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BOUNDARY THEORY OF MARKOV PROCESSES (THE DISCRETE CASE)

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BOUNDARY THEORY OF MARKOV PROCESSES
(THE DISCRETE CASE)

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The paper contains a detailed account of the theory of Martin boundaries for Markov processes with a countable number of states and discrete time. The probabilistic method of Hunt is used as a basis. This method is modified so as not to go outside the limits of the usual notion of a Markov process. The generalization of this notion due to Hunt is discussed in the concluding section.

Contents

Introduction

The boundary theory of Markov processes permits the investigation of the "final" behaviour of the paths of such processes, that is, the behaviour as the time t tends to infinity (or to the moment of cut-off). Knowledge of the final behaviour is in its turn a prerequisite for the investigation of general boundary conditions (from the probabilistic point of view this is reduced to the study of possible continuations of the process after the moment of cut-off). Another important application of boundary theory is the description of all harmonic and superharmonic positive (excessive) functions connected with the process. This problem motivated the creation of the theory of the Martin boundary in 1941 [4]. Martin investigated the set of positive solutions of Laplace's equation in an arbitrary domain of euclidean space.

The probability interpretation of Martin's results was proposed by Doob [1]: these results are directly related to the Wiener process, but Doob proved that they can also be extended to discrete Markov chains.

A new approach to the theory of the Martin boundary was proposed by Hunt [6]. In the Martin-Doob theory first an integral representation of excessive functions is deduced by probability methods, and then from it a theorem on the final behaviour of the paths is obtained. Hunt proved a theorem on the final behaviour directly by means of probability arguments, and then, applying this theorem to h-processes, he obtained a simple derivation of the integral representation of excessive functions.

The reading of Hunt's important paper is made more difficult because it is written in terms of a generalization, due to the author, of the idea of a Markov process (approximate Markov chains).[1] This may give the impression that the success of the methods applied depends significantly on this generalization. Actually this is not so, and in this paper Hunt's method is modified so that we need not go outside the classes of usual Markov chains.

Problems of boundary theory admit a natural dual formulation.

Instead of harmonic (excessive) functions we can investigate harmonic (excessive) measures. In view of the self-adjointness of Laplace's operator in the case considered by Martin, this dual problem does not contain in itself anything really new. The situation changes in the general case, and now, instead of one Martin boundary, two are constructed in Doob's theory: the exit boundary and the entrance boundary. The role played by the exit boundary in the study of the final behaviour of the paths must now be played by its dual, the entrance boundary, in investigating the "initial" behaviour. However, to give this latter term a meaning, we have to widen the usual interpretation of a Markov chain. One of the possible extensions consists in considering stationary processes defined for values of the time from $-\infty$ to $+\infty$. For such processes the "initial" behaviour means the behaviour as $t \to -\infty$. Another possibility is to consider the generalization of Markov processes proposed by Hunt: Hunt processes begin at a random instant $\zeta \geqslant -\infty$, and the "initial" behaviour for these is the behaviour as $t \to \zeta$. It is not necessary to construct the dual boundary again, since it can be obtained from the previously constructed one by inversion of the time in the process.

[1] Chapter 10 of the recent book of Kemeny, Snell and Knapp [3] is written in these terms. This chapter contains a well-considered and polished account of Hunt's paper.

These questions are dealt with in the concluding sections of this paper.

Thus, the reader can gain a first acquaintance with the idea of inversion of time and its application to boundary theory from the simpler and more usual material of stationary processes. At the same time it must be emphasized that stationary processes are not exhausted by Hunt's theory, since they do not satisfy the Hunt requirement of finiteness of the mean number of hits on each state. The actual problem remaining is the construction of a theory including both stationary processes and Hunt processes.

The present paper treats (as also the papers of Doob and Hunt) only discrete Markov chains. It can serve as an introduction to boundary theory for general Markov processes, to which the author intends to devote a subsequent paper.

For an understanding of this paper only a knowledge of elementary probability and measure theory is needed.

§1. Harmonic and excessive functions and measures

We take as starting point a *transition function* in a countable space E. This is a non-negative function $p(x, y)$, $(x, y \in E)$, satisfying the condition

$$\sum_y p(x, y) \leqslant 1 \qquad (x \in E). \tag{1}$$

Let f and μ be any functions on E. We denote by $\mathbf{P}f$ and $\mu\mathbf{P}$ functions given by the formulae[1]

$$\left.\begin{aligned} \mathbf{P}f(x) &= \sum_y p(x, y) f(y) \qquad (x \in E), \\ (\mu\mathbf{P})(y) &= \sum_x \mu(x) p(x, y) \qquad (y \in E). \end{aligned}\right\} \tag{2}$$

Since the right-hand sides contain infinite series, these formulae do not have a meaning for all f and μ. However, they have a meaning if f and μ are non-negative. (By a non-negative function we always mean are with values in the extended number half-line $[0, +\infty]$.)

The transition function $p(x, y)$ can be interpreted as a matrix of countable order. Here $\mathbf{P}f$ is the product of this matrix by a countable vector column f, and $\mu\mathbf{P}$ is the product of a vector row by $\mathbf{P}$. From another point of view, the first formula in (2) describes the effect of the kernel $p(x, y)$ on functions, and the second describes its effect on measures in E.[2] The integral of f with respect to the measure μ is denoted by the scalar product

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[1] If the domain of summation is not indicated, this means that it is E.

[2] We have to deal almost exclusively with non-negative f and μ. Note that in the general case the first formula in (2) has a meaning if f is bounded, and the second if $\sum_x |\mu(x)| < \infty$, that is, if the signed measure has bounded variation.

$$(f, \mu) = \sum_y f(y) \mu(y).$$

If $(f, \mu) < \infty$, then we say that f is *μ-integrable*, and also that μ is *f-finite*.

A non-negative function f is called *excessive* if $Pf \leqslant f$,[1] and *harmonic* if $f(x) < \infty$ for all x and $Pf = f$. Similarly, a measure is called *excessive* if $\mu P \leqslant \mu$, and *harmonic* if $\mu(x) < \infty$ for all x and $\mu P = \mu$.

One of the central problems before us is the description of all harmonic and excessive functions and measures connected with the transition function $p(x, y)$. It is expedient here to consider only γ-integrable functions h and l-finite measures ν, where γ and l are, respectively, a previously selected measure and function on E (they are called *standard*). On fundamental interest is the case when $p(x, y)$ is transient (see the definition in §3). In this case we are able to attach to each point $y \in E$ a γ-integrable excessive function k_y and a l-finite excessive measure κ_y. By means of the kernels $k_y(x)$ and $\kappa_y(x)$ we construct two compactifications E^* and $\hat{E}$ of E, such that $k_y(x)$ is extended for each $x \in E$ continuously to E^*, and $\kappa_y(x)$ is extended continuously to $\hat{E}$. From the sets $E^* \setminus E$ and $\hat{E} \setminus E$ we single out Borel sets B and $\hat{B}$, where k_y is a harmonic function and $(k_y, \gamma) = 1$, for $y \in B$, and κ_y is a harmonic measure and $(l, \kappa_y) = 1$ for $y \in \hat{B}$.

It can be proved that every γ-integrable excessive function h is representable uniquely in the form

$$h(x) = \int_{E \cup B} k_y(x) \mu_h(dy),$$

and any l-finite excessive measure ν is expressible uniquely in the form

$$\nu(x) = \int_{E \cup \hat{B}} \kappa_y(x) \mu^\nu(dy).$$

Here μ_h and μ^ν are finite measures which are determined uniquely by h and ν, respectively. We call them *spectral measures*.

The set B is called a *space of exits*, and the set $\hat{B}$ a *space of entries*. The origin of these terms becomes clear in the following section.

§2. Markov processes

Suppose that a particle moves in a space E, going through a sequence of states $a_0, a_1, a_2, \ldots$ The path $a_0 a_1 a_2 \ldots$ may be terminate or may continue unboundedly. The set of all (terminating or non-terminating) paths is denoted by Ω. The set of all non-terminating paths is denoted by Ω_∞.

[1] By $f \leqslant g$ we mean that $f(x) \leqslant g(x)$ for all $x \in E$.

Among the subsets of Ω the so-called simple sets play a special role. A *simple set* $[a_0a_1a_2 \ldots a_n]$ is composed of all paths beginning with the states $a_0, a_1, \ldots, a_n$ and continuing in any manner after the moment n. We denote by $\mathscr{F}$ the σ-algebra in Ω generated by all simple sets.

We depend on the following theorem on measures in Ω.

THEOREM A. *Suppose that for any n and any $a_0, a_1, \ldots, a_n \in E$ a non-negative number $p(a_0, a_1, \ldots, a_n)$ is given, where*

$$\sum_{a_n} p(a_0, a_1, \ldots, a_n) \leqslant p(a_0, a_1, \ldots, a_{n-1}). \tag{3}$$

Then there exists a measure $\mathbf{P}$, *which is moreover unique, on the σ-algebra $\mathscr{F}$ such that*

$$\mathbf{P}[a_0a_1 \ldots a_n] = p(a_0, a_1, \ldots, a_n).$$

This theorem will be proved in the Appendix.[1]

From Theorem A it follows that for any measure ν on E with $\nu(x) < \infty$ for all x, there exists a measure $\mathbf{P}_\nu$ on Ω such that

$$\mathbf{P}_\nu [a_0a_1 \ldots a_n] = \nu(a_0)\, p(a_0, a_1) \ldots p(a_{n-1}, a_n). \tag{4}$$

An important role is played by the particular case when ν is the unit measure concentrated at the point x (when $\nu(y) = \delta(x, y)$, where $\delta(x, y) = 1$ if $x = y$, $\delta(x, y) = 0$ if $x \neq y$). The corresponding measure in Ω is denoted by $\mathbf{P}_x$, so that

$$P_x [a_0a_1 \ldots a_n] = \delta(x, a_0)\, p(a_0, a_1) \ldots p(a_{n-1}, a_n), \tag{4'}$$

$\mathbf{P}_x$ is the probability measure concentrated on the paths starting from x. We note that for any ν.

$$\mathbf{P}_\nu = \sum_x \nu(x)\, \mathbf{P}_x$$

and that $\mathbf{P}_\nu(\Omega) = \nu(E)$.

Each measure in the space of paths Ω determines a random process.[2] The process determined by the measure $\mathbf{P}_\nu$ is called the *Markov process with initial distribution* ν *and transition function* $p(x, y)$. The process corresponding to the measure $\mathbf{P}_x$ is called the *Markov process with initial state* x *and transition function* $p(x, y)$.

One of the basic results of boundary theory states that almost every non-terminating path tends to some point of the exit space B. The measure of the set of paths for which this limit belongs to the Borel set $\Gamma \subseteq B$ is

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1 The necessity of the condition (3) is obvious, since $[a_0, a_1, \ldots, a_n] \subseteq [a_0, a_1, \ldots, a_{n-1}]$ for any a_n, and since distinct $[a_0, a_1, \ldots, a_n]$ do not intersect.

2 If $p(\Omega) = 1$, then $\mathbf{P}(A)$ can be interpreted as the probability that the trajectories of motion belong to A. In the general case $\mathbf{P}(A)$ may prove to be greater than 1 and even equal to ∞.

$$\int_{\Gamma} (k_y, \nu)\, \mu_1 (dy),$$

where k_y is the harmonic function corresponding to the point $y \in B$, and μ_1 is the spectral measure of the excessive function 1.

To explain the role of the exit space, we have to introduce into the discussion paths without beginning or end. These are functions a_t with range in E defined for all integers t from $-\infty$ to $+\infty$. The set of such paths is denoted by $\hat{\Omega}$. We use the term simple sets in $\hat{\Omega}$ for the sets $[a_m, a_{m+1}, \ldots, a_n]_m^n$ consisting of all paths passing at the moments $m, m+1, \ldots, n$ through the points $a_m, a_{m+1}, \ldots, a_n$. (Before the moment m and after the moment n they can behave arbitrarily). We denote by $\hat{\mathscr{F}}$ the σ-algebra in $\hat{\Omega}$ generated by all simple sets.

For the construction of measures in the space $\hat{\Omega}$ we can use the following modification of Theorem A.

THEOREM B. *Suppose that for any integers $m \leqslant n$ and any $a_m, a_{m+1}, \ldots, a_n \in E$, non-negative numbers $p_m^n(a_m, a_{m+1}, \ldots, a_n)$ are given, where*

$$\sum_{a_n} p(a_m, a_{m+1}, \ldots, a_n) = p(a_m, a_{m+1}, \ldots, a_{n-1}), \tag{3'}$$

$$\sum_{a_m} p(a_m, a_{m+1}, \ldots, a_n) = p(a_{m+1}, \ldots, a_n). \tag{3''}$$

Then there exists a unique measure P *on the σ-algebra $\hat{\mathscr{F}}$ such that*

$$\mathbf{P}\,[a_m a_{m+1} \ldots a_n]_m^n = p_m^n (a_m, a_{m+1}, \ldots, a_n).$$

The necessity of the conditions (3′), (3″) is evident. For the proof of Theorem B see the Appendix.

We suppose that the transition function $p(x, y)$ satisfies (1) with the equality sign and that ν is a harmonic measure. Then the function

$$p_m^n (a_m, a_{m+1}, \ldots, a_n) = \nu (a_m)\, p(a_m, a_{m+1}) \ldots p(a_{n-1}, a_n)$$

satisfies the conditions (3′) – (3″), and by Theorem B there exists a measure P_ν on $\hat{\mathscr{F}}$ such that

$$\mathbf{P}_\nu\, [a_m a_{m+1} \ldots a_n]_m^n = \nu (a_m)\, p(a_m, a_{m+1}) \ldots p(a_{n-1}, a_n). \tag{4''}$$

A random process determined by P_ν in $\hat{\Omega}$ is called *a stationary Markov process with stationary distribution ν and transition function $p(x, y)$.*

In boundary theory it is proved that for such a process almost all paths converge as $t \to -\infty$ to some point of the space of entrances $\hat{B}$. The measure of the set of paths for which this limit belongs to the Borel set $\Gamma \subseteq \hat{B}$ is

$$\int_{\hat{\Gamma}} (1, \varkappa_y)\, \mu^\nu (dy),$$

where $\varkappa_y$ is the harmonic measure corresponding to the point $y \in \hat{B}$ and μ^ν is the spectral measure for ν.

This result can be further generalized in several directions.

Random variables connected with Markov processes are $\hat{\mathcal{F}}$-measurable functions defined on Ω or on some subset of this space (or $\hat{\mathcal{F}}$-measurable functions on $\hat{\Omega}$ or a subset of $\hat{\Omega}$). The integral of such a function ξ on its domain of definition with respect to the measure $\mathbf{P}_\nu$ is denoted by $M_\nu \xi$, and with respect to $\mathbf{P}_x$ by $M_x \xi$.

Here are some examples.

ζ is the terminal moment of a path: if the last moment at which the path ω is defined is n, then $\zeta(\omega) = n$; if the path does not terminate, then $\zeta(\omega) = +\infty$.

x_n is the position of a particle at the moment n. This function is defined on the set $\{\omega : \zeta(\omega) \geqslant n\}$. In the case of a stationary process

$$\mathbf{P}_\nu\, [x_n = y] = \nu\, (y)$$

for any n. For a process with the initial distribution ν

$$\mathbf{P}_\nu\, [x_n = y] = \sum_z \mathbf{P}_\nu\, [x_{n-1} = z]\, p\,(z, y). \tag{5}$$

To prove this equation it suffices to note that $\{\omega: x_n = y\} = \bigcup_z \{\omega: x_{n-1} = z, x_n = y\}$, to decompose the sets occurring here into simple sets, and to use (4).

We put

$$p\,(n, x, y) = \mathbf{P}_x\,\{x_n = y\}.$$

From (5) it follows that

$$p\,(n, x, y) = \sum_z p\,(n-1, x, z)\, p\,(z, y),$$

and in view of the obvious relation

$$\sum_y \delta\,(x, y) = 1$$

we have

$$M_x f\,(x_n) = M_x \sum_y \delta\,(x_n, y)\, f\,(x_n) = \sum_y M_x \delta\,(x_n, y)\, f\,(y) = \sum_y p\,(n, x, y)\, f\,(y) = \mathbf{P}^n f\,(x), \tag{6}$$

where $\mathbf{P}$ is an operator given by the first formula in (2). (6) is valid also for $n = 0$, if $\mathbf{P}^0$ is taken as the unit operator.

§3. The Green's function

We return to the problem raised in §1 of describing all excessive functions corresponding to a transition function $p(x, y)$. From (1) it follows that the non-negative constants always belong to the set of excessive functions. It may happen that no other excessive functions exist.

For example, let E be the set of all integers and $p(x, y) = \frac{1}{2}$ if $|x - y| = 1$, and $p(x, y) = 0$ for remaining pairs x, y. (The corresponding Markov processes are called *simple random walks.*) It is evident that

$$\mathbf{P}f(x) = \frac{1}{2} f(x+1) + \frac{1}{2} f(x-1),$$

and the condition that f is an excessive function can be written

$$\varphi(x+1) \leqslant \varphi(x),$$

where $\varphi(x) = f(x+1) - f(x)$. For any natural number k

$$f(x+k) = f(x) + \varphi(x) + \varphi(x+1) + \ldots + \varphi(x+k-1) \leqslant \leqslant f(x) + k\varphi(x),$$

$$f(x) = f(x-k) + \varphi(x-k) + \ldots + \varphi(x-1) \geqslant f(x-k) + k\varphi(x).$$

Since f is non-negative, it follows from the first inequality that $f(x) \geqslant -k\varphi(x)$ and from the second that $f(x) \geqslant k\varphi(x)$. Since k is arbitrary, it follows that $\varphi(x) = 0$, and therefore f is a constant.

Relying on the notion of a Green's function we derive a class of processes for which sufficiently many excessive functions exist. The *Green's function* is defined by the series

$$g(x, y) = \sum_{n=0}^{\infty} p(n, x, y). \tag{7}$$

The process is called *transient* if $g(x, y) < \infty$ for arbitrary x and y.

We note that by (5) $p(n, x, y) = \mathsf{P}_x\{x_n = y\} = M_x\delta(x_n, y)$. Hence

$$g(x, y) = M_x \sum_{n=0}^{\zeta} \delta(x_n, y). \tag{8}$$

Under the sign of mathematical expectation there stands the number of paths that go through the point y. The condition of being transient implies that this number is almost certainly finite. Thus, for a transient process almost all paths go only a finite number of times through one and the same state. Hence, if the states are enumerated in any order, for almost all non-terminating paths the number of states tends to infinity.

The simple random walk considered above has the opposite property: almost all paths go infinitely often through any point.[1] Processes with this property are called *recurrent*. It can be proved that every connected Markov process is either transient or recurrent. (We say that a Markov

[1] See, for example, [5]. Ch. XIII, §3.

process is connected if for any two states x and y there exists n such that $p(n, x, y) > 0$; in other words, if $g(x, y) > 0$ for any x and y.) For any recurrent process, as for the simple random walk, there do not exist non-constant excessive functions. *Henceforth, without saying so each time, we only discuss transient processes.*

The Green's function corresponds to the operators

$$Gf(x) = \sum_y g(x, y) f(y),$$

$$\mu G(y) = \sum_x \mu(x) g(x, y).$$

From (6) and (7) it is clear that

$$G = \sum_{n=0}^{\infty} \mathbf{P}^n. \tag{9}$$

Hence, for non-negative f and μ

$$f + \mathbf{P}Gf = Gf, \qquad \mu + \mu G\mathbf{P} = \mu G. \tag{10}$$

Therefore it is evident that Gf is an excessive function and μG an excessive measure.

We put $\delta_y(x) = \delta(x, y)$. It is obvious that

$$g(x, y) = (G\delta_y)(x) = (\delta_x G)(y).$$

Hence $g(x, y)$ is an excessive function of x for fixed y and an excessive measure with respect to y for fixed x. Thus, the Green's function permits us to connect an excessive measure and an excessive function with each point of E. It is this initial store of excessive functions and measures from which subsequently all excessive functions and measures are obtained.

We derive one important property of the Green's function.

LEMMA 1. *For any states x and y*

$$g(x, y) = \pi(x, y) g(y, y), \tag{11}$$

where $\pi(x, y) = \mathbf{P}_x\{x_n = y \text{ for some } n\}$ is the probability of reaching y starting from x.

PROOF. We put

$$A_m = \{x_0 \neq y,\ x_1 \neq y,\ \ldots,\ x_{m-1} \neq y,\ x_m = y\}.$$

We note that

$$\mathbf{P}_x\{A_m,\ x_{m+k} = y\} = \mathbf{P}_x(A_m)\, p(k, y, y). \tag{12}$$

To see this we have to decompose the set $\{A_m, x_{m+k} = y\}$ into simple sets and use (4).

The sum on the right in (8) is evidently equal to[1]

[1] χ_A denotes the indicator of A, that is, the function equal to 1 on A and to 0 outside A.

$$\sum_{m=0}^{\infty}{}' \chi_{A_m} \sum_{n=m}^{\zeta} \delta(x_n, y)^{1)}.$$

Hence

$$g(x, y) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} M_x \chi_{A_m} \delta(x_n, y) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{P}_x \{A_m, x_{n+k} = y\}.$$

Bearing (12) and (7) in mind we obtain (11).

REMARK. We put $B_k^m = \{x_n = z \text{ for some } n \in [m, m+k]\}$. Decomposing the set $A_m \cap B_k^m$ into simple sets, we can prove that $\mathbf{P}_x \{A_m, B_k^m\} = \mathbf{P}_x(A_m) \mathbf{P}_y(B_k^0)$. Letting $k \to \infty$, we obtain $\mathbf{P}_x(A_m, B_\infty^m) = \mathbf{P}_x(A_m) \mathbf{P}_y(B_\infty^0)$. Hence

$$\pi(x, z) \geqslant \mathbf{P}_x \{ \bigcup_{m=0}^{\infty} [A_m \cap B_\infty^m] \} = \sum_{m=0}^{\infty} \mathbf{P}_x(A_m) \mathbf{P}_y(B_\infty^0) = \pi(x, y) \pi(y, z). \quad (13)$$

This remark will be used in §9.

§4. Supermartingales

The investigation of excessive functions and paths of Markov processes is conducted most conveniently by means of the apparatus of supermartingales. In this section we introduce the notion and present some properties of supermartingales. The presentation will be in a most elementary form, fully sufficient, however, for our purpose.

DEFINITION. Let P be a measure on the σ-algebra F in the space Ω. Suppose that in Ω there are given F-measurable functions $y_0, y_1, \ldots, y_N$ with values belonging to a countable set E and numerical functions $z_0, z_1, \ldots, z_N$. We say that $z_0, z_1, \ldots, z_N$ is *a supermartingale with respect to* $y_0, y_1, \ldots, y_N$ if for any $n = 0, 1, \ldots, N$,

1) z_n is a function of $y_0, y_1, \ldots, y_n : z_n = f_n(y_0, y_1, \ldots, y_n)$;

2)[1] for any $a_0, a_1, \ldots, a_{n-1}$ из E

$$\sum_{a_n} \mathbf{P}\{y_0 = a_0, \ldots, y_{n-1} = a_{n-1}, y_n = a_n\} f_n(a_0, a_1, \ldots, a_n) \leqslant$$
$$\leqslant \mathbf{P}\{y_0 = a_0, \ldots, y_{n-1} = a_{n-1}\} f_{n-1}(a_0, a_1, \ldots, a_{n-1}). \quad (14)$$

From this definition the following two properties follow at once:

4.A. If d is a constant, then, together with $\{z_n\}$ the sequence[2] $\{z_n \wedge d\}$ is also a supermartingale with respect to $\{y_n\}$.

4.B. For any non-negative function φ

$$M\varphi(y_0, \ldots, y_{n-1}) z_n \leqslant M\varphi(y_0, \ldots, y_{n-1}) z_{n-1}.$$

1 In terms of conditional expectation condition 2) can be restated in the form $M(z_n \mid y_0, y_1, \ldots, y_{n-1}) \leqslant z_{n-1}$ almost certainly.

2 We denote by $a \wedge b$ the smaller of the two numbers a and b, and by $a \vee b$ the larger.

To deduce 4.B from 2) it is sufficient to note that

$$\varphi(y_0, \ldots, y_{n-1}) = \sum_{a_0, \ldots, a_{n-1}} \varphi(a_0, \ldots, a_{n-1}) \delta(a_0, y_0) \ldots \delta(a_{n-1}, y_{n-1}).$$

The most important property of supermartingales is stated in terms of Markov moments. A random variable[1] τ, taking the values 0, 1, 2, ..., is called a Markov moment (with respect to the sequence $y_0, y_1, y_2, \ldots$) if for any n

$$\delta(\tau, n) = \varphi_n(y_0, \ldots, y_n) \tag{15}$$

(φ_n is some function). Intuitively this definition means that observing the values $y_0, y_1, \ldots, y_n, \ldots$, we can answer uniquely until the moment n the question whether the equation $\tau = n$ is true. It is easy to verify that along with τ the function $\tau \vee m$ is also a Markov moment, where m is a non-negative integer.

LEMMA 2. *Let $z_0, z_1, \ldots, z_N$ be a supermartingale with respect to $y_0, y_1, \ldots, y_N$ and let two Markov moments $\sigma \leqslant \tau \leqslant N$ be given. Then*

$$Mz_\sigma \leqslant Mz_\tau. \tag{16}$$

PROOF. First we prove that if the Markov moment satisfies $n \leqslant \tau \leqslant N$, then for any non-negative function φ

$$M\varphi(y_0, \ldots, y_n) z_\tau \leqslant M\varphi(y_0, \ldots, y_n) z_n. \tag{17}$$

This is obvious for $n = N$. Hence it is sufficient to verify that if it holds for $n = m$, then it holds for $n = m - 1$. Thus, let $m - 1 \leqslant \tau \leqslant N$. We have

$$M\varphi(y_0, \ldots, y_{m-1}) z_\tau = M\varphi(y_0, \ldots, y_{m-1}) \delta(\tau, m-1) z_{m-1} + \\ + M\varphi(y_0, \ldots, y_{m-1}) [1 - \delta(\tau, m-1)] z_{\tau \vee m}. \tag{18}$$

By (15) we have $\delta(\tau, m-1) = \tilde{\varphi}(y_0, \ldots, y_{m-1})$, and applying the inductive hypothesis to the Markov moment $\tau \vee m \geqslant m$, we find that the second term in (18) does not exceed

$$M\varphi(y_0, \ldots, y_{m-1}) [1 - \delta(\tau, m-1)] z_m.$$

By 4.B the last expression is not diminished if z_m is replaced by z_{m-1}. Making this change and substituting the estimate so obtained in (18) we see that (17) holds for $n = m - 1$.

To complete the proof of the lemma we note that by (17) and (18)

$$M\delta(\sigma, n) z_\tau = M\delta(\sigma, n) z_{\tau \vee n} \leqslant M\delta(\sigma, n) z_n = M\delta(\sigma, n) z_\sigma.$$

Summing this inequality for $n = 0, 1, \ldots, N$, we obtain (16).

Relying on Lemma 2 we now prove a fundamental lemma about the number of crossings of a fixed interval $[c, d]$ for a positive supermartingale.

[1] In certain cases it is useful also to allow the value $+\infty$ for τ. Here, as before, it is required that (15) be satisfied for all finite n.

The *number of down-crossings of* $[c, d]$ in the sequence $z_0, z_1, \ldots, z_N$ is the largest number k for which numbers $0 \leqslant t_1 < t_2 < \ldots < t_{2k-1} < t_{2k} \leqslant N$ can be chosen so that $z_{t_1} \geqslant d,\ z_{t_2} \leqslant c,\ z_{t_3} \geqslant d,\ z_{t_4} \leqslant c,\ \ldots,\ z_{t_{2k-1}} \geqslant d,\ z_{t_{2k}} \leqslant c.$

LEMMA 3. *Suppose that the non-negative random variables* $z_0, z_1, \ldots, z_N$ *form a supermartingale with respect to* $y_0, y_1, \ldots, y_N$. *Then the number of down-crossings of* $[c, d]$ *in the sequence* $z_0, z_1, \ldots, z_N$ *satisfies the inequality*

$$M\nu \leqslant \frac{1}{d-c} M z_0. \tag{19}$$

PROOF. We put $\tau_0 = 0$ and define τ_n $(n = 1, 2, \ldots)$ inductively as follows: τ_n for odd n is the smallest value $k \geqslant \tau_{n-1}$ for which $z_k > d$, or, if there are no such values of k, then $\tau_n = N$; τ_n for even n is the smallest value $k \geqslant \tau_{n-1}$ for which $z_k \leqslant c$, or, if there are no such values of k, then $\tau_n = N$. It is easily verified that $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ are Markov moments and $\tau_n = N$ for $n \geqslant 2\nu + 2$.

According to 4.A. $\tilde{z}_n = z_n \wedge d$ is a supermartingale. We choose m so that $2m \geqslant N$, and put

$$S = (\tilde{z}_{\tau_1} - \tilde{z}_{\tau_2}) + (\tilde{z}_{\tau_3} - \tilde{z}_{\tau_4}) + \ldots + (\tilde{z}_{\tau_{2\nu-1}} - \tilde{z}_{\tau_{2\nu}}) + \\ + (\tilde{z}_{\tau_{2\nu+1}} - \tilde{z}_{\tau_{2\nu+2}}) + \ldots + (\tilde{z}_{\tau_{2m-1}} - \tilde{z}_{\tau_{2m}}).$$

We note that

$$\tilde{z}_{\tau_1} = d, \quad \tilde{z}_{\tau_2} \leqslant c, \quad \tilde{z}_{\tau_3} = d, \quad \tilde{z}_{\tau_4} \leqslant c, \ \ldots, \ \tilde{z}_{\tau_{2\nu-1}} = d, \quad \tilde{z}_{\tau_{2\nu}} \leqslant c, \\ \tilde{z}_{\tau_{2\nu+1}} \geqslant \tilde{z}_{\tau_{2\nu+2}} = \tilde{z}_{\tau_{2\nu+3}} = \ldots = \tilde{z}_{\tau_{2m}}.$$

Therefore

$$S \geqslant \nu (d-c). \tag{20}$$

On the other hand,

$$S = \tilde{z}_{\tau_1} + (\tilde{z}_{\tau_3} - \tilde{z}_{\tau_2}) + \ldots + (\tilde{z}_{\tau_{2m-1}} - \tilde{z}_{\tau_{2m-2}}) - \tilde{z}_{\tau_{2m}}.$$

By Lemma 2,

$$M\tilde{z}_0 = M\tilde{z}_{\tau_0} \geqslant M\tilde{z}_{\tau_1} \geqslant M\tilde{z}_{\tau_2} \geqslant \ldots \geqslant M\tilde{z}_{\tau_{2m-1}}.$$

Noting that $\tilde{z}_{\tau_{2m}} \geqslant 0$, we find

$$MS \leqslant M\tilde{z}_{\tau_1} \leqslant M\tilde{z}_0 \leqslant M z_0. \tag{21}$$

Now (19) follows from (20) and (21).

§5. Excessive functions and supermartingales

Let f be a non-negative function in the space E and let $x_0, x_1, x_2, \ldots$ be a path of a Markov process with initial state x and transition function $p(x, y)$. We add to E one further point, which we denote by $*$, and we take $x_n = *$ if $n > \zeta$. We put $f(*) = 0$. With these conventions, the functions $x_n(\omega)$ and $f(x_n(\omega))$ are defined for all n and ω. We show that if f is excessive, then the sequence $f(x_0), f(x_1), \ldots, f(x_N)$ is a supermartingale with respect to $x_0, x_1, \ldots, x_N$. For if at least one of the points $a_0, a_1, \ldots, a_n$ is $*$, then all subsequent ones also are $*$. In this case $f(a_n) = 0$. Hence the left-hand side of (14) is zero, while the right is non-negative. If all the points $a_0, a_1, \ldots, a_n \in E$, then by (4) we can write (14) in the form

$$\sum_{a_n} \delta(x, a_0) p(a_0, a_1) \ldots p(a_{n-1}, a_n) f(a_n) \leqslant$$
$$\leqslant \delta(x, a_0) p(a_0, a_1) \ldots p(a_{n-2}, a_{n-1}) f(a_{n-1}).$$

For $n = 1$ these inequalities coincide with the condition that f is excessive, and from their validity for $n = 1$ the validity for all n follows.

Let ν_N be the number of down-crossings of $[c, d]$ in the sequence $f(x_0), f(x_1), \ldots, f(x_N)$. By Lemma 3 of §4

$$M_x \nu_N \leqslant \frac{1}{d-c} M_x f(x_0) = \frac{f(x)}{d-c}. \tag{22}$$

Now let ν be the number of down-crossings of $[c, d]$ in the infinite sequence $f(x_0), f(x_1), \ldots$ Evidently, $\nu_N \uparrow \nu$, and hence it follows from (22) that

$$M_x \nu \leqslant \frac{f(x)}{d-c}.$$

We assume that $f(x) < \infty$. Then $M_x \nu < \infty$ and consequently $\nu < \infty$ ($\mathbf{P}_x$.a.e.).

However, it is easy to see that the following elementary proposition is true.

If a numerical sequence makes only a finite number of down-crossings of any interval $[c, d]$ with rational ends, then this sequence tends to a finite or infinite limit.

By what has been proved this theorem is applicable to $f(x_0), f(x_1), \ldots,$ along almost all paths $x_0, x_1, \ldots$ Thus, almost certainly there exists the limit

$$\xi = \lim_{n\to\infty} f(x_n).$$

By Fatou's lemma, from the inequality

$$M_x f(x_n) = \mathbf{P}^n f(x) \leqslant f(x)$$

it follows that

$$M_x \xi \leqslant f(x).$$

Hence ξ is almost surely finite.

Evidently $\xi = 0$ if $\zeta < \infty$. Hence there is interest only in the value of ξ on the set Ω_∞ of all non-terminating paths.

So we have proved the following theorem.

THEOREM 1. *If f is an excessive function and if $f(x) < \infty$, then* *the finite limit*

$$\lim_{n\to\infty} f(x_n)$$

exists $\mathbf{P}_x$*.a.e. on* Ω_∞.

We leave it to the reader to show that if $f(x) = \infty$, then $\mathbf{P}_x$*.a.e.* on Ω_∞ one of two possibilities holds: either $f(x_n) = +\infty$ for all n, or $f(x_n)$ tends to a finite limit.

In the following sections analogous properties will be established for the densities of excessive measures.

§6. Position of a particle before the moment of leaving a set D

Let $D \subseteq E$. We define *the moment of leaving the set D as*

$$\tau = \sup\{t : x_t \in D\}.$$

The random variable τ takes the values 0, 1, 2, ... and the value $+\infty$. It is taken to be undefined if $x_t \bar{\in} D$ for all t.

We put

$$L_D(x) = \mathbf{P}_x\{\tau = 0\} = \mathbf{P}_x\{x_0 \in D,\ x_t \bar{\in} D \text{ for } t > 0\}$$

We note that

$$\mathbf{P}_x\{x_\tau = y\} = \sum_{m=0}^{\infty} \mathbf{P}_x\{\tau = m,\ x_m = y\} = \\ = \sum_{m=0}^{\infty} p(m,\ x,\ y) L_D(y) = g(x,\ y) L_D(y). \quad (23)$$

It is clear that

$$\sum_y g(x,\ y) L_D(y) = \sum_y \mathbf{P}_x\{x_\tau = y\} \leqslant 1. \quad (24)$$

Let n be a non-negative integer. We investigate the distribution of the point $x_{\tau-n}$. This point is not defined if $\tau < n$ or $\tau = +\infty$ or if τ is not defined. In all three cases we put $x = *$. Let $a_0, a_1, \ldots, a_n \in E$. We have

$$\mathbf{P}_x\{x_\tau = a_0,\ x_{\tau-1} = a_1,\ \ldots,\ x_{\tau-n} = a_n\} = \\ = \sum_{m=n}^{\infty} \mathbf{P}_x\{\tau = m,\ x_m = a_0,\ x_{m-1} = a_1,\ \ldots,\ x_{m-n} = a_n\} = \\ = \sum_{m=n}^{\infty} p(m-n,\ x,\ a_n) p(a_n,\ a_{n-1}) \ldots p(a_1,\ a_0) L_D(a_0) = \\ = g(x,\ a_n) p(a_n,\ a_{n-1}) \ldots p(a_1,\ a_0) L_D(a_0).$$

Multiplying this equation by $\gamma(x)$ and summing over x we obtain

$$\mathbf{P}_\gamma\{x_\tau = a_0,\ x_{\tau-1} = a_1,\ \ldots,\ x_{\tau-n} = a_n\} = \\ = \eta(a_n)\, p(a_n,\ a_{n-1}) \ldots p(a_1,\ a_0)\, L_D(a_0), \tag{25}$$

where

$$\eta = \gamma G. \tag{26}$$

In particular,

$$\mathbf{P}_\gamma\{x_\tau = y\} = \eta(y)\, L_D(y). \tag{27}$$

§7. Densities of excessive measures and supermartingales

Let τ be the moment of leaving the set D and let f be a non-negative function in E. We put $f(*) = 0$. We ask the question: when is $f(x_T)$, $f(x_{T-1})$, ..., $f(x_{T-N})$ a supermartingale with respect to x_T, x_{T-1}, ..., x_{T-N} (for measure $\mathbf{P}_\gamma$)?

By (25) we can write (14) as follows:

$$\sum_{a_n} \eta(a_n)\, p(a_n,\ a_{n-1}) \ldots p(a_1,\ a_0)\, L(a_0)\, f(a_n) \leqslant \\ \leqslant \eta(a_{n-1})\, p(a_{n-1},\ a_{n-2}) \ldots p(a_1,\ a_0)\, L(a_0)\, f(a_{n-1}).$$

Obviously it is sufficient that

$$\sum_{a_n} \eta(a_n)\, f(a_n)\, p(a_n,\ a_{n-1}) \leqslant \eta(a_{n-1})\, f(a_{n-1}),$$

that is, that the measure $f\eta$ is excessive.

Let ν_N be the number of down-crossings of $[c,\ d]$ in the sequence $f(x_T)$, $f(x_{T-1})$, ..., $f(x_{T-N})$ or, what is equivalent, the number of up-crossings of $[c,\ d]$ in the sequence $f(x_{T-N})$, ..., $f(x_T)$. By Lemma 3 of §4

$$M_\gamma \nu_N \leqslant \frac{1}{d-c} M_\gamma f(x_\tau).$$

We denote by ν_D the number of up-crossings in the sequence $f(x_0)$, $f(x_1)$, ..., $f(x_T)$. If D is finite, then almost certainly $\tau < \infty$ and $\nu_N \uparrow \nu_D$ as $N \to \infty$. Hence $M_\gamma \nu_D \leqslant \frac{1}{d-c} M_\gamma f(x_{\tau_D})$. Now we consider an expanding sequence of finite sets D_n whose sum is the whole of E. Then $\nu_{D_n} \uparrow \nu$, where ν is the number of up-crossings of $[c,\ d]$ in the infinite sequence $f(x_0)$, $f(x_1)$, ... It is evident that

$$M_\gamma \nu \leqslant \frac{1}{d-c} \sup_D M_\gamma f(x_{\tau_D}). \tag{28}$$

We say that f belongs to the class K_γ if

$$Q = \sup_D M_\gamma f(x_{\tau_D}) < \infty. \tag{29}$$

From (28) it is clear that if $f \in K_\gamma$, then $\nu < \infty (P_\gamma. a.e.)$. Hence, as in §5, there follows the existence $P_\gamma. a.e.$ of the finite or infinite limit

$$\xi = \lim_{n\to\infty} f(x_n).$$

Let $\nu(c)$ be the number of up-crossings of $[c, 2c]$ in the infinite sequence $f(x_0)$, $f(x_1)$, ... and let $\lim\limits_{c\to+\infty} \nu(c) = \bar{\nu}$. Evidently $\{\xi = \infty\} \subseteq \{\bar{\nu} \geqslant 1\}$. Hence

$$P_\gamma\{\xi = \infty\} \leqslant P_\gamma\{\bar{\nu} \geqslant 1\} \leqslant M_\gamma \bar{\nu}.$$

By Fatou's lemma and (28)

$$P_\gamma\{\xi = \infty\} \leqslant \varliminf_{c\to\infty} M_\gamma \nu(c) \leqslant \varliminf_{c\to\infty} \frac{Q}{c} = 0.$$

So we have proved the following theorem:

THEOREM 2. *If the density f of the excessive measure μ with respect to the measure $\eta = \gamma G$ belongs to the class K_γ, then $P_\gamma. a.e.$ on Ω_∞ there exists the finite limit*

$$\lim_{n\to\infty} f(x_n).$$

§8. Excessive measures with density classes K_γ. The Martin kernel

Let γ be a finite measure, that is, $(1, \gamma) < \infty$. Then, by Lemma 1, for any $y \in E$

$$\eta(y) = \sum_x \gamma(x) g(x, y) \leqslant \sum_x \gamma(x) g(y, y) = (1, \gamma) g(y, y) < \infty.$$

We put $E_\gamma = \{y: \eta(y) > 0\}$. It is easily seen that E contains the set $\{y: \gamma(y) > 0\}$ and consists of all points that a particle of this set hits with positive probability. The probability of going out of E_γ is zero. Hence the Markov process may be considered only on the set E_γ. The most interesting case is when $E_\gamma = E$. In this case we say that the measure γ is *standard*.

Henceforth we assume that γ is a standard measure.

According to §3 the measure μG is excessive if $\mu \geqslant 0$. The density of this measure with respect to $\eta = \gamma G$ is given by[1]

$$f(y) = \sum_x \mu(x) k_y(x) = (k_y, \mu), \tag{30}$$

[1] If γ is not standard, then the measure μG has a density with respect to γG if and only if $E_\mu \subseteq E_\gamma$. Formula (30) remains valid if $k_y(x)$ is defined by (31) for $y \in E_\gamma$ and $k_y(x)$ is given arbitrarily for $y \notin E_\gamma$.

where

$$k_y(x) = \frac{g(x, y)}{\eta(y)}. \tag{31}$$

The kernel $k(x, y) = k_y(x)$ is called the *Martin kernel*.

When does the density (30) belong to K_ν? By (27)

$$M_\gamma f(x_\tau) = \sum_y f(y)\, \mathbf{P}_\gamma\{x_\tau = y\} = \sum_y f(y)\,\eta(y)\,L(y) = (\mu G, L) = \sum_{x, y} \mu(x)\, g(x, y)\, L(y),$$

and by (24)

$$M_\gamma f(x_\tau) \leqslant \sum_x \mu(x).$$

Thus, if μ is a finite measure, then the measure μG has with respect to η a density of class K_γ. The following proposition follows from Theorem 3.

THEOREM 3. *For any finite measure μ there exists P_γ.a.e. on Ω_∞ the finite limit*

$$\lim_{n\to\infty} (k_{x_n}, \mu).$$

In particular, for any y there exists P_γ.a.e. on Ω_∞ the finite limit

$$\lim_{n\to\infty} k_{x_n}(y).$$

§9. Martin compactification

As already stated in §3, the Green's function $g(x, y)$ determines for each point $y \in E$ an excessive function $g(x, y)$. The function $k_y(x)$ differs from it only by a factor not depending on x and satisfies the normalizing relation

$$(k_y, \gamma) = 1. \tag{32}$$

By Lemma 1

$$k_y(x) = \frac{\pi(x, y)\, g(y, y)}{\sum_z \gamma(z)\,\pi(z, y)\, g(y, y)} = \frac{\pi(x, y)}{\sum_z \gamma(z)\,\pi(z, y)}$$

(because $g(y, y) \geqslant p(0, y, y) = 1$). According to (13) $\pi(z, y) \geqslant \pi(z, x)\, \pi(x, y)$. Hence for all y

$$k_y(x) \leqslant \frac{1}{a(x)}, \tag{33}$$

where

$$a(x) = \sum_z \gamma(z)\,\pi(z, x)$$

(since γ is a standard measure, $a(x) > 0$ for all x).

We enumerate the points of E in arbitrary order by the integers and let $N(x)$ be the number of the point x. We put

$$\rho(y, z) = | 2^{-N(y)} - 2^{-N(z)} | + \sum_x | k_y(x) - k_z(x) | \underset{\cdot}{a}(x) 2^{-N(x)}.$$

This defines a metric in E, and the distance between any two points does not exceed 3. Forming the completion of E with respect to this metric we obtain the compactum E^*. The set E is open in E^*, so that the boundary ∂E of E in E^* is $E^* \setminus E$. The compactification so constructed is called the *Martin compactification* and the boundary ∂E the *Martin boundary*.

It should be noted that the metric $\rho(y, z)$ can be chosen with a certain degree of arbitrariness. It is essential only that y_n is a Cauchy sequence if and only if:

a) the functions $k_{y_n}(x)$ converge at any point x and

b) $N(y_n) \to \infty$ or y_n remains constant from some number n_0 on.

For each $x \in E$, $k_y(x)$ as a function of y can be extended continuously to E^*. Suppose that the sequence $y_n \in E$ converges to $y \in \partial E$. Then $k_{y_n}(x) \to k_y(x)$ for any x. Hence it follows that for $y \in \partial E$, k_y is excessive and satisfies the condition

$$(k_y, \gamma) \leqslant 1 \tag{34}$$

(equality in (32) need not hold).

The topology in E^* constructed on the metric $p(x, y)$ is called M_+ (or the M_+-topology).

The fundamental role of Martin boundaries in the theory of Markov processes is determined by the following theorem.

THEOREM 4. *With any initial state x, for almost all non-terminating paths there exists in the topology M_+ the limit*

$$\lim_{n\to\infty} x_n = x_\infty \in \partial E.$$

PROOF. For a process with initial distribution γ this statement follows at once from Theorem 3 and the Remark in §3, according to which for a transient chain $N(x_n) \to \infty$ almost certainly.

We denote by A the set of all non-terminating paths for which the theorem does not hold and put $h(x) = \mathsf{P}_x(A)$. It will be proved later (see Corollary to Theorem 8), that h is harmonic. By what has been proved, $(h, \gamma) = \mathsf{P}_\gamma(A) = 0$. Hence Theorem 4 follows from the following lemma.

LEMMA 4. *If γ is a standard measure and h an excessive function, then from $(h, \gamma) = 0$ it follows that $h = 0$ everywhere.*

PROOF OF LEMMA 4. For any n we have $\mathsf{P}^n h \leqslant h$ and therefore

$$0 \leqslant (\gamma \mathbf{P}^n, h) = (\gamma, \mathbf{P}^n h) \leqslant (\gamma, h) = 0.$$

Thus, $(\gamma \mathsf{P}^n, h) = 0$. Summing over n we have $(\eta, h) = 0$, where $\eta = \gamma G$. By definition of a standard measure η is everywhere positive, hence h is zero everywhere.

§10. Distribution of x_ζ

If $\zeta < \infty$, then x_ζ is a point of E at which a path terminates. If $\zeta = \infty$, then $x_\zeta = x_\infty$ is defined in Theorem 4 and belongs to ∂E.

Let τ be the moment of leaving D. Comparing (23), (27) and (31) we have

$$\mathbf{P}_x\{x_\tau = y\} = k_y(x)\,\mathbf{P}_\gamma\{x_\tau = y\}.$$

Hence for any function

$$M_x f(x_\tau) = M_x \sum_y \delta(x_\tau, y) f(y) = \sum_y f(y)\,\mathbf{P}_x\{x_\tau = y\} = \sum_y f(y)\,k_y(x)\,\mathbf{P}_\gamma\{x_\tau = y\} =$$
$$= M_\gamma \sum_y f(y)\,k_y(x)\,\delta(x_\tau, y) = M_\gamma f(x_\tau)\,k(x, x_\tau) \quad (35)$$

(we recall that $k(x, y) = k_y(x)$). We consider now a sequence of finite moments $D_n \uparrow E$ and denote by τ_n the moment leaving D_n. On the set Ω_∞ we have $\tau_n \to \infty$, and hence $x_{\tau_n} \to x_\infty$ and $k(x, x_{\tau_n}) \to k(x, x_\infty) = k(x, x_\zeta)$ almost certainly. On the set $\{\zeta < \infty\}$, we have $\tau_n = \zeta$ beginning with some $n_0(\omega)$; hence $k(x, x_{\tau_n}) \to k(x, x_\zeta)$. Suppose that[1] $f \in C(E^*)$. Then $f(x_{\tau_n}) \to f(x_\zeta)$. Bearing (33) in mind, we can take the limit in (35). Thus,

$$M_x f(x_\zeta) = M_\gamma f(x_\zeta)\,k(x, x_\zeta). \quad (36)$$

On the Borel sets of the compactum E^* we consider the measure μ_1 defined by

$$\mu_1(\Gamma) = \mathbf{P}_\gamma\{x_\zeta \in \Gamma\}. \quad (37)$$

By (36)

$$M_x f(x_\zeta) = \int_{E^*} k_y(x)\,f(y)\,\mu_1(dy). \quad (38)$$

The formula (38), which has been proved for continuous functions f, extends in an obvious way to all Borel non-negative functions. Putting $f = \chi_\Gamma$, we get

$$\mathbf{P}_x\{x_\zeta \in \Gamma\} = \int_\Gamma k_y(x)\,\mu_1(dy). \quad (39)$$

Thus, $k_y(x)$ $(x \in E,\ y \in E^*)$ can be interpreted as the density of the distribution for the point x_ζ (with respect to μ_1) for the initial state x.

We note that by (4′)

$$\mathbf{P}_x\{x_n = y, \zeta = n\} = p(n, x, y)\,[1 - \mathbf{P}1](y).$$

---

[1] $C(E^*)$ is the space of all continuous functions on the compactum E^*.

Hence

$$\mathbf{P}x\{x_\zeta = y\} = \sum_{n=0}^{\infty} p(n, x, y)(1-\mathbf{P}1)(y) = g(x, y)[1-\mathbf{P}1](y). \tag{40}$$

From (37) and (40)

$$\mu_1(y) = \eta(y)[1-\mathbf{P}1](y) \qquad (y \in E). \tag{41}$$

Next, if $f \in C(E^*)$, then

$$M_x f(x_\infty) = \lim_{n\to\infty} M_x f(x_n) = \lim_{n\to\infty} \mathbf{P}^n f(x)$$

and by (37) and (41)

$$\int_{E^*} f(y)\mu_1(dy) = M_\gamma f(x_\zeta) = (1-\mathbf{P}1, f\eta) + \lim_{n\to\infty} \sum_{x, y} \gamma(x) p(n, x, y) f(y). \tag{42}$$

We note that by (38) and (41)

$$M_x f(x_\zeta)\chi_{\zeta<\infty} = M_x f(x_\zeta)\chi_E(x_\zeta) = \int_E k_y(x) f(y)\mu_1(dy) = G[f(1-\mathbf{P}1)](x) \tag{43}$$

and by (38)

$$M_x f(x_\infty) = M_x f(x_\zeta)\chi_{\partial E}(x_\zeta) = \int_{\partial E} k_y(x) f(y)\mu_1(dy). \tag{44}$$

§11. *h*-processes. Martin representation of excessive functions

Let γ be a standard measure and h a γ-integrable excessive function. We prove that h is everywhere finite. For each $y \in E$ we can find n and x such that $\gamma(x)p(n, x, y) > 0$. Since $\mathbf{P}^n h \leqslant h$, we have

$$\gamma(x)\, p(n, x, y)\, h(y) \leqslant (\mathbf{P}^n h, \gamma) \leqslant (h, \gamma) < \infty.$$

Hence $h(y) < \infty$. We put $E^h = \{x: 0 < h(x)\}$ and define on E^h the transition function

$$p^h(x, y) = \frac{1}{h(x)} p(x, y) h(y). \tag{45}$$

A Markov process answering to the transition function $p^h(x, y)$ is called an *h-process*. All characteristics of this process are denoted by the same letters as for the initial process but with the upper suffix h: $\mathbf{P}^h_x$, $g^h(x, y)$, etc. For the Green's function the following relation holds:

$$g^h(x, y) = \frac{1}{h(x)} g(x, y) h(y).$$

The measure γh is standard for an h-process. Here $(\gamma h)G^h = \eta h$ and the

Martin kernel corresponding to γh is given by

$$k_y^h(x) = \frac{g^h(x, y)}{(\eta h)(y)} = \frac{k_y(x)}{h(x)}. \tag{46}$$

From this it is evident that the Martin topology for an h-process coincides with the Martin topology of the initial process, and the Martin compactification E^{h^*} for the h-process leads to the closure of E^h in the space E^*. The Martin boundary ∂E^h is simply the boundary of E^h in E^*.

We put

$$\mu_h(\Gamma) = \mathbf{P}_{\gamma h}^h\{x_\zeta \in \Gamma\}. \tag{47}$$

Evidently for any Borel function $f \geqslant 0$

$$\int_{E^*} f(y)\,\mu_h(dy) = M_{\gamma h}^h f(x_\zeta). \tag{48}$$

Hence

$$M_{\gamma h}^h f(x_\infty) = M_{\gamma h}^h(f\chi_{\partial E})(x_\zeta) = \int_{\partial E} f(y)\,\mu_h(dy). \tag{49}$$

Applying to the h-process the formulae (38), (44), (41) and (42) we have

$$M_x^h f(x_\zeta) = \frac{1}{h(x)} \int_{E^*} k_y(x) f(y)\,\mu_h(dy), \tag{50}$$

$$M_x^h f(x_\infty) = \frac{1}{h(x)} \int_{\partial E} k_y(x) f(y)\,\mu_h(dy), \tag{51}$$

$$M_x^h f(x_\zeta)\,\chi_{\zeta<\infty} = \frac{1}{h(x)} G[(h - \mathbf{P}h) f], \tag{52}$$

$$\mu_h(y) = \eta(y)[h(y) - \mathbf{P}h(y)], \tag{53}$$

$$\int_{E^*} f(y)\,\mu_h(dy) = (h - \mathbf{P}h, f\eta) + \lim_{n\to\infty} \sum_{x,\,y} \gamma(x)\,p(n, x, y)\,h(y)\,f(y) \qquad (f \in C(E^*)). \tag{54}$$

(In (50) - (51) we enlarge the domain of integration by taking $\mu_h(E^* \setminus E^{h^*}) = 0$).

In (50) putting $f = 1$ and noting (46), we observe that for any $x \in E^h$

$$h(x) = \int_{E^*} k_y(x)\,\mu_h(dy). \tag{55}$$

Outside the set E^h both sides of this equation are zero, (if $x \bar{\in} E^h$, $y \in E^h$, then $p(n, x, y) = 0$ for all n; hence, $g(x, y) = 0$ and $k_y(x) = 0$. Thus, $k_y(x) = 0$ also for y in the set E^{h^*} on which the measure μ_h is concentrated). Thus, the representation (55) holds for all $x \in E$. It is called *the Martin representation of the excessive function h*. The measure μ_h is called *the spectral measure of the function h*.

By (53)

$$\int_E k_y(x)\,\mu_h(dy) = G(h - \mathbf{P}h).$$

Hence the Martin decomposition can be put in the form

$$h(x) = G(h - \mathbf{P}h)(x) + \int_{\partial E} k_y(x)\,\mu_h(dy). \tag{56}$$

§12. The spectral measure of k_z. The space of exits

First let $z \in E$. Then

$$\mathbf{P}k_z(y) = \sum_u p(y, u)\frac{g(u, z)}{\eta(z)} = \frac{g(y, z) - \delta(y, z)}{\eta(z)} = k_z(y) - \frac{\delta(y, z)}{\eta(z)}, \tag{57}$$

and according to (54)

$$\mu_{k_z}(y) = \eta(y)\,[k_z(y) - \mathbf{P}k_z(y)] = \delta_z(y), \tag{58}$$

where $\delta_z(y) = \delta(y, z)$ is the unit measure concentrated at z. Thus, $\mu_{k_z} = \delta_z$ for all $z \in E$.

The set of all $z \in \partial E$ for which $\mu_{k_z} = \delta_z$ is called, by convention, *the space of exits* and is denoted by B.

THEOREM 5. *The space of exits B is a Borel subset of ∂E. For any γ-integrable excessive function h we have $\mu_h(\partial E \setminus B) = 0$. If $z \in B$, then k_z is a harmonic function and $(k_z, \gamma) = 1$.*

PROOF. If $z \in B$, then evidently

$$\mu_{k_z}\{z\} = 1. \tag{59}$$

On the other hand, for any $z \in \partial E$ by (47) and (34)

$$\mu_{k_z}(E^*) = (k_z, \gamma) \leqslant 1. \tag{60}$$

Hence, if (59) is satisfied, then $\mu_{k_z} = \delta_z$ and $z \in B$. Thus, B is given by (59). By (54), for $z \in \partial E$,

$$\mu_{k_z}\{z\} = \lim_{m\to\infty}\int_{E^*} e^{-m\rho(x, z)}\mu_{k_z}(dx) = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{x, y}\gamma(x)\,p(n, x, y)\,k_z(y)\,e^{-m\rho(x, y)}.$$

Hence it is evident that B is a Borel set.

We write $k = k_z$, omitting the subscript z when no confusion can arise.

Let $z \in \partial E$ and φ, $\psi \in C(E^*)$. For any $n \geqslant 0$, $m \geqslant 0$

$$M^h_{h\gamma}\varphi(x_n)\,\psi(x_{n+m}) = \sum_{x, y, z}\gamma(x)\,h(x)\,p^h(n, x, y)\,\varphi(y)\,p^h(m, y, z)\,\psi(z) =$$
$$= \sum_{x, y}\gamma(x)\,p(n, x, y)\,h(y)\,\varphi(y)\,M^h_y\psi(x_m).$$

Taking the limit as $m \to \infty$ and using (51) we have

$$M_{h\gamma}^{h}\varphi(x_n)\psi(x_\infty)=\sum_{x,y}\gamma(x)\,p(n,x,y)\,h(y)\,\varphi(y)\int_{\partial E}k_z^h(y)\,\psi(z)\,\mu_h(dz)=$$
$$=\int_{\partial E}\sum_{x,y}\gamma(x)\,h(x)\,p^k(n,x,y)\,k(y)\,\varphi(y)\,\psi(z)\,\mu_h(dz)=\int_{\partial E}M_{k\gamma}^{k}\varphi(x_n)\,\psi(z)\,\mu_h(dz).$$

Now letting $n\to\infty$ and noting (48) we have

$$\int_{\partial E}\varphi(z)\,\psi(z)\,\mu_h(dz)=\int_{\partial E}\left[\int_{\partial E}\varphi(u)\,\mu_k(du)\right]\psi(z)\,\mu_h(dz).$$

Since ψ is an arbitrary continuous function, it follows that μ_h – *a.e.*

$$\varphi(z)=\int_{\partial E}\varphi(u)\,\mu_{k_z}(du).$$

It is clear that for μ_h-almost all z this equation holds simultaneously for the sequence of functions

$$\varphi_m(y)=e^{-m\rho(y,z)}\qquad(m=1,2,\ldots)$$

and hence in the limit as $n\to\infty$ the resulting equation (59) is satisfied. Hence

$$\mu_h(\partial E\setminus B)=0.$$

For $z\in B$ it follows from (58) that k_z is harmonic, and from (60) that $(k_z,\gamma)=1$.

REMARK. From (50) it is clear that if $z\in E\cup B$, then $\mathsf{P}_x^{k_z}\{x_\zeta=z\}=1$, so that for any initial state x almost all paths of a k_z-process terminate at z.

§13. The Uniqueness Theorem

THEOREM 6. *Every γ-integrable excessive function h has a unique representation of the form*

$$h(x)=\int_{E\cup B}k_z(x)\,\mu(dz),\tag{61}$$

where μ is a measure on the Borel subsets of $E\cup B$. The measure μ is finite.

For any finite measure μ (61) defines a γ-integrable excessive function. This function is harmonic if and only if $\mu(E)=0$.

From Theorem 6 it follows, in particular, that if h has a representation of the form (61), then μ coincides with the spectral measure μ_h, and (61) coincides with the Martin representation.

PROOF. From Theorem 5, the Martin representation (55) of the γ-integrable excessive function h can be rewritten in the form (61), where $\mu=\mu_h$. Since $\mathsf{P}k_z\leqslant k_z$ for $z\in E$ and $\mathsf{P}k_z=k_z$ for $z\in B$, every function h obtained by (61) is excessive, and if $\mu(E)=0$, it is harmonic. Since

$(k_z, \gamma) = 1$ for all $z \in E \cup B$, we see that $(h, \gamma) = \mu(E \cup B) < \infty$.

We show now that if h is given by (61), then μ coincides with the spectral measure μ_h. Let $f \in C(E)$. Applying (54) to h and to k_z we note that

$$\int_{E*} f(y)\,\mu_h(dy) = \int_{E\cup B} \Big[\int_{E*} f(y)\,\mu_{k_z}(dy)\Big]\,\mu(dz).$$

But $\mu_{k_z} = \delta_z$ for $z \in E \cup B$. Therefore

$$\int_{E*} f(y)\,\mu_h(dy) = \int_{E\cup B} f(z)\,\mu(dz).$$

Since μ_h is concentrated on $E \cup B$, it follows that $\mu_h = \mu$.

We observe finally that if $Ph = h$, then by (53) $\mu_h(y) = 0$ for all $y \in E$.

§14. Minimal excessive functions

A non-zero excessive function h is called *minimal* if from $h = h_1 + h_2$, where h_1 and h_2 are excessive functions, it follows that $h_1 = c_1 h$, $h_2 = c_2 h$ (c_1 and c_2 are constants). It is easily proved that a harmonic function h is minimal if and only if every harmonic function h_1 satisfying $0 \leqslant h_1 \leqslant h$, is proportional to h.

THEOREM 7. *The general form of γ-integrable minimal excessive functions is ck_z, where $z \in E \cup B$ and c is a positive constant.*

PROOF. From (54) it is clear that $\mu_{h_1+h_2} = \mu_{h_1} + \mu_{h_2}$. Let $z \in E \cup B$ and $k_z = h_1 + h_2$, where h_1 and h_2 are excessive. Then $\mu_{h_1} + \mu_{h_2} = \mu_{k_z} = \delta_z$. Hence $0 = \delta(E^* \setminus z) = \mu_{h_1}(E^* \setminus z) + \mu_{h_2}(E^* \setminus z)$ and $\mu_{h_i}(E^* \setminus z) = 0$. From (55)

$$h_i(z) = \int_{E*} k_y \mu_h(dy) = k_z \mu_{h_i}(z).$$

Thus, k_z is minimal.

Next, let h be any γ-integrable minimal excessive function and μ_h its spectral measure. Then $\mu_h(E \cup B) = (h, \gamma)$. By Lemma 5 this quantity is positive if $h \neq 0$. Hence there exists a point $z \in E \cup B$, any neighbourhood of which has positive measure μ_h. We put $U_n = \{y\colon \rho(y, z) < 1/n\}$,

$$h_n = \int_{U_n} k_y \mu_h(dy).$$

It is obvious that h_n and $h - h_n$ are excessive. Since h_n is minimal, we have $h_n = c_n h$. Since $(h_n, \gamma) = \mu_h(U_n)$ and $(h, \gamma) = \mu_h(E \cup B) = (h, \gamma)$, we have $c_h = \mu_h(U_n) / \mu_h(E \cup B)$ and consequently

$$h = \frac{(h, \gamma)}{\mu_h(U_n)} \int_{U_n} k_y \mu_h(dy).$$

Taking the limit as $n \to \infty$ we have $h = (h, \gamma) k_z$.

§15. The Operator θ. Final random variables

We consider in the space of paths Ω the mapping θ that is defined on the set $\Omega_1 = \{\omega: \zeta(\omega) \geqslant 1\}$ and carries the path $a_0 a_1 a_2 \ldots$ into $a_1 a_2 a_3 \ldots$ For any function $\xi(\omega)$ we put

$$\theta\xi(\omega) = \begin{cases} \xi(\theta\omega) & \text{if} \quad \omega \in \Omega_1; \\ 0 & \text{if} \quad \omega \notin \Omega_1. \end{cases} \tag{62}$$

The random variable ξ is called *final* if $\theta\xi = \xi$. We note that final random variables are different from zero only on Ω_∞ and do not change their value if arbitrarily many initial intervals of the paths are changed.

We denote by A the set of non-terminating paths for which the limit $x_\infty = \lim_{n\to\infty} x_n$ does not exist or does not belong to ∂E. An example of a final random variable is the function that is equal to some constant b on A, to $f(x_\infty)$ on $\Omega_\infty \setminus A$ (f is an arbitrary Borel function on ∂E), and to 0 outside Ω_∞. In particular, the random variable χ_A is final.

LEMMA 5. *For any non-negative random variable ξ*

$$M_x \{\delta(x_0, a_0)\,\delta(x_1, a_1) \ldots \delta(x_n, a_n)\,\theta^n \xi\} = \\ = \delta(x, a_0)\, p(a_0, a_1) \ldots p(a_{n-1}, a_n)\, M_{a_n}\xi. \tag{63}$$

PROOF. We denote by $\mu_1(A)$ and $\mu_2(A)$ the values of the right- and left-hand sides of (63) for $\xi = \chi_A$. Obviously it is sufficient to prove that $\mu_1(A) = \mu_2(A)$ for any $A \in \mathscr{F}$. The functions μ_1 and μ_2 are measures. By (63) for the simple set $A = [a_0 a_1 \ldots a_n]$

$$\mu_1(A) = \mu_2(A) = \delta(x, a_0)\, p(a_0, a_1) \ldots \\ \ldots p(a_{n-1}, a_n)\, \delta(a_n, b_0)\, p(b_0, b_1) \ldots p(b_{m-1}, b_m).$$

In accordance with Theorem A, from the fact that two measures coincide on simple sets it follows that they coincide on the σ-algebra $\mathscr{F}$.

THEOREM 8. *If $\xi \geqslant 0$ is a final random variable and $h(x) = M_x\xi$ is finite for each x, then $h(x)$ is a harmonic function, and measures corresponding to an h-process are given by the formula*

$$\mathbf{P}_x^h(A) = \frac{\int_A \xi\, d\mathbf{P}_x}{h(x)} \qquad (x \in E^h). \tag{64}$$

PROOF. By (62) and (63)

$$h(x) = M_x\xi = M_x \sum_y \delta(x, y)\,\theta\xi = \sum_y p(x, y)\, M_y\xi = \sum_y p(x, y)\, h(y).$$

Hence h is harmonic.

To prove the second statement it is sufficient to note that (64) defines a measure on the σ-algebra $\mathscr{F}$, where the measure on the simple set $[a_0 a_1 a_2 \ldots a_n]$ is

$$\frac{1}{h(x)} M_x \delta(x_0, a) \ldots \delta(x_n, a_n)\,\xi = \frac{1}{h(x)} M_x \delta(x_0, a) \ldots \delta(x_n, a)\,\theta^n\xi = \\ = \frac{1}{h(x)} \delta(x, a_0)\, p(a_0, a_1) \ldots p(a_{n-1}, a_n)\, h(a_n),$$

that is, coincides with $\mathbf{P}_x^h$.

COROLLARY. *If $M_x\xi = M_x\eta$ for any two final random variables ξ and η, then $\xi = \eta$ ($\mathbf{P}_x$.a.e.).*

PROOF. From (64), for any $A \in \mathscr{F}$

$$\int_A \xi \, d\mathbf{P}_x = \int_A \eta \, d\mathbf{P}_x.$$

THEOREM 9. *Let h be a bounded harmonic function. Then*

$$h(x) = \int_B k_y(x)\, \varphi(y)\, \mu_1(dy), \tag{65}$$

where φ is a bounded Borel function. Also $\mathbf{P}_x$.a.e. on Ω_∞,

$$\lim_{n\to\infty} h(x_n) = \varphi(x_\infty) \tag{66}$$

and

$$M_x\varphi(x_\infty) = h(x). \tag{67}$$

PROOF. Let $0 \leqslant h \leqslant a$, where a is a constant. The functions h and $f = a - h$ are excessive. By Lemma 4 and Theorem 7

$$a = \int_B k_y(x)\, \mu_a(dy) = \int_B k_y(x)\, (\mu_h + \mu_f)(dy).$$

We assume that $a > 0$. Dividing the equation by a we find by Theorem 7 that $\mu_1 = (1/a)(\mu_h + \mu_f)$. Hence μ_h has a bounded density with respect to μ_1: $\mu_h(dy) = \varphi(y)\ \mu_1(dy)$. Therefore (61) can be written in the form (65). By (44) and (65) $M_x\varphi(x_\infty) = h(x)$. On the other hand, by Theorem 1 there exists $\mathbf{P}_x$.a.e. the limit

$$\xi = \lim_{n\to\infty} h(x_n).$$

By (6)

$$M_x\xi = \lim_{n\to\infty} M_x h(x_n) = \lim_{n\to\infty} \mathbf{P}^n h(x) = h(x).$$

Thus, $M_x\varphi(x_\infty) = M_x\xi$. But $\varphi(x_\infty)$ and ξ are final random variables, and by the Corollary to Theorem 8 $\varphi(x_\infty) = \xi$ $\mathbf{P}_x$.a.e. (we must take $\varphi(x_\infty) = 0$ outside Ω_∞).

COROLLARY. *Every final random variable ξ coincides almost certainly with $\varphi(x_\infty)$, where φ is some Borel function on B.*

PROOF. Let ξ be bounded. Then $h(x) = M_x\xi$ is a bounded harmonic function, and by Theorem 9 $\mathbf{P}_x$.a.e. on Ω_∞ we have $\lim h(x_n) = \varphi(x_\infty)$ and $M_x\varphi(x_\infty) = h(x) = M_x\xi$. By the Corollary to Theorem 8, $\mathbf{P}_x$.a.e. $\xi = \varphi(x_\infty)$. If ξ is arbitrary, then $\xi \wedge a$ is bounded, a being a constant. By what has been proved, $\xi \wedge a = \varphi_a(x_\infty)$ $\mathbf{P}_x$.a.e. Obviously it follows from this that $\xi = \varphi(x_\infty)$ $\mathbf{P}_x$.a.e., where $\varphi(x) = \lim_{a\to\infty} \varphi_a(x)$.

THEOREM 10. *Let* φ *be a non-negative Borel* μ_1*-integrable function on* ∂E. *Then*

$$h(x)=\int_B k_y(x)\,\varphi(y)\,\mu_1(dy). \tag{68}$$

defines a harmonic function h *such that* P_x.*a.e. on* Ω_∞,

$$\lim_{n\to\infty} h(x_n)=\varphi(x_\infty).$$

PROOF. The function $h(x)$ is finite everywhere by (33). We select a positive constant c and put

$$\varphi_1(x)=\begin{cases}\varphi(x) & \text{if} \quad \varphi(x)\leqslant c,\\ 0 & \text{if} \quad \varphi(x)>c,\end{cases} \qquad \varphi_2(x)=\begin{cases}0 & \text{if} \quad \varphi(x)\leqslant c,\\ \varphi(x) & \text{if} \quad \varphi(x)>c,\end{cases}$$

$$h_i(x)=\int_B k(x,y)\,\varphi_i(y)\,\mu_1(dy).$$

From (44)

$$M_x\varphi(x_\infty)=h(x), \qquad M_x\varphi_i(x_\infty)=h_i(x), \tag{69}$$

and by Theorem 8 h_1, h_2, h are harmonic functions. h_1 is bounded, and by Theorem 9 P_x.*a.e.*

$$\lim_{n\to\infty} h_1(x_n)=\varphi_1(x_\infty).$$

By Theorem 1 P_x*a.e.* there exist also the limits

$$\xi=\lim_{n\to\infty} h_1(x_n), \qquad \xi_2=\lim_{n\to\infty} h_2(x). \tag{70}$$

By Fatou's lemma

$$M_x\xi_2\leqslant \lim_{n\to\infty} M_x h_2(x_n)=\lim \mathbf{P}^n h_2(x)=h_2(x).$$

Hence by (69) and (70) we find

$$M_x\mid \xi-\varphi(x_\infty)\mid = M_x\mid \xi_2-\varphi_2(x_\infty)\mid \leqslant M_x\xi_2+M_x\varphi_2(x_\infty)\leqslant 2h_2(x).$$

The left-hand side does not depend on c and the right-hand side tends to zero as $c\to\infty$. Hence $\xi=\varphi(x_\infty)$, P_x.*a.e.*.

REMARK 1. According to Theorem 10, (68) gives a generalized solution of Dirichlet's problem with boundary function $\varphi(x)$; the boundary values are taken along almost all non-terminating paths.

REMARK 2. Theorems 9 and 10 can be "relativized" in an obvious way; selecting any harmonic function H we can replace the measure μ_1 by μ_H, the condition of boundedness of h by $h\leqslant H$ and the measure P_x by P_x^H.

§16. The Space of entries. Decomposition of excessive measures.

We say that a non-negative function $l(x)$ is standard if $s = Gl$ is everywhere positive and finite. We put

$$\varkappa_y(x)=\frac{g(y,x)}{s(y)}. \qquad (71)$$

We note that $\varkappa_y$ is an excessive measure and $(l, \varkappa_y) = 1$. According to Lemma 1 $g(y, x) = \pi(y, x)g(x, x)$. By (13) we have $g(y,z)=\pi(y,z)g(z,z)\geqslant \geqslant\pi(y,x)\pi(x,z)g(z,z)=\pi(y,x)g(x,z)$. Hence $s(y)\geqslant\pi(y,x)s(x)$ and $\varkappa_y(x) \leqslant 1/c(x)$, where $c(x) = s(x)/g(x, x) > 0$.

We consider in E the metric

$$\hat\rho(y,z)=|2^{-N(y)}-2^{-N(z)}|+\sum_x|\varkappa_y(x)-\varkappa_z(x)|c(x)2^{-N(x)},$$

where $N(x)$ has the same meaning as in §9. Forming the completion of E with respect to this metric we obtain a compactum $\hat E$. The topology of $\hat E$ determined by the metric $\hat\rho$ is called M. The boundary of E in the topology M is $\hat E\setminus E$. Repeating the arguments of §9 we extend $\varkappa_y(x)$ continuously to $y\in\hat E\setminus E$.

We now prove the following theorem.

THEOREM 11. *With each l-finite excessive measure α we can associate a finite measure μ^α on the Borel subsets of $\hat E$ such that for $f\in C^*(E)$*

$$\int_{\hat E} f(y)\mu^\alpha(dy)=(f_s,\alpha-\alpha\mathbf{P})+\lim_{n\to\infty}\sum_{x,y}f(y)\alpha(y)p(n,x,y)l(x). \qquad (72)$$

The space of entries $\hat B$ is the set of all $z\in\hat E\setminus E$ for which the spectral measure of the excessive measure $\varkappa_z$ is δ_z. For $z\in\hat B$ the measure $\varkappa_z$ is harmonic and $(l, \varkappa_z) = 1$.

Every l-finite excessive measure α has a unique representation in the form

$$\alpha(x)=\int_{E\cap\hat B}\varkappa_y(x)\mu(dy). \qquad (73)$$

Here μ coincides with the spectral measure μ^α. For α to be harmonic it is necessary and sufficient that $\mu^\alpha(E) = 0$.

Let ν be an l-finite excessive measure[1] *and $0 < \nu(x) < \infty$ for all x. The formula*

$$p_\nu(x,y)=\frac{\nu(y)p(y,x)}{\nu(x)} \qquad (74)$$

[1] If the standard measure γ and the standard function l are chosen so that

$$\sum_{x,y}g(x,y)\gamma(x)l(y)<\infty,$$

then the measure $\nu = \gamma G$ satisfies the necessary conditions.

defines in E a transition function. The measure $\hat{\gamma} = l\nu$ is standard for it, and the corresponding Martin kernel is $\kappa_y(x)/\nu(x)$. The Martin compactification corresponding to this kernel coincides with $\hat{E}$, and the space of exits coincides with $\hat{B}$. h is excessive (harmonic) with respect to $p_\nu(x, y)$ if and only if the measure $\alpha = h\nu$ is excessive (harmonic) with respect to $p(x, y)$. Here $(l, \alpha) = (h, \hat{\gamma})$, so that α is l-finite if and only if h is $\hat{\gamma}$-integrable. The spectral measure of the excessive function $h = \alpha/\nu$ with respect to $p_\nu(x, y)$ coincides with μ^α.

PROOF. The statements about the Martin kernel for the transition function $p_\nu(x, y)$ and the corresponding compactification are verified directly. It is also evident that if $\alpha = h\nu$, then $(l, \alpha) = (h, \hat{\gamma})$ and $Ph \leqslant h$ $(Ph = h)$ if and only if $\alpha P_\nu \leqslant \alpha$ $(\alpha P_\nu = \alpha)$. We define the spectral measure of the excessive measure α as that of the excessive function $h = \alpha/\nu$ (with respect to the transition function $p_\nu(x, y)$). (72) follows from (42). From the definition of $\hat{B}$ it is clear that $\hat{B}$ is the space of exits for $p_\nu(x, y)$, and from (72) that μ^a and B do not depend on the choice of ν. The representation (73) follows from the Martin representation for the p_ν-excessive function $h = \alpha/\nu$. The remaining parts of the theorem are verified without difficulty.

REMARK. If $\nu(x)$ becomes zero or infinite at some points, then (74) defines a transition function on $E^\nu = \{x\colon\ 0 < \nu(x) < +\infty\}$. In this case the Martin compactification for $p_\nu(x, y)$ coincides with the closure of E^ν in $\hat{E}$.

§17. The behaviour of the stationary process as $t \to -\infty$

We consider a stationary Markov process with transition function $p(x, y)$ and stationary distribution ν. The formula $a_t = \hat{a}_{-t}$ $(t = 0, 1, \ldots)$ defines a mapping r of $\hat{\Omega}$ into Ω_∞. The complete inverse image of A under r is denoted by A^r. We define in Ω_∞ the measure

$$\mathbf{P}(A) = \mathbf{P}_\nu(A^r) \qquad (A \in F).$$

If $A = [a_0 a_1 \ldots a_n]$, then $A^r = [a_n a_{n-1} \ldots a_0]^0_{-n}$ and by (4″)

$$\mathbf{P}(A) = \mathbf{P}_\nu(A^r) = \nu(a_n)\, p(a_n, a_{n-1}) \ldots p(a_1, a_0).$$

On the other hand, let $p_\nu(x, y)$ be a transition function defined by (74). We denote by ${}^\nu\mathbf{P}_\alpha$ the measure in Ω corresponding to the Markov process with transition function $p_\nu(x, y)$ and initial distribution α. (Since $\sum_y p_\nu(x, y) = 1$, the measure ${}^\nu\mathbf{P}_\alpha$ is concentrated on Ω_∞.) We note that

$${}^\nu\mathbf{P}_\nu[a_0 a_1 \ldots a_n] = \nu(a_n)\, p(a_n, a_{n-1}) \ldots p(a_1, a_0).$$

Therefore $\mathbf{P}(A) = {}^\nu\mathbf{P}_\nu(A)$ for all simple sets A. By Theorem A this equation is satisfied on all $A \in \mathscr{F}$. Thus,

$$\mathbf{P}_\nu(A^r) = {}^\nu\mathbf{P}_\nu(A) \qquad (A \in F). \tag{75}$$

We denote by A the set of all paths $\{a_n\}$ in Ω_∞ for which $\lim\limits_{n\to\infty} a_n$ does not exist, or exists but is not in $\hat{B}$, and by $\hat{A}$ the set of all paths $\{\hat{a}_t\}$ in $\hat{\Omega}$ for which $\lim\limits_{t\to-\infty} \hat{a}_t$ does not exist, or exists but is not in $\hat{B}$. It is easy to see that $A^r = \hat{A}$, and by (75)

$$\mathbf{P}_\nu(\hat{A}) = {}^\nu\mathbf{P}_\nu(A).$$

By Theorems 4 and 5 and (39) we have ${}^\nu\mathrm{P}_x(A) = 0$ for any $x \in E$. Hence $\mathrm{P}_\nu(\hat{A}) = {}^\nu\mathrm{P}_\nu(A) = 0$.

Next, from (44) and Theorem 5

$${}^\nu M_x f(x_\infty) = \int\limits_{\hat{B}} \frac{\varkappa_y(x)}{\nu(x)} f(y)\,\mu(dy), \tag{76}$$

where μ is the spectral measure of the function 1 with respect to $p_\nu(x, y)$. By Theorem 11 it coincides with the measure μ^ν. From (75) and (76)

$$M_\nu f(x_{-\infty}) = {}^\nu M_\nu f(x_\infty) = \int\limits_{\hat{B}} (1, \varkappa_y) f(y)\,\mu^\nu(dy). \tag{77}$$

Putting $f = \chi_\Gamma$, where $\Gamma \subseteq \hat{B}$, we have

$$\mathbf{P}_\nu\{x_{-\infty} \in \Gamma\} = \int\limits_{\Gamma} (1, \varkappa_y)\,\mu^\nu(dy). \tag{78}$$

So we have proved the following theorem.

THEOREM 12. *For a stationary process with l-finite stationary measure* ν *the limit*

$$x_{-\infty} = \lim_{t\to-\infty} x_t \tag{79}$$

in the topology M_- *exists almost certainly and belongs to the space of entries* $\hat{B}$. *The distribution* $x_{-\infty}$ *is given by* (78). *In particular, if* $\nu = \varkappa_y$, *where* $y \in B$, *then almost certainly* $x_{-\infty} = y$.

Applying Theorem 12 to an h-process we can obtain a more general statement in which the function 1 ceases to play an exceptional role.

THEOREM 13. *Let* ν *be an l-finite harmonic measure,* h *a* γ*-integrable harmonic function. Then for the stationary process with transition function* $p^h(x, y) = (h(x))^{-1} p(x, y) h(y)$ *and stationary measure* $h\nu$, *the limit* (79) *in the topology* M_- *exists almost certainly and belongs to the space of entries* $\hat{B}$. *The measure of the set of paths for which* $x_{-\infty} \in \Gamma \subseteq \hat{B}$ *is*

$$\int_{\Gamma} (h, \varkappa_y)\, \mu^{\nu}(dy). \tag{80}$$

REMARK. From Theorems 4 and 5 and (51) it follows that almost all paths of the process considered in Theorem 13 have in the topology M_+ the limit

$$\lim_{t \to +\infty} x_t = x_{+\infty} \in B,$$

where the measure of the set of paths for which $x_{+\infty} \in \Gamma \subseteq B$ has the value

$$\int_{\Gamma} (k_y, \nu)\, \mu_h(dy) \tag{81}$$

(μ_h is the spectral measure of the excessive function h).

§18. Stationary processes with random moments of birth and cut-off

We attempt to extend the results of the previous section to the case where the measure ν and the function h are excessive, but not necessarily harmonic. The transition from $h = 1$ to an arbitrary excessive function h does not cause serious difficulties. Therefore we first take $h = 1$.

We construct in some countable space $\bar{E} \supset E$ a transition function and harmonic measure coinciding on E with $p(x, y)$ and ν and defining in $\bar{E}$ a stationary process. Almost all paths of this process remain in E from the moment ξ of first hitting E to the moment ζ of leaving E. In E there arises a random process with random moments of birth and cut-off. We call this a stationary process with transition function $p(x, y)$ and stationary measure ν.

Let Z be the set of all integers. We define in the space $\bar{E} = Z \times E$ the transition function $\bar{p}$ for which

$$\begin{gathered} \bar{p}(0 \times x, 0 \times y) = p(x, y), \\ \bar{p}(0 \times x, 1 \times x) = 1 - \sum_y p(x, y), \\ \bar{p}(m \times x, (m+1) \times x) = 1 \quad \text{if} \quad m \neq 0, \end{gathered}$$

and the remaining values are zero. It is clear that (1) is satisfied for $\bar{p}$ with equality. Next, let ν be an excessive measure for $p(x, y)$. The formulae

$$\begin{gathered} \bar{\nu}(0 \times x) = \nu(x), \quad \bar{\nu}(m \times x) = \nu(x) - \sum_y \nu(y)\, p(y, x) \quad \text{for} \quad m > 0, \\ \bar{\nu}(m \times x) = 0 \quad \text{for} \quad m < 0, \end{gathered}$$

define in $\bar{E}$ a harmonic measure with respect to $\bar{p}$. Identifying $0 \times x$ with x we may assume that $E \subset \bar{E}$. In this case $\bar{p}$ coincides with p and $\bar{\nu}$ with ν on E. Obviously it may be assumed that the points $m \times x$ lie above x if $m > 0$ and below if $m < 0$. We consider in $\bar{E}$ the process $\bar{x}_t$ with transition function $\bar{p}$ and stationary measure $\bar{\nu}$. Subtracting from $\hat{\Omega}$ a set of measure zero we may assume that the paths behave as follows. After unit time a particle not lying in E moves one unit upwards; a particle lying in E moves in E in correspondence with the transition probability $p(x, y)$; if the process in E guided by this law must terminate, then instead of this there is a move in $\bar{E}$ of one unit upwards. Let ξ be the moment of first reaching E and ζ the moment of leaving E. Rejecting on each path the part not belonging to E we obtain in E a random process x_t with moment of birth ξ and moment of cut-off ζ. For this process the measure of the set $\{x_m = a_m, \ldots, x_n = a_n\}$ is

$$\nu(a_m)\, p(a_m, a_{m+1}) \ldots p(a_{n-1}, a_n). \tag{82}$$

We have the same expression as in (4″). It is natural therefore to call the process so constructed *stationary with transition function $p(x, y)$ and stationary measure ν*.

The paths of this process are functions with values in E defined on all possible intervals of the form $[m, n]$, $(-\infty, n]$, $[m, +\infty)$ and $(-\infty, +\infty)$. The set of all paths is denoted by Ω', and the σ-algebra generated by all simple sets $[a_m \ldots a_n]_m^n$ by $\mathscr{F}'$. We have constructed on $\mathscr{F}'$ a measure P_ν that is equal to (82) on the simple set $[a_m \ldots a_n]_m^n$. The measure P_ν is defined uniquely by this condition, as follows from the following lemma, which we prove in the Appendix.

LEMMA A. *If two measures on the σ-algebra $\mathscr{F}'$ in Ω' coincide and are finite on all simple sets, then they coincide everywhere.*

From

$$\begin{aligned}\nu(a_m)\, h(a_m)\, p^h(a_m, a_{m+1}) \ldots p^h(a_{n-1}, a_n) &= \\ = \nu(a_m)\, p(a_m, a_{m+1}) \ldots p(a_{n-1}, a_n)\, h(a_n) &= \\ = h(a_n)\, \nu(a_n)\, p_\nu(a_n, a_{n-1}) \ldots p(a_{m+1}, a_m)&\end{aligned}$$

it follows that if x_t is a stationary process with transition function p^h and stationary measure $h\nu$, then $\hat{x}_t = x_{-t}$ is a stationary process with transition function p_ν and stationary measure $h\nu$. Using these remarks it is easy to prove the following theorem, a generalization of Theorem 13.

THEOREM 14. *Let ν be an l-finite excessive function, and h a γ-integrable excessive function. Then for the stationary process with transition function $p^h(x, y)$ and stationary measure $h\nu$ almost all non-terminating paths have in the topology M_+ a limit $x_{+\infty} \in B$. The measure of the set of paths for which $x_\zeta \in \Gamma \subseteq E \cup B$ is*

$$\int_\Gamma (k_y, \nu)\, \mu_h(dy). \tag{83}$$

Hence all paths without a beginning have in M_- a limit $x_{-\infty} \in \hat{B}$. The measure of the sets $\{x_\xi \in \Gamma\}$, where $\Gamma \subseteq E \cup \hat{B}$, is

$$\int_\Gamma (h, \varkappa_y)\, \mu^\nu (dy). \tag{84}$$

PROOF. We consider only the case $h = 1$. (The passage to the general case is effected just as in the derivation of Theorem 13 from Theorem 12.) Apart from the transition function p in E we consider the transition function $\bar{p}$ in $\bar{E}$, where all mappings constructed from it are distinguished by a bar on top. We introduce in $\bar{E}$ the standard measure $\bar{\gamma}$, putting $\bar{\gamma}(m \times x) = 0$ of $m < 0$ and selecting the value of $\bar{\gamma}(m, x)$ for $m \geqslant 0$ so that

$$\sum_m \bar{\gamma}(m \times x) = \gamma(x).$$

It is easy to calculate that

$$\bar{k}_{n \times y}(m \times x) = k_y(x) \quad \text{for} \quad n \geqslant 0. \tag{85}$$

We denote by E^+ the set of all points $n \times y$, where $n \geqslant 0$. By (85) the "divergent to infinity" sequence $n_r \times y_r \in E^+$ converges in the M^+-topology corresponding to $\bar{p}(x, y)$ if and only if y_r converges in the M^+-topology connected with $p(x, y)$, or $n_r \to \infty$ and $y_r = y$ beginning with some r. Hence the Martin boundary ∂E^+ is contained in $\partial\bar{E}$ and $\partial E^+ = \partial E \cup E'$, where E' is in natural correspondence with E.

Let $\bar{h}$ be a $\bar{p}$-harmonic function and $\mu_{\bar{h}}$ its spectral measure. By (54)

$$\int_{\bar{E}^*} f(y)\, \mu_{\bar{h}}(dy) = \lim_{n \to \infty} M_{\bar{\gamma}} f(\bar{x}_n)\, \bar{h}(\bar{x}_n) \qquad (f \in C(\bar{E}^*)).$$

We put $f(y) = \rho(y, E^+)$. Then the right-hand side is evidently zero. Hence $\mu_{\bar{h}}$ is concentrated on $E^+ \cup \partial E^+$. On the other hand, $\mu_{\bar{h}}$ is concentrated on $\partial\bar{E}$. This means that it is concentrated on ∂E^+. Hence it is easy to deduce that the space of exits $\bar{B}$ is $B \cup E'$ and may be naturally identified with $B \cup E$. We apply to the process $\bar{x}_t$ the remark at the end of §17. It is obvious that $\{\bar{x}_\infty = y\} = \{x_\zeta = y\}$. Hence the first part of Theorem 14 follows.

To prove the second part it is sufficient to apply the part just proved to the inverted process $\hat{x}_t = x_{-t}$.

§19. Hunt processes

Hunt noticed that boundary theory is applicable to a class of processes wider than Markov processes. Roughly speaking, these are processes which behave like Markov processes with transition function $p(x, y)$ after the moment of first reaching any finite set D.

We denote (as in §18) by Ω' the set of all functions a_t with values in

E, defined on all possible intervals $(-\infty, n]$, $[m, n]$, $[m, +\infty)$ and $(-\infty, +\infty)$. The left end of an interval is called the moment of birth and the right the moment of cut-off. In the space Ω' we consider the τ-algebra F' generated by all simple sets. Important examples of F'-measurable functions are: a) the moment of birth ξ; b) the moment of cut-off ζ; c) the position x_t of the path at the moment t (the domain of definition of x_t is the set $\{\xi \leqslant t \leqslant \eta\}$; d) the moment σ_D of first hitting D and the moment τ_D of leaving D (the domain of definition is the set Ω'_D of paths hitting D); e) $\mathbf{N}(x)$, the number of hits in the point x (the domain of definition is $-\Omega'$).

Let $\mathbf{P}$ be a measure on the τ-algebra F'. The process determined by $\mathbf{P}$ is called *a Hunt process with transition function $p(x, y)$ and characteristic measure β if:*

19.A. For any finite D, any $n = 0, 1, \ldots,$ and any $a_0, a_1, \ldots, a_n \in E$

$$\mathbf{P}\{x_{\sigma_D} = a_0, x_{\sigma_D+1} = a_1, \ldots, x_{\sigma_D+n} = a_n\} = \\ = \nu_D(a_0)\, p(a_0, a_1) \ldots p(a_{n-1}, a_n).$$

19.B. $M\mathbf{N}(x) = \beta(x) < \infty$ for any $x \in E$.

For $n = 0$ from 19.A we have $\nu_D(a_0) = \mathbf{P}\{x_{\sigma_D} = a_0\}$. Condition 19.A means that $y_t = x_{\sigma_D+t}$ is a Markov process with transition function $p(x, y)$ and initial distribution ν_D.

We note that by 19.B for almost all paths $\mathbf{N}(x)$ is finite, hence x_{σ_D} and x_{τ_D} are defined[1] on Ω'_D. We denote by $\mathbf{N}_D(x)$ the number of hits of a path in the state x, starting with the moment σ_D. By 19.A, $M\mathbf{N}_D(x)$ is the mean number of hits in the point x for a path of the Markov process with transition function $p(x, y)$ and initial distribution ν_D. Hence from (3) it follows that

$$M\mathbf{N}_D(x) = (\nu_D G)(x). \tag{86}$$

Evidently $\mathbf{N}_D(x) \uparrow \mathbf{N}(x)$ for $D \uparrow E$. Hence

$$(\nu_D G)(x) \uparrow \beta(x) \tag{87}$$

for $D \uparrow E$. From (87) it is clear that β is an excessive measure (with respect to $p(x, y)$). We note that

$$\beta(x) = \sum_t \mathbf{P}\{x_t = x\}.$$

Since $g(x, x) \geqslant 1$,

$$\nu_D(x) \leqslant (\nu_D G)(x) \leqslant \beta(x) < \infty.$$

If any state is attained only by a set of paths of measure zero, such a state may, without difficulty, be removed from E. Hence, without restricting significantly the generality, it may be assumed that the

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[1] Here, as in the whole of §19, D is taken as a finite set.

following additional condition is satisfied:

19.C. $\beta(x) > 0$ for any $x \in E$.

EXAMPLE 1. The Markov process with transition function $p(x, y)$ and starting distribution ν can be regarded as a process with moment of birth $\xi = 0$ and moment of cut-off ζ. Condition 19.A is always satisfied.[1] Condition 19.B is satisfied if the process is transient.

EXAMPLE 2. The stationary process with transition function $p(x, y)$ and stationary distribution ν satisfies condition 19.A, but not 19.B (if $\nu \neq 0$).

EXAMPLE 3. Let x_t be a Hunt process and $\nu(\omega)$ any integer-valued random variable. Then $\tilde{x}_t = x_{\nu+t}$ is a Hunt process with the same transition function and characteristic measure. We call this transformation a *random translation in time.*

For the process x_t the inverted process is defined by the formula $\hat{x}_t = x_{-t}$. Its moment of birth is $-\zeta$, and moment of cut-off $-\xi$.

THEOREM 15. *By inverting a Hunt process with transition function $p(x, y)$ and characteristic measure β a Hunt process is obtained with the same characteristic measure and the transition function*

$$p_\beta(x, y) = \frac{\beta(y)\, p(y, x)}{\beta(x)}.$$

PROOF. Let σ' be the moment of first hitting the finite set D' for the process $\hat{x}_t$. Then $\tau = -\sigma'$ is the moment of leaving D' for the process x_t. We have

$$\mathbf{P}\{\hat{x}_{\sigma'} = a_0,\ \hat{x}_{\sigma'+1} = a_1,\ \ldots,\ \hat{x}_{\sigma'+n} = a_n\} = \mathbf{P}\{x_\tau = a_0,\ x_{\tau-1} = a_1,\ \ldots,\ x_{\tau-n} = a_n\}. \tag{88}$$

Let σ_D be the moment of first hitting D for the process x_t. By 19.A $y_t = x_{\sigma_D+t}$ is a Markov process with transition function $p(x, y)$ and initial distribution ν_D. Let $\bar{\tau}$ be the moment of leaving D' for the process y_t. According to (25)

$$\mathbf{P}\{y_{\bar\tau} = a_0,\ y_{\bar\tau-1} = a_1,\ \ldots,\ y_{\bar\tau-n} = a_n\} = \\ = (\nu_D G)(a_n)\, p(a_n, a_{n-1}) \ldots p(a_1, a_0)\, L_D(a_0). \tag{89}$$

Obviously outside the set $A_D = \{\sigma_D > \tau\}$ we have $y_{\bar\tau} = x_\tau,\ \ldots,\ y_{\bar\tau-n} = x_{\tau-n}$. Hence the left-hand side of (89) differs by not more than $\mathbf{P}(A_D)$ from the probability (88). But as $D \uparrow E$ $\mathbf{P}(A_D) \downarrow 0$ and $\nu_D G \uparrow \beta$ by (87). Hence, taking the limit in (89), we have

$$\mathbf{P}\{x_\tau = a_0,\ x_{\tau-1} = a_1,\ \ldots,\ x_{\tau-n} = a_n\} = \\ = \beta(a_n)\, p(a_n, a_{n-1}) \ldots p(a_1, a_0)\, L_D(a_0). \tag{90}$$

The right-hand side is equal to $L_D(a_0)\, \beta(a_0)\, p_\beta(a_0, a_1) \ldots p_\beta(a_n, a_{n-1})$

[1] It is left to the reader to verify this.

and by (88) the process $\hat{x}_t$ satisfies 19.A. That 19.B is satisfied for this process is obvious.

COROLLARY. *Let* ${}^{\beta}\mathbf{P}_x$ *be the measure corresponding to the Markov process with transition function* $p_\beta(x, y)$ *and initial state* x. *Let*

$$L_D^\beta(x) = {}^{\beta}\mathbf{P}_x\{x_0 = x,\ x_t \notin D \text{ for } t > 0\}.$$

Then for the Hunt process with transition function $p(x, y)$ *and characteristic function* β

$$\nu_D(x) = \beta(x) L_D^\beta(x). \tag{91}$$

PROOF. Putting $n = 0$ in (91), (88) and (90), we have

$$\mathbf{P}\{\hat{x}_{\sigma'} = a_0\} = \mathbf{P}\{x_\tau = a_0\} = \beta(a_0) L_D(a_0). \tag{92}$$

Considering now $\hat{x}_t$ as an initial process and x_t as an inverted one, on applying (92) we get (91).

(91) shows that the distribution ν_D can be uniquely reconstructed from the transition function $p(x, y)$ and the characteristic measure β.

LEMMA 6. *For* $D \uparrow E$ *there exist the limits*

$$(k_y, \nu_D) \uparrow \mathscr{H}_\beta(y), \qquad (y \in E^*),$$
$$(L_D, \varkappa_y) \uparrow S(y), \qquad (y \in \hat{E}).$$

Here for $y \in E$

$$\mathscr{H}_\beta(y) = \frac{\beta(y)}{\eta(y)},$$
$$S(y) = \frac{1}{s(y)}.$$

(Outside E $\mathscr{H}_\beta$ *and* S *may tend to* $+\infty$. S *depends only on the transition function* $p(x, y)$, *and* $\mathscr{H}_\beta$ *depends only on* $p(x, y)$ *and* β.*)*

PROOF. By (31) and (71) for $y \in E$

$$(k_y, \nu_D) = \frac{(\nu_D G)(y)}{\eta(y)}, \qquad (L_D, \varkappa_y) = \frac{GL_D(y)}{s(y)}.$$

By (87) it follows from this that

$$(k_y, \nu_D) \uparrow \frac{\beta(y)}{\eta(y)} \quad \text{for} \quad y \in E.$$

By (24), $GL_D(y)$ is the probability that the path of the Markov process with initial state y and transition function $p(x, y)$ hits[1] D. Hence it is clear that $GL_D(y) \uparrow 1$ and $(L_D, \varkappa_y) \uparrow 1/s(y)$. It remains to note that

$$(k_y, \nu_D) = \sum_{x \in D} \nu_D(x) k_y(x)$$

is a continuous function of y on E^* and hence that for $D \uparrow E$ this function does not diminish for $y \in \partial E$. Similar arguments are applicable to $(L_D, \varkappa_y)$.

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[1] We recall that being transient almost all paths hitting D leave D.

THEOREM 16. *For the Hunt process with transition function $p(x, y)$ and characteristic measure β almost all non-terminating paths have in the topology M_+ a limit $x_{+\infty} \in B$. Almost all paths not having a beginning have in the topology M_- a limit $x_{-\infty} \in \hat{B}$. Here*

$$\mathbf{P}\{x_{\zeta} \in \Gamma\} = \int_{\Gamma} \mathscr{H}_{\beta}(y)\, \mu_1(dy), \qquad (\Gamma \subseteq E \cup B), \tag{93}$$

$$\mathbf{P}\{x_{\xi} \in \Gamma\} = \int_{\Gamma} S(y)\, \mu^{\beta}(dy), \qquad (\Gamma \subseteq E \cup \hat{B}) \tag{94}$$

(μ_1 is the spectral measure of the excessive function I, and μ^{β} is the spectral measure of the excessive measure β).

PROOF. By 19.A $y_t = x_{\sigma_D + t}$ is a Markov process with transition function $p(x, y)$ and initial distribution ν_D. By Theorems 4 and 5 for almost all non-terminating paths the limit of this process $y_{+\infty}$ exists and is in B. By (83) the probability that $y_{\bar{\zeta}} \in \Gamma$ is

$$\int_{\Gamma} (k_y, \nu_D)\, \mu_h(dy) \tag{95}$$

($\bar{\zeta}$ is the moment of cut-off of y_t). We denote by C_D the set of paths of the process x_t not meeting D. Evidently outside D $y_{\bar{\zeta}} = x_{\zeta}$. Hence (95) differs from $\mathbf{P}\{x_{\zeta} \in \Gamma\}$ by not more than $\mathbf{P}(C_D)$. But for $D \uparrow E$ we have $\mathbf{P}(C_D) \downarrow 0$, and hence the limit (95) is equal to $\mathbf{P}\{x_{\zeta} \in \Gamma\}$. This proves (93).

The remaining statements of the theorem are obtained by applying the part already proved to the inverse process. Here we have to use Theorem 11 and the Corollary to Theorem 15.

So far the question of the existence and uniqueness of a Hunt process with given transition function $p(x, y)$ and characteristic measure β has remained open. First we prove the uniqueness theorem.

THEOREM 17. *The transition function and characteristic measure determine a Hunt process uniquely to within a random translation of time.*

PROOF. As in §9, let $N(x)$ denote the number of the state x. For each path $\{a_t\}$ we consider the smallest of the numbers $N(a_t)$ and call the first moment for which this smallest number is attained canonical. The canonical moment v is a function of the path that can be defined by the conditions

$$N(x_t) \geqslant N(x_v) \quad \text{for all} \quad t,$$

$$N(x_t) > N(x_v) \quad \text{for} \quad t < v.$$

By means of a random translation of time we can make sure that $v = 0$. We assume that this condition is satisfied and put
$D_k = \{x : N(x) \leqslant k\}$, $\sigma_k = \sigma_{D_k}$, $\nu_k = \nu_{D_k}$. We note that $\sigma_k \leqslant v = 0$ for any k. Next, $N(x_t) > N(x_{\sigma_k})$ for $t < \sigma_k$. Hence

$$\{\sigma_k = -m\} = \{\sigma_k + m = 0\} = \{N(x_{\sigma_k+t}) > N(x_{\sigma_k+m}) \text{ for } t = 0, 1, \ldots, \ldots, m-1, N(x_{\sigma_k+t}) \geqslant N(x_{\sigma_k+m}) \text{ for } t > m\} = \{v' = m\},$$

where v' is the canonical moment for the process $y_t = x_{\sigma_k+t}$ $(t \geqslant 0)$. By 19.A

$$\mathbf{P}\{\sigma_k = -m, x_{\sigma_k} = x, x_n = a_n, x_{n+1} = a_{n+1}, \ldots, x_r = a_r\} = \\ = \mathbf{P}\{y_0 = x, y_{n+m} = a_n, \ldots, y_{r+m} = a_r, v' = m\}. \tag{96}$$

Let $a_n \in D_k$. Then the left-hand side of (96) is zero for $n < -m$. Hence, summing over all $x \in D_k$ and over all $m \geqslant 0 \vee (-n)$, we get

$$\mathbf{P}\{x_n = a_n, x_{n+1} = a_{n+1}, \ldots, x_r = a_r\} = \\ = \mathbf{P}_{\nu_k}\{v' + n \geqslant 0, y_{n+v'} = a_n, \ldots, y_{r+v'} = a_r\}. \tag{97}$$

Thus, the measures of simple sets in Ω' can be reconstructed from the transition function $p(x, y)$ and the measures ν_k. By the Corollary to Theorem 15 the latter is defined uniquely by $p(x, y)$ and the measure β. It remains to use Lemma A of §18.

The following existence theorem holds.

THEOREM 18. *Let β be an excessive measure for the transition function $p(x, y)$, where $\beta(x) < \infty$ for all $x \in E$. Then there exists a Hunt process corresponding to $p(x, y)$ and β.*

To prove this theorem measures ν_D can be defined by (92) and then the measures of simple sets can be given by (97). The detailed execution of this plan is somewhat unwieldy (it is carried through in [3], Chapter 10, §12).

If $\beta = \alpha G$, the required Hunt process is obtained by considering a Markov process with initial distribution α.

The general case can be treated as follows: It is easy to verify that if $D_k \uparrow E$ and $\nu_k = \nu_{D_k}$ are defined by (92), then $\nu_k G \uparrow \beta$. We construct the Hunt process with characteristic measures $\beta_k = \nu_k G$, produce in each of them arbitrary displacements of time, making the canonical moment zero, and then take the limit as $k \to \infty$.

Appendix
Measures in spaces of paths

We prove Theorems A and B stated in §2 and Lemma A of §18. First we prove the propositions on the *uniqueness* of a measure. Here we depend on a simple lemma from set theory.

A system $\mathscr{C}$ of subsets of a set Ω is called a π-system if the intersection of two sets of $\mathscr{C}$ also belongs to $\mathscr{C}$. The system $\mathscr{H}$ is called a λ-system if: λ_1) the sum of two disjoint sets of $\mathscr{H}$ also belongs to $\mathscr{H}$; λ_2) if $A, B \in \mathscr{H}$ and $A \supseteq B$, then $A \setminus B \in \mathscr{H}$; λ_3) if $A_1, \ldots, A_n, \ldots \in \mathscr{H}$ and $A_n \uparrow A$, then $A \in \mathscr{H}$; λ_4) $\Omega \in \mathscr{H}$.

LEMMA B. *If the λ-system $\mathscr{H}$ contains the π-system $\mathscr{C}$, then $\mathscr{H}$ contains the σ-algebra $\sigma(\mathscr{C})$, generated by $\mathscr{C}$.*

This lemma is proved in the first few pages of [2] (see Lemma 1.1).

COROLLARY. *Suppose that two measures given on a σ-algebra $\mathscr{F}$ in a space Ω coincide and are finite on a π-system $\mathscr{C}$ generated by $\mathscr{F}$, If Ω can be partitioned into the sum of a countable number of pairwise disjoint sets $\Omega_n \in \mathscr{C}$, then the two measures coincide everywhere on $\mathscr{F}$.*

PROOF. We denote by $\mathscr{H}$ the family of all sets $A \in \mathscr{F}$ on which the measures coincide and are finite. We put $A \in \mathscr{H}_n$ if $A \in \mathscr{H}$ and $A \subseteq \Omega_n$; $A \in \mathscr{C}_n$ if $A \in \mathscr{C}$ and $A \subseteq \Omega_n$. Let $\mathscr{S}_n$ be the σ-algebra in the space Ω_n generated by $\mathscr{C}_n$. Evidently $\mathscr{C}_n$ is a π-system, $\mathscr{H}_n$ is a λ-system in Ω_n and $\mathscr{C}_n \subseteq \mathscr{H}_n$. By Lemma B $\mathscr{S}_n \subseteq \mathscr{H}_n$.

We put $A \in \tilde{\tilde{\mathscr{F}}}$ if $A_n \cap \Omega_n \in \mathscr{S}_n$ for all n. Evidently $\tilde{\tilde{\mathscr{F}}}$ contains $\mathscr{C}$ and is a σ-algebra. Hence $\tilde{\tilde{\mathscr{F}}} \supseteq \mathscr{F}$. Thus, if $A \in \mathscr{F}$, then for all n, $A \cap \Omega_n \in \mathscr{S}_n \subseteq \mathscr{H}_n \subseteq \mathscr{H}$. But if two measures coincide on $A \cap \Omega_n$ for all n, they coincide on A.

It is now quite simple to prove the uniqueness of the measure in Theorems A and B. It is sufficient to apply the Corollary just proved to the π-system of all simple sets and to note that the simple sets $[a_0]$ $(a_0 \in E)$ are pairwise disjoint and that their sum is the entire space of paths.

To prove Lemma A of §18 we denote by $\mathscr{C}$ the family of sets of the form

$$\{\xi = s,\ x_{m_1} = a_m,\ x_{m_2} = a_{m_2},\ \ldots,\ x_{m_k} = a_{m_k},\ \zeta = t\}, \tag{0.1}$$

where $-\infty \leqslant s \leqslant m_1 < m_2 < \ldots < m_n \leqslant t \leqslant +\infty$; $k = 1, 2, \ldots$ We note that the sets

$$\{\xi = s,\ x_s = x,\ \zeta = t\} \qquad (-\infty < s \leqslant t \leqslant +\infty,\ x \in E),$$

$$\{\xi = -\infty,\ x_t = x,\ \zeta = t\} \qquad (-\infty < t < +\infty,\ x \in E),$$

$$\{\xi = -\infty,\ x_0 = x,\ \zeta = +\infty\} \qquad (x \in E)$$

belong to $\mathscr{C}$, are pairwise disjoint, and that their sum is the whole space of paths. It remains to verify that the given measures are finite and coincide on $\mathscr{C}$. For the sets

$$[a_m \ldots a_n]_m^n = \{\xi \leqslant m,\ x_m = a_m,\ \ldots,\ x_n = a_n, \geqslant \zeta\ n\} \tag{0.2}$$

this is true by the conditions of the Lemma. Hence this is true also for the sets

$$\{\xi \leqslant s,\ x_{m_1} = a_{m_1},\ x_{m_2} = a_{m_2},\ \ldots,\ x_{m_k} = a_{m_k},\ \zeta \geqslant t\} \tag{0.3}$$
$$(-\infty < s \leqslant m_1 < m_2 < \ldots < m_k \leqslant t < +\infty),$$

which can be expressed as a countable sum of pairwise disjoint sets (0.2). Letting $s \downarrow -\infty$ or $t \uparrow +\infty$ we conclude that the measures coincide on the sets (0.3) also for $s = -\infty$ and for $t = +\infty$. Hence it is clear that the measures coincide on all sets (0.1).

The proof of the existence of the measures described in Theorems A and

B is based on the following general theorem from measure theory.

THEOREM B. *Let $\mathcal{A}$ be an algebra of sets in the space Ω (that is, a family of sets containing together with any two sets their sum and together with any set its complement). Let $\mathbf{P}(A)$ be a non-negative function on $\mathcal{A}$, satisfying the conditions:*

α. *If A_1, A_2 belong to $\mathcal{A}$ and are disjoint, then*

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2).$$

β. *The space Ω can be partitioned into a countable union of disjoint subsets $\Omega_n \in A$ such that $P(\Omega_n) < \infty$.*

γ. *If $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ are sets in $\mathcal{A}$ and* $\lim \mathbf{P}(A_n) > 0$, *then $\bigcap_n A_n$ is non-void.*

Then, on the σ-algebra $\mathcal{F}$ generated by $\mathcal{A}$ there exists a measure coinciding with $\mathbf{P}$ on $\mathcal{A}$.

The sequence of sets $A_n \in \mathcal{A}$ is called a *nest*, if $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$, $\lim \mathbf{P}(A_n) > 0$ and $\mathbf{P}(A_1) < \infty$. By virtue of β it is easy to prove that the condition γ is equivalent to the following:

γ'. *Every nest has a non-void intersection.*

Theorem A will be deduced from the following theorem:

THEOREM D. *Let $\mathcal{F}$ be a σ-algebra in the space Ω_∞ of non-terminating paths generated by the simple sets $[a_0 \ldots a_n]$. For any n and any $a_0 \ldots a_n \in E$ suppose that a non-negative number $p(a_0, a_1, \ldots, a_n)$ is given, where*

$$\sum_{a_n} p(a_0, a_1, \ldots, a_n) = p(a_0, a_1, \ldots, a_{n-1}). \tag{0.4}$$

Then there exists a measure $\mathbf{P}$ on the σ-algebra $\mathcal{F}$ for which

$$\mathbf{P}[a_0, a_1 \ldots a_n] = p(a_0, a_1, \ldots, a_n). \tag{0.5}$$

PROOF. We call a simple set in Ω_∞ of the form $[a_0, a_1, \ldots, a_n]$ a *simple n-set*. A set that can be represented as the sum of simple n-sets is called a *cylinder n-set*.[1]

We note that for $m \geqslant n$:

a) If a simple m-set A intersects a simple n-set B, then $A \subseteq B$. (Hence it follows that for $m = n$ $A = B$.)

b) If a simple m-set intersects a cylindrical n-set, then it is contained in it.

c) A simple n-set is a cylindrical m-set.

d) A cylindrical n-set is a cylindrical m-set.

e) The sum and the complement of a cylindrical n-set is also a cylindrical n-set.

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1 Theorem D, as the Theorem E deduced below, is a particular case of a well known theorem of Kolmogorov on measures in products. In this particular case the general proof is simplified significantly. In this account of a simplified proof we follow the book of Kemeny, Snell and Knapp [3].

Properties a), c), e), are evident, b) follows from a), and d) from c). By d) and e) the family A of all cylindrical sets is an algebra. We define on this algebra a measure, relating to each cylindrical n-set A the sum of the values $p(a_0, \ldots, a_n)$ over all simple sets $[a_0, \ldots, a_n]$ contained in A. Although every cylindrical n-set A is at the same time an $(n+1)$-set, by (0.4) the number associated with A does not depend on whether it is regarded as an n-set or as an $(n+1)$-set. It is easily seen that the function on A introduced in this way satisfies α and β. If we prove that it satisfies also condition γ', then Theorem D follows from Theorem C.

We require the following:

LEMMA C. *If the sets C_n form a nest, then for each m there exists a simple m-set B such that $B \cap C_n$ also forms a nest.*

We note that

$$0 < \lim \mathbf{P}(C_n) = \lim \sum \mathbf{P}(B \cap C_n) = \sum \lim \mathbf{P}(B \cap C_n) \tag{0.6}$$

(the sum is taken over all simple m-sets B; the signs of summation and limit can be interchanged, since $\mathbf{P}(B \cap C_n) \leqslant \mathbf{P}(B \cap C_0)$ for all n and $\Sigma \mathbf{P}(B \cap C_0) = \mathbf{P}(C_0) < \infty$. From (0.6) it follows that for some B $\lim \mathbf{P}(B \cap C_n) > 0$ and hence $B \cap C_n$ is a nest.

We come to the proof of γ'. Let A_n be a nest. By d) we may regard A_n as a cylindrical n-set. By Lemma C a simple 0-set B_0 can be chosen so that $A_n^1 = A_n \cap B_0$ form a nest. Next, a simple 1-set B_1 can be chosen so that $A_n^2 = A_n^1 \cap B_1 = A_n \cap B_0 \cap B_1$ form a nest. Continuing this construction, for each m we construct the nest $A_n^m = A_n^{m-1} \cap B_m = A_n \cap B_0 \cap B_1 \cap \ldots \cap B_m$. Obviously $\mathbf{P}(A_n \cap B_0 \cap B_1 \cap \ldots \cap B_n) \neq 0$. Hence $A_n \cap B_0 \cap B_1 \cap \ldots \cap B_n$ is non-void. In view of a) $B_0 \supseteq B_1 \supseteq \ldots \supseteq B_n$, and in view of b) $B_n \subseteq A_n$. Obviously there exists a path $\omega = b_0 b_1 \ldots b_n \ldots$ such that $B_n = [b_0 b_1 \ldots b_n]$. It is clear that $\omega \in B_n \subseteq A_n$, and hence the intersection of A_n is non-void.

PROOF OF THEOREM A. We extend the space E to $\bar{E}$ adding to E one more point $*$. We put

$$\bar{p}(a_0, a_1, \ldots, a_n) = \begin{cases} p(a_0, a_1, \ldots, a_n), & \text{when } a_0, a_1, \ldots, a_n \in E, \\ p(a_0, a_1, \ldots, a_m), & \text{when } a_0, a_1, \ldots, a_m \in E,\ a_{m+1} = \ldots = a_n = *, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that $\bar{p}$ satisfies (0.4). Let $\bar{\Omega}_\infty$ be a space of non-terminating paths and $\bar{\mathscr{F}}$ a σ-algebra in this space generated by cylindrical sets. We reject from each path in $\bar{\Omega}_\infty$ containing the element $*$ the part of it beginning with the first asterisk (paths not containing $*$ are left unchanged). So we obtain a mapping α of $\bar{\Omega}_\infty$ into Ω.

By Theorem D there exists a measure $\bar{\mathbf{P}}$ on the σ-algebra $\bar{\mathscr{F}}$ such that $\bar{\mathbf{P}}[a_0 a_1 \ldots a_n] = \bar{p}(a_0, a_1, \ldots, a_n)$. The formula

$$\mathbf{P}(A) = \bar{\mathbf{P}}[\alpha^{-1}(A)], \qquad (A \in \mathscr{F})$$

defines a measure on $\mathscr{F}$, where $\mathbf{P}[a_0 \dots a_n] = \overline{p}(a_0, a_1, \dots, a_n) = p(a_0, a_1, \dots, a_n)$ for $a_0, a_1, \dots, a_n \in E$.

PROOF OF THEOREM B. The formula

$$\hat{a}_n = \begin{cases} a_n & \text{if} \quad n = 0, \\ a_{2n-1} & \text{if} \quad n > 0, \\ a_{-2n} & \text{if} \quad n < 0 \end{cases}$$

defines a mapping of Ω_∞ into $\hat{\Omega}$. (Under this mapping the path $a_0a_1a_2a_3a_4 \dots$ *goes into* $\dots a_4a_2a_0a_1a_3 \dots$). We put

$$p'(a_0, a_1, a_2, \dots, a_{2k}) = p^k_{-k}[a_{2k}, a_{2k-2}, \dots, a_2, a_0, a_1, \dots, a_{2k-1}],$$
$$p'(a_0, a_1, a_2, \dots, a_{2k-1}) = p^k_{-k+1}[a_{2k-2}, \dots, a_2, a_0, a_1, \dots, a_{2k-1}].$$

By (3′) - (3″) the function p' satisfies (0.4). By Theorem D a measure $\mathbf{P}'$ can be constructed in Ω_∞ such that $\mathbf{P}'[a_0a_1 \dots a_n] = p'(a_0, a_1, \dots, a_n)$. We define in $\hat{\Omega}$ the measure

$$\mathbf{P}(A) = \mathbf{P}'[\alpha^{-1}(A)].$$

It is easy to see that for it

$$\mathbf{P}[a_{-k}, \dots, a_0, \dots, a_k]^k_{-k} = p^k_{-k}(a_{-k}, \dots, a_0, \dots, a_k).$$

Using the additivity of p and the properties (3′) - (3″) of p^n_m, it is easy to prove that for any $m \leqslant n$

$$\mathbf{P}[a_m \dots a_n]^n_m = p^n_m(a_m, \dots, a_n).$$

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