is, V_t is continuous and maps bounded sets into precompact ones.

The resulting semigroup $V_t : X \to X,\ t \in \mathbf{R}^+$, satisfies all the conditions of Theorem 1.1, and its minimal global B-attractor $\mathfrak{M}$ is the set $\omega(B) = \bigcap_{t \geqslant 0} [\gamma_t^+(B)]_{D^s}$. This semigroup has all the other properties established in [1] for the case $n = 2$.

These properties also hold for one of the modifications of the three-dimensional N.-S. equations that I proposed. In [3], where these equations are examined, the space $\overset{0}{J}(\Omega)$ is taken as X; any of the spaces $D^s,\ s > 0$, could be used as well. As I mentioned in [1], the same properties are shared by the semigroups generated by the N.-S. equations with other classical boundary conditions, including the non-homogeneous boundary condition $v\,|_{\partial\Omega} = \alpha$ and the periodicity conditions on x_k. The same is true for the heat convection equations and for the equations of magnetohydrodynamics of viscous incompressible fluids. The main attribute of the semigroup $V_t : X \to X,\ t \in \mathbf{R}^+$, which enables us to find the true (minimal) attractors corresponding to it in the form of ω-limit sets, is the complete continuity property of the operators V_t for $t > 0$. Let us call such semigroups *class* 1 *semigroups.*

To the same class there belong semigroups generated by quasi-linear equations and systems of parabolic type (of course, only by those for which unique global solubility has been established; on this subject see [4], [5], and so on). This fact is also emphasized in [1]. For a certain subclass of such equations (the so-called gradient equations) there exists a "good" Lyapunov function. Its presence makes certain aspects of the proof of existence of a bounded global attractor $\hat{\mathfrak{M}}$ easier, and also substantially simplifies the structure of $\mathfrak{M}$.

In the early summer of 1986 I received from the American mathematician J. Hale a preprint [6], which deals with the construction of the attractor $\mathfrak{M}$ for semilinear wave equations with weak dissipation. From [6] I learned that Hale and a number of his colleagues from the Lefschetz Centre had for a long time been developing a theory of ω-limit sets for various dynamical systems. At the end of the sixties and beginning of the seventies they were led to investigate semigroups on spaces X that are not locally compact. This was deemed necessary in their research into ODE's with retarded argument. This topic proved to be a good source of questions which allowed them to formulate a number of propositions generalizing hitherto known ones. They singled out a class of semigroups (class 1) which was exactly the one I was led to from my investigations of the N.-S. equations. The results they obtained in the early seventies had been systematically presented in Hale's book [7], which appeared in 1977 and was translated into Russian in 1984. Its author was unaware of my work [1], just as I did not know of the work of Hale and his colleagues. This is understandable: the problems that led us to consider class 1 semigroups belong to different fields of mathematics.

The larger part of Hale's book [7] is devoted to differential equations with retarded argument. It seems that the Soviet PDE experts were not familiar with the work of American mathematicians and with Hale's book [7], and that those who were, and who knew about my paper [1], did not see the underlying connection between the two.

At the next stage the same group of American mathematicians started applying their results to semilinear parabolic equations, and at the end of the seventies and beginning of the eighties, to semilinear wave equations with added dissipative terms.

The results pertaining to semilinear parabolic equations are systematized in Henry's book [8], which was published by Springer in 1981 and translated into Russian in 1985. Henry could have referred to [1], since the work of Mallet-Paret [9], which does refer to [1], appeared in 1976. However, this did not happen: in Chapter 2, §7, he lists problems of hydrodynamics of a viscous incompressible fluid as problems to which the results of the theory he develops should be extended. As the authors of the famous "Satyricon" would have put it: "We have got ourselves into a fine mess": in [1] a basic problem of that kind is investigated and on the first page it says that other hydrodynamic problems and parabolic equations can be treated in a similar manner.

I expected the dynamical systems experts pondering the problems of turbulence to be interested in [1] and to continue the study of the dynamics V_t on $\mathfrak{M}$. For this is "almost the same" as the dynamics described by ODE's on compact manifolds. Namely, V_t, $t \in \mathbf{R}^+$, can be extended on $\mathfrak{M}$ to a continuous group V_t, $t \in \mathbf{R}$. $\mathfrak{M}$ itself is compact and connected, and every full trajectory $\gamma(v) = \{V_t(v),\ t \in \mathbf{R}\}$, $v \in \mathfrak{M}$, is determined by its orthogonal projection on some fixed finite-dimensional subspace $\mathbf{R}^N$ of the space X. The number N is determined only by the constant data of the problem—by the coefficient ν, the domain Ω in which the flow is studied, external factors (the volume force $f = f(x)$, $x \in \Omega$), and the boundary conditions, which are assumed to be time-independent. In the dynamics, the number N itself is of considerable interest, being the first quantitative characteristic of the complexity of the regimes (flows) possible under given conditions. The main question on the agenda is that of the "degree" of hyperbolicity of the flow on $\mathfrak{M}$, or more precisely, on some "significant" part of it $\mathfrak{M}_0$. Invariant measures on $\mathfrak{M}$ can be constructed by the well-known methods of Krylov and Bogolyubov. The question as to which of these measures is to be preferred also deserves serious consideration.

Having these directions for research in mind, I conveyed them and the results of [1] to Arnol'd, Sinai, Oseledets, and also Ruelle. Ruelle started publishing his results on the large time behaviour of the N.-S. equations in 1979 ([11], [12], and so on). By that time Foiaş and Temam had carried out the work reported in [13]. For reasons that I do not fully comprehend, Ruelle did not read [1] (at least in an English translation), and familiarized himself with what had been done about the N.-S. equations through [13]. However, in [13] the set $\mathfrak{M}$ is not discussed, and neither are any other invariant sets except for the sets $\omega(v)$. Also, neither this paper nor later publications of Foiaş and Temam refer to [1], though they refer to a number of my papers, including [22], which supplements [1]. It could be that from his conversation with me Ruelle remembered only the assertion that rigorous (unconditional) results had been obtained for the two-dimensional case, which, of course, I told him. He might have thought that the principles I found are applicable to two-dimensional problems only, while most hydrodynamics experts consider turbulence to be a characteristic of three-dimensional flows, and think that only three-dimensional models should be investigated. I disagree with this judgement. One should not confuse the dimension of the phase space X with the dimension of the domain Ω filled with the fluid. For any PDE the phase space is infinite-dimensional for any dimension of the domain Ω. If, on the other hand, one studies models described by systems of ODE's, then one should consider at least three-dimensional systems.

The work of Foiaş and Temam [13] contains results concerning three-dimensional N.-S. equations, but they are of a conditional nature. Namely, it is assumed that these equations (with no-slip boundary conditions) are globally uniquely soluble in the space $X = \mathscr{D}^1 \equiv \overset{\circ}{W}{}_2^1(\Omega) \cap \overset{\circ}{J}(\Omega)$ or in a bounded subdomain B of this space that satisfies $V_t(B) \subset B$ for all $t \in \mathbf{R}^+$. Ruelle works under the same assumption, examining the dynamics V_t, $t \in \mathbf{R}^+$, on the whole space X (which is either $\mathscr{D}^1$ or $B \subset \mathscr{D}^1$, and trying to determine the "degree" of its hyperbolicity. As one can see from Corollary 1.1, it makes sense to restrict oneself to the minimal global B-attractor for X ($=\mathscr{D}^1$ or B).

Arnol'd did not adopt the ideological framework and the results of [1] either. In his 1982 lecture [49], which was sent to the Swedish Academy of Sciences on the occasion of his being awarded the Grafoord prize, he tells in detail about his approaches to the study of problems of turbulence in hydrodynamics, and refers to Leray and me as follows: "Leray, and later Ladyzhenskaya, tried to convince me that two-dimensional turbulence is impossible, that turbulence is either non-existence or non-uniqueness of solutions of the N.-S. equations, and that a correct description of turbulence requires a modification of these equations" (see p.4). As far as I am concerned, only the last part of this statement, that is, the need for a modification of the N.-S. equations, is correct. All the rest does not correspond to the real state of affairs, either now, or in the past. In 1965, which is the date Arnol'd gives for our discussion of turbulence problems, I did not yet have the framework presented in [1], but then, and even in 1958, when unique global solubility of initial boundary-value problems for the two-dimensional N.-S. equations was proved and the results presented at a joint session of two departments of the Academy of Sciences of USSR (the departments of mathematics and physics), I objected to the interpretation of these results as a rigorous argument for non-existence of two-dimensional turbulent flows. Such a point of view was advanced immediately, during the discussion of my lecture, and was persistently repeated subsequently by a number of experts in hydrodynamics. These results show only that two-dimensional N.-S. equations (with boundary conditions) provide a deterministic description of the dynamics (evolution) for any Reynolds number Re. As the parameter Re is increased, the quantitative characteristics of the flow (as expressed by the quantities $|v(t, x)|$, $|v_x(t, x)|$, and so on) grow, which by itself reflects the growing complexity of the flow. My attitude to the three-dimensional N.-S. equations is quite another matter. I now think that they do not provide a deterministic description of the dynamics, and therefore, if one wants to construct a theory of turbulence not in the spirit of the theory of probability, one must start by modernizing the N.-S. equations for large values of $|v_x(t, x)|$. I have frequently put forward such suggestions, and this is not the place for a discussion on the subject. For one of the modernizations I proposed there exists a compact global B-attractor with the same properties as $\mathfrak{M}$ for the two-dimensional equations ([3]).

But let us go back to the work of Foiaş and Teman [13]. In contrast to [1], this paper deals with the non-homogeneous boundary condition $v(t, x)|_{\partial\Omega} = \alpha(x)$. This can, with the help of the technique I used in the study of the stationary problem ([2]), be reduced to the case of the homogeneous boundary condition $v|_{\partial\Omega} = 0$. The resulting system of equations differs from the N.-S. system of equations only by the lower order terms, which influence neither the arguments nor the final conclusions. (I shall review this technique again in §2.) The homogeneous case is covered by Theorem 1.1 and Corollary 1.1. However, in [13] the authors only prove the existence of the sets $\omega(v)$ for single semitrajectories $\gamma^+(v)$, while the attraction of v to $\omega(v)$ is guaranteed only in a norm that is weaker than the norm of the space to which v belongs. It should be noted that the paper [13] does not pose precise questions. Nowhere does it say which space has been chosen as the basic one, that is, the phase space for the problem. From the arguments and results it appears that all the semi-trajectories are taken to be in the space $X = D^1 \equiv \overset{0}{W}{}_2^1(\Omega) \cap \overset{0}{J}(\Omega)$, both when $n = 2$ and $n = 3$, that is, for two-dimensional and three-dimensional problems, while the attraction of these sets to $\omega(v)$ is established only in the norm of $X = D^0 \equiv J(\Omega)$. The same remark applies to the authors' estimates of the Hausdorff dimension $d_H(A)$ of invariant sets. These are assumed to be compact in D^1 (and even bounded in D^2), while $d_H(A)$ is estimated for A regarded as a subset of the space D^0. However, in the case $n = 3$ the space D^0 is inadmissible in the sense that in this space there is no determinacy. It is not clear why the authors did not use the paper [1] in order to get, with its help, all the results obtainable for the problem they consider.

Let us move on now to the work of Babin and Vishik, which deals with attractors for PDE's. Starting in 1982, they published a number of papers on the subject ([17]-[21], and so on). In the beginning they considered the two-dimensional N.-S. equation and parabolic equations, and later studied semilinear wave equations. The analysis of the first topic is done in full accordance with the treatment of the N.-S. equations in [1]. The paper [1] is mentioned in the following words: "Thus, for example, in the work [1] Ladyzhenskaya proved the existence of an invariant set for the two-dimensional N.-S. system and showed that any trajectory in this set can be reconstructed from its finite-dimensional projection." No reference to the origin of the questions that they are studying is given, neither is there any indication of what is really done in [1]. In their other publications the authors omit altogether any comparisons of their results with the results in [1]. Nevertheless, a whole series of propositions, which the authors formulate as their own, are proved in [1] or are direct corollaries of statements in [1]. Thus, for example, Theorem 3 of [17] on the existence of a compact invariant set $\mathfrak{M}$ for the two-dimensional equations is one of the first results of [1]. Theorem 1 of §1 of [17] is a corollary of the results concerning semigroups that correspond to N.-S. equations, which appear in the early pages of [1], and so on.

Even if the reader can follow their analysis of the N.-S. and parabolic equations, he will find their studies of semilinear wave equations with weak dissipation much harder to understand. The desired goal, which is the construction of the attractor $\mathfrak{M}$, is not stated clearly. Instead, the authors introduce different definitions and concepts that vary from one paper to another.

They introduce into the definition of an attractor some additional properties that are not inherent in that concept. The authors could not construct a minimal global B-attractor $\mathfrak{M}$ as we did in [1] for the N.-S. equations, since this required showing that the sets $\gamma_t^+(B)$ are precompact. An answer to the question of which points of the phase space comprise $\mathfrak{M}$ could have been obtained from an alternative description of $\mathfrak{M}$ which appears in [1], that is, the set $\mathfrak{M}$ consists of full trajectories $\gamma(v)$, each of which belongs to some bounded set. Moreover, in the hyperbolic case there is no problem with continuing the semitrajectories $\gamma^+(v)$ in the direction of negative t. Thanks to the existence of a good Lyapunov function, that is, one that is continuous in X and decreases strictly along all trajectories apart from stationary points, in their chosen problem both "ends" of such trajectories (as $t \to +\infty$ and as $t \to -\infty$) have to approach somehow the set Z of all stationary points. In view of this, a natural candidate for $\mathfrak{M}$ is the set $\mathcal{A}$ consisting of the set Z and the points of all full trajectories connecting elements of Z. To justify this, one has to show that all sets bounded in the phase space $X_0 = \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega)$ are attracted to $\mathcal{A}$ in the norm of X_0. This is not done in the work of Babin and Vishik. There the following is proved: $\mathcal{A}$ attracts the semitrajectories $\gamma^+(\vec{v})$, $\vec{v} \in X_0$ (here $\vec{v}$ stands for the pair $(v, \partial_t v)$) in the weak topology only. Strong attraction to $\mathcal{A}$ (that is, in the norm of X_0) is proved only for $\gamma^+(v)$ with $\vec{v}$ belonging to the space $X_1 \equiv \overset{0}{W}{}_2^2(\Omega) \times \overset{0}{W}{}_2^1(\Omega)$, which is compactly embedded in X_0.

The authors construct the set $\mathcal{A}$ under the assumption that the set Z is finite and that there is a "good" extension of invariant unstable manifolds $\mathcal{M}^-(z_i)$, $\vec{z}_i \in Z$, $i = 1, \ldots, k$, from small neighbourhoods of the points on which they are defined. (It is not clear how to use their procedure without these assumptions; they claim that it can be done.) $\mathcal{A}$ is the union of these extended manifolds. The authors call it "the maximal regular attractor of the problem". Haraux [39] showed that this set coincides with the set $\mathfrak{M}$ for the given problem.

In [38], under the same analytic conditions on the non-linear term as the ones used by Babin and Vishik, but without their additional assumptions, such as finiteness of Z and so on, Hale proved the existence of the set $\mathfrak{M}$ as well as its compactness and connectedness. Similar results appear in [44]. I shall quote these results and their extension in §3. There I shall also introduce and study semigroups of class 2 (or AC semigroups). This class includes semigroups of solution operators for semilinear wave equations, which are investigated in the works just mentioned, and many others.

Let us consider further some questions of terminology. These questions are important in the formation of a new direction in the study of PDE's. Babin and Vishik introduced the concept of a "maximal attractor for the problem", sometimes adding to it the symbol (X', X), which indicates that attraction takes place in the weak topology only. (I shall only consider attraction in the metric of X itself.) This concept seems unfortunate to me for a number of reasons. The statement "$\mathcal{A}$ is an attractor" means traditionally that it attracts. What and how it attracts has then to be specified, while the attribute "maximal" is universally understood to mean "the biggest". However, such an object is easily found, being X itself.

It is desirable, on the contrary, to look for the smallest attractor of a problem, that is, its "minimal attractor". If we are talking about the search for a set that attracts every point of X, the term "global" is used, and the question should be posed as one in which "the global attractor of the problem (or the semigroup)" is to be found. If, on the other hand, we want to find a set that attracts uniformly every bounded subset of X, then it makes sense to call this set a "global B-attractor of the problem", and to try to find the "minimal global B-attractor". What this set will turn out to be depends on the problem. It is not natural to prescribe any of its properties in advance, and, as Babin and Vishik do in their papers, to imply these in the term "maximal attractor of the problem".

The epithet "maximal" is appropriate in the search for invariant sets of a problem (or a semigroup). Here the following questions are of interest: to find the maximal compact invariant set of a problem, or to find the maximal bounded set of a problem. As is shown in [1], for the N.-S. equations the set $\mathfrak{M}$ of all limiting regimes turns out to be the minimal global B-attractor, the maximal compact invariant set, as well as the maximal bounded set for the problem. Here it is assumed that the equations are supplemented by some kind of boundary conditions. It would appear that the fact that the answers to such different questions are actually identical provoked the shift of the attribute "maximal" from invariant sets to attractors.

But let us return to more interesting subjects. New properties of the attractor $\mathfrak{M}$ for the N.-S. equations and for other PDE's mentioned above were discovered in the process of determining the finiteness of various dimensions of invariant sets. I shall use the symbols $d_\tau(\mathfrak{A})$, $d_H(\mathfrak{A})$, and $d_f(\mathfrak{A})$, respectively, for the topological, Hausdorff, and fractal (information) dimensions of $\mathfrak{A}$. It is known that $d_\tau(\mathfrak{A}) \leqslant d_H(\mathfrak{A}) \leqslant d_f(\mathfrak{A})$, and that finiteness of $d_\tau(\mathfrak{A})$ does not guarantee finiteness of $d_H(\mathfrak{A})$, nor does finiteness of $d_H(\mathfrak{A})$ guarantee finiteness of $d_f(\mathfrak{A})$. As far as I know, the first such study of invariant sets in locally non-compact spaces is the paper [9] of Mallet-Paret. In this paper, finiteness of $d_\tau(\mathfrak{A})$ is established for a compact set $\mathfrak{A}$ that is invariant, or at least "negatively invariant", with respect to a smooth map whose differentials have a certain contraction property. This work led to the appearance of papers in which estimates of $d_H(\cdot)$ for the N.-S. equations were found ([13], [17], [22], [24]). Better upper bounds (with respect to the parameter ν) of $d_H(\mathfrak{M})$ for the N.-S. equations were established in [18] using the results of the paper [25] of Douady and Oesterlé. Estimates for $d_f(\cdot)$ were tackled next ([16]). Unfortunately, it is impossible to enlarge here on the results obtained in this field; this is the topic of my paper [50]. In §2 I quote only a simple and easily applicable theorem on the estimates of $d_H(\mathfrak{A})$ and $d_f(\mathfrak{A})$ for bounded invariant sets. This result comes from [22], although in [22] only an estimate for $d_H(\mathfrak{A})$ is discussed. However, it follows from the argument there that if the contraction coefficient $\delta < 1/2$, then the same upper bound works also for $d_f(\mathfrak{A})$). It is shown in §§2 and 3 how to verify the condition of this theorem in actual problems.

To conclude the Introduction, let me return to the beginning. I did not arrive immediately at the idea of studying limiting regimes of the N.-S. equations, that is, of looking for true global attractors under some boundary conditions. Initially I tried to prove the conjecture of Landau that all their solutions are quasi-periodic in t, that the increase of their complexity (transition to turbulence) is nothing but the appearance of new quasi-periods that are incommensurate with the existing ones, and that the number of quasi-periods grows as the Reynolds number increases. But at some stage I began to doubt the truth of this conjecture and embarked on the search for ways of describing the sets of all points in the phase space in which the system can find itself after very long times, without deciding in advance on the character (dependence) of the dynamics V_t for large t.

The results obtained by this approach confirm the following part of Landau's conjecture: the number of quasi-periods of all possible quasi-periodic regimes (solutions) does not exceed the number $d_H(\mathfrak{M})$; that is, it really is controlled by the stationary factors of the problem. I drew this conclusion in [22]. But I think that it is not these solutions, or more precisely the trajectories $\gamma(v)$ that correspond to them, that determine $d_H(\mathfrak{M})$, that is, $d_H(\mathfrak{A})$ is probably larger than the maximum number of quasi-periods of all quasi-periodic solutions. Some justification for this opinion is provided by the Lorenz attractor.

§2. Problems with dissipation of parabolic type (semigroups of class 1)

Most processes in nature involve some loss (that is, dissipation) of energy. This dissipation can take many different forms. In the framework of equations that describe the process, it is expressed by the inclusion of terms called dissipative. In some cases these terms are to be found among the principal (from the point of view of the theory of equations) terms and change the type of equation, in other cases they are to be found among lower order terms. The influence of these terms grows stronger as time progresses. Experts in mechanics and physics have for a long time held the opinion that after a long time autonomous dissipative systems "forget" their initial states and enter regimes "determined by stationary factors". Such effects are definitely observed in fluids, both in nature and particularly well in special experiments. I shall only discuss the case of incompressible fluids, that is, when div $v = 0$ for the velocity field v. The dynamics of such fluids are usually described by the Navier-Stokes (N.-S.) equations and it is believed that these equations do it in a deterministic way, that is, the solution $v(t) = V_t(v_0)$ is uniquely determined for all $t \in \mathbf{R}^+$ by the initial state v_0 and the boundary condition. In the case $n = 2$ (that is, for two-dimensional problems), this is really so. Moreover, for X we can choose one of many different spaces. However, in the case $n = 3$ with large Reynolds numbers no "interesting" spaces X have been found. Therefore, all propositions below that deal with this case are of a conditional nature.

The remark above about "forgetting" the initial state was the only idea from the theory of turbulent flows that I found useful, and which led me to start investigating ω-limit sets for N.-S. equations. In fact, if B is that part of the phase space X from which the experimenter chooses the initial states of the system, then starting from some time τ (and later) he can only observe states that correspond to points of the set $\gamma_\tau^+(B)$ (see §1), and after "infinitely long time" only $\omega(B)$ is left of this set (more precisely, of its closure). In real experiments one is given the domain Ω that is filled with fluid, the volume forces $f = f(x)$, $x \in \Omega$, and the boundary condition

$$v(t, x) = \alpha(x), \quad x \in \partial\Omega, \quad t \in \mathbf{R}^+, \tag{2.1}$$

while the initial conditions

$$v(t, x)|_{t=0} = v_0(x), \quad x \in \Omega, \tag{2.2}$$

are to be varied. In experiments in which the fluid is set in motion by, for example, rotating the walls of the vessel (but in such a way that the shape of Ω is invariant in $\mathbf{R}^3$), α in (2.1) depends initially on t, but stabilizes (that is, becomes independent of t) after some time t_0. Different rates of convergence of $\alpha(t, x)$ to $\alpha(t_0, x)$ give rise to different values of $v(t_0, x)$. Taking t_0 as the starting time, the problem is reduced to the formulation above. We also note that in this experiment the normal component $(\alpha, n)|_{\partial\Omega}$ of the field v on $\partial\Omega$ is 0.

To investigate the questions we are interested in, it is useful to reformulate the problem (2.1), (2.2) in terms of the function $u(t, x) = v(t, x) - \alpha(x)$, $(t, x) \in \mathbf{R}^+ \times \Omega$, which satisfies, as does v, the incompressibility condition $\operatorname{div} u = 0$ (so that $\operatorname{div} \alpha = 0$). For u the boundary condition becomes homogeneous: $u|_{\partial\Omega} = 0$. The equations satisfied by $u(t, x)$ when $(t, x) \in \mathbf{R}^+ \times \Omega$ differ slightly from the N.-S. equations—their left-hand side now includes linear terms of the form $\alpha_k u_{x_k} + u_k \alpha_{x_k} \equiv j(u)$, while the free term changes as follows: $f(x) \to f(x) + \nu\,\Delta\alpha(x) - \alpha_k(x)\alpha_{x_k}(x) \equiv \tilde{f}(x)$. In general, linear terms can have a profound influence on the behaviour of the solution $u(t) \equiv U_t(u_0)$ as $t \to \infty$, but in this case one can extend $\alpha|_{\partial\Omega}$ from $\partial\Omega$ to Ω in such a way that these terms are completely dominated by the term $-\nu\Delta u$. This can be done if $\alpha|_{\partial\Omega}$ is such that

$$j_k \equiv \int_{S_k} (\alpha, n)\, ds = 0 \qquad (k = 1, \ldots, m) \tag{2.3}$$

for each connected component S_k of the boundary $\partial\Omega$. We also remark that $\sum_{k=1}^{m} j_k = 0$ due to the equation $\operatorname{div} \alpha(x) = 0$. This extension has the following properties: $\operatorname{div} \alpha(x) = 0$, $x \in \Omega$, and for every $w \in D^1$ (the space D^1 will be defined below) we have the inequality

$$\left| \int_{\Omega} w_k(x)\, \alpha_{x_k}(x)\, w(x)\, dx \right| \leqslant \frac{\nu}{2} \| w_x \|^2, \tag{2.4}$$

where $\|\cdot\|$ is always the $L_2(\Omega)$ or the $\vec{L}_2(\Omega)$ norm. In fact, the extension of $\alpha|_{\partial\Omega}$ can be constructed in such a way that on the right-hand side of (2.4) we have an arbitrary $\xi > 0$ instead of $\nu/2$, but for our purposes $\xi = \nu/2$ suffices. The smoothness of $\alpha(x)$, $x \in \Omega$, is determined by the smoothness of $\partial\Omega$ and of $\alpha|_{\partial\Omega}$ (an explicit construction of such extensions $\alpha(x)$, $x \in \Omega$, is given in the second Russian edition of [2]—see Chapter V, §4). For simplicity of exposition, I shall assume that both $\partial\Omega$ and $\alpha|_{\partial\Omega}$ are "sufficiently" smooth, without indicating precisely what degree of smoothness is required in each and every case. In fact, for the main propositions, not much smoothness is required.

Inequality (2.4) ensures that the following inequality holds:

$$(2.5)\quad \int_\Omega (-\nu\,\Delta u + \alpha_k u_{x_k} + u_k \alpha_{x_k})\, u\,dx = \int_\Omega (\nu u_x^2 + u_k \alpha_{x_k} u)\,dx \geqslant \frac{\nu}{2}\|u_x\|^2.$$

Once an estimate of $\|u(t)\|^2$ is found for all t, (2.5) allows us to make the same qualitative conclusion as in the case of $\alpha(x) \equiv 0$, that is, to conclude that there exists a bounded absorbing set B_0. This is quite sufficient for all subsequent estimates of stronger norms of the solutions $u(t)$, $t > 0$. From these estimates will follow complete continuity of the solution operators U_t for $t > 0$. The methods of [1] can be used to show that all the properties of the semigroups V_t, $t \in \mathbf{R}^+$ (corresponding to the case $\alpha \equiv 0$) which are described in §1 and in more detail in this section, are shared by the semigroups U_t, $t \in \mathbf{R}^+$. The only exception is provided by the injectivity of the operators U_t for $t > 0$ if $(\alpha, n)|_{\partial\Omega} \not\equiv 0$. In [1] (§2) this property is proved under the assumption that $(\alpha, n)|_{\partial\Omega} \equiv 0$; moreover, a qualified estimate of the modulus of continuity of the operators U_t^{-1} is given. The case $(\alpha, n)|_{\partial\Omega} \not\equiv 0$ under conditions (2.3) is covered by Lemma 3.1 in [13], in which it is proved that the semigroup U_t is analytic for $t > 0$. From this result it follows that U_t is injective on all our ω-limit sets.

In view of the above, I shall restrict myself to the case $\alpha \equiv 0$. For the case $\alpha \not\equiv 0$ such that condition (2.3) is satisfied, and for the case of periodic boundary conditions, the results are the same.

Thus, the first object of our inquiry is the following problem for the N.-S. equations:

$$(2.6)\quad \partial_t v - \nu\,\Delta v + \sum_{k=1}^{n} v_k v_{x_k} = \nabla p + f, \quad \operatorname{div} v = 0,$$

$$(2.7)\quad v|_{\partial\Omega} = 0,$$

$$(2.8)\quad v|_{t=0} = v_0$$

where $f = f(x)$; $v_k = v_k(t, x)$ $(k = 1, \dots, n)$ are the components of the vector $v(t, x) \in \mathbf{R}^n$; x runs through a bounded domain $\Omega \subset \mathbf{R}^n$, $n = 2$ or $n = 3$. Let me briefly review that part of the results concerning the problem (2.6)–(2.8) that I obtained in the fifties, and which are necessary for our discussion (see [2] and so on).

The most important spaces for this problem are $\overset{0}{J}(\Omega)$ and $H(\Omega)$ (in the notation of [2]). For brevity, let us denote them by $\overset{0}{J}$ and D^1. (The letter H is used in many works to denote the space $L_2(\Omega)$; therefore I replace

$H(\Omega)$ by D^1.) $\overset{0}{I} \equiv \overset{0}{I}(\Omega)$ is a subspace of the vector space $\vec{L}_2(\Omega)$. The set $\dot{I}^\infty(\Omega)$ of all infinitely differentiable divergence-free (solenoidal) vector fields $u(x)$, $x \in \Omega$, with support in Ω is dense in $\overset{0}{I}$. The scalar product in $\vec{L}_2(\Omega)$ and in $\overset{0}{I}$ is defined by

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx, \tag{2.9}$$

and the norm is denoted by $\|\cdot\|$. The orthogonal complement of $\overset{0}{I}$ in $\vec{L}_2(\Omega)$ consists of gradient vector fields. Let us denote the orthogonal projection of $\vec{L}_2(\Omega)$ onto $\overset{0}{I}$ by P.

The space D^1 is the closure of the set $\dot{I}^\infty(\Omega)$ in the norm of the Dirichlet integral, that is,

$$\|u\|_1 = \left(\int_\Omega u_x^2(x)\, dx\right)^{1/2}. \tag{2.10}$$

The scalar product in it is given by

$$(u, v)_1 = \int_\Omega u_x(x)\, v_x(x)\, dx \tag{2.11}$$

(in (2.9)–(2.11) summation over vector indices is assumed). The set D^1 is dense in the space $\overset{0}{I}$.

A conceptually new stage in the study of the problem (2.6)–(2.8) was inaugurated in the fifties by the complete exclusion from the equations of the pressure p, the formulation of a closed form problem for the field $v(t, x)$, $(t, x) \in \mathbf{R}^+ \times \Omega$, and the development of tools for the treatment of such problems. (Once the field v is found, it serves to determine p; this part of the problem is well researched, and I shall not touch on it.) This is the spirit of one of the first works of this period—the 1951 paper [26] of Hopf. Its significance was realized only later, after the results on unique global solubility of the Cauchy problem for linear operator equations ([27], [28]) and on unique solubility of the non-linear problem (2.6)–(2.8) (which are global for $n = 2$ and local, in various senses, for $n = 3$) ([29], [30], [2]) became available. Exclusion of the pressure p is obtained by applying the projection operator P to both sides of (2.6). In view of the above, the result is the following equation for v:

$$\frac{dv}{dt} + \nu A v + B(v) = f, \tag{2.12}$$

where $Av = -P\Delta v$, $B(v) = P(v_k v_{x_k})$, while f, without loss of generality, is assumed to belong to $\overset{0}{I}$, so that $Pf = f$. We supplement (2.12) with the initial condition (2.8), while the boundary condition (2.7) is "hidden" in the description of the "maximal" domain of definition $D(A)$ of the operator A

considered as an unbounded operator on $\overset{0}{I}$. In this case this domain is $D(A) = W_2^2(\Omega) \cap D^1$, and on this domain A is self-adjoint. Moreover, A defines a one-to-one map of $D(A)$ onto $\overset{0}{I}$, A is positive definite, and A^{-1} is compact. The non-linear operator B is defined on $D(A)$ and is bounded. In a certain sense B is "weaker" than A (that is, it is dominated by A).

The spectrum of A is discrete and positive. Let us denote the eigenvalues of A by λ_k and let us order them in increasing order $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$. We denote the corresponding eigenfunctions of A by φ_k $(k = 1, \dots)$ and assume that they are orthonormal in $\overset{0}{I}$. The system $\{\varphi_k\}_{k=1}^{\infty}$ forms an orthogonal basis in the spaces $\overset{0}{I}$ and D^1. The same is true for all the spaces D^s, $s \in \mathbf{R}$, which are defined as follows: the elements of D^s are vector fields $u(x)$ of the form $u(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x)$ for which $\sum_{k=1}^{\infty} \lambda_k^s a_k^2 < \infty$. The scalar product in D^s is defined by $(u, v)_s = \sum_{k=1}^{\infty} \lambda_k^s a_k b_k$, where $u(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x)$ and $v(x) = \sum_{k=1}^{\infty} b_k\varphi_k(x)$. We shall denote the norm in D^s by $\|\cdot\|_s$; $D^0 = \overset{0}{I}$. In particular, the elements of D^2 are vector fields in $D(A) = W_2^2(\Omega) \cap D^1$, and the norm $\|\cdot\|_2$ is equivalent to the usual Sobolev norm of the space $W_2^2(\Omega)$; this is a result of Solonnikov in 1960. The numbers $\lambda_k \to \infty$ as $k \to \infty$; moreover, $\lambda_k = O(k)$ for $n = 2$ and $\lambda_k = O(k^{2/3})$ for $n = 3$. The precise asymptotics of λ_k are given in [31]. Let us denote by P_N the orthogonal projection of $\overset{0}{I}$ onto the subspace $\mathbf{R}^N$ spanned by the eigenfunctions $\varphi_1, \dots, \varphi_N$.

Let us first consider the case $n = 2$. In this case it has been proved that the solution operators V_t of the problem (2.12), (2.8) form a continuous semigroup $X \to X$, $t \in \mathbf{R}^+$, in D^s for any $s \geqslant 0$. Let us take the largest of these spaces, $\overset{0}{I}$. The solutions $v(t) = V_t(v_0)$, $v_0 \in \overset{0}{I}$, satisfy for almost all $t \in \mathbf{R}^+$ the energy equality

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|v_x(t)\|^2 = (f, v(t)). \tag{2.13}$$

From this relation we can easily conclude that the ball

$$B_0 = \{v \in \overset{0}{I}: \ \|v\| \leqslant R_0\}, \quad R_0 > (\nu\lambda_1)^{-1}\|f\|,$$

is an absorbing set for all $B \subset \mathscr{B}$, and that $V_t(B_0) \subset B_0$. (All the solutions $v(t)$ of the three-dimensional equations (2.12), even the "weak" solutions of Hopf, enter the ball B_0 in finite time.) Moreover, for all $\varepsilon > 0$, any solution v belongs to $C([\varepsilon, \infty), D^1)$ and

$$\sup_{t \in [\varepsilon, \infty)} \|v(t)\|_1^2 \leqslant \varepsilon^{-1}\mathscr{M}(\|v(0)\|) \quad \text{for all } \varepsilon \in (0, 1], \tag{2.14}$$

where $\mathcal{M}(\cdot)$ is a function continuous on $\mathbf{R}^+$. I shall not give its precise form here, neither shall I indicate the dependencies on ν, f, and $\partial\Omega$, since these are considered to be fixed in this problem. Similar estimates uniform in $t \geqslant \varepsilon > 0$ hold also for stronger norms $\|v(t)\|_s$, $s > 1$, if f and $\partial\Omega$ are sufficiently smooth.

From (2.14) and from the continuity of V_t it follows that for $t > 0$ the operators V_t are completely continuous. In view of this and the existence of a bounded global B-attractor B_0 we have the following theorem.

***Theorem* 2.1.** *In the phase space $X = \overset{0}{I}$ there exists a minimal global B-attractor $\mathfrak{M}$ for the two-dimensional N.-S. equations with the boundary condition* (2.7) *on the boundary of a bounded domain $\Omega \subset \mathbf{R}^2$. $\mathfrak{M}$ is a non-empty connected set in $\overset{0}{I}$ which is bounded in D^s, $s \geqslant 1$.*

The quantity s in Theorem 2.1 is limited only by the smoothness of f and $\partial\Omega$. Theorem 2.1 is the first result in [1] concerning the problem (2.6)–(2.8) (connectedness of $\mathfrak{M}$ is not mentioned in [1]). The existence of $\mathfrak{M}$ hinges on the following two properties of the semigroup V_t, $t \in \mathbf{R}^+$: the complete continuity of V_t for $t > 0$, and the existence of a bounded absorbing set B_0. In § 1 I highlighted this fact, which follows from the reasoning of [1], in the form of Theorem 1.1. Below I shall prove Theorem 2.5, which deals with existence of $\mathfrak{M}$ for class 1 semigroups under weaker assumptions on the set B_0 that attracts points of X.

The absorbing set B_0 for the problem (2.6)–(2.8) has the property $V_t(B_0) \subset B_0$, $t \in \mathbf{R}^+$, and therefore for this problem

$$\mathfrak{M} = \omega(B_0) = \bigcap_{t \geqslant 0} [V_t(B_0)]_X.$$

Moreover, it is proved in [1] that $V_t([B]_X) = [V_t([B]_X)]_X$ and therefore $\mathfrak{M} = \omega(B_0) = \bigcap_{t\geqslant 0} V_t(B_0)$. Because $\mathfrak{M}$ is invariant, every semitrajectory $\gamma^+(v)$, $v \in \mathfrak{M}$, lies in $\mathfrak{M}$ and can be continued in the following way: from v we can find $v_1 \in \mathfrak{M}$ such that $v = V_1(v_1)$, from v_1 we can find $v_2 \in \mathfrak{M}$ such that $v_1 = V_1(v_2)$, and so on. We shall call the union of the points $\gamma^-(v) \equiv \{V_t(v_k),\ t \in [0, 1),\ k = 1, 2, \ldots\}$ the negative semitrajectory, which continues $\gamma^+(v)$ in the direction of negative t (down to $-\infty$), and $\gamma(v) = \gamma^+(v) \cup \gamma^-(v)$ the full trajectory. Let us use the notation $V_t(v_k) \equiv V_{t-k}(v_0)$, $t \in [0, 1)$ $(k = 1, 2, \ldots)$. Then $\{V_t(v_0),\ t \in \mathbf{R}\} = \gamma(v)$ is a continuous curve in $\mathfrak{M}$, and $V_{\tau+t}(v_0) = V_\tau(V_t(v_0))$ for all $t \in \mathbf{R}$ and all $\tau \in \mathbf{R}^+$.

In general such a continuation may be non-unique. However for the problem (2.6)–(2.7) we proved in [1] that $V_t(v_0) \neq V_t(\tilde{v}_0)$ for all $t \geqslant 0$ if $v_0 \neq \tilde{v}_0$ and $v_0, \tilde{v}_0 \in D^{3+\varepsilon}$, $\varepsilon > 0$. (The same line of argument can be used to lower $3 + \varepsilon$ to $2 + \varepsilon$, $\varepsilon > 0$.) In particular, this ensures that V_t is invertible on $\mathfrak{M}$, and from the estimates given in [1] a certain "qualified" continuity of V_t^{-1} on $\mathfrak{M}$ can be inferred. The family of operators $\{V_t,\ t \in \mathbf{R}\}$ with

$V_t = V_{-t}^{-1}$ for $t < 0$ forms a continuous group on $\mathfrak{M}$. The construction of backward continuation of $\gamma^+(v)$, $v \in \mathfrak{M}$, to $\gamma(v)$ described above gives in this case the only possible continuation, since the v_k ($k = 1, 2, \ldots$) are uniquely determined by v; moreover, $\gamma(v) = \{V_t(v_0),\ t \in \mathbf{R}\}$. This is one of the most interesting facts proved in [1]. Let us formulate it as a theorem for emphasis.

***Theorem* 2.2.** *The semigroup* $V_t\colon \overset{0}{I} \to \overset{0}{I}$, $t \in \mathbf{R}^+$ *for the problem* (2.6), (2.7) *in* $\Omega \subset \mathbf{R}^2$ *can be continued on* $\mathfrak{M}$ *to a continuous group* $V_t\colon \mathfrak{M} \to \mathfrak{M}$, $t \in \mathbf{R}$; $v(t) = V_t(v_0)$, $t \in \mathbf{R}$, $v_0 \in \mathfrak{M}$, *is the solution of equation* (2.12).

The solutions $v(t)$, $t \in R$, in $\mathfrak{M}$ are very smooth functions of $(t, x) \in \mathbf{R} \times \overline{\Omega}$ if f and $\partial\Omega$ have the requisite smoothness.

The set $\mathfrak{M}$ is characterized in [1] also in the following way.

***Theorem* 2.3.** *Any solution* $v(t)$ *of the problem* (2.12), (2.8) *that exists for all* $t \in \mathbf{R}$ *and is contained in some* $B \subset \overset{0}{I}$ *is also contained in* $\mathfrak{M}$ *(that is,* $v(t) \in \mathfrak{M}$, $t \in \mathbf{R}$). *For all* $v_0 \in \mathfrak{M}$ *there exists a unique solution* $v(t)$, $t \in \mathbf{R}$, *of the problem* (2.12), (2.8) *that lies in* $\mathfrak{M}$ *and is equal to* v_0 *at* $t = 0$.

The second statement is a consequence of Theorem 2.2. The first statement is true because the set $\gamma(v) \equiv \{v(t),\ t \in \mathbf{R}\}$ is bounded in $\overset{0}{I}$ and invariant, and since all $B \subset \mathscr{B}$ are attracted to $\mathfrak{M}$, we have $\gamma(v) \subset \mathfrak{M}$.

I shall now describe, following [1], results that can be obtained for the three-dimensional problem (2.6)–(2.8) in a bounded domain $\Omega \subset \mathbf{R}^3$, with $\partial\Omega$ and f smooth.

$$(2.15)\quad \left\{\begin{array}{l}\text{Suppose that in } D^1 \text{ there is a bounded set } B \text{ such that for all}\\ v_0 \in B \text{ the problem (2.12), (2.8) has a solution } v(t) \equiv V_t(v_0),\\ t \in \mathbf{R}^+, \text{ in } C[\mathbf{R}^+, D^1), \text{ that is, } v \in C[\mathbf{R}^+, D^1), \text{ and } B_1 \equiv \gamma^+(B) \equiv\\ \equiv \{V_t(v_0),\ t \in \mathbf{R}^+,\ v_0 \in B\} \text{ is a bounded set in } D^1.\end{array}\right.$$

Then we have the following result.

***Theorem* 2.4.** *Suppose that* (2.15) *holds for the three-dimensional problem* (2.12), (2.8). *Then the solution operators* V_t, $t \in \mathbf{R}^+$, *form in* $X \equiv [B_1]_{D^1}$ *a continuous semigroup of class* 1. *The set* $\omega(X)$ *is the minimal global B-attractor for this semigroup. This set is an invariant compact set in* D^1. *On* $\omega(X)$ *the semigroup can be continued to a continuous group. Any solution* $v(t)$ *that exists for all* $t \in \mathbf{R}$ *and is contained in* X *belongs to* $\omega(X)$. *If* f *and* $\partial\Omega$ *are sufficiently smooth, then* $\omega(X)$ *is a bounded set in* D^2 *and all* D^s, $s > 2$. *If* B *is connected, so is* $\omega(X)$.

In this case we regard $[B_1]_{D^1} \equiv X$ as a complete metric space whose metric is induced by the metric (norm) of D^1. To prove the first statement about the set $\omega(X)$ it is sufficient to show that the V_t, $t \in \mathbf{R}^+$, form on X a continuous semigroup of class 1. All analytical tools necessary for this are contained in [2]. Since $V_t(X) \subset X$, we have $\omega(X) = \bigcap_{t \geq 0} [V_t(X)]_X$ In this case also we can prove that $V_t(X) = [V_t(X)]_X$. The fact that the operators V_t are injective, from which it follows that the semigroup V_t, $t \in \mathbf{R}^+$, can be continued to a continuous group on $\omega(X)$, is proved by the same argument as in [1].

In fact, particular features of the N.-S. equations played no role in the proof of Theorems 2.1, 2.3, and 2.4; only certain properties of the semigroups that they generate were of importance. To make the reader appreciate this, let us isolate each of the features shared by semigroups of class 1 in the form of a theorem.

We shall use the following concepts: a semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$, is called *bounded* if for all $B \subset \mathscr{B}$ the set $\gamma^+(B) \subset \mathscr{B}$. A semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$, is called *point-dissipative* if there is a set $B_0 \subset \mathscr{B}$ that attracts all points of X. Moreover, we shall use the following well-known fact.

Lemma 2.1. *$\omega(B)$ consists of precisely those points that are limits of sequences of the form $\{V_{t_k}(v_k)\}$, $t_k \uparrow \infty$, $v_k \in B$.*

The proof simply uses the definition (1.2) of the set $\omega(B)$.

The following generalization of Theorem 1.1 is true.

Theorem 2.5. *Let $V_t : X \to X$, $t \in \mathbf{R}^+$, be a semigroup of class* 1 (*that is, the V_t are completely continuous for all $t > 0$*) *in a complete metric space X. Then it has the following properties*:

1. *If $\gamma^+(B) \subset \mathscr{B}$, then $\omega(B)$ is a non-empty compact set that attracts B.*

2. *Let the semigroup be bounded, and let there exist a set $B_0 \subset \mathscr{B}$ that attracts all $B \subset \mathscr{B}$* (*that is, there exists a bounded global B-attractor*). *Then the minimal global B-attractor $\mathfrak{M}$ of the semigroup is a non-empty compact invariant set. Any bounded invariant set $\mathscr{A}$, including the full trajectories*

$$\gamma(v_0) \equiv \{v(t),\ t \in R,\ v(0) = v_0;\quad V_\tau(v(t)) = v(\tau + t) \quad \text{for all } \tau \in \mathbf{R}^+ \\ \text{and all } t \in \mathbf{R}\},$$

that lie in $\mathscr{B}_0$, is contained in $\mathfrak{M}$.

3. *All the statements of part* 2 *are true if the semigroup is bounded and point-dissipative.*

4. *If V_t for all $t > 0$ establishes a one-to-one correspondence between the points of $\omega(B)$ in part* 1, *or the points of $\mathfrak{M}$ in parts* 2 *and* 3, *then the inverse operators V_t^{-1} are continuous on $\omega(B)$ or on $\mathfrak{M}$, respectively, and on these sets the semigroup V_t, $t \in \mathbf{R}^+$, can be continued to the group V_t, $t \in \mathbf{R}$, where $V_t \equiv V_{-t}^{-1}$ for $t < 0$. If the semigroup V_t, $t \in \mathbf{R}^+$, is continuous, then so is the group V_t, $t \in \mathbf{R}$.*

5. *If X is a connected complete metric space, and $V_t : X \to X$, $t \in \mathbf{R}^+$, is a continuous semigroup, then $\omega(B)$ in part* 1 *is connected if B is, and so is $\mathfrak{M}$ in parts* 2 *and* 3.

Remark 2.1. 1. To prove that $V_t(\omega(B)) \subset \omega(B)$ and that $V_t(\mathfrak{M}) \subset \mathfrak{M}$ we only use the continuity of V_t.

2. Under the conditions of parts 2 or 3 of the theorem, $\mathfrak{M}$ consists of all full trajectories $\gamma(v_0)$ each of which lies in some $B \subset \mathscr{B}$.

3. We shall prove Theorem 2.5 under the assumption that X is unbounded. If X is bounded, many of the conditions of the theorem are satisfied automatically. Thus, one can choose the set B_0 required in parts 2 and 3 to be X itself. The corresponding changes in the argument and in the construction of $\mathfrak{M}$ are obvious.

4. All the statements of parts 1–3 of Theorem 2.5 are also true for discrete semigroups $\{V^n\}$ $(n = 0, 1, \ldots)$, where V^n is the n-th power of a completely continuous operator V, with $V^0 = I$. In the case corresponding to part 4 above, the semigroup can be continued to the group $\{V^n\}$, $n \in \mathbf{Z}$, where $V^{-n} \equiv (V^n)^{-1}$ $(n = 1, 2, \ldots)$.

The proof of Theorem 2.5 follows the lines of the proofs of all the theorems we have formulated above. We quote this proof here, since we have not given the proofs of any of the other theorems.

1. The sets $\gamma_t^+(B) = V_t(\gamma^+(B))$ are precompact for all $t > 0$, and $\gamma_{t_2}^+(B) \subset \gamma_{t_1}^+(B)$ for $t_2 > t_1$. Therefore the set

$$\omega(B) \equiv \bigcap_{t\geqslant 0} [\gamma_t^+(B)]_X = \bigcap_{t\geqslant T} [\gamma_t^+(B)]_X$$

for all $T > 0$ is the intersection of nested compact sets. Therefore, $\omega(B)$ is a non-empty compact set, and $\sup\limits_{u\in[\gamma_t^+(B)]_X} \inf\limits_{v\in\omega(B)} \rho(u, v) \to 0$ as $t \to \infty$, that is, the set B is attracted to $\omega(B)$. This last statement is proved by a standard argument of ad absurdum reduction.

Let us show that $V_t(\omega(B)) \subset \omega(B)$ for $t > 0$. In fact, as we said in Lemma 2.1, the points $v \in \omega(B)$ are completely characterized by being limits of sequences of the form $V_{t_k}(v_k)$, $t_k \uparrow \infty$, $v_k \in B$. By continuity of V_t, the point $V_t(v)$ is the limit of the sequence $V_{t+t_k}(v_k)$, $t + t_k \uparrow \infty$, of the same form and therefore $V_t(\omega(B)) \subset \omega(B)$.

To prove the reverse inclusion $\omega(B) \subset V_t(\omega(B))$ we shall use the complete continuity of V_t, $t > 0$. Let $v \in \omega(B)$; then $v = \lim\limits_{t_k\uparrow\infty} V_{t_k}(v_k)$, $v_k \in B$, and without loss of generality we assume that $1 + t < t_1 < t_2 < \ldots$. The set $\{y_k \equiv V_{t_k - t}(v_k)\}_{k=1}^{\infty}$ belongs to the precompact set $\gamma_1^+(B)$. Let us pick out from this set a convergent subsequence $y_{k_j} \to y$, $k_j \uparrow \infty$. Its limit y belongs to $\omega(B)$, and therefore $v = \lim\limits_{k_j\uparrow\infty} V_{t_{k_j}}(v_{k_j}) = \lim\limits_{k_j\uparrow\infty} V_t(y_{k_j}) = V_t(y)$. Thus, $\omega(B) \subset V_t(\omega(B))$ for all $t > 0$, and therefore $V_t(\omega(B)) = \omega(B)$, that is, $\omega(B)$ is invariant. As we showed above (before Theorem 2.2), it follows that every semitrajectory $\gamma^+(v) \subset \omega(B)$ can be continued to a full trajectory $\gamma(v) \subset \omega(B)$.

2. Suppose that the conditions of part 2 hold. Let us take any ε-neighbourhood of B_0, $O_\varepsilon(B_0) \equiv B_1$, $\varepsilon > 0_\varepsilon$, where $O_\varepsilon(B_0)$ is the union of all open balls of radius ε centred at points of B_0 and let us check that $\mathfrak{M} \equiv \omega(B_1)$ has all the properties indicated in part 2. We have only to verify that $\omega(B_1)$ attracts all $B \subset \mathscr{B}$. The set B_1 absorbs B, that is, $V_t(B) \subset B_1$ for all $t \geqslant t_1(B)$ (here we do not indicate the dependence of t_1 on ε and B_1, since these are fixed), and B_1 is attracted to $\omega(B_1)$. Therefore, $\omega(B) \subset \omega(B_1)$, and $\omega(B_1)$ attracts B. If moreover $V_t(\mathscr{A}) = \mathscr{A}$ for all $t \in \mathbf{R}^+$, then $\mathscr{A} \subset \omega(B_1)$.

3. Suppose that the conditions of part 3 hold, that is, there is a set $B_0 \subset \mathscr{B}$ that attracts all points $v \in X$. Then $V_t(v) \in B_1 \equiv O_\varepsilon(B_0)$ for all $t \geqslant t_1(v)$. From the continuity of $V_{t_1(v)}$ and the fact that $O_\varepsilon(B_0)$ is an open set, there exists a neighbourhood $O(v)$ of the point v for which $V_{t_1(v)}(O(v)) \subset B_1$. Hence it follows that $V_{t+t_1(v)}(O(v)) \subset V_t(B_1)$ for all $t \in \mathbf{R}^+$, and since $\omega(B_1)$ attracts B_1, $V_{t+t_1(v)}(O(v)) \subset O_{\varepsilon_1}(\omega(B_1))$ for all $t \geqslant t_2(\varepsilon_1)$, where ε_1 is an arbitrary positive number. Since from every covering $\bigcup_{v\in K} O(v)$ of the compact set K one can choose a finite subcovering $O(K) \equiv \bigcup_{i=1}^{m} O(v_i)$, $v_i \in K$, $m = m(K)$, we have $V_t(O(K)) \subset O_{\varepsilon_1}(\omega(B_1))$ for all

$$t \geqslant t_2(\varepsilon_1) + \max_{i=1,\dots,m} t_1(v_i) \equiv t_3(\varepsilon_1, K).$$

In particular, $V_t(O(\omega(B))) \subset O_{\varepsilon_1}(\omega(B_1))$ for all $t \geqslant t_3(\varepsilon_1, \omega(B))$. If one takes into account that $V_t(B)$ is absorbed by $O(\omega(B))$ in finite time (note that any neighbourhood $O(K)$ of a compact set K contains some neighbourhood of the form $O_{\varepsilon_2}(K)$, $\varepsilon_2 > 0$), then it follows from the argument above that $V_t(B) \subset O_{\varepsilon_1}(\omega(B_1))$ for all $t \geqslant t_4(\varepsilon_1, B)$, that is, $\omega(B_1)$ attracts B.

4. By hypothesis V_t $(t > 0)$ is invertible on $\omega(B)$; since $\omega(B)$ is a compact set, and V_t is continuous, it is well known that V_t^{-1} is also continuous (and even uniformly continuous) on $\omega(B)$. The group property of the family $\{V_t, t \in \mathbf{R}\}$ is verified directly from the definition $V_t \equiv V_{-t}^{-1}$, $t < 0$, and the semigroup property of $\{V_t, t \in \mathbf{R}^+\}$. It only remains to verify the continuity of $V_t(v)$ in $(t, v) \in \mathbf{R}^- \times \omega(B)$. Thus, let v and $\tilde{v}$ be in $\omega(B)$ and $t < 0$. It is clear that $j \equiv \rho(V_t(v), V_{t+\tau}(\tilde{v})) \leqslant j_1 + j_2$, where $j_1 = \rho(V_t(v), V_t(\tilde{v}))$ and $j_2 = \rho(V_t(\tilde{v}), V_\tau(V_t(\tilde{v}))) = \rho(V_{-\tau}(V_{t+\tau}(\tilde{v})), V_{\tau+t}(\tilde{v}))$. By uniform continuity of V_t on $\omega(B)$, $j_1 < \varepsilon/2$ if $\rho(v, \tilde{v}) < \delta_1(\varepsilon, t, \omega(B))$. The second term j_2 also satisfies $j_2 < \varepsilon/2$ if $|\tau| \leqslant \delta_2 = \delta_2(\varepsilon, \omega(B))$, since $\rho(V_\tau(w), w) < \varepsilon/2$ for $\tau \in [0, \delta_2)$ and all $w \in \omega(B)$.

5. Let us recall that the connectedness of X means that it is impossible to represent X as the union of two non-trivial (that is, different from X and $\emptyset$) open subsets, while the connectedness of a subset $\mathscr{A} \subset X$ is the

connectedness of $\mathscr{A}$ regarded as a topological space with the induced topology of X. Let B be a connected subset of X and $\gamma^+(B) \subset \mathscr{B}$. Then all the sets $\gamma_t^+(B)$ are connected and precompact for all $t > 0$, and so $[\gamma_t^+(B)]_X$ are connected compact sets for all $t > 0$. Moreover, we know that $\omega(B) = \bigcap_{t>0} [\gamma_t^+(B)]$ and that $[\gamma_t^+(B)]_X \subset O_\varepsilon(\omega(B))$ for all $t \geqslant t(\varepsilon, B)$, where ε is any number. Hence it follows that $\omega(B)$ is connected. The connectedness of $\mathfrak{M}$, whose existence is guaranteed by other parts of Theorem 2.5, follows from the fact that $\mathfrak{M} = \omega(B)$ for all $B \supset B_1$, and in particular for a connected $B \supset B_1$.

The statement of part 3 of Theorem 2.5 is useful, for example, in the case when the continuous semigroup admits a *"good" Lyapunov function* $L : X \to \mathbf{R}$, that is, a continuous function on X that is a strictly decreasing function of t on any semitrajectory $\gamma^+(v) = \{V_t(v),\ t \in \mathbf{R}^+\}$, $v \in X$, except of course, the fixed points z, for which $V_t(z) = z$. Let us denote the set of all fixed points of the semigroup by Z and assume that it is finite. Then Z can serve as the set B_0 of part 3 of Theorem 2.5. In fact, since $L(V_t(v))$ strictly decreases as t increases for all $v \notin Z$, there exists $\lim_{t\to\infty} L(V_t(v)) \equiv l_+(v)$, and it is clear that $L|_{\omega(v)} = l_+(v) = \mathrm{const}$. Therefore $\omega(v) \in Z$ for all $v \in X$, and since v is attracted to $\omega(v)$, Z attracts all $v \in X$.

In view of the invariance of $\mathfrak{M}$, through each $v \in \mathfrak{M}$ there passes at least one continuous trajectory $\gamma(v) = \{v(t),\ t \in R;\ v(0) = v\} \subset \mathfrak{M}$. Since $\mathfrak{M}$ is compact, its segments $\gamma_\tau^-(v) = \{v(t), t \in (-\infty, \tau]\}$ are precompact, and they define the α-limit set: $\alpha(\gamma^-(v)) \equiv \bigcap_{\tau\leqslant 0} [\gamma_\tau^-(v)]_X$. Just as for $\omega(B)$, we can prove that $\alpha(\gamma^-(v))$ is a connected invariant compact set, and $v(t)$ is attracted to $\alpha(\gamma^-(v))$ as $t \downarrow -\infty$. The function $L(v(t))$ increases as $t \downarrow -\infty$, and is bounded on the real line $t \in \mathbf{R}$. Therefore there exists $\lim_{t\downarrow-\infty} L(v(t)) \equiv l_-(\gamma^-(v))$, $L|_{\alpha(\gamma^-(v))} = l_-(\gamma^-(v)) = \mathrm{const}$, and $\alpha(\gamma^-(v)) \subset Z$. Thus both "ends" of $\gamma(v) \subset \mathfrak{M}$ approach Z. If $Z = \bigcup_{i=1}^m z_i$, then since $\omega(v)$ and $\alpha(\gamma^-(v))$ are connected, each of these sets coincides with an element of Z. If Z consists of only one point z_1, then $\mathfrak{M} = \{z_1\}$ and z_1 attracts any $B \subset \mathscr{B}$, so that the point z_1 is globally stable. If Z consists of a larger number of points, then $\mathfrak{M}$ must, because it is connected, contain trajectories connecting points of Z.

The case considered above (when there is a "good" Lyapunov function) shows just how much larger is the minimal global B-attractor $\mathfrak{M}$ of a semigroup compared with its minimal global attractor $\hat{\mathfrak{M}}$. Let us formulate the argument above as a theorem.

***Theorem* 2.6.** *Suppose that the class* 1 *semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$, is continuous and bounded, and that there exists on X a continuous Lyapunov function $L : X \to \mathbf{R}$ that decreases strictly on each $\gamma^+(v) = \{V_t(v),\ t \in \mathbf{R}^+\}$, $v \in X$, as $t \uparrow \infty$, apart from $\gamma^+(z) = z \in Z$, the set of fixed points of the semigroup. Then $\omega(v) \subset Z$ for all $v \in X$, and therefore the set Z is non-*

empty and attracts all $v \in X$. If Z is a bounded set, then the semigroup has a global B-attractor $\mathfrak{M}$ with all the properties listed in part 2 of Theorem 2.5. Both ends of full trajectories $\gamma(v) \subset \mathfrak{M}$ are attracted (as $t \to \pm\infty$, respectively) to Z. If $Z = \bigcup_{i=1}^{m} z_i$ ($m < \infty$), then each $\omega(v)$ coincides with a point of Z, and $\mathfrak{M}$ consists of full trajectories connecting these points.

I have not introduced into this theorem statements relating to unstable manifolds $\mathcal{M}^-(z_i)$, since for their construction some additional conditions have to be satisfied, and I am not able to enlarge on this topic here. The foundation for the study and construction of $\mathcal{M}^-(z_i)$ (and $\mathcal{M}^-$ for periodic solutions) for PDE's and a certain class of semigroups has been laid in the work of Yudovich [32] and Ladyzhenskaya and Solonnikov [33]. In the cases they considered, as well as in the case of parabolic equations with sufficiently smooth terms, $\mathcal{M}^-(z_i)$ in a neighbourhood of z_i is a smooth finite-dimensional manifold. Much of Henry's book and the work of Babin and Vishik and others deals with the analysis of these manifolds.

Let us also consider the question of "finite-dimensionality of the dynamics V_t, $t \in \mathbf{R}$, on $\mathfrak{M}$" or of "finite-dimensionality" of $\mathfrak{M}$ (or of $\omega(B)$). I introduced the first concept "finite-dimensionality of the dynamics V_t, $t \in \mathbf{R}$, on $\mathfrak{M}$", in [1]. It has the following meaning: there exists a natural number N such that any full trajectory $\gamma(v) = \{V_t(v),\ t \in \mathbf{R}\}$ on $\mathfrak{M}$ is uniquely determined by its orthogonal projection $P_N(\gamma(v))$ on some finite-dimensional subspace $\mathbf{R}^N = P_N X$ of the phase space X, where in this case X is some Hilbert space. This fact was proved in [1] for (2.12) with $n = 2$. It holds for any invariant set $\mathfrak{A}$ that is bounded in the metric of D^2 (or D^1 only), and not only for $n = 2$, but for $n = 3$ also. (The proof is the same for both cases, but the magnitude of N is of course different in the cases $n = 2$ and $n = 3$.) The orthogonal projection onto the linear subspace $\mathbf{R}^N$ spanned by $\varphi_1, \dots, \varphi_N$ serves as the projection P_N. Let us reproduce this proof here.

Let $\mathfrak{A}$ be the invariant set for equation (2.12) that is bounded in the metric of D^2. If $n = 2$, $\mathfrak{M}$ can serve as $\mathfrak{A}$; if $n = 3$, we can take $\omega(X)$ from Theorem 2.4. Then there exists a common upper bound μ for all its elements v and for $\max_{x \in \Omega} |v(x)|$. In view of this, any two solutions $v(t) = V_t(v_0)$ and $\tilde{v}(t) = V_t(\tilde{v}_0)$, $t \in \mathbf{R}$, of (2.12) contained in $\mathfrak{A}$ satisfy the inequalities

$$(2.16) \qquad |(B(v(t)) - B(\tilde{v}(t)),\ u(t))| \leq \nu \| A^{1/2} u(t) \|^2 + \mu^2 (4\nu)^{-1} \| u(t) \|^2$$

and

$$(2.17) \qquad |(B(v(t)) - B(\tilde{v}(t)),\ Q_N u(t))| \leq \frac{\nu}{2} \| A^{1/2} Q_N u(t) \|^2 + \frac{2\mu^2}{\nu} \| u(t) \|^2,$$

in which $u(t) = v(t) - \tilde{v}(t)$, $Q_N = I - P_N$. Let us recall that in the case $n = 2$ the number $\mu = \sup_{v \in \mathfrak{M}} \max_{x \in \Omega} |v(x)|$ is determined by ν, f, and Ω.

The solutions $v(t)$ and $\widetilde{v}(t)$, $t \in \mathbf{R}$, are smooth: $v, \widetilde{v} \in C(\mathbf{R}, D^2)$, $\partial_t v$, $\partial_t \widetilde{v} \in C(\mathbf{R}, \mathring{J})$ and satisfy (2.12) for all $t \in \mathbf{R}$. The difference $u(t) = v(t) - \widetilde{v}(t)$ satisfies the relation

$$\frac{du(t)}{dt} + \nu A u(t) + B(v(t)) - B(\widetilde{v}(t)) = 0, \quad t \in \mathbf{R}. \tag{2.18}$$

Taking its scalar product with $u(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|A^{1/2}u(t)\|^2 = -(B(v(t)) - B(\widetilde{v}(t)), u(t)).$$

In view of (2.16) it follows that $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \leqslant m\|u(t)\|^2$, $m = \mu^2(4\nu)^{-1}$, and therefore

$$\|u(t)\| \leqslant e^{mt}\|u(0)\|. \tag{2.19}$$

Taking the scalar product of (2.18) and $Q_N u(t)$, using the fact that Q_N and A commute and (2.17), we obtain also the estimate

$$\frac{1}{2}\frac{d}{dt}\|Q_N u(t)\|^2 + \frac{\nu}{2}\|A^{1/2}Q_N u(t)\| \leqslant 8m\|u(t)\|^2. \tag{2.20}$$

Let us now use the fact that every $w \in Q_N D^1$ satisfies the inequality

$$\|A^{1/2}w\| \geqslant \lambda_{N+1}\|w\|^2, \tag{2.21}$$

where $\lambda_{N+1} \to \infty$ as $N \to \infty$. If $P_N v(t) = P_N \widetilde{v}(t)$, then $u(t) = Q_N u(t)$, and we obtain from (2.20) the inequality

$$\frac{1}{2}\frac{d}{dt}\|Q_N u(t)\|^2 + \left(\frac{\nu}{2}\lambda_{N+1} - 8m\right)\|Q_N u(t)\|^2 \leqslant 0. \tag{2.22}$$

Let us take N so large that

$$\frac{\nu}{2}\lambda_{N+1} - 8m \equiv \varepsilon_1 > 0. \tag{2.23}$$

Then by (2.22)

$$\|Q_N u(t_1)\| \leqslant e^{-\varepsilon_1(t_1 - t)}\|Q_N u(t)\| \tag{2.24}$$

for all $t_1 \geqslant t$. The norms $\|Q_N u(t)\|$ are bounded uniformly in $t \in \mathbf{R}$. Therefore, taking in (2.24) the limit as $t \to -\infty$, we see that $Q_N u(t_1) = 0$ for all $t_1 \in \mathbf{R}$, that is, $v(t_1) = \widetilde{v}(t_1)$ for all $t_1 \in \mathbf{R}$. Let us summarize this in a theorem.

***Theorem* 2.7.** *If N is taken so large that* (2.23) *holds with some $\varepsilon_1 > 0$, then from the equality of the projections $P_N v(t)$ and $P_N \widetilde{v}(t)$ of any two full trajectories in $\mathfrak{A}$, namely $v(t)$ and $\widetilde{v}(t)$, $t \in \mathbf{R}$, it follows that $v(t) = \widetilde{v}(t)$ for all $t \in \mathbf{R}$.*

We can also draw the following conclusions from (2.19)–(2.21):

$$\frac{d}{dt}\|Q_N u(t)\|^2 + \nu\|A^{1/2}Q_N u(t)\|^2 \leqslant 16me^{2mt}\|u(0)\|^2,$$

and therefore

$$\|Q_N u(t)\| \leqslant \delta(t, N)\|u(0)\|, \tag{2.25}$$

where

$$\delta(t, N) = \left(e^{-\nu\lambda_{N+1}t} + \frac{16m}{2m+\nu\lambda_{N+1}}e^{2mt}\right)^{1/2}. \tag{2.26}$$

It is clear that for a suitable choice of N and $t = t_1$ the magnitude of $\delta(t_1, N)$ will be less than 1. This fact combined with (2.19) makes it possible to find an upper bound for the Hausdorff dimension of $\mathfrak{A}$ considered as a subset of the space $\overset{0}{I}$. More precisely, I proved the following theorem in [22].

***Theorem* 2.8.** *Let B be a bounded set in a Hilbert space X, and let there be defined a map $V: B \to X$ such that $B \subseteq V(B)$ and for all $v, \tilde{v} \in B$*

$$\|V(v) - V(\tilde{v})\|_X \leqslant l\|v - \tilde{v}\|_X, \tag{2.27}$$

and

$$\|Q_N V(v) - Q_N V(\tilde{v})\|_X \leqslant \delta\|v - \tilde{v}\|_X, \quad \delta < 1, \tag{2.28}$$

where Q_N is the orthogonal projection of X onto the subspace $X_N^{\perp}$ of codimension N. Then

$$\mathrm{d}_H(B) \leqslant N \log\left(\frac{8\varkappa^2 l^2}{1-\delta^2}\right) \Big/ \log\frac{2}{1+\delta^2} \equiv d, \tag{2.29}$$

where $\varkappa$ is the Gauss constant.

This useful statement is relatively easy to prove. By constructing suitable coverings we see directly that the Hausdorff $(d+\varepsilon)$-measure of B is zero for all $\varepsilon > 0$. This theorem gives an upper bound d_1 for the Hausdorff dimension of the invariant sets $\mathfrak{A}$ of Theorem 2.7 considered as subsets of $\overset{0}{I}$. Namely, let us choose N and t_1 so that

$$e^{-\nu\lambda_{N+1}t_1} = \frac{1}{4} \quad \text{and} \quad \frac{16m}{2m+\nu\lambda_{N+1}}e^{2mt_1} \leqslant \frac{1}{4}. \tag{2.30}$$

Then $\delta^2(t_1, N) \leqslant 2^{-1}$ and

$$\mathrm{d}_H(\mathfrak{A}) \leqslant N(\log 4/3)^{-1}\log\frac{\varkappa^2(2m+\nu\lambda_{N+1})}{4m} \equiv d_1. \tag{2.31}$$

Here N has to be s large that (2.30) is satisfied, namely;

$$\nu\lambda_{N+1}\log\frac{2m+\nu\lambda_{N+1}}{64m} \geqslant 2m\log 4. \tag{2.32}$$

We recall that $\lambda_N = O(N)$ for $n = 2$ and $\lambda_N = O(N^{2/3})$ for $n = 3$. Thus, we have proved the following result.

***Theorem* 2.9.** *The Hausdorff dimension of the set $\mathfrak{A}$ of Theorem* 2.7, *considered as a subset of the space $\overset{0}{I}$, does not exceed the number d_1 of* (2.31), *where N is such that* (2.32) *holds. In particular, for the two-dimensional problem* (2.7), (2.8), $\mathrm{d}_H(\mathfrak{M}) \leqslant d_1$.

The proof of finiteness of the Hausdorff dimension of invariant sets $\mathfrak{A}$ regarded as subsets of the spaces D^s, $s > 0$ (in particular, of the set $\omega(X)$ of Theorem 2.4 as a subset of D^1) follows the same lines.

Remark 2.2. From the arguments of [22] leading to the proof of Theorem 2.8 it also follows that the d of (2.29) provides an upper bound for $d_f(B)$ if $\delta < 1/2$. As can be seen from (2.19) and (2.25)–(2.26), the condition $\delta < 1/2$ can be satisfied by the choice of suitable N and t_1. In this way, upper bounds for the fractal dimensions of the sets of Theorem 2.9 can be obtained.

The paper [50] is planned for publication in 1987. It contains better upper bounds of $d_H(\cdot)$ and $d_f(\cdot)$ for invariant sets for PDE's of different types, including hyperbolic equations, as well as of the number N of Theorem 2.7.

§3. Semigroups of class 2. Equations of hyperbolic type

The dynamics of problems of hyperbolic type are of a qualitatively different character. The solution operators V_t that correspond to these problems are not completely continuous for $t > 0$, and therefore most of the arguments used in §2 for the problem (2.6)–(2.8) and other problems of parabolic type are not directly applicable. Difficulties appear from the very start when we try to establish that ω-limit sets for single points and for sets $B \subset \mathscr{B}$ are non-empty. However, a more careful analysis of what we have done for linear hyperbolic equations shows that the methods for finding true attractors for problems of parabolic type would still work in the semilinear hyperbolic case. The reason for this is that an operator V_t can be represented as a sum $W_t + U_t$ of a contraction operator W_t and a completely continuous operator U_t. Let us illustrate the process on the example of a semilinear wave equation

$$\partial_t^2 v + \varepsilon\partial_t v - \Delta v + f(v) = h, \quad h = h(x), \qquad f(0) = 0, \quad \varepsilon = \mathrm{const} > 0, \tag{3.1}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$ with the boundary condition

$$v|_{\partial\Omega} = 0. \tag{3.2}$$

The functions f and h are regarded as fixed, with $h \in L_2(\Omega)$; precise conditions on f will be formulated below. The initial data

$$v|_{t=0} = \varphi, \quad \partial_t v|_{t=0} = \dot{\varphi} \tag{3.3}$$

run through the linear phase space X.

Let us start with the homogeneous linear problem

$$\partial_t^2 w + \varepsilon\partial_t w - \Delta w = 0, \quad w|_{\partial\Omega} = 0, \qquad w|_{t=0} = \varphi, \quad \partial_t w|_{t=0} = \dot{\varphi}. \tag{3.4}$$

It has been thoroughly studied in the spaces $D^s = D(A^{s/2})$, $s \in \mathbf{R}$, where A is the unbounded self-adjoint positive definite operator in $L_2(\Omega)$ generated by the Laplacian multiplied by (-1) under the zero boundary condition: $w|_{\partial\Omega} = 0$. I showed as early as 1950 (see [34], [35]) that its domain of definition is the subset $D(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ of the space $L_2(\Omega)$ if $\partial\Omega$ is C^2; in all that follows we shall implicitly consider $\partial\Omega$ to be sufficiently smooth. Let us denote by D^s the domain of definition of the power $A^{s/2}$ of A, equipped with the scalar product $(u, v)_s \equiv (A^{s/2}u, A^{s/2}v)$, where $(\,,)$ is the scalar product in $L_2(\Omega)$, and the norm $\|u\|_s \equiv (u, u)_s^{1/2}$. This is a complete Hilbert space, which we shall denote by the same symbol D^s, $D^0 = L_2(\Omega)$. The space D^s (for $s = 1, 2, \ldots$) consists of those elements $u \in W_2^s(\Omega) \equiv H^s(\Omega)$ for which $\Delta^{(m)}u|_{\partial\Omega} = 0$ ($m = 0, 1, \ldots [(s-1)/2]$); its norm $\|\cdot\|_s$ is equivalent to the usual Sobolev norm of $H^s(\Omega)$ [35]. The spaces $H^s(\Omega)$ for all $s \in \mathbf{R}^+$ and their subspaces $H_0^s(\Omega)$ were introduced in the late fifties, while in the sixties their connections with the spaces D^s were established. For example, $D^s \approx H^s(\Omega) = H_0^1(\Omega)$ for $s \in [0, 1/2)$; $D^s \approx H_0^s(\Omega)$ for $s \in (1/2, 3/2)$; $D^s \approx H^s(\Omega) \cap H_0^1(\Omega)$ for $s \in (3/2, 5/2)$, and so on. For more details on $H^s(\Omega)$, $H_0^s(\Omega)$, and D^s see [36]. We denote the norm in $L^2(\Omega)$ by $\|\cdot\|$.

To investigate the problem (3.1)–(3.3) we only need the continuity of the embedding of D^s into $L_p(\Omega)$, where $p = 2n/(n-2s)$ for $2s \in [0, n)$; p can be any number in $(1, \infty)$ for $2s = n$, and $p = \infty$ for $2s > n$. Hence it follows that $L_{p'}(\Omega)$, where $p' = p/(p-1)$, is continuously embedded in D^{-s} for $2s \in [0, n)$.

Let us interpret the problem (3.4) as the problem of finding the pair $\vec{w}(t) = \begin{bmatrix} w(t) \\ \partial_t w(t) \end{bmatrix}$ with $\vec{w}(0) = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} \equiv \vec{\varphi}$. Appropriate phase spaces for this problem are the spaces $X_s \equiv D^{s+1} \times D^s$, $s \in \mathbf{R}$. The elements in X_s are the pairs $\vec{u} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$; the scalar product in X_s is given by

$$\left(\begin{bmatrix} u \\ \dot{u} \end{bmatrix}, \begin{bmatrix} v \\ \dot{v} \end{bmatrix}\right)_{X_s} = (u, v)_{s+1} + (\dot{u}, \dot{v})_s,$$

while the norm in X_s is denoted by $\|\cdot\|_{X_s}$. In any of the spaces X_s there exists a unique solution $\vec{w} \in C(\mathbf{R}, X_s)$ of the problem (3.4) that depends continuously on $\vec{\varphi} \in X_s$.

Thus, in X_s we can define a linear continuous group $W_t : X_s \to X_s$, $t \in \mathbf{R}$. (We do not display explicitly the dependence of W_t on the index s.) The solution $\vec{w}(t) = W_t(\vec{w}_0)$, $t \in \mathbf{R}$, forms a continuous curve in X_s, the full trajectory $\gamma(\vec{w}_0)$. As before, we denote by $\gamma_\tau^+(\vec{w}_0)$ the following part of this trajectory:

$$\gamma_\tau^+(\vec{w}_0) \equiv \{W_t(\vec{w}_0),\ t \in [\tau, \infty)\}; \quad \gamma_0^+(\vec{w}_0) \equiv \gamma^+(\vec{w}_0);$$

$$\gamma_{\tau_1, \tau_2}^+(\vec{w}_0) \equiv \{W_t(\vec{w}_0),\ t \in [\tau_1, \tau_2]\}.$$

The operators $W_t : X_s \to X_s$, $t \in \mathbf{R}^+$, satisfy the estimates

$$\|W_t\|_{\mathscr{L}(X_s, X_s)} \leqslant m_s e^{-\alpha t}, \quad t \in \mathbf{R}^+, \tag{3.5}$$

for some $\alpha > 0$.

These can be proved by Fourier methods, as I did in the late forties and early fifties (see [37], [35]) for the equations (3.1) with $\varepsilon = 0$, as well as for any equation of hyperbolic type, in which instead of Δ we have a general symmetric elliptic operator with coefficients independent of t. The presence of the term $\varepsilon\,\partial_t w$ does not change anything in the argument, introducing into the result only a factor $e^{-\alpha t}$ which decays with time.

Instead of Fourier methods, it is possible to apply the procedures for obtaining a priori estimates for solutions of general hyperbolic equations that were introduced in [35]. The same methods will also work for non-linear equations. To exhibit the exponential decay of solutions $w(t)$ as $t \to \infty$, one has first to make the substitution $w(t) = \tilde{w}(t)e^{-\alpha t}$. Then $\tilde{w}(t)$ satisfies the equation

$$\mathscr{M}\tilde{w} \equiv \partial_t^2 \tilde{w} + (\varepsilon - \alpha)\,\partial_t \tilde{w} + A_1 \tilde{w} = 0 \tag{3.6}$$

where $A_1\tilde{w} = -\Delta\tilde{w} - \alpha(\varepsilon - \alpha)\tilde{w}$. For $\alpha > 0$ sufficiently small we have $\varepsilon - \alpha > 0$ and the operator A_1 has all the properties of the operator $-\Delta$, including positive definiteness. To prove (3.5) it is enough to verify that

$$\|\tilde{w}(t)\|_{s+1}^2 + \|\partial_t \tilde{w}(t)\|_s^2 \leqslant m_s^2 \quad \text{for all } t \in \mathbf{R}^+. \tag{3.7}$$

The same estimates can be obtained by considering the equalities

$$0 = (A_1^{s/2}(\mathscr{M}\tilde{w}(t)),\ A_1^{s/2}\partial_t \tilde{w}(t)) = \frac{1}{2}\frac{d}{dt}\|A_1^{s/2}\partial_t \tilde{w}(t)\|^2 + \tag{3.8}$$

$$+ (\varepsilon - \alpha)\|A_1^{s/2}\partial_t \tilde{w}(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|A^{\frac{s+1}{2}}\tilde{w}(t)\|^2.$$

We only need to take into account the fact that $\|A_1^{s/2}v\| \approx \|(-\Delta)^{s/2}v\| = \|v\|_s$.

Estimates of the derivatives $\partial_t^k \tilde{w}(t)$ ($k = 2, 3, \ldots$) follow from the estimates (3.7) and the equation (3.6). The same estimates for $\tilde{w}(t)$ can be obtained in "reverse" order: first estimate $\|\partial_t^{k+1}\tilde{w}(t)\|^2 + \|A_1^{1/2}\,\partial_t^k \tilde{w}(t)\|^2$ using the equalities

$$0 = (\partial_t^k(\mathscr{M}\tilde{w}(t)),\ \partial_t^{k+1}\tilde{w}(t)) = \frac{1}{2}\frac{d}{dt}\|\partial_t^{k+1}\tilde{w}(t)\|^2 +$$

$$+ (\varepsilon - \alpha)\|\partial_t^{k+1}\tilde{w}(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t^k A_1^{1/2}\tilde{w}(t)\|^2,$$

and from these estimate $\|A_1^{l/2}\tilde{w}(t)\|$ and $\|A_1^{l/2}\,\partial_t^m \tilde{w}(t)\|$ with the help of (3.6).

To get a priori estimates for the solutions of hyperbolic equations of general form we have to use either the forward or the reverse method; sometimes the two even have to be combined. By the same methods we obtain estimates of the norms for solutions of semi-linear problems, provided, of course, that some restrictions are imposed on $f(v)$.

The solution of the non-homogeneous linear problem

$$(3.9_1)\qquad \partial_t^2 u+\varepsilon\partial_t u-\Delta u = g(t),\quad u\,|_{\partial\Omega}=0,$$

$$(3.9_2)\qquad u\,|_{t=0}=0,\quad \partial_t u\,|_{t=0}=0,$$

is, as is well known, determined with the help of the operators W_t by

$$(3.10)\qquad \vec{u}(t)\equiv\begin{bmatrix} u(t)\\ \partial_t u(t)\end{bmatrix}=\int_0^t W_{t-\tau}\left(\begin{bmatrix}0\\ g(\tau)\end{bmatrix}\right)d\tau.$$

In view of (3.5), this solution satisfies the estimate

$$(3.11)\qquad \|\vec{u}(t)\|_{X_s}\leqslant m_s\int_0^t e^{-\alpha(t-\tau)}\|g(\tau)\|_s\,d\tau.$$

The derivative $\partial_t u(t)$ can be considered as a solution of a problem of the form (3.9_1), but with free term $\partial_t g(t)$ instead of $g(t)$, and with initial data $\partial_t u|_{t=0}=0$, $\partial_t(\partial_t u)|_{t=0}=g(0)$. Therefore

$$\partial_t\vec{u}(t)=\begin{bmatrix}\partial_t u(t)\\ \partial_t^2 u(t)\end{bmatrix}=W_t\left(\begin{bmatrix}0\\ g(0)\end{bmatrix}\right)+\int_0^t W_{t-\tau}\left(\begin{bmatrix}0\\ \partial_\tau g(\tau)\end{bmatrix}\right)d\tau,$$

and in view of (3.5)

$$(3.12)\qquad \|\partial_t\vec{u}(t)\|_{X_{s-1}}\leqslant m_{s-1}e^{-\alpha t}\|g(0)\|_{s-1}+m_{s-1}\int_0^t e^{-\alpha(t-\tau)}\|\partial_\tau g(\tau)\|_{s-1}\,d\tau.$$

Let us now study the non-linear problem (3.1)–(3.3). Sufficient conditions for its global unique solubility in the spaces X_s with $s\geqslant 0$ an integer have been known since the fifties. The same line of argument also provides sufficient conditions for such solubility in spaces X_s for all $s\geqslant 0$. These conditions ensure that the solution operators $V_t: X_s\to X_s$ form a continuous group V_t, $t\in\mathbf{R}$, and not a semigroup, as in problems of parabolic type. This important fact has not, for some reason, been used in a number of studies of the problem (3.1)–(3.3). It immediately guarantees both the existence of unique continuations of semitrajectories $\gamma^+(\vec{v})$ to full trajectories $\gamma(\vec{v})$ on invariant sets, and the invertibility of V_t on $\omega(B)$ and on $\mathfrak{M}$.

Thus, suppose we know that the problem (3.1)–(3.3) corresponds to a continuous group V_t, $t\in\mathbf{R}$, in X, where X is one of the spaces X_s, $s\geqslant 0$, so that

$$\vec{v}(t)=\begin{bmatrix}v(t)\\ \partial_t v(t)\end{bmatrix}=V_t(\vec{\varphi}),\qquad t\in\mathbf{R}.$$

Let us represent the solution $v(t)$, $t\in\mathbf{R}^+$, as a sum $v(t)=w(t)+u(t)$, where $w(t)$ is a solution of the problem (3.4), while $u(t)$ is the solution of problem (3.9) with $g(t)=-f(v(t))+h$. In the phase space X this can be written as

$$(3.13)\qquad V_t(\vec{\varphi})=W_t(\vec{\varphi})+U_t(\vec{\varphi}),\qquad t\in\mathbf{R}^+.$$

The operators U_t are thus introduced in the following way:

$$(3.14) \qquad U_t(\vec{\varphi}) = \int_0^t W_{t-\tau}\left(\begin{bmatrix} 0 \\ -f(v(\tau)+h) \end{bmatrix}\right) d\tau,$$

where $v(t)$ is the solution of the problem (3.1)–(3.3). These operators are non-linear, and the family U_t, $t \in \mathbf{R}^+$, does not have the semigroup property. On the other hand, as we shall presently show, the U_t are completely continuous if $f(v)$ depends smoothly on v and if it and its derivatives do not grow "too fast" as $|v| \to \infty$.

Suppose that $X = X_s$ for some $s \geqslant 0$, $h \in D^s$, and that for the solution $\vec{v}(t)$ of the problem (3.1)–(3.3) we know that $\vec{v} \in C(\mathbf{R}^+, X_s)$ and

$$(3.15) \qquad \|\vec{v}(t)\|_{X_s} \leqslant \mathcal{M}_1(\|\vec{\varphi}\|_{X_s}), \quad t \in \mathbf{R}^+,$$

where $\mathcal{M}_1(\cdot)$ is a continuous non-decreasing function of its argument; we do not explicitly indicate the dependence of $\mathcal{M}_1$ on s, h, and f, since these are fixed. Let us further assume that $f(\varphi_1) \in D^s$ for all $\varphi_1 \in D^{s+1}$, and that for some $\delta \in (0, 1]$ we have $f'(\varphi_1)\varphi_2 \in D^{s+\delta-1}$ for all $\varphi_1 \in D^{s+1}$ and all $\varphi_2 \in D^s$, and moreover

$$(3.16) \qquad \|f(\varphi_1)\|_s \leqslant \mathcal{M}_2(\|\varphi_1\|_{s+1}),$$

while

$$(3.17) \qquad \|f'(\varphi_1)\varphi_2\|_{s+\delta-1} \leqslant \mathcal{M}_3(\|\varphi_1\|_{s+1}, \|\varphi_2\|_s), \quad \delta \in (0, 1],$$

where $\mathcal{M}_i(\cdot)$ (and all the functions $\mathcal{M}_i(\cdot)$) introduced below) are some continuous non-decreasing functions of their arguments. From (3.11), (3.14), (3.15), and the condition (3.16) we conclude that

$$(3.18) \quad \|U_t(\vec{\varphi})\|_{X_s} \leqslant m_s \alpha^{-1}[\|h\|_s + \mathcal{M}_2(\mathcal{M}_1(\|\vec{\varphi}\|_{X_s}))] \equiv \mathcal{M}_4(\|\vec{\varphi}\|_{X_s}).$$

The inequality (3.12), with $s+\delta$ instead of s, and (3.14)–(3.17) give us the following:

$$\begin{aligned}(3.19) \quad \|\partial_t U_t(\vec{\varphi})\|_{X_{s+\delta-1}} &\leqslant m_{s+\delta-1}e^{-\alpha t}\|-f(\varphi)+h\|_{s+\delta-1} + \\ &+ m_{s+\delta-1}\alpha^{-1}\sup_{\tau\in[0,t]}\|f'(v(\tau))\partial_\tau v(\tau)\|_{s+\delta-1} \leqslant \\ &\leqslant m_{s+\delta-1}e^{-\alpha t}[\mathcal{M}_2(\|\varphi\|_{s+\delta}) + \|h\|_{s+\delta-1}] + \\ &+ m_{s+\delta-1}\alpha^{-1}\mathcal{M}_3(\sup_{\tau\in[0,t]}\|v(\tau)\|_{s+1}, \sup_{\tau\in[0,t]}\|\partial_\tau v(\tau)\|_s) \leqslant \\ &\leqslant m_{s+\delta-1}[\mathcal{M}_2\|\varphi\|_{s+\delta}) + \|h\|_{s+\delta-1}] + \\ &+ m_{s+\delta-1}\alpha^{-1}\mathcal{M}_3(\mathcal{M}_1(\|\vec{\varphi}\|_{X_s}), \mathcal{M}_1(\|\vec{\varphi}\|_{X_s})) \equiv \mathcal{M}_5(\|\vec{\varphi}\|_{X_s}\end{aligned}$$

(where I have assumed without loss of generality that $\|u\|_{s_1} \leqslant \|u\|_{s_2}$ for $s_1 < s_2$). From the estimates (3.18), (3.19) and from equation (3.9$_1$) we have the following estimate for $U(t)$:

$$(3.20) \qquad \|U_t(\vec{\varphi})\|_{X_{s+\delta}} \leqslant \mathcal{M}_6(\|\vec{\varphi}\|_{X_s}), \quad \delta > 0.$$

Since the space $X_{s+\delta}$ with $\delta > 0$ is compactly embedded in X_s, we conclude from (3.2) that the operators U_t, $t \in \mathbf{R}^+$, viewed as operators from X_s into X_s, are compact.

Thus, we have demonstrated that under conditions (3.16), (3.17) an operator V_t, $t \in \mathbf{R}^+$, can be represented as a sum $W_t + U_t$ (see (3.13)), where W_t has the contraction property (3.5) and U_t is compact. We are only interested in the cases in which the $V_t : X_s \to X_s$, $t \in \mathbf{R}$, form a continuous semigroup. In these cases the U_t are completely continuous for all $t \in \mathbf{R}^+$. As we show below, true attractors for such semigroups can be found by using the methods we applied to semigroups of class 1.

Next we introduce a class of semigroups that includes the class of semigroups just described, semigroups of class 1, and more generally all semigroups with the property that for all $B \subset \mathscr{B}$ such that $\gamma^+(B) \subset \mathscr{B}$ there exists a compact attractor that attracts B.

Let us call a semigroup $V_t : X \to X$, $t \in \mathbf{R}^+$, acting on a complete metric space X *asymptotically compact* or a *semigroup of class* 2 if for each $B \subset \mathscr{B}$ such that $\gamma^+(B) \subset \mathscr{B}$ any sequence of the form $\{V_{t_k}(v_k)\}_{k=1}^{\infty}$, $t_k \uparrow \infty$, $v_k \in B$, is precompact.

We shall restrict ourselves to continuous semigroups of class 2. Our aim is to prove for these semigroups propositions similar to those proved in §2 for class 1 semigroups. We first give two propositions that are true for any continuous semigroups. We shall use K to denote compact sets.

Proposition 3.1. *For any compact set K the sets*

$$\gamma^+_{0,T}(K) \equiv \{V_t(v);\ t \in [0, T],\ v \in K\},\ T < \infty,$$

are compact.

In fact, the V_t, $t \in \mathbf{R}^+$, define a continuous map $V : Y \equiv \mathbf{R}^+ \times X \to X$, and $\gamma^+_{0,T}(K)$ is the V-image of the compact set $[0, T] \times K$. Thus, as is well known, $\gamma^+_{0,T}(K)$ is also compact.

Proposition 3.2. *If $\gamma^+(K)$ is precompact, then $\omega(K) = \bigcap_{t \geqslant 0} [\gamma_t^+(K)]_X$ is a non-empty invariant compact set that attracts K.*

The set $\omega(K)$, being the intersection of nested non-empty compact sets, is a non-empty compact set. The inclusion $V_t(\omega(B)) \subset \omega(B)$ for all $t \geqslant 0$ is true for any B and any continuous V_t, as we remarked in §2. The reverse inclusion, $\omega(B) \subset V_t(\omega(B))$, has in fact also been proved in §2. The compact set $[\gamma^+(K)]_X$ can be regarded as the phase space for the semigroup V_t, $t \in \mathbf{R}^+$, since $V_t([\gamma^+(K)]_X) \subset [\gamma^+(K)]_X$. In view of this, the semigroup on this set is a semigroup of class 1, and in §2 we proved invariance of ω-limit sets for such semigroups.

Proposition 3.3. *If V_t, $t \in \mathbf{R}^+$, is a continuous semigroup of class* 2, *then for any compact set K the boundedness of $\gamma^+(K)$ entails the precompactness of $\gamma^+(K)$, and therefore all the statements of Proposition* 3.2.

To prove this, let us take an arbitrary sequence y_k $(k = 1, 2, ...)$ of elements in $\gamma^+(K)$, that is, $y_k = V_{t_k}(v_k)$, $v_k \in K$. If all t_k belong to the interval $[0, T]$ for some $T < \infty$, then $\{y_k\}_{k=1}^{\infty}$ is precompact by Proposition 3.1. If on the other hand $\{t_k\}_{k=1}^{\infty}$ form an unbounded sequence, then a subsequence $\{t_{k_j}\}_{j=1}^{\infty}$ can be chosen such that $t_{k_j} \uparrow \infty$, and $\{V_{t_{k_j}}(v_{k_j})\}_{j=1}^{\infty}$ is precompact by the assumption that the semigroup is asymptotically compact.

***Proposition* 3.4.** *Let V_t, $t \in \mathbf{R}^+$, be a continuous semigroup of class 2. Then for all B such that $\gamma^+(B)$ is bounded, $\omega(B)$ is a non-empty invariant compact set that attracts B.*

It is clear that $\omega(B)$ is closed, bounded, and non-empty, since $\omega(B)$ includes non-empty sets $\omega(v)$, $v \in B$. Moreover, as we remarked in the proof of Proposition 3.2, $V_t(\omega(B)) \subset \omega(B)$ for all $t \geqslant 0$. Let us show that $\omega(B) \subset V_t(\omega(B))$ for all $t \geqslant 0$. Let $v \in \omega(B)$; there exists a sequence of the form $\{V_{t_k}(v_k)\}_{k=1}^{\infty}$, $t_k \uparrow \infty$, $v_k \in B$, that converges to v (see Lemma 2.1). Each of the $V_{t_k}(v_k)$ with $t_k \geqslant t$ can be written as $V_t(V_{t_k-t}(v_k))$. The sequence $\{V_{t_k-t}(v_k)\}_{t_k \geqslant t}$ is precompact because V_t, $t \in \mathbf{R}^+$, is asymptotically compact, and all its limit points belong to $\omega(B)$. Let us choose from this sequence a convergent subsequence and denote its limit by $\widetilde{v}$. From what we have said and the continuity of V_t it follows that $v = V_t(\widetilde{v})$, that is, that $\omega(B) \subset$ $\subset V_t(\omega(B))$.

It remains to prove that $\omega(B)$ attracts B. Suppose not; then there exists an $\varepsilon > 0$ and a sequence $\{V_{t_k}(v_k)\}_{k=1}^{\infty}$, $t_k \uparrow \infty$, $v_k \in B$, such that $\operatorname{dist}\{V_{t_k}(v_k);\ \omega(B)\} \geqslant \varepsilon$. But this contradicts the fact that all such sequences are precompact and all their limit points belong to $\omega(B)$.

***Proposition* 3.5.** *If V_t, $t \in \mathbf{R}^+$, is a bounded point-dissipative semigroup and the V_t are continuous for all $t \in \mathbf{R}^+$, then there exists a bounded set B_1 that absorbs small neighbourhoods $O(K)$ of any compact set K in finite time; moreover, $V_t(B_1) \subset B_1$.*

Proof. Let B_0 be a bounded set that attracts all $v \in X$. Let us take a small neighbourhood $O_{\varepsilon_0}(B_0)$ of B_0 with $\varepsilon_0 > 0$ fixed, and consider the set $B_1 = \gamma^+(O_{\varepsilon_0}(B_0))$. Since the semigroup is bounded, $B_1 \subset \mathscr{B}$. It is clear that $V_t(B_1) \subset B_1$. For all $v \in X$ there exists a $t(v)$ such that $V_t(v) \subset O_{\varepsilon_0}(B_0)$ for all $t \geqslant t(v)$. Since the operator $V_{t(v)}$ is continuous there exists a neighbourhood $O(v)$ of the point v for which $V_{t(v)}(O(v)) \subset O_{\varepsilon_0}(B_0)$. Hence it follows that $V_{t(v)+t}(O(v)) \subset V_t(O_{\varepsilon_0}(B_0)) \subset B_1$ for all $t \geqslant 0$, that is, B_1 absorbs $O(v)$ after time $t(v)$. Since from a covering $\bigcup_{v \in K} O(v)$ of a compact set K we can choose a finite subcovering $\bigcup_{i=1}^{m} O(v_i) \equiv O(K)$, $v_i \in K$, $m = m(K) < \infty$, it follows that $O(K)$ is absorbed by B_1 after time $t(K) = \max_{i=1,\ldots,m} t(v_i)$.

Let us note that boundedness of the semigroup was only needed to ensure the boundedness of the set B_1.

***Theorem* 3.1.** *Let $V_t : X \to X$, $t \in \mathbf{R}^+$, be a continuous bounded point-dissipative semigroup of class 2. Then for this semigroup there exists a non-empty minimal global B-attractor $\mathfrak{M}$. It is compact and invariant. If X is connected, so is $\mathfrak{M}$.*

The attractor $\mathfrak{M}$ is constructed by the formula

$$\mathfrak{M} = \omega(B_1) = \bigcap_{t \geqslant 0} [V_t(B_1)]_X, \tag{3.21}$$

where B_1 is the set from Proposition 3.5. As we showed in Proposition 3.4, $\omega(B_1)$ is a non-empty invariant compact set that attracts B_1. The same properties are shared by $\omega(B)$ for all $B \subset \mathscr{B}$. Let us show that every $B \subset \mathscr{B}$ is attracted by the set $\omega(B_1)$. We take an arbitrary ε-neighbourhood $O_\varepsilon(\omega(B_1))$, $\varepsilon > 0$. It absorbs B_1 in some finite time $t_1 = t_1(\varepsilon)$. In its turn, B_1 absorbs the compact set $\omega(B)$ and even some neighbourhood of it, $O(\omega(B))$, in finite time $t_2 = t_2(\omega(B))$ (see Proposition 3.5). Finally, $O(\omega(B))$ absorbs B in finite time $t_3 = t_3(B, O(\omega(B)))$. Thus, an arbitrary $B \subset \mathscr{B}$ is absorbed by $O_\varepsilon(\omega(B_1))$ in finite time $t_1 + t_2 + t_3$, which of course depends on ε and B. From this and the invariance of the sets $\omega(B)$ it follows that $\omega(B) \subset \omega(B_1)$ for all B, and for $B \supset B_1$ we have $\omega(B) = \omega(B_1)$. If X is connected, then by taking for B some connected set $B \subset \mathscr{B}$ containing B_1 we get a connected $\omega(B)$, which coincides with $\omega(B_1)$.

A useful criterion for ascertaining whether or not a semigroup belongs to class 2 is given by the following proposition.

***Proposition* 3.6.** *If a semigroup V_t, $t \in \mathbf{R}^+$, acts on a Banach space X, and V_t can be represented as a sum $W_t + U_t$ in which W_t, $t \in \mathbf{R}^+$, is a family of contraction operators, or more precisely, of operators such that*

$$\|W_t(B)\|_X \leqslant m_1(t) m_2(\|B\|_X), \tag{3.22}$$

where $m_k(\cdot)$ are continuous functions on $\mathbf{R}^+$ and $m_1(t) \to 0$ as $t \to \infty$, $\|B\|_X \equiv \sup_{v \in B} \|v\|_X$, while U_t for all $t \in \mathbf{R}^+$ maps bounded sets into precompact sets, then V_t, $t \in \mathbf{R}^+$, is a semigroup[1] *of class* 2.

Let $\gamma^+(B)$ be a bounded set. Let us check that an arbitrary sequence $\mathscr{M} = \{V_{t_k}(v_k)\}_{k=1}^\infty$, $t_k \uparrow \infty$, $V_k \in B$, can be covered by a finite ε-net for all $\varepsilon > 0$. For a fixed $\varepsilon > 0$ let us choose the number l such that $m_1(l) \leqslant \varepsilon[2m_2(\|B\|_X)]^{-1}$ and let us split the set $\mathscr{M}$ into two parts: $\mathscr{M}_1 = \{V_{t_k}(v_k)\}_{k=1}^{k_1}$ with $t_k < 1$ and $\mathscr{M}_2 = \{V_{t_k}(v_k)\}_{k=k_1+1}^\infty$ with $t_k \geqslant 1$. The set $\mathscr{M}_2$ is a subset of the set $V_l(\gamma^+(B))$, whose elements can be represented in

[1] The semigroup V_t, $t \in \mathbf{R}^+$, may act only on a part $\mathscr{M}$ of the Banach space X. In that case V_t: $\mathscr{M} \to \mathscr{M}$, $t \in \mathbf{R}^+$, is a semigroup of class 2.

the form $W_l(v)+U_l(v)$, $v \in \gamma^+(B)$. The set $U_l(\gamma^+(B))$ is precompact and can therefore be covered by a finite $\varepsilon/2$-net, and since the norms of the elements $W_l(v)$, $v \in \gamma^+(B)$, do not exceed $\varepsilon/2$, the set $V_l(\gamma^+(B))$ can be covered by a finite ε-net. Thus it is clear that the set $\mathscr{M}$ can also be covered by a finite ε-net.

From the analysis of problem (3.1)–(3.3) above it is clear that the semigroups V_t, $t \in \mathbf{R}^+$, corresponding to it in spaces $X = X_s$, $s \geqslant 0$, satisfy the conditions of Proposition 3.6 if $h \in D^s$ and if the requirements (3.16) and (3.17) hold for f. To check that class 2 semigroups are point-dissipative, the following theorem is useful.

***Theorem* 3.2.** *All the statements of Theorem* 2.6 *are true for a continuous bounded class* 2 *semigroup* $V_t : X \to X$, $t \in \mathbf{R}^+$, *that admits a "good" Lyapunov function.*

In view of Theorem 3.1, the proof of this theorem is exactly the same as the proof of Theorem 2.6 for semigroups of class 1.

Let us next formulate some results concerning attractors for the problem (3.1)–(3.3). For brevity, let us restrict ourselves to the most interesting case of $\Omega \subseteq \mathbf{R}^3$. We shall begin by quoting results concerning the solubility of the problem (3.1)–(3.3) (which have been known to me since the mid-fifties).

1. Let f be a differentiable function, and

$$f(0)=0, \quad |f'(u)| \leqslant c_1(|u|^2+1) \tag{3.23}$$

for some $c_1 \in \mathbf{R}^+$, and let its primitive $\mathscr{F}(u) = \int_0^u f(s)ds$ satisfy the condition

$$(1, \mathscr{F}(u)) \geqslant -\left(\frac{1}{2}-\varepsilon_1\right) \|u_x\|^2 - c_2 \tag{3.24}$$

for some $\varepsilon_1 > 0$, where $c_2 \in \mathbf{R}^+$ for all $u \in D^1$. Then the problem (3.1)–(3.3) is uniquely soluble in $C(\mathbf{R}, X_0)$ for all $h \in L_2(\Omega)$, and the solution operators V_t, $t \in \mathbf{R}$, form a continuous group $V_t : X_0 \to X_0$, $t \in \mathbf{R}$. The corresponding semigroup $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, is bounded.

2. If for every $u \in D^1$

$$(f(u), u) \geqslant -(1-\varepsilon_2)\|u_x\|^2 - c_3 \tag{3.25}$$

with $\varepsilon_2 > 0$ and $c_3 \in \mathbf{R}^+$, $h \in L_2(\Omega)$, and $\partial\Omega \in C^2$, then the stationary problem

$$-\Delta z + f(z) = h, \quad z|_{\partial\Omega} = 0, \tag{3.26}$$

is soluble in D^1. The set of its solutions is bounded and closed in the spaces D^1 and D^2. Therefore the set Z of all stationary points $\vec{z} = \begin{bmatrix} z \\ 0 \end{bmatrix}$ of the semigroup is bounded and closed in the spaces X_0 and X_1.

There exists a "good" Lyapunov function for the problem (3.1)–(3.3) under conditions (3.23)–(3.24) in the phase space X_0. It has the form

$$(3.27)\qquad \mathcal{L}\left(\begin{bmatrix} u \\ \dot{u} \end{bmatrix}\right)=\frac{1}{2}(\|\dot{u}\|^2+\|u_x\|^2)+(1,\ \mathcal{F}(u))-(h,\ u).$$

Along solutions of the problems (3.1)–(3.3) we have

$$(3.28)\qquad \frac{d}{dt}\mathcal{L}(\vec{v}(t))=-\varepsilon\|\partial_t v(t)\|^2.$$

This equality is none other than the basic energy relation for (3.1)–(3.3).

We note that all the conditions (3.23)–(3.26) are satisfied by functions $f(v) = cv^3+P_2(v)$, where c = const > 0 and $P_2(v)$ is an arbitrary quadratic polynomial. The following is a new result for the problem (3.1)–(3.3).

3. If $\mathcal{F}$ satisfies (3.24) and

$$(3.29)\qquad f(0)=0,\quad |f'(u)|\leqslant c_4(|u|^{2-\gamma}+1)$$

for some $\gamma > 0$, then conditions (3.16), (3.17) are satisfied with $s = 0$ and $\delta = \gamma/2$, and therefore $U_t(\vec{\varphi})$ satisfy (3.20) with $s = 0$ and $\delta = \gamma/2$. The same fact guarantees that the semigroup $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, belongs to class 2 by Proposition 3.6. In view of all we have said above concerning the problem (3.1)–(3.3), Theorems 3.1 and 3.2 can be used to obtain the following result.

Theorem 3.3. *If f satisfies conditions* (3.29), (3.24), (3.25) *and* $h \in L_2(\Omega)$, *then the solution operators* $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, *of the problem* (3.1)–(3.3) *for a bounded domain* $\Omega \subset \mathbf{R}^3$ *with* $\partial\Omega \in C^2$ *form a continuous bounded point-dissipative semigroup of class* 2, *and therefore the statements of Theorem* 3.1 *hold with* $X = X_0$. *The set Z of all stationary points is its minimal global attractor. It is a compact set in X_0 and a bounded closed subset of X_1. The minimal global B-attractor $\mathfrak{M}$ is compact, invariant, connected and consists of full trajectories connecting points of Z. Each full trajectory $\gamma(v)$ in a bounded set $B \subset X_0$ belongs to $\mathfrak{M}$. If Z consists of finitely many points: $Z = \bigcup_{i=1}^{m} \vec{z}_i$, then for each $\vec{v} \in \mathfrak{M}\setminus Z$ the points $V_t(\vec{v})$ converge as $t \to \infty$ to one of the points $\vec{z}_i$, and $V_t(\vec{v}) \to \vec{z}_j \neq \vec{z}_i$ as $t \to -\infty$. The set $\mathfrak{M}$ is bounded in the space X_1.*

All the statements of Theorem 3.3, apart from the claims that $V_t(v)$ is attracted to Z as $t \to -\infty$ and that $\mathfrak{M}$ is bounded in X_1, have already been proved. For $\vec{v} \in \mathfrak{M}$ the set $\gamma(\vec{v})$ is precompact. Therefore for this set there exists a non-empty α-limit set $\alpha(\vec{v}) \equiv \bigcap_{t\leqslant 0} \gamma_t^-(v)$, $\gamma_t^-(v) \equiv \{V_\tau(\vec{v}),\ \tau \in (-\infty,\ t]\}$; $\alpha(\vec{v})$ is a compact invariant connected set, which attracts $\vec{v}$ as $t \to -\infty$.

It is clear that $\mathcal{L}|_{\alpha(\vec{v})} = \lim_{t\to-\infty} \mathcal{L}(V_t(\vec{v})) \equiv l_-(v) = \text{const}$ and therefore $\alpha(\vec{v}) \subset Z$. If $Z = \bigcup_{i=1}^{m} \vec{z}_i$, then since the sets $\omega(\vec{v})$ and $\alpha(\vec{v})$ are connected, each of them coincides with a point of Z, and these points are different for $\vec{v} \in \mathfrak{M}\setminus Z$.

It remains only to show that $\mathfrak{M}$ is a bounded subset of X_1. Let $\vec{v}_0 \in \mathfrak{M}$; taking $t_1 < t$, let us represent $\vec{v}(t) = V_t(\vec{v}_0)$, $t \in \mathbf{R}$, in the form

$$\vec{v}(t) = V_{t-t_1}(\vec{v}(t_1)) = W_{t-t_1}(\vec{v}(t_1)) + U_{t-t_1}(\vec{v}(t_1)) =$$
$$= W_{t-t_1}(\vec{v}(t_1)) + \int_0^{t-t_1} W_{t-t_1-\tau}\left(\begin{bmatrix} 0 \\ -f(V_\tau(\vec{v}(t_1))+h \end{bmatrix}\right) d\tau =$$
$$= W_{t-t_1}(\vec{v}(t_1)) + \int_{t_1}^{t} W_{t-\xi}\left(\begin{bmatrix} 0 \\ -f(V_\xi(\vec{v}_0))+h \end{bmatrix}\right) d\xi.$$

Making t_1 tend to $-\infty$ in this equality and using the fact that $\|\vec{v}(t_1)\|_{X_0}$ is uniformly bounded in t_1, we obtain

$$\text{(3.30)} \qquad \vec{v}(t) = V_t(\vec{v}_0) = \int_{-\infty}^{t} W_{t-\xi}\left(\begin{bmatrix} 0 \\ -f(V_\xi(\vec{v}_0)) + h \end{bmatrix}\right) d\xi, \quad t \in \mathbf{R}, \quad \vec{v}_0 \in \mathfrak{M}.$$

Using this representation, we can estimate $\|\vec{v}(t)\|_{X_{\gamma/2}}$ just as we estimated $\|\vec{U}_t(\varphi)\|_{X_{\gamma/2}}$ above (see (3.20)), and reach the conclusion that

$$\sup_{t\in\mathbf{R}} \|v(t)\|_{X_{\gamma/2}} \leqslant \mathcal{M}_7(\|\vec{v}(0)\|_{X_0}).$$

This new information on $\vec{v}(t)$ and the representation (3.30) allow us to repeat the calculations and to arrive, after finitely many steps, at an estimate of $\sup_{t\in R} \|\vec{v}(t)\|_{X_1}$ in terms of $\|\vec{v}(0)\|_{X_0}$.

This procedure can be extended to establish the boundedness of $\mathfrak{M}$ in the spaces X_s for s arbitrarily large if f, h, and $\partial\Omega$ satisfy appropriate smoothness and compatibility conditions.

I think that Theorem 3.3 has, in essence, been proved in Hale's paper [38]. However, the paper [38] has not reached Leningrad yet, and I judge its contents from the preprint [6], which refers to it, and from the book [7], which contains a number of propositions that could be used for this purpose.

The claim 3 above about the validity of (3.20) with $s = 0$ and $\delta = \gamma/2$ under condition (3.29) has also been proved by Haraux [39]. He effectively found a set B_0 that is bounded in X_0 and absorbs any $B \subset \mathcal{B}$. Had he used the results obtained earlier by American mathematicians, he could have proved all the statements of Theorem 3.3. But it appears that Haraux was unaware of these papers, since they are not referred to in his work, and he only concludes from his results that the "maximal (X_1, X_0)-attractor" presented in the papers [20] and [21] of Babin and Vishik attracts (in the norm of X_0) all bounded sets $B \subset \mathscr{B}$. I have placed inverted commas around the words "maximal (X_1, X_0)-attractor", since this term, which is introduced in the work of Babin and Vishik, is not defined here. These authors construct it with the help of unstable manifolds under a number of conditions additional to the ones needed here. In fact, of course, it coincides with $\mathfrak{M}$.

In their preprint [6] the authors tried to remove the restriction (3.29), that is, they tried to consider the case (3.23). However, there is a mistake in their argument, and it cannot be corrected with the means they have at their disposal. Hale himself found the mistake, and notified me of this fact in a letter. The mistake can be corrected if one has available the following estimate for solutions of the linear problem (3.9_1) with initial data in X_0:

$$(3.31)\qquad \sup_{t\in\mathbf{R}^+}\Big(\int_t^{t+1}\|u(\tau)\|^q_{L_r(\Omega)}\,d\tau\Big)^{1/q}\leqslant$$

$$\leqslant \mathscr{M}_8\big(r, q, \sup_{t\in\mathbf{R}^+}\|\vec{u}(t)\|_{X_0}\big)\Big(1+\sup_{t\in\mathbf{R}^+}\int_t^{t+1}\|g(\tau)\|\,d\tau\Big),\ r>6,\ q\geqslant 2.$$

Suppose that (3.31) holds with some $r > 6$ and $q = 2$. Let us consider a solution $\vec{v}\in C(\mathbf{R}^+, X_0)$ of the problem (3.1)–(3.3) as the solution of the linear problem of the form (3.9_1) with $g(t) = -f(v(t))+h$ and $\vec{v}(0) = \vec{\varphi}\in X_0$. For this solution, in view of (3.31) and (3.23), we have

$$(3.32)\qquad \sup_{t\in\mathbf{R}^+}\Big(\int_t^{t+1}\|v(\tau)\|^2_{L_r(\Omega)}\,d\tau\Big)^{1/2}\leqslant \mathscr{M}_9(r, 2, \|\vec{\varphi}\|_{X_0});$$

$$(3.33)\qquad \|\partial_t g(t)\|_{L_p(\Omega)}=\|-f'(v(t))\,\partial_t v(t)\|_{L_p(\Omega)}\leqslant$$

$$\leqslant c_2\,\|\,|v(t)|+1\,\|^2_{L_r(\Omega)}\,\|\partial_t v(t)\|\leqslant$$

$$\leqslant c_2\big(\|v(t)\|_{L_r(\Omega)}+(\operatorname{mes}\Omega)^{1/r}\big)^2\,\mathscr{M}_1(\|\vec{\varphi}\|_{X_0})\equiv j(t),$$

with $p = 2r/(4+r) > 6/5$ and

$$(3.34)\qquad \sup_{t\in\mathbf{R}^+}\int_t^{t+1} j(\tau)\,d\tau\leqslant \mathscr{M}_{10}(r, 2, \|\vec{\varphi}\|_{X_0}).$$

Since $L_p(\Omega)\subset D^{-\beta}$, $\beta = 3(1/p-1/2) < 1$, and $\|\partial_t g(t)\|_{(-\beta)}\leqslant c\|\partial_t g(t)\|_{L_p(\Omega)}$, we have

$$(3.35)\qquad \sup_{t\in\mathbf{R}^+}\int_t^{t+1}\|\partial_\tau g(\tau)\|_{(-\beta)}\,d\tau\leqslant c\,\mathscr{M}_{10}(r, 2, \|\vec{\varphi}\|_{X_0}).$$

Now we can estimate $\|\partial_t U_t(\vec{\varphi})\|_{X_{-\beta}}$, using (3.35) and the representation (3.12) in which we have to take $g(0) = -f(\varphi)+h$ and $\partial_t g(t) = -f'(v(t))\partial_t v(t)$, to obtain

$$(3.36)\quad \| \partial_t U_t(\vec{\varphi}) \|_{X_{-\beta}} \leqslant m_{(-\beta)} e^{-\alpha t} \| -f(\varphi)+h \|_{(-\beta)} + \\ + m_{(-\beta)} \int_0^t e^{-\alpha(t-\tau)} \| f'(v(\tau))\, \partial_\tau v(\tau) \|_{(-\beta)} d\tau \leqslant \mathcal{M}_{11}(r, 2, \| \vec{\varphi} \|_{X_0}).$$

From this and (3.9_1) with $g(t) = -f(v(t))+h$ we get

$$(3.37)\qquad \sup_{t\in \mathbf{R}^+} \| U_t(\vec{\varphi}) \|_{X_{1-\beta}} \leqslant \mathcal{M}_{12}(r, 2, \| \vec{\varphi} \|_{X_0}),\quad \beta < 1.$$

Thus, U_t maps sets bounded in X_0 into sets bounded in $X_{1-\beta}$, $1-\beta > 0$, that is, U_t are compact operators, and $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, is a semigroup of class 2. Therefore the following (conditional) theorem holds.

***Theorem* 3.4.** *Suppose that f satisfies conditions* (3.23)–(3.25), $h \in L_2(\Omega)$, *and that the solutions $\vec{u} \in C(\mathbf{R}^+, X_0)$ of the linear problem* (3.9_1) *with initial data in X_0 satisfy the estimate* (3.31) *with some $r > 6$ and $q = 2$. Then all the statements of Theorem* 3.3 *are valid for the semigroup $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, of solution operators of the problem* (3.1)–(3.3).

The estimate (3.31) is very plausible. It holds for periodic boundary conditions; in this case, however, the norm in X_0 is defined slightly differently:

$$\left\| \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \right\|_{X_0} = (\| \dot{u} \|^2 + \| u_x \|^2 + \| u \|^2)^{1/2},$$

while the role of the "principal" linear operator A of the stationary part of the equation is played by the operator $-\Delta+I$ with periodic boundary conditions. For these boundary conditions the estimate (3.31) with any $r \in (6, \infty)$ and $q = 2r/(r-6)$ has been proved by Kapitanski, by using analytic tools developed by a number of authors for the Cauchy problem for the wave equation (see [40], [42], and so on). (Let me again remind the reader that I only quote results for $n = 3$; analogues of all these results have been established for all n.) Therefore the following theorem holds.

***Theorem* 3.5.** *If f satisfies conditions* (3.23)–(3.25) *and* $h \in L^2(\Omega)$, *then all the statements of Theorem* 3.3 *are true for equation* (3.1) *with periodic (in x_k) boundary conditions in a parallepiped $\Omega \subset \mathbf{R}^3$.*

In [43] I obtained the following result for the problem (3.1)–(3.3) with $\gamma = 0$:

***Theorem* 3.6.** *If f satisfies conditions* (3.23)–(3.25), $h \in L_2(\Omega)$, *and $\partial\Omega \in C^2$, then the solution operators V_t form a group in X_1; the corresponding semigroup $V_t : X_1 \to X_1$, $t \in \mathbf{R}^+$, is bounded; the set Z is bounded in X_1. If moreover f is twice differentiable and*

$$(3.38)\qquad | f''(v) | \leqslant \mathcal{M}(|v|),$$

where $\mathcal{M}(\cdot)$ is a function bounded on any bounded subset of the positive real line $\mathbf{R}^+$, then the group $V_t : X_1 \to X_1$, $t \in \mathbf{R}^+$, is continuous. If $h \in D^1$ and $\partial\Omega \in C^3$, then the semigroup $V_t : X_1 \to X_1$, $t \in \mathbf{R}^+$, is point-dissipative and of class 2. All the statements of Theorem 3.3 hold for this semigroup with the space X_1 instead of X_0, and with X_2 instead of X_1.

The first statements of this theorem are proved with the help of some results concerning the linear problem and the a priori estimate

$$\sup_{t\in\mathbf{R}^+} \|\vec{v}(t)\|_{X_1} \leqslant M_1(\|\vec{v}(0)\|_{X_1})$$

for exact solutions of the problem (3.1)–(3.3) and their Galërkin approximations. This estimate is obtained by the method described at the beginning of this section and with the use of an optimal embedding theorem. (In [20] and other papers, such an estimate is obtained under more severe restrictions on f, which include the condition (3.29) with $\gamma > 0$.) To check that the semigroup is of class 2, we use the same line of argument as in the preceding theorems which deal with the problem (3.1)–(3.3). Some of the terms (definitions) that appear in this paper are still absent from [43].

Let us show how, with the help of Theorem 2.8, we can obtain an upper bound for $\dim_H$ of any set $\mathfrak{A}$ that is bounded in the metric of X_1 and invariant with respect to the group $V_t : X_0 \to X_0$, $t \in \mathbf{R}$. Thus, let $\mathfrak{A}$ be invariant and $\|\mathfrak{A}\|_{X_1} \leqslant c$. Then for all $\vec{u} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \in \mathfrak{A}$ there exists a common upper bound for the quantities

$$(3.39) \qquad \|u_x\|,\ \|u_{xx}\|,\ \max_{x\in\Omega} |u(x)|,\ \|\dot{u}\|,\ \|\dot{u}_1\|,\ \|\dot{u}\|_{L_6(\Omega)} \leqslant c_1.$$

The trajectories $\gamma(\vec{v})$ on $\mathfrak{A}$ are smooth curves: solutions $\vec{v}(t) = V_t(\vec{v}(0))$, $t \in \mathbf{R}$, of the problem (3.1)–(3.3) with $\vec{v}(0) \in \mathfrak{A}$ are elements of $C(\mathbf{R}, X_1)$, $\partial_t v \in C(\mathbf{R}, X_0)$; the estimates (3.39) hold for $u = v(t)$ and $\dot{u} = \partial_t v(t)$ for all $t \in \mathbf{R}$, and for $\partial_t^2 v(t)$ we have the estimate

$$(3.40) \qquad \|\partial_t^2 v(t)\| \leqslant c_2 \quad \text{for all } t \in \mathbf{R}.$$

In view of this,

$$(3.41) \qquad |f(v)|,\quad |f'(v)|,\quad |f''(v)| \leqslant c_3$$

for all $\vec{v} \in \mathfrak{A}$, where $c_3 = \sup\limits_{|v|\leqslant c_1} \left\{\left|\frac{d^k f(v)}{dv^k}\right|,\ k = 0, 1, 2\right\}$.

For any two solutions $v(t)$ and $\tilde{v}(t)$ of the problem (3.1)–(3.3) with initial data in $\mathfrak{A}$

$$f(v(t)) - f(\tilde{v}(t)) = \int_0^1 \frac{d}{d\xi} f[\xi v(t) + (1-\xi)\tilde{v}(t)]\, d\xi = \mathscr{E}(t)\, u(t),$$

where $\mathscr{E}(t) = \int_0^1 f'[\xi v(t) + (1-\xi)\tilde{v}(t)]d\xi$, and $u(t) = v(t) - \tilde{v}(t)$. From (3.41) it follows that

$$(3.42) \qquad |\mathscr{E}(t)| \leqslant c_3 \quad \text{for all } t \in \mathbf{R}.$$

Moreover,

$$\frac{d\mathscr{E}(t)}{dt} = \int_0^1 f''[\xi v(t) + (1-\xi)\tilde{v}(t)]\cdot[\xi \partial_t v(t) + (1-\xi)\partial_t \tilde{v}(t)]d\xi$$

and

$$(3.43) \qquad \left|\frac{d\mathscr{E}(t)}{dt}\right| \leqslant c_4 (|\partial_t v(t)| + |\partial_t \tilde{v}(t)|) \quad \text{for all } t \in \mathbf{R}.$$

To verify the conditions of Theorem 2.8, let us decompose X_0 into the orthogonal sum of two subspaces X_0^N and $X_0^{N\perp}$. The elements of X_0^N are the pairs $\begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ in which u and $\dot{u}$ are of the form $u = u(x) = \sum_{k=1}^{N} a_k \varphi_k(x)$, $\dot{u} = \dot{u}(x) = \sum_{k=1}^{N} b_k \varphi_k(x)$, where $\varphi_k(x)$ are the eigenfunctions of the operator $-\Delta$, namely,

$$(3.44) \qquad -\Delta\varphi_k = \lambda_k \varphi_k, \quad \varphi_k|_{\partial\Omega} = 0, \quad (\varphi_k, \varphi_l) = \delta_k^l,$$

and a_k and b_k are arbitrary numbers. The elements of $X_0^{N\perp}$ are the pairs $\begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ with $u = \sum_{k=N+1}^{\infty} a_k \varphi_k$ and $\dot{u} = \sum_{k=N+1}^{\infty} b_k \varphi_k$. The norms in X_0^N and $X_0^{N\perp}$ are naturally the same as in X_0. Let us denote orthogonal projections onto these spaces by P_N and Q_N, respectively. For $u(t) = v(t) - \tilde{v}(t)$ we have

$$(3.45) \qquad \partial_t^2 u(t) + \xi \partial_t u(t) - \Delta u(t) + \mathscr{E}(t) u(t) = 0, \quad t \in \mathbf{R}.$$

Let us denote the pair $\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix}$ by $\vec{u}(t)$. Taking the scalar product of (3.45) with $\partial_t u(t)$, we obtain the following estimates:

$$\frac{1}{2}\frac{d}{dt}(\|\partial_t u(t)\|^2 + \|u_x(t)\|^2) + \xi\|\partial_t u(t)\|^2 = -(\mathscr{E}(t), u(t), \partial_t u(t)) \leqslant$$
$$\leqslant c_3 \|u(t)\|\,\|\partial_t u(t)\| \leqslant c_3 \lambda_1^{-1/2} \|u_x(t)\|\,\|\partial_t u(t)\| \leqslant c_5 (\|\partial_t u(t)\|^2 + \|u_x(t)\|^2),$$

where $c_5 = c_3(2\lambda_1^{1/2})^{-1}$. Hence it follows that

$$(3.46) \qquad \|\vec{u}(t)\|_{X_0}^2 \leqslant e^{2c_5 t} \|\vec{u}(0)\|_{X_0}^2.$$

Let us take the scalar product of (3.45) with $\partial_t u''(t)$, where $u''(t) \equiv Q_N u(t)$, and bring the result into the following form:

$$\frac{1}{2}\frac{d}{dt}(\|\partial_t u''(t)\|^2 + \|u''_x(t)\|^2) + \varepsilon\|\partial_t u''(t)\|^2 = -(\mathscr{E}(t) u(t), \partial_t u''(t)) =$$
$$= -\frac{d}{dt}(\mathscr{E}(t) u(t), u''(t)) + \left(\frac{d\mathscr{E}(t)}{dt} u(t) + \mathscr{E}(t) \partial_t u(t), u''(t)\right).$$

From this and (3.39)-(3.43) it follows that

$$(3.47)\quad \frac{d}{dt}\left[\frac{1}{2}\|\partial_t u''(t)\|^2+\frac{1}{2}\|u'_x(t)\|^2+(\mathscr{E}(t)\,u(t),\ u''(t))\right]+$$
$$+\varepsilon\|\partial_t u''(t)\|^2\leqslant[c_4\|(|\partial_t v(t)|+|\partial_t\tilde{v}(t)|)\,|u(t)|\,\|+c_3\|\partial_t u(t)\|]\,\|u''(t)\|\leqslant$$
$$\leqslant[c_4(\|\partial_t v(t)\|_{L_3(\Omega)}+\|\partial_t\tilde{v}(t)\|_{L_3(\Omega)})\,\|u(t)\|_{L_6(\Omega)}+$$
$$+c_3\|\partial_t u(t)\|]\,\|u''(t)\|\leqslant c_6\|\vec{u}(t)\|_{X_0}\|u''(t)\|.$$

Let us introduce the function

$$\Psi(t)\equiv\frac{1}{2}(\|\partial_t u''(t)\|^2+\|u''_x(t)\|^2)+(\mathscr{E}(t)\,u(t),\ u''(t))+\varepsilon_1(u''(t),\ \partial_t u''(t)),$$

where ε_1 is a small positive number to be chosen later. In view of (3.47) and (3.45),

$$(3.48)\quad \frac{d\Psi(t)}{dt}\leqslant-\varepsilon\|\partial_t u''(t)\|^2+\varepsilon_1\|\partial_t u''(t)\|^2+\varepsilon_1(u''(t),\ \partial_t^2 u''(t))+$$
$$+c_6\|\vec{u}(t)\|_{X_0}\|u''(t)\|=(\varepsilon-\varepsilon_1)\|\partial_t u''(t)\|^2+\varepsilon_1[-\varepsilon\|\partial_t u''(t)\|^2-$$
$$-\|u''_x(t)\|^2-(\mathscr{E}(t)\,u(t),\ u''(t))]+c_6\|\vec{u}(t)\|_{X_0}\|u''(t)\|;$$

$$(3.49)\quad \frac{d\Psi(t)}{dt}+\varepsilon_1\Psi(t)\leqslant$$
$$\leqslant-\left(\varepsilon-\frac{3\varepsilon_1}{2}+\varepsilon_1\varepsilon\right)\|\partial_t u''(t)\|^2-\frac{\varepsilon_1}{2}\|u''_x(t)\|^2+$$
$$+c_6\|\vec{u}(t)\|_{X_0}\|u''(t)\|+\varepsilon_1^2\|u''(t)\|\,\|\partial_t u''(t)\|\leqslant$$
$$\leqslant-\left(\frac{\varepsilon}{2}-\frac{3\varepsilon_1}{2}+\varepsilon_1\varepsilon\right)\|\partial_t u''(t)\|^2-\frac{\varepsilon_1}{2}\|u''_x(t)\|^2+$$
$$+c_6\|\vec{u}(t)\|_{X_0}\|u''(t)\|+\frac{\varepsilon_1^4}{2\varepsilon}\|u''(t)\|^2.$$

Let us take, for example, $\varepsilon_1=\varepsilon/3$, so that $\frac{\varepsilon}{2}-\frac{3\varepsilon_1}{2}+\varepsilon_1\varepsilon>0,\frac{\varepsilon_1^4}{2\varepsilon}=\varepsilon^3(162)^{-1}$. We now use the inequality

$$(3.50)\qquad \|w_x\|^2\geqslant\lambda_{N+1}\|w\|^2,$$

which holds for any $w\in Q_N D^1$. In particular, it holds for $u''(t)$. With the help of this inequality, it follows from (3.49) that

$$(3.51)\quad \frac{d\Psi(t)}{dt}+\varepsilon\Psi(t)\leqslant$$
$$\leqslant-\frac{\varepsilon}{6}\left(\frac{1}{2}-\frac{\varepsilon^2}{27}\lambda_{N+1}^{-1}\right)\|u''_x(t)\|^2+\frac{3}{\varepsilon}c_6^2\lambda_{N+1}^{-1}\|\vec{u}(t)\|_{X_0}^2.$$

Let us take N so large that

$$(3.52)\qquad \frac{1}{2}-\frac{\varepsilon^2}{27}\lambda_{N+1}^{-1}\geqslant 0.$$

Then the first term on the right-hand side of (3.51) can be neglected, and after the resulting inequality is integrated with respect to t we get

$$\Psi(t)\leqslant e^{-\varepsilon_1 t}\Psi(0)+\frac{3}{\varepsilon}c_6^2\lambda_{N+1}^{-1}e^{-\varepsilon_1 t}\int_0^t\|\vec{u}(\tau)\|_{\mathfrak{X}_0}^2 e^{\varepsilon_1\tau}\,d\tau. \tag{3.53}$$

Because of (3.46) it follows from (3.53) that

$$\Psi(t)\leqslant e^{-\varepsilon_1 t}\Psi(0)+c_6^2(\varepsilon_1\lambda_{N+1})^{-1}(\varepsilon_1+2c_5)^{-1}e^{2c_5 t}\|\vec{u}(0)\|_{\mathfrak{X}_0}^2, \tag{3.54}$$

where $\varepsilon_1=\varepsilon/3$. Furthermore,

$$\begin{aligned}\Psi(t)&\geqslant\frac{1}{2}\|\vec{u}''(t)\|_{\mathfrak{X}_0}^2-c_3\|u(t)\|\,\|u''(t)\|-\varepsilon_1\|u''(t)\|\,\|\partial_t u''(t)\|\geqslant\\ &\geqslant\frac{1}{2}\|\vec{u}''(t)\|_{\mathfrak{X}_0}^2-c_3\lambda_{N+1}^{-1/2}\|u(t)\|\,\|u''_x(t)\|-\varepsilon_1\lambda_{N+1}^{-1/2}\|u''_x(t)\|\,\|\partial_t u''(t)\|\geqslant\\ &\geqslant\left(\frac{1}{2}-\varepsilon_1\lambda_{N+1}^{-1/2}\right)\|\partial_t u''(t)\|^2+\\ &\quad+\frac{1}{2}\left(\frac{1}{2}-\varepsilon_1\lambda_{N+1}^{-1/2}\right)\|u''_x(t)\|^2-c_3^2\|u(t)\|^2\lambda_{N+1}^{-1},\end{aligned} \tag{3.55}$$

while

$$\begin{aligned}\Psi(0)&\leqslant\\ &\leqslant\frac{1}{2}\|\vec{u}''(0)\|_{\mathfrak{X}_0}^2+c_3\|u(0)\|\,\|u''(0)\|+\varepsilon_1\|u''(0)\|\,\|\partial_t u''(0)\|\leqslant c_7\|\vec{u}(0)\|_{\mathfrak{X}_0}^2.\end{aligned} \tag{3.56}$$

Let us impose another restriction on the choice on N:

$$\frac{1}{2}-\frac{\varepsilon}{3}\lambda_{N+1}^{-1/2}\geqslant\frac{1}{4}. \tag{3.57}$$

Then from (3.54)-(3.57) and (3.46) we deduce the estimate

$$\begin{aligned}\frac{1}{8}\|\vec{u}''(t)\|_{\mathfrak{X}_0}^2&\leqslant c_3^2\|u(t)\|^2\lambda_{N+1}^{-1}+c_7e^{-\varepsilon_1 t}\|\vec{u}(0)\|_{\mathfrak{X}_0}^2+\\ &+\frac{c_6^2}{\varepsilon_1\lambda_{N+1}(\varepsilon_1+2c_5)}e^{2c_5 t}\|\vec{u}(0)\|_{\mathfrak{X}_0}^2\leqslant\\ &\leqslant\|\vec{u}(0)\|_{\mathfrak{X}_0}^2\{e^{2c_5 t}\lambda_{N+1}^{-1}[c_3^2+9c_6^2\varepsilon^{-1}(\varepsilon+6c_5)^{-1}]+c_7e^{-\varepsilon t/3}\}.\end{aligned} \tag{3.58}$$

Let us choose $t=t_1$ so that

$$c_7e^{-\xi t_1/3}=(32)^{-1},\quad\text{that is,}\quad t_1=3\xi^{-1}\log(32c_7), \tag{3.59}$$

and take N so large that

$$e^{2c_5 t_1}\lambda_{N+1}^{-1}[c_3^2+9c_6^2\varepsilon^{-1}(\varepsilon+6c_5)^{-1}]\leqslant(32)^{-1}. \tag{3.60}$$

Then

$$\|\vec{u}''(t_1)\|_{\mathfrak{X}_0}^2\leqslant\frac{1}{2}\|\vec{u}(0)\|_{\mathfrak{X}_0}^2. \tag{3.61}$$

The inequalities (3.61) and (3.46) with $t=t_1$ guarantee that the conditions of Theorem 2.8 hold, and therefore we have the following theorem:

***Theorem* 3.7.** *The Hausdorff dimension of any set $\mathfrak{A}$ that is invariant with respect to the group $V_t : X_0 \to X_0$, $t \in \mathbf{R}$, and bounded in the metric of X_1 does not exceed some finite number determined by $\|\mathfrak{A}\|_{X_1} \equiv \sup_{\vec{v} \in \mathfrak{A}} \|\vec{v}\|_{X_1} \equiv c$, ε, Ω and c_3 of* (3.41).

Remark 3.1. From Remark 2.2 it follows that the fractal dimension of the set $\mathfrak{A}$ of Theorem 3.7 is also finite. Moreover, the set $\mathfrak{A}$, just as in Theorems 2.7 and 2.9, does not have to be an attractor in any sense whatsoever. Only its invariance and boundedness in the metric of X_1 are required. For groups $V_t : X \to X$, $t \in \mathbf{R}$, such sets can also be found by constructing ω-limit sets. For these sets, less information on the group is required than in the case of attractors. Namely, the invariance of $\omega(B)$ follows directly from the fact that the operators V_t are continuous for all $t \in \mathbf{R}$. The fact that $\omega(B)$ is bounded in X follows from the fact that the set $\gamma^+(B)$ is bounded in X. (The facts that $\omega(B)$ is non-empty and B is attracted to $\omega(B)$ are an entirely different matter, and are not implied by the conditions above.) For example, for the problem (3.1)–(3.3) the continuity of the group V_t, $t \in \mathbf{R}$, on X_0 holds already if conditions (3.23)–(3.25) are satisfied. Moreover, the corresponding semigroup $V_t : X_0 \to X_0$, $t \in \mathbf{R}^+$, is then bounded and has a bounded absorbing set B_0, which of course contains the set Z. Therefore $\omega(B_0)$ is a non-empty invariant bounded subset of X_0. All the other invariant bounded sets are contained in $\omega(B_0)$. If we choose a subset of B_0, $\widetilde{B}_0 \supset Z$, that is bounded in the metric of X_1, then $[\gamma^+(\widetilde{B}_0)]_{X_0}$ will be bounded in X_1, and $\omega(\widetilde{B}_0)$ will be an invariant set bounded in X_1. (All the properties of solution operators for the problem (3.1)–(3.3) listed here have already been mentioned above.) The finiteness of $d_H\omega(\widetilde{B}_0)$ is ensured by Theorem 3.7, and that of $d_f\omega(\widetilde{B}_0)$ by Remark 3.1.

Better upper bounds for $d_H(\cdot)$ and $d_f(\cdot)$ for sets $\mathfrak{A}$ in the space X_0 that are invariant and bounded in X_1 can be obtained by making use of [25] (see [44] and [50]).

The presence of a "good" Lyapunov function is not a necessary condition for a system to have a compact global B-attractor $\mathfrak{M}$. For example, we can introduce in (3.1) a term of the form $\sum_{i=1}^{n} a_i(x)v_{x_i}$ with $a_i(x)$ sufficiently small. This will destroy the Lyapunov function, but will not prevent the existence of a compact $\mathfrak{M}$. The general theorems of §§2, 3 cover very diverse equations and systems (not only of parabolic and hyperbolic type). In §1 I indicated a number of systems of equations of hydrodynamics for which attractors $\mathfrak{M}$ can be found with the help of the theorems of §2. To this one can add the equations of motion of Oldroyd fluids (see [45] and so on). The semigroups generated by the Navier-Stokes-Voigt equations belong to class 2, and for these existence of the attractor $\mathfrak{M}$ is guaranteed by theorems of §3 [46]. To this class there belong also the semigroups corresponding to semilinear equations with strong dissipation in which the dissipative term is of the form $-\varepsilon\Delta\partial_t v$, $\varepsilon > 0$. It appears that this was the first type of PDE for which a solution operator V_t was represented as the sum $W_t + U_t$ of a linear contraction operator W_t and a compact operator U_t. These equations are the subject of research by Webb, Massatt, and others ([47], [48], [46], and so on).

It is not possible for me to give here anything like a complete list of papers dealing with the search for attractors for PDE's. A great number of these have been published in recent years, and this number is growing all the time. Let me just stress again that the class of semigroups introduced in §3 (semigroups of class 2) includes all the semigroups for which the sets $\omega(B)$ are compact and attract B for all $B \subset \mathscr{B}$.

References

[1] O.A. Ladyzhenskaya, On the dynamical system generated by the Navier-Stokes equations, Zap. Nauch. Sem. LOMI **27** (1972), 91-114. MR **48** # 6720. = J. Soviet Math. **3** (1975), 458-479.

[2] ———, *Matematicheskie voprosy dinamiki vyazkoi neszhimaemoi zhidkosti,* 1st ed., Fizmatgiz, Moscow 1961, 2nd ed., Nauka, Moscow 1970. MR **27** # 5034a, **42** # 6442.
Translation: The mathematical theory of viscous incompressible flow, 2nd ed., Gordon and Breach, New York-London-Paris 1969. MR **40** # 7610.

[3] ———, Limit states for modified Navier-Stokes equations in three-dimensional space, Zap. Nauch. Sem. LOMI **84** (1979), 131-146. MR **81f**:35094. = J. Soviet Math. **21** (1983), 345-356.

[4] ———, V.A. Solonnikov, and N.N. Uraltseva, *Lineinye i kvazilineinye uravneniya parabolicheskogo tipa,* Nauka, Moscow 1967. MR **39** # 3159a.
Translation: Linear and quasi-linear equations of parabolic type, Translations of Math. Monographs vol. 23, Amer. Math. Soc. Providence, RI, 1968. MR **39** # 3159b.

[5] ——— and N.N. Uraltseva, A survey of results on the solubility of boundary-value problems for second-order uniformly elliptic and parabolic quasi-linear equations having unbounded singularities, Uspekhi Mat. Nauk **41**:5 (1986), 59-83. = Russian Math. Surveys **41**:5 (1986), 1-31.

[6] J.K. Hale and N. Stavrakakis, Limiting behaviour of linearly damped hyperbolic equations, LCDS Preprint 86-6.

[7] ———, Theory of functional differential equations, Springer-Verlag, Heidelberg-New York-Berlin 1977.
Translation: *Teoriya funktsional'no-differentsial'nykh uravnenii*, Mir, Moscow 1984.

[8] D. Henry, Geometric theory of semi-linear parabolic equations, Lecture Notes in Math. **840** (1981). MR **83j**:35130.
Translation: *Geometricheskaya teoriya polulineinikh parabolicheskikh uravnenii,* Mir, Moscow 1985.

[9] J. Mallet-Paret, Negatively invariant sets of compact maps and an extension of a theorem of Cartwright, J. Differential Equations **22** (1976), 331-348. MR **54** # 11378.

[10] C. Foiaş and G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova **39** (1967), 1-34. MR **36** # 6764.
[11] D. Ruelle, Measures describing a turbulent flow, Preprint IHES/P/79/313.
[12] ———, Large volume limit of the distribution of characteristic exponents in turbulence, Preprint IHES/P/82/45.
[13] C. Foiaş and R. Temam, Some analytic and geometric properties of the solutions of the Navier-Stokes equations, J. Math. Pures Appl., **58** (1979), 339-368. MR **81k**:35130.
[14] O.A. Ladyzhenskaya, A study of the Navier-Stokes equations in the case of stationary flow of an incompressible fluid, Uspekhi Mat. Nauk **14**:3 (1959), 75-97. MR **22** # 10437.
[15] R. Temam, Navier-Stokes equations and non-linear functional analysis, Preprint 82-T-14, Univ. de Paris-Sud.
[16] P. Constantin, C. Foiaş, and R. Temam, Attractors representing turbulent flows, Preprint 84-T-35, Univ. de Paris-Sud.
[17] A.V. Babin and M.I. Vishik, Attractors for the Navier-Stokes system and for parabolic equations, and estimates of their dimension, Zap. Nauch. Sem. LOMI **115** (1982), 3-15.
= J. Soviet Math. **28** (1985), 619-627.
[18] ——— and ———, Attractors of partial differential evolution equations and estimates of their dimension, Uspekhi Mat. Nauk **38**:4 (1983), 133-187.
= Russian Math. Surveys **38**:4 (1983), 151-213.
[19] ——— and ———, Upper and lower bounds of the dimension of attractors of evolution partial differential equations, Sibirsk. Mat. Zh. **24**:5 (1983), 15-30. MR **85j**:35088.
= Siberian Math. J. **24** (1985), 658-671.
[20] ——— and ———, Regular attractors of semigroups and evolution equations, J. Math. Pures Appl. **62** (1983), 441-491. MR **85g**:58058.
[21] ——— and ———, Maximal attractors of semigroups corresponding to evolution differential equations, Mat. Sb. **126** (1985), 397-419.
= Math. USSR-Sb. **54** (1986), 387-408.
[22] O.A. Ladyzhenskaya, On finite-dimensionality of bounded invariant sets for the Navier-Stokes equations and for other dissipative systems, Zap. Nauch. Sem. LOMI **115** (1982), 137-155. MR **84e**:35020.
= J. Soviet Math. **28** (1985), 714-726.
[23] Yu.S. Ilyashenko, Weakly contracting systems and attractors of the Galërkin approximations of the Navier-Stokes equations, Uspekhi Mat. Nauk **36**:3 (1981), 243-244.
[24] ———, On the dimension of attractors of k-contracting systems in an infinite-dimensional spaces, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1983**, no. 3, 52-59. MR **84m**:58080.
= Moscow Univ. Math. Bull. **38**:3 (1983), 61-69.
[25] A. Douady and J. Oesterlé, Dimension de Hausdorff des attracteurs, C.R. Acad. Sci. **290** (1980), 1135-1138. MR **82a**:58033.
[26] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachrichten **4** (1950-51), 213-231.
[27] O.A. Ladyzhenskaya, On the solution of non-stationary operator equations of various types, Dokl. Akad. Nauk SSSR **102** (1955), 207-210. MR **17** # 161.
[28] ———, On non-stationary operator equations and their applications to linear problems of mathematical physics, Mat. Sb. **45** (1958), 123-158. MR **22** # 12290.

[29] A.A. Kiselev and O.A. Ladyzhenskaya, On the existence and uniqueness of solutions of the non-stationary problem for a viscous incompressible fluid, Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 655-680.
[30] O.A. Ladyzhenskaya, Solution "in the large" of a boundary-value problem for the Navier-Stokes equation in the case of two space variables, Dokl. Akad. Nauk SSSR **123** (1958), 427-429.
[31] G. Métivier, Étude asymptotique des valeurs propres et de la fonction spectrale de problèmes aux limites, Thèse, Université de Nice 1976.
[32] V.I. Yudovich, Mathematical theory of stability of fluid flow, Doctoral dissertation, Institute of problems of mechanics of the Academy of Sciences of the USSR, Moscow 1972.
[33] O.A. Ladyzhenskaya and V.A. Solonnikov, On the principle of linearization and invariant sets for problems of magnetohydrodynamics, Zap. Nauch. Sem. LOMI **38** (1973), 46-93.
= J. Soviet Math. **8** (1977), 384-422.
[34] ———, On the closure of an elliptic operator, Dokl. Akad. Nauk SSSR **79** (1951), 723-725.
[35] ———, *Smeshannaya zadacha dlya giperbolicheskogo uravneniya* (The mixed problem for hyperbolic equations), Gostekhizdat, Moscow 1953.
[36] H. Triebel, Interpolation theory. Functional spaces. Differential operators, North Holland, Amsterdam-New York-Oxford 1978.
Translation: *Teoriya interpolyatsii. Funktsional'nye prostranstva. Differentsial'nye operatory*, Mir, Moscow 1980.
[37] O.A. Ladyzhenskaya, On the convergence of Fourier series defining the solution of the mixed problem for hyperbolic equations, Dokl. Akad. Nauk SSSR **85** (1952), 481-484.
[38] J.K. Hale, Asymptotic behaviour and dynamics in infinite dimensions, in: Non-linear differential equations, Res. Notes in Math. vol. 132, Pitman, London 1985, 1-42.
[39] A. Haraux, Two remarks on hyperbolic dissipative problems, Preprint no. 85029, Collège de France 1984.
[40] R. Strichartz, A priori estimates for the wave equation and some applications, J. Funct. Anal. **5** (1970), 150-183.
[41] H. Pecher, L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen, Math. Z. **150** (1976), 150-183.
[42] J. Ginibre and G. Velo, The global Cauchy problem for the non-linear Klein-Gordon equation, Math. Z. **189** (1985), 487-505.
[43] O.A. Ladyzhenskaya, On the attractors of non-linear evolution problems with dissipation, Zap. Nauch. Sem. LOMI **152** (1986), 72-85.
[44] J.M. Ghidaglia and R. Temam, Attractors for damped non-linear hyperbolic equations, Preprint 85-T-39, Univ. de Paris-Sud.
[45] A.A. Cotsiolis and A.P. Oskalov, On limiting behaviour and the attractor for the equations of the flow of Oldroyd fluids, Zap. Nauch. Sem. LOMI **152** (1986), 67-71. MR **87k**:35031.
[46] V.K. Kalantarov, On the attractors for some non-linear problems of mathematical physics, Zap. Nauch. Sem. LOMI **152** (1986), 50-54.
[47] G. Webb, Existence and asymptotic behaviour for strongly damped non-linear wave equations, Canad. J. Math. **32** (1980), 631-643.

[48] P. Massat, Limiting behaviour for strongly damped non-linear wave equations, J. Differential Equations **48** (1983), 334-349.
[49] V.I. Arnol'd, On some non-linear problems, Grafoord lectures, The Royal Swedish Acad. Sci., Stockholm 1982, 1-7.
[50] O.A. Ladyzhenskaya, On estimates of fractal dimension and the number of determining modes for invariant sets of dynamical systems, Zap. Nauch. Sem. LOMI **163**:19 (1987).

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