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Maximally reducible monodromy of bivariate hypergeometric systems

T. M. Sadykov and S. Tanabé

Abstract. We investigate the branching of solutions of holonomic bivariate Horn-type hypergeometric systems. Special attention is paid to invariant subspaces of Puiseux polynomial solutions. We mainly study Horn systems defined by simplicial configurations and Horn systems whose Ore–Sato polygons are either zonotopes or Minkowski sums of a triangle and segments proportional to its sides. We prove a necessary and sufficient condition for the monodromy representation to be maximally reducible, that is, for the space of holomorphic solutions to split into a direct sum of one-dimensional invariant subspaces.

Keywords: hypergeometric system of equations, monodromy representation, monodromy reducibility, intertwining operator.

§ 1. Introduction

Computing the monodromy group of a differential equation or a system of such equations is a notoriously difficult problem in the analytic theory of linear differential equations. One of the reasons for this is that the computation of the monodromy group requires a complete understanding of the structure of the solution space of the system, including the dimension of this space, a basis of it, the fundamental group of the complement of the singularities of the system as well as the analytic continuation and branching properties of the chosen basis in the complement of the singularities.

The purpose of this paper is to compute the monodromy groups of certain families of bivariate systems of partial differential equations of hypergeometric type and to investigate their properties. It uses and extends results in [1] and [2]. While the monodromy group of the classical second-order Gauss hypergeometric differential equation has been computed by Schwarz and the monodromy of the ordinary generalized hypergeometric equation has been described in [3], the problem of finding the

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monodromy group of a general hypergeometric system of partial differential equations remains unsolved despite all the effort and several well-understood special cases (see [4], [5] and the references therein).

The original motivation for our results goes back to [6] where the authors posed the problem of describing those Gelfand–Kapranov–Zelevinsky non-confluent hypergeometric systems (see [7]) whose solution space contains a non-zero rational function for a suitable choice of its parameters. In terms of monodromy, this is equivalent to the existence of a one-dimensional subspace of the space of holomorphic solutions of the Gelfand–Kapranov–Zelevinsky (GKZ) system on which the monodromy acts trivially.

In this paper, we solve the closely related problem of describing all holonomic bivariate hypergeometric systems in the sense of Horn (see [8] and the references therein) whose solution space splits into a direct sum of one-dimensional monodromy-invariant subspaces (Theorem 6.1). Throughout the paper, we will call such a monodromy representation *maximally reducible*.

The relation between the GKZ and Horn hypergeometric systems was studied in detail in [8], § 5: for any GKZ system there is a canonically defined Horn system and a naturally defined bijective map from a subspace of the space of its analytic solutions to the space of solutions of the GKZ system. The solutions of the Horn system that are not taken into account by this map are its persistent Puiseux polynomial solutions in the sense of Definition 2.10 below. Here and throughout the paper, by a Puiseux polynomial we mean a finite linear combination of monomials with (in general) arbitrary complex exponents. As announced in [8], Theorem 5.3, the persistent polynomial solutions span the cokernel of the map from GKZ solutions to the solutions of the associated Horn system.

In our set-up, the above-mentioned question in [6] can be answered as follows. The dimension of the space of non-persistent Puiseux polynomial solutions of a Horn system is equal to that of the space of Puiseux polynomial solutions of the corresponding GKZ system. For a bivariate Horn system, a full characterization of its persistent polynomial solutions is given in Proposition 2.12 and Corollary 4.2.

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§ 2. Notation, definitions and preliminaries

The following notation will be used throughout the paper: n is the dimension of the complex space with the coordinate x; m is the number of rows in the matrix defining the Horn system;

$$\nu(a_1, b_1; a_2, b_2) \equiv \nu \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

is the index of the two vectors (a_1, b_1), (a_2, b_2) (see Definition 2.6); $|m| = \sum_{i=1}^{n} m_i$ and $m! = m_1! \cdots m_n!$ for $m = (m_1, \dots, m_n)$; $x^m = x_1^{m_1} \cdots x_n^{m_n}$ for $x = (x_1, \dots, x_n)$

and $m = (m_1, \ldots, m_n)$; $\mathbb{Z}_{\geqslant 0}$ is the set of non-negative integers, $\mathbb{Z}_{\leqslant 0}$ is the set of non-positive integers; $\mathrm{Horn}(\varphi)$ is the Horn hypergeometric system defined by the Ore–Sato coefficient φ (see Definition 2.3); $\mathrm{Horn}(A, c)$ stands for the Horn hypergeometric system defined by the Ore–Sato coefficient (2.2), where $t_i = 1$ for all $i = 1, \ldots, n$ and $U(s) \equiv 1$ (see the construction after Definition 2.3); $\Psi(\varphi)$ is the subspace of Puiseux polynomial solutions of the Horn system defined by the Ore–Sato coefficient φ (see Definition 2.3); $\Psi_0(\varphi) \subset \Psi(\varphi)$ is the subspace of persistent Puiseux polynomial solutions of the Horn system defined by the Ore–Sato coefficient φ (see Definition 2.10); $\mathcal{F}$ is the set of all pure fully supported solutions of a Horn system (observe that it is in general not a vector subspace since the intersection of the domains of convergence of all elements in $\mathcal{F}$ may be empty); $\mathcal{F}_{x^{(0)}}$ is the vector space of fully supported solutions of a Horn system which converge at a non-singular point $x^{(0)}$; $\mathcal{A}(\varphi)$ is the amoeba of the singularity of an Ore–Sato coefficient φ; ${}^c\mathcal{A}(\varphi)$ is the complement of this amoeba (see Definition 5.1); $C^\vee$ is the dual of a convex cone C; $\mathcal{P}(\varphi)$ is the polygon of the Ore–Sato coefficient φ (see Definition 2.5); for an Ore–Sato coefficient φ and $\zeta \in \mathbb{R}^n$ we set

$$M(\varphi, \zeta) = \begin{cases} \text{the connected component of} & \\ \text{the set } {}^c\mathcal{A}(\varphi)\text{, which contains } \zeta, & \text{if } \zeta \in {}^c\mathcal{A}(\varphi), \\ \mathbb{R}^n, & \text{if } \zeta \in \mathcal{A}(\varphi); \end{cases}$$

$S(\mathrm{Horn}(A, c))$ is the space of solutions of the system $\mathrm{Horn}(A, c)$ that are holomorphic outside the singular hypersurface.

Definition 2.1. A formal Laurent series

$$\sum_{s \in \mathbb{Z}^n} \varphi(s) x^s \tag{2.1}$$

is said to be *hypergeometric* if for every $j = 1, \ldots, n$ the quotient of its adjacent coefficients $\varphi(s + e_j)/\varphi(s)$ is a rational function of the indices of summation $s = (s_1, \ldots, s_n)$. Throughout the paper we denote this rational function by $P_j(s)/Q_j(s + e_j)$. Here $\{e_j\}_{j=1}^n$ is the standard basis of the lattice $\mathbb{Z}^n$. By the *support* of this series we mean the subset of $\mathbb{Z}^n$ on which $\varphi(s) \neq 0$. We say that such a series is *fully supported* if the convex hull of its support contains (a shift of) an open n-dimensional cone (see Theorem 7 in [9] and Theorem 5.3 below for the description of the domain of convergence of a hypergeometric Laurent series).

A *hypergeometric function* is a (multi-valued) analytic function obtained by means of the analytic continuation of a hypergeometric series with a non-empty domain of convergence along all possible paths.

Theorem 2.2 (O. Ore, M. Sato [10], [11]). *The coefficients of a hypergeometric series are given by the formula*

$$\varphi(s) = t^s U(s) \prod_{i=1}^{m} \Gamma(\langle \mathbf{A}_i, s \rangle + c_i), \tag{2.2}$$

where $t^s = t_1^{s_1} \cdots t_n^{s_n}$, $t_i, c_i \in \mathbb{C}$, $\mathbf{A}_i = (A_{i,1}, \ldots, A_{i,n}) \in \mathbb{Z}^n$, $i = 1, \ldots, m$, *and* $U(s)$ *is a product of some rational function and a periodic function* $\phi(s)$ *such that* $\phi(s + e_j) = \phi(s)$ *for all* $j = 1, \ldots, n$.

A precise description of the rational function factor of $U(s)$ is available in [11], Appendix (A.3).

We will call any function of the form (2.2) *the Ore–Sato coefficient of a hypergeometric series.* We remark that in view of the formula

$$\sin(\pi z)\Gamma(1-z)\Gamma(z)=\pi,$$

the following functions are admitted as Ore–Sato coefficients:

$$\varphi(s)=t^s\prod_{i\in\mathbf{I}}\Gamma(\langle\mathbf{A}_i,s\rangle+c_i)\prod_{j\notin\mathbf{I}}\frac{e^{\pi\sqrt{-1}(\langle\mathbf{A}_j,s\rangle+c_j)}}{\Gamma(1-\langle\mathbf{A}_j,s\rangle-c_j)},$$

where $\mathbf{I}\subset\{1,\ldots,m\}$.

Given the above data $(t_i,c_i,\mathbf{A}_i,U(s))$ which determine the coefficient of a hypergeometric series, it is straightforward to compute the rational functions $P_j(s)/Q_j(s+e_j)$ by means of the Γ-function identity. The converse requires solving a system of difference equations which is soluble only under certain compatibility conditions on P_j, Q_j. A careful analysis of this system was performed in [12].

In this paper, the Ore–Sato coefficient (2.2) plays the role of a primary object which generates everything else: the series, the system of differential equations, the algebraic hypersurface containing the singularities of its solutions, the amoeba of its defining polynomial and, finally, the most complicated object with the richest structure: the monodromy group of the hypergeometric system of differential equations. We also assume that $m\geqslant n$ since otherwise the corresponding hypergeometric series (2.1) is just a linear combination of hypergeometric series in fewer variables (multiplied by an arbitrary function of the remaining variables that makes the system non-holonomic) and n can be decreased to satisfy the inequality above.

Definition 2.3. A formal Laurent series $\sum_{s\in\mathbb{Z}^n}\varphi(s)x^s$ whose coefficients satisfy the relations $\varphi(s+e_i)/\varphi(s)=P_j(s)/Q_j(s+e_j)$, is a formal solution of the following system of partial differential equations of hypergeometric type:

$$(x_jP_j(\theta)-Q_j(\theta))f(x)=0,\qquad j=1,\ldots,n. \tag{2.3}$$

Here $\theta=(\theta_1,\ldots,\theta_n)$, $\theta_j=x_j\frac{\partial}{\partial x_j}$. The system (2.3) will be referred to as *the Horn hypergeometric system defined by the Ore–Sato coefficient* $\varphi(s)$ (see [10]) and denoted by $\mathrm{Horn}(\varphi)$. We denote the space of solutions of $\mathrm{Horn}(\varphi)$ by $S(\mathrm{Horn}(\varphi))$. Unless otherwise stated, we consider only holonomic Horn hypergeometric systems, that is, $\mathrm{rank}(\mathrm{Horn}(\varphi))$ is always assumed to be finite. A necessary and sufficient condition for the system $\mathrm{Horn}(\varphi)$ to be holonomic was established in [13], Theorem 6.3.

We will often be dealing with the important special case of the Ore–Sato coefficient (2.2) when $t_i=1$ for all $i=1,\ldots,n$ and $U(s)\equiv 1$. The Horn system associated with such an Ore–Sato coefficient will be denoted by $\mathrm{Horn}(A,c)$, where A is the matrix with rows $\mathbf{A}_1,\ldots,\mathbf{A}_m\in\mathbb{Z}^n$ and $c=(c_1,\ldots,c_m)\in\mathbb{C}^m$.

In this case the following differential operators $P_j(\theta)$ and $Q_j(\theta)$ explicitly determine the system (2.3):

$$P_j(s) = \prod_{i\colon A_{i,j}>0} \prod_{\ell_j^{(i)}=0}^{A_{i,j}-1} (\langle \mathbf{A}_i, s\rangle + c_i + \ell_j^{(i)}),$$

$$Q_j(s) = \prod_{i\colon A_{i,j}<0} \prod_{\ell_j^{(i)}=0}^{|A_{i,j}|-1} (\langle \mathbf{A}_i, s\rangle + c_i + \ell_j^{(i)}).$$

Throughout the paper we shall write $H_j(A,c) = x_j P_j(\theta) - Q_j(\theta),\ j = 1, \dots, n$.

Definition 2.4. The Ore–Sato coefficient (2.2), the corresponding hypergeometric series (2.1), and the associated hypergeometric system (2.3) are said to be *non-confluent* if

$$\sum_{i=1}^{m} \mathbf{A}_i = 0. \tag{2.4}$$

It is well known that a non-confluent holonomic hypergeometric system is regular (see, for example, [13], Theorem 6.3), that is, every solution has at most polynomial growth under a sectorial approach to any of its singular loci.

Definition 2.5. Using, if necessary, the following version of the Gauss multiplication formula for the Γ-function and $N \in \mathbb{N}$:

$$\Gamma(\langle \mathbf{A}_i, s\rangle + c_i) = \frac{N^{\langle \mathbf{A}_i, s\rangle + c_i}}{(2\pi)^{(N-1)/2}\sqrt{N}} \times \Gamma\left(\frac{\langle \mathbf{A}_i, s\rangle + c_i}{N}\right)\Gamma\left(\frac{\langle \mathbf{A}_i, s\rangle + c_i + 1}{N}\right)\cdots\Gamma\left(\frac{\langle \mathbf{A}_i, s\rangle + c_i + N - 1}{N}\right),$$

we may assume without loss of generality that the non-zero components of the vector $\mathbf{A}_i$ are relatively prime for every $i = 1, \dots, m$.

Let l_i be the generator of the sublattice $\{s \in \mathbb{Z}^2 \colon \langle \mathbf{A}_i, s\rangle = 0\}$ and let k_i be the number of those elements in $\{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ that coincide with $\mathbf{A}_i$. By Minkowski's theorem, the non-confluence condition (2.4) implies that there is a uniquely determined (up to a shift) integer convex polygon whose sides are shifts of the vectors $k_i l_i$. Note that the vectors $\mathbf{A}_1, \dots, \mathbf{A}_m$ are the outer normals to the sides of this polygon. The number of sides of this polygon coincides with the number of different elements in the set of vectors $\{\mathbf{A}_1, \dots, \mathbf{A}_m\}$. We call this polygon *the polygon of the Ore–Sato coefficient* (2.2) and denote it by $\mathcal{P}(\varphi)$.

Conversely, any convex integer polygon determines an $m \times 2$ matrix whose rows add up to the zero vector. Hence, together with a vector of parameters, it determines a non-confluent hypergeometric system of equations. We will denote this system by $\mathrm{Horn}(A(\mathcal{P}), c)$. This relation is illustrated by Example 4.5.

Definition 2.6. For a pair of vectors (a_1, b_1), $(a_2, b_2) \in \mathbb{Z}^2$ we set

$$\nu(a_1, b_1; a_2, b_2) = \begin{cases} \min(|a_1 b_2|, |b_1 a_2|), & \text{if } (a_1, b_1),\ (a_2, b_2) \\ & \text{are in opposite open quadrants} \\ & \text{of the lattice } \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

The number $\nu(a_1, b_1; a_2, b_2)$ is called the *index* associated with the lattice vectors (a_1, b_1) and (a_2, b_2). The index of the rows of a 2×2 matrix M will be denoted by $\nu(M)$.

Definition 2.7. By the *initial exponent* of a multiple hypergeometric series

$$x^{\alpha} \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s$$

we mean the vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. Observe that the initial exponent of such a series is only defined up to shifts by integer vectors. However, in view of Proposition 3.11 and Corollary 3.13 (to be proved in § 3), this is exactly what we need for computing the monodromy of hypergeometric systems.

Definition 2.8. The support S of a series solution of the system (2.3) is said to be *irreducible* if there are no series solutions of (2.3) supported on a non-empty proper subset of S.

Definition 2.9. A series solution $f(x) = \sum_{\alpha \in \Lambda} c_{\alpha} x^{\alpha}$ with irreducible support is said to be *pure* if we have $\alpha \equiv \beta \mod \mathbb{Z}^n$ for all $\alpha, \beta \in \Lambda$. In other words, a Puiseux series solution (in particular, a Puiseux polynomial solution) centred at the origin and with irreducible support is said to be pure if it is given by the product of a monomial and a Laurent series. A set $\{f_k(x)\}_{k=1}^{r}$ of linearly independent series is called a *pure basis* of the solution space of a Horn system in a neighbourhood of a non-singular point $x \in \mathbb{C}^n$ if all series f_k converge in a neighbourhood of x, determine pure solutions, and together span a vector space whose dimension equals the holonomic rank of the Horn system.

Since a Horn system has polynomial coefficients, it follows that every Puiseux series solution (centred at the origin) of a holonomic Horn system can be written as a finite linear combination of pure solutions. Here the holonomic property is used to ensure that the linear combination is finite. Moreover, in a neighbourhood of a non-singular point, a pure basis in the local solution space of a Horn system is defined uniquely up to a permutation and multiplication of its elements by non-zero constants. In this paper we will neglect this inessential difference between pure bases of solutions to hypergeometric systems. If necessary, we will explicitly specify the ordering of the elements of a pure basis and the way they are normalized. A pure basis of a hypergeometric system is especially convenient for monodromy computations since the monodromy matrices are diagonal in the domain of convergence of the basis series.

Definition 2.10. A Puiseux polynomial solution of a hypergeometric system $\mathrm{Horn}(A, c)$ is said to be persistent if its support remains finite under arbitrary small perturbations of the vector c of parameters.

For example, the first solution of the hypergeometric system (3.5) is a persistent Puiseux monomial since it remains monomial for any $(c_1, c_2, c_3) \in \mathbb{C}^3$. The second solution of (3.5) is a (Puiseux) polynomial only for $-(c_1 + c_2 + c_3) \in \mathbb{N}$ and, therefore, is not a persistent polynomial solution. This notion is also illustrated in Examples 4.5, 6.8, 6.9.

We write $\Psi(\varphi)$ for the vector space of all (not necessarily persistent) Puiseux polynomial solutions of the Horn system defined by the Ore–Sato coefficient $\varphi(s)$, and $\Psi_0(\varphi)$ for the space of all persistent polynomial solutions of this system. The following proposition is an immediate consequence of Definition 2.10.

Proposition 2.11. *For the Ore–Sato coefficient φ defined by (2.2) with a generic vector $c = (c_1, \ldots, c_m) \in \mathbb{C}^m$ of parameters, every Puiseux polynomial solution of the corresponding hypergeometric system* $\mathrm{Horn}(\varphi)$ *is persistent. In other words, $\Psi(\varphi) = \Psi_0(\varphi)$ for almost all $c \in \mathbb{C}^m$.*

The next proposition is proved by analyzing the difference equations for the coefficients of a hypergeometric polynomial (see [8]).

Proposition 2.12. *Let $\varphi(s)$ be an Ore–Sato coefficient and let $f(x)$ be a Puiseux polynomial solution of* $\mathrm{Horn}(\varphi)$. *If this polynomial solution is persistent, then there is a multi-index $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ with distinct components such that for any $s \in \operatorname{supp} f$ and $\ell = 1, \ldots, n$ there are $j \in I$ and $k \in \{0, \ldots, |A_{j,\ell}| - 1\}$ with $\langle \mathbf{A}_j, s\rangle + c_j + k = 0$.*

Definition 2.13. We say that the Ore–Sato coefficient $\varphi(s) = \prod_{i=1}^{m} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i)$ (as well as the corresponding hypergeometric Horn system $\mathrm{Horn}(\varphi(A, c))$) is *resonant* if there is a multi-index $I = (i_1, \ldots, i_k)$, $1 \leqslant i_1 < \cdots < i_k \leqslant m$, $1 \leqslant k \leqslant m$, such that for every linear relation $a_{i_1}\mathbf{A}_{i_1} + \cdots + a_{i_k}\mathbf{A}_{i_k} = 0$ with integer and relatively prime coefficients $a_{i_1}, \ldots, a_{i_k} \in \mathbb{Z}$ we have $a_{i_1}c_{i_1} + \cdots + a_{i_k}c_{i_k} \in \mathbb{Z}$. The hypergeometric system $\mathrm{Horn}(\varphi(A, c))$ is said to be *maximally resonant* if the above holds for any multi-index $I = (i_1, \ldots, i_k)$ such that the corresponding integer vectors $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_k}$ are linearly dependent.

The notion of resonance is illustrated by the following example of a hypergeometric system of the smallest possible rank.

Example 2.14. To simplify the notation here and throughout the paper, systems of linear homogeneous differential equations will be specified by giving a set of their generating operators. The Horn system

$$\begin{gathered} x_1(\theta_1 + \theta_2 + c_3) - (\theta_1 + c_1), \\ x_2(\theta_1 + \theta_2 + c_3) - (\theta_2 + c_2) \end{gathered} \tag{2.5}$$

is the only (up to a monomial change of variables defined by a unimodular matrix) bivariate hypergeometric system whose holonomic rank equals 1 for all values of the parameters $c_1, c_2, c_3 \in \mathbb{C}$. The only solution of (2.5) is $x_1^{-c_1} x_2^{-c_2} (1 - x_1 - x_2)^{c_1 + c_2 - c_3}$. This system is resonant (and also maximally resonant since it has holonomic rank 1) if and only if $c_1 + c_2 - c_3 \in \mathbb{Z}$. The monodromy of (2.5) depends only on the values of c_1, c_2, c_3 modulo $\mathbb{Z}$ and is a subgroup of $\mathbb{C}$ with the three generators $\{\exp(2\pi\sqrt{-1}\, c_1), \exp(2\pi\sqrt{-1}\, c_2), \exp(2\pi\sqrt{-1}\, c_3)\}$ in the non-resonant case. In the resonant case it has two generators, and in the case of trivial monodromy it consists of the identity.

The crucial importance of the notion of resonance will be revealed in the theorems and examples that follow. Roughly speaking, the parameters of a hypergeometric system are non-resonant if all solutions are either fully supported Puiseux series (centred at the origin) or persistent Puiseux polynomials. Resonant parameters may correspond to non-holonomic systems, systems with non-persistent polynomial solutions, non-fully supported series solutions, or logarithmic solutions which admit no Puiseux series expansions (centred at the origin) at all. For example, the hypergeometric system (2.6) is maximally resonant.

Definition 2.15. A solution $f(x)$ of the system of differential equations $\mathrm{Horn}(\varphi)$ at a non-singular point $x^{(0)} \in \mathbb{C}^n$ is said to *generate* a vector subspace $L \subset S(\mathrm{Horn}(\varphi), V(x^{(0)}))$ of the space of all holomorphic solutions of $\mathrm{Horn}(\varphi)$ in a sufficiently small simply connected neighbourhood $V \ni x^{(0)}$ if every element of L can be represented as a linear combination of branches of $f(x)$ on $V(x^{(0)})$. A function is called a *generating solution* of a system of equations if it generates the whole space of its holomorphic solutions at any non-singular point. In § 4 we will construct generating solutions for two important families of hypergeometric systems (see Propositions 4.4 and 4.7).

Example 2.16. The maximally resonant Horn system defined by the Ore–Sato coefficient $\varphi(s_1, s_2) = \Gamma(s_1)\Gamma(s_2)\Gamma(s_1+s_2)\Gamma(-s_1)^2\Gamma(-s_2)^2$ is generated by the differential operators

$$x_1\theta_1(\theta_1+\theta_2)-\theta_1^2, \qquad x_2\theta_2(\theta_1+\theta_2)-\theta_2^2. \tag{2.6}$$

This system has holonomic rank 4. Its space of holomorphic solutions is spanned by

$$1, \quad \log x_1, \quad \log x_2, \quad \log x_1 \log x_2 + \mathrm{PolyLog}(2, x_1) + \mathrm{PolyLog}(2, x_2).$$

Here $\mathrm{PolyLog}(2, z) = \sum_{k=1}^{\infty} z^k/k^2$. The resultant of the principal symbols of (2.6) equals $x_1x_2(x_1-1)(x_2-1)(x_1+x_2-1)$. Using the properties of $\mathrm{PolyLog}(2, z)$ (see [14]), we conclude that the monodromy group of (2.6) is generated by the four matrices

$$M_{x_1=0} = \begin{pmatrix} 1 & 0 & 2\pi\sqrt{-1} & 0 \\ 0 & 1 & 0 & 2\pi\sqrt{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_{x_2=0} = \begin{pmatrix} 1 & 2\pi\sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi\sqrt{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{x_1=1} = \begin{pmatrix} 1 & -2\pi\sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_{x_2=1} = \begin{pmatrix} 1 & 0 & -2\pi\sqrt{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This monodromy representation shows that $\log x_1 \log x_2 + \mathrm{PolyLog}(2, x_1) + \mathrm{PolyLog}(2, x_2)$ is a generating solution of $S(\mathrm{Horn}(\varphi))$.

If the monodromy representation of the entire solution space $S(\mathrm{Horn}(\varphi))$ is irreducible, then the system admits a generating solution. On the other hand, the monodromy representation can be reducible even when $S(\mathrm{Horn}(\varphi))$ contains a generating solution (see Example 2.16).

Our main result (Theorem 6.1) describes bivariate hypergeometric systems whose solution spaces split into one-dimensional invariant subspaces for almost all values of the parameters. The following definition will be used throughout the paper.

Definition 2.17. We say that the monodromy representation of a system of equations is *maximally reducible* if its solution space splits into a direct sum of one-dimensional invariant subspaces.

§ 3. The structure of the space of holomorphic solutions of a Horn system

3.1. Integral representations and calculation of multidimensional residues. Our main tool for computing the analytic continuation of a hypergeometric series is the Mellin–Barnes integral. The following theorem gives an integral representation for solutions of a hypergeometric system.

Theorem 3.1 [12]. *Let*

$$\psi(s)=\prod_{j=1}^{m}\Gamma(\langle\mathbf{A}_j,s\rangle+c_j)$$

be a non-confluent Ore–Sato coefficient. We put $\varphi(s)=\psi(s)\phi(s)$, where $\phi(s)$ is a periodic meromorphic function with period 1 *in every coordinate direction. Then the Mellin–Barnes integral*

$$\mathrm{MB}(\varphi,\mathcal{C}):=\int_{\mathcal{C}}\varphi(s)x^s\,ds \tag{3.1}$$

represents a solution of $\mathrm{Horn}(A,c)$*. Here $\mathcal{C}$ is any n-dimensional contour which is homologous to its unitary shifts in any real direction in the complement of the singularities of the integrand in* (3.1).

The next proposition is proved in the same way as Theorem 3.1, by computing the multidimensional residues at simple singularities. It enables us to convert a multiple hypergeometric series into an iterated Mellin–Barnes integral.

Proposition 3.2. *Suppose that $\psi(k)/k!$ is a non-confluent Ore–Sato coefficient with generic parameters, A is a non-singular square integer matrix with rows $\mathbf{A}_1,\dots,\mathbf{A}_n$, and $c\in\mathbb{C}^n$. For a sufficiently small $\varepsilon>0$ and $k\in\mathbb{N}^n$ put $\tau(k)=\{s\in\mathbb{C}^n\colon |\langle\mathbf{A}_j,s\rangle+c_j+k_j|=\varepsilon$ for all $j=1,\dots,n\}$ and define $\mathcal{C}=\sum_{k\in\mathbb{N}^n}\tau(k)$. Then*

$$\sum_{k\in\mathbb{N}^n}\frac{(-1)^{|k|}}{k!}\,\psi(k)x^{Ak+c}\\ =\frac{1}{(2\pi\sqrt{-1}\,)^n|A|}\int_{\mathcal{C}}\prod_{j=1}^{n}\Gamma\bigl((-A^{-1}(s-c))_j\bigr)\psi(A^{-1}(s-c))x^s\,ds.$$

The following theorem gives a solution of the hypergeometric system $\mathrm{Horn}(A, c)$ in the form of a multiple Mellin–Barnes integral and enables us to convert it into a hypergeometric (Puiseux) series by computing the residues at a distinguished family of singularities of the integrand.

Theorem 3.3 [12]. *Let A be an $m \times n$ integer matrix of maximal rank n with rows $\mathbf{A}_1, \dots, \mathbf{A}_m$, and let $I = (i_1, \dots, i_n) \subset \{1, \dots, m\}$ be a multi-index such that the matrix $\mathbf{A}_I$ with rows $\mathbf{A}_{i_1}, \dots, \mathbf{A}_{i_n}$ is non-singular. For a sufficiently small $\varepsilon > 0$ and $k \in \mathbb{N}^n$ put $\tau_I(k) = \{s \in \mathbb{C}^n \colon |\langle \mathbf{A}_{i_j}, s\rangle + c_{i_j} + k_j| = \varepsilon \text{ for all } j = 1, \dots, n\}$ and define $\mathcal{C}_I = \sum_{k \in \mathbb{N}^n} \tau_I(k)$. Then for generic $c \in \mathbb{C}^m$ and $c_I = (c_{i_1}, \dots, c_{i_n})$ the following Mellin–Barnes integral satisfies the system of equations* $\mathrm{Horn}(A, c)$ *and can be represented in the form of a hypergeometric* (*Puiseux*) *series*:

$$\frac{1}{(2\pi\sqrt{-1}\,)^n} \int_{\mathcal{C}_I} \prod_{j=1}^{m} \Gamma(\langle \mathbf{A}_j, s\rangle + c_j) x^s \, ds$$
$$= \sum_{k \in \mathbb{N}^n} \frac{(-1)^{|k|}}{k!\, |\mathbf{A}_I|} \prod_{j \notin I} \Gamma\bigl(\langle \mathbf{A}_j, -\mathbf{A}_I^{-1}(k + c_I)\rangle + c_j\bigr) x^{-\mathbf{A}_I^{-1}(k + c_I)}. \tag{3.2}$$

3.2. Holonomic rank formulae. To state the main result (Theorem 3.7) of this section, we introduce the following notion.

Definition 3.4. For $m \geqslant n$ let A be an $m \times n$ integer matrix of rank n with rows $\mathbf{A}_1, \dots, \mathbf{A}_m$ and let $c \in \mathbb{C}^m$ be a vector of parameters. Let $I = (i_1, \dots, i_n)$ be a multi-index such that the square matrix $\mathbf{A}_I$ with rows $\mathbf{A}_1, \dots, \mathbf{A}_m$ is non-singular. Denote the vector $(c_{i_1}, \dots, c_{i_n})$ by c_I. The hypergeometric system $\mathrm{Horn}(\mathbf{A}_I, c_I)$ will be referred to as *an atomic system associated with the system* $\mathrm{Horn}(A, c)$. The number of atomic systems associated with a hypergeometric system $\mathrm{Horn}(A, c)$ is equal to the number of maximal non-singular square submatrices of A.

It follows from Theorem 1.3 in [15] that, as far as the supports of series solutions are concerned, a generic hypergeometric system is made up of associated atomic systems. More precisely, the set of supports of solutions of a hypergeometric system with generic parameters consists of supports of solutions of the associated atomic systems. This is illustrated in Example 6.8 below (see also Fig. 3 in § 6). In particular, the initial exponents of Puiseux polynomial solutions of a hypergeometric system are precisely the initial exponents of Puiseux polynomials which satisfy the associated atomic systems.

Proposition 3.5. *For every solution $v(x)$ of an atomic system associated with a non-confluent holonomic system* $\mathrm{Horn}(A, c)$ *with a generic vector of parameters $c \in \mathbb{C}^m$, there is a solution $u(x) \in S(\mathrm{Horn}(A, c))$ whose support coincides with the support of $v(x)$.*

Proof. Consider the non-confluent holonomic system $\mathrm{Horn}(A, c)$ defined by the Ore–Sato coefficient

$$\varphi(s) = \phi(s) \prod_{i=1}^{m} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i)$$

with a suitable meromorphic periodic function $\phi(s)$.

All solutions of the associated atomic system $\mathrm{Horn}(A_{\mathbf{I}}, c_{\mathbf{I}})$, $\mathbf{I} = (i_1, \dots, i_n) \subset \{1, \dots, m\}$ admit an integral representation

$$v(x) = \int_{C_{\mathbf{I}}} \prod_{i \in \mathbf{I}} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i)\phi(s)x^s \, ds$$

for a suitable choice of the contour $C_{\mathbf{I}}$ and the periodic function $\psi(s)$.

Using this integral representation, we obtain the following solution of $\mathrm{Horn}(A, c)$:

$$u(x) = \int_{C_{\mathbf{I}}} \prod_{i \in \mathbf{I}} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i) \prod_{j \notin \mathbf{I}} \Gamma(\langle \mathbf{A}_j, s\rangle + c_j)\phi(s)x^s \, ds.$$

Since the vector of parameters $c \in \mathbb{C}^m$ is generic, we may assume that the contour $C_{\mathbf{I}}$ contains only poles of multiplicity n of the product $\prod_{i \in \mathbf{I}} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i)$, which are moreover disjoint from the poles of the product $\prod_{j \notin \mathbf{I}} \Gamma(\langle \mathbf{A}_j, s\rangle + c_j)\phi(s)$. Thus in a small neighbourhood of the poles of the factor $\prod_{i \in \mathbf{I}} \Gamma(\langle \mathbf{A}_i, s\rangle + c_i)$ the meromorphic function $\prod_{j \notin \mathbf{I}} \Gamma(\langle \mathbf{A}_j, s\rangle + c_j)\phi(s)$ is holomorphic. It follows immediately that the support of $u(x)$ coincides with the support of $v(x)$. □

Remark 3.6. If the vector of parameters $c \in \mathbb{C}^m$ is not generic, then the support of a solution $u(x) \in S(\mathrm{Horn}(A, c))$ of a hypergeometric system can be a proper subset of the support of any solution $v(x) \in S(\mathrm{Horn}(A_I, c_I))$ of the associated atomic system.

Consider the following example:

$$A = ((-1, 2), (2, -1), (-1, -1)), \qquad c = (0, 0, -2).$$

With every solution

$$w(x) = \sum_{m,n \geqslant 0} \mathrm{Res}_{\substack{-s_1+2s_2=-m \\ 2s_1-s_2=-n}} \Gamma(-s_1 + 2s_2)\Gamma(-s_1 - s_2 - 2)\Gamma(2s_1 - s_2)x^s$$

of the hypergeometric system $\mathrm{Horn}(A, c)$ we associate the following solution of an atomic system:

$$v(x) = \sum_{m,n \geqslant 0} \mathrm{Res}_{\substack{-s_1+2s_2=-m \\ 2s_1-s_2=-n}} \Gamma(-s_1 + 2s_2)\Gamma(2s_1 - s_2)x^s.$$

Since the solution space $S(\mathrm{Horn}(A, c))$ is invariant under the action of monodromy, the function

$$u(x) = \frac{1}{2\pi\sqrt{-1}}\bigl(w(x_1 e^{2\pi\sqrt{-1}}, x_2) - w(x_1, x_2)\bigr)$$

is a solution of $\mathrm{Horn}(A, c)$. A straightforward computation shows that

$$u(x) = \frac{\bigl(x_1^{2/3}x_2^{2/3} + \sqrt[3]{x_1} + \sqrt[3]{x_2}\bigr)^2}{3x_1^{4/3}x_2^{4/3}},$$

whence the support of $u(x)$ consists of the six points $\{s \in \mathbb{C}^2 : s_1 - 2s_2 \in \mathbb{Z}_{\geqslant 0}, -2s_1 + s_2 \in \mathbb{Z}_{\geqslant 0}, \ -2 \leqslant s_1 + s_2 \leqslant 0\}$. Observe that the meromorphic function $\Gamma(-s_1 + 2s_2)\Gamma(-s_1 - s_2 - 2)\Gamma(2s_1 - s_2)x^s$ has triple poles at these six points while its other poles are simple.

The following theorem summarizes the main properties of the space of holomorphic solutions of Horn systems that will be used in what follows.

Theorem 3.7. *Assume that the hypergeometric system* $\mathrm{Horn}(A,c)$ *is non-confluent and holonomic. Then the following assertions hold for almost all values of the parameter vector* $c \in \mathbb{C}^m$.

1) *The space of local holomorphic solutions of* $\mathrm{Horn}(A,c)$ *at a non-singular point* $x^{(0)}$ *admits the decomposition*

$$S(\mathrm{Horn}(A,c)) = \Psi \oplus \mathcal{F}_{x^{(0)}}.$$

Here Ψ *is the subspace of persistent Puiseux polynomial solutions and* $\mathcal{F}_{x^{(0)}}$ *is the subspace of fully supported Puiseux series solutions converging at* $x^{(0)}$.

2) *The dimension of the space* $\mathcal{F}_{x^{(0)}}$ *of the Puiseux series (centred at the origin) satisfying* $\mathrm{Horn}(A,c)$ *and converging at* $x^{(0)} \in {}^{c}\mathcal{A}(\varphi(A,c))$ *equals*

$$\dim_{\mathbb{C}} \mathcal{F}_{x^{(0)}} = \sum_{I=(i_1,\dots,i_n)\subset\{1,\dots,m\}} |\det \mathbf{A}_I|,$$

where I *ranges over all multi-indices that satisfy* $M(\varphi(A,c), \mathrm{Log}\, x^{(0)}) \subset \mathrm{Log}\, x^{(1)} - (\mathbf{A}_I^{-1}\mathbb{R}_+^n)^{\vee}$ *for some point* $x^{(1)} \in \mathbb{C}^n$.

3) *The dimension of the space* Ψ_0 *of persistent Puiseux polynomial solutions of the bivariate system* $\mathrm{Horn}(A,c)$ *is given by* $\dim_{\mathbb{C}} \Psi_0 = \sum_{\mathbf{A}_i,\mathbf{A}_j \text{ lin. indep.}} \nu(\mathbf{A}_i,\mathbf{A}_j)$.

Proof. 1) Any Puiseux series solution (centred at the origin) of a Horn system with generic parameters is either a fully supported series or a persistent Puiseux polynomial because all polynomial solutions of such a system are persistent by Proposition 2.11. Indeed, for a polynomial to be a solution of a hypergeometric system, the vector of its exponents must satisfy a system of linear algebraic equations, and the number of these equations is not smaller than the dimension of the space of variables. The assumption about generic parameters implies that the right-hand sides of these equations are also generic. Such a system of equations is soluble only when it is given by a non-singular square matrix. The corresponding solutions of the hypergeometric system are precisely the persistent polynomials. In particular, this means that $\Psi(\varphi) = \Psi_0(\varphi)$ for all Ore–Sato coefficients φ with generic parameters. Since no finite linear combination of elements in $\Psi(\varphi)$ can yield a fully supported Puiseux series, we see that the sum is direct.

2) This follows from part 1 combined with the two-sided Abel lemma (see [9], Lemma 11), which describes a geometric duality between the domain of convergence of a non-confluent hypergeometric series and its support. By part 1, the assumption of generic parameters implies that all non-polynomial solutions of the Horn system in question are fully supported. It is therefore sufficient to consider all such series for each of the atomic hypergeometric systems associated with $\mathrm{Horn}(A,c)$.

3) This is proved in [8], Theorem 6.6. $\square$

The following result (see [8]) gives the holonomic rank of a bivariate non-confluent Horn system with generic parameters.

Theorem 3.8 [8]. *Let A be an $m \times 2$ integer matrix of full rank such that its rows $\mathbf{A}_1, \dots, \mathbf{A}_m$ satisfy $\mathbf{A}_1 + \dots + \mathbf{A}_m = 0$. If $c \in \mathbb{C}^m$ is a generic parameter vector, then the ideal* $\mathrm{Horn}(A, c)$ *is holonomic. Moreover,*

$$\mathrm{rank}(\mathrm{Horn}(A,c)) = \Bigg(\sum_{i\colon A_{i,1}>0} A_{i,1} \Bigg) \Bigg(\sum_{i\colon A_{i,2}>0} A_{i,2} \Bigg) - \sum_{\mathbf{A}_i, \mathbf{A}_j \text{ lin. dep.}} \nu(\mathbf{A}_i, \mathbf{A}_j),$$

where the sum is taken over all linearly dependent pairs $\mathbf{A}_i$, $\mathbf{A}_j$ of rows of A that lie in opposite open quadrants of the lattice $\mathbb{Z}^2$.

Remark 3.9. The conclusion of Theorem 3.8 holds only when the matrix A is non-confluent. For example, the confluent Horn system generated by the operators $x_1(\theta_1 + \theta_2)(\theta_1 + \theta_2 - a) - \theta_1$ and $x_2(\theta_1 + \theta_2)(\theta_1 + \theta_2 - a) - \theta_2$ is holonomic of rank 2. Indeed, if f lies in the kernel of each of these operators, then $f'_{x_1} = f'_{x_2}$ and hence $f = g(x_1 + x_2)$ for a suitable univariate function g. Moreover, $g(t)$ is a solution of the ordinary differential equation $t^2 g''(t) + ((1-a)t-1)g'(t) = 0$. A fundamental system of solutions of this equation is 1, $\Gamma(-a, 1/t)$, where $\Gamma(p, q)$ is the incomplete gamma-function. Thus a basis of the solution space of the Horn system is 1, $\Gamma\big(-a, \frac{1}{x_1+x_2}\big)$. Observe that $\Gamma\big(1, \frac{1}{x_1+x_2}\big) = e^{-1/(x_1+x_2)}$. Thus the holonomic rank of a confluent system can be smaller than the product of the degrees of the operators even if there are neither parallel rows $\mathbf{A}_i, \mathbf{A}_j$ nor persistent polynomial solutions (see Definition 2.10).

Remark 3.10. Although Theorem 3.8 is essentially bivariate, it can be generalized to spaces of variables of arbitrary dimension. Theorems 6.10, 7.13 in [13] provide an explicit combinatorial formula for the holonomic rank of a non-confluent hypergeometric system $\mathrm{Horn}(A, c)$. We choose an $(m - n) \times m$ submatrix B of A with integer coefficients and whose columns span $\mathbb{Z}^{m-n}$ as a lattice, satisfying $B \cdot A = 0 \in \mathbb{Z}^{m-n} \times \mathbb{Z}^n$. Let $g = |\ker(B)/\mathbb{Z}A|$ be the index of the integer lattice generated by the columns of A in its saturation. Then the following formula holds for generic $c \in \mathbb{C}^m$:

$$\mathrm{rank}(\mathrm{Horn}(A,c)) = g \, \mathrm{vol}(B) + \mathrm{rank}(\Psi_0(\varphi)),$$

where $\mathrm{vol}(B)$ is the normalized volume of the convex hull of the columns of B. This formula is a numerical counterpart of the decomposition Theorem 3.7, 1) for the space of holomorphic solutions of a hypergeometric system.

In Example 3.14 we will see that $\mathrm{rank}(\Psi_0) = 1$ since Ψ_0 is generated by f_1, and $\mathrm{rank}(\mathrm{Horn}(A, (c_1, c_2, c_3))) = 2$. In fact, if $-(c_1 + c_2 + c_3) \notin \mathbb{N}$, then the space of fully supported solutions has dimension 1, but if $-(c_1 + c_2 + c_3) \in \mathbb{N}$, then the dimension of the quotient space Ψ/Ψ_0 is 1.

3.3. Monodromy action on the invariant subspace of Puiseux polynomial solutions. Recall that by a Puiseux polynomial we mean a finite linear combination of monomials with (in general) arbitrary complex exponents. Such a polynomial may only have singularities on the union of the coordinate hyperplanes $\{x \in \mathbb{C}^n\colon x_1 \cdots x_n = 0\}$. The set of all Puiseux polynomial solutions of a Horn system is a vector subspace Ψ of the space of its local holomorphic solutions. This subspace is clearly invariant under the action of monodromy.

Let $\{p_k(x)\}_{k=1}^{p}$ be a pure basis of the vector space Ψ (see Definition 2.9). In other words, suppose that $p_k(x) = x^{v_k}\tilde{p}_k(x)$, where $v_k \in \mathbb{C}^n$ and $\tilde{p}_k(x)$ is a Laurent polynomial (that is, a polynomial with integer exponents). Since a Laurent polynomial has no branching, it follows that the branching of this basis is the same as that of a system of monomials $x^{v_1}, \dots, x^{v_p}$, where $v_k \in \mathbb{C}^n$. Thus the branching locus for the solutions of such a Horn system is $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, and the generators of the fundamental group of its complement with base point $(1, \dots, 1)$ are $\gamma_j = (1, \dots, 1, e^{2\pi\sqrt{-1}\,t}, 1, \dots, 1)$, $t \in [0, 1]$, $j = 1, \dots, n$. The corresponding monodromy matrix is given by $M_j = \operatorname{diag}(e^{2\pi\sqrt{-1}\,v_j})$.

3.4. Intertwining operators for Horn systems. The purpose of this subsection is to compute the intertwining operators for the monodromy representations of Horn systems whose parameters differ by integers. This will enable us to conclude that certain monodromy representations are equivalent. The intertwining operators for the monodromy representations of ordinary hypergeometric differential equations were computed in [3].

Recall that $S(\mathrm{Horn}(A, c))$ stands for the vector space of (local) solutions of the hypergeometric system $\mathrm{Horn}(A, c)$. The class of hypergeometric functions is closed under multiplication by Puiseux monomials. More precisely, the operator $x^{\lambda}\bullet$ which multiplies a function by the monomial $x^{\lambda} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ is an isomorphism between vector spaces:

$$x^{\lambda}\bullet\colon S\big(\mathrm{Horn}(A, A\lambda + c)\big) \to S(\mathrm{Horn}(A, c)).$$

Since multiplication by a Laurent monomial does not alter the branching of a function, we conclude that the hypergeometric systems $\mathrm{Horn}(A, c)$ and $\mathrm{Horn}(A, A\lambda + c)$ have the same monodromy for all $\lambda \in \mathbb{Z}^n$.

Proposition 3.11. *Let $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{Z}^n$ be the rows of an integer matrix A of full rank n and let $c \in \mathbb{C}^m$. Then the differential operator*

$$\langle \mathbf{A}_j, \theta \rangle + c_j - 1\colon S\big(\mathrm{Horn}(A, c - e_j)\big) \to S(\mathrm{Horn}(A, c)) \tag{3.3}$$

is an intertwining operator for the monodromy representations of the corresponding Horn systems.

Proof. Let $H_i(A, c)$ be the differential operator defining the ith equation of the hypergeometric system $\mathrm{Horn}(A, c)$ given by (2.3).

The desired result follows since when $A_{i,j} \leqslant 0$ we have

$$\big(\langle \mathbf{A}_j, \theta - e_i \rangle + c_j - 1\big) H_i(A, c - e_j) = H_i(A, c)\big(\langle \mathbf{A}_j, \theta \rangle + c_j - 1\big)$$

while when $A_{i,j} > 0$ we have

$$\big(\langle \mathbf{A}_j, \theta \rangle + c_j - 1\big) H_i(A, c - e_j) = H_i(A, c)\big(\langle \mathbf{A}_j, \theta \rangle + c_j - 1\big). \qquad \square$$

Using these intertwining operators, we establish an analogue of Proposition 2.7 in [3].

Proposition 3.12. *Suppose that $S(\mathrm{Horn}(A,c+\ell)) \supset \Psi_0 \neq \{0\}$ for $\ell \in \mathbb{Z}^n$. Assume that each column of the matrix A with rows $\mathbf{A}_1,\ldots,\mathbf{A}_m$ contains at least one positive and one negative element (see [13], Convention 1.4). Then there is a non-trivial monodromy-invariant subspace of $S(\mathrm{Horn}(A,c))$ of codimension greater than 1. In particular, the monodromy representation of $S(\mathrm{Horn}(A,c))$ is reducible.*

Proof. Let $J \subset \{1,\ldots,m\}$ be the set of indices such that $\ker(\langle \mathbf{A}_j,\theta\rangle + c_j + \ell_j) \cap \Psi_0 \ni x^\alpha \neq 0$ for $j \in J$. We remark here that we can always find a monomial element in Ψ_0 whenever $\Psi_0 \neq \{0\}$. This can be seen as follows. The exponent α_j of any persistent polynomial solution $\sum_{j=1}^{p} c_{\alpha_j} x^{\alpha_j}$ satisfies the relation $-(\mathbf{A}_I \cdot \alpha_j + c_I) \in \mathbb{Z}_{\geqslant 0}$, where we use the same notation $\mathbf{A}_I$, c_I as in Theorem 3.2. Since $-(\mathbf{A}_I \cdot (\alpha_1 \pm e_k) + c_I) \in \mathbb{Z}_{\geqslant 0}$ for the 'initial exponent' $\alpha_1 = \alpha_2 \pm e_k$ and some $k \in \{1,\ldots,n\}$, we obtain that $-(\mathbf{A}_I \cdot e_k) \in \mathbb{Z}_{\geqslant 0}$. This contradicts our assumption on the presence of both positive and negative entries in each column of A. Therefore the operator

$$\langle \mathbf{A}_j,\theta\rangle + c_j + \ell_j \colon S(\mathrm{Horn}(A,c+\ell)) \to S(\mathrm{Horn}(A,c+\ell+e_j))$$

has a non-trivial kernel. Assume that $\ell_j < 0$ and choose a maximal number k_j with $\ell_j \leqslant k_j \leqslant -1$ such that the operator

$$\langle \mathbf{A}_j,\theta\rangle + c_j + k_j \colon S\bigl(\mathrm{Horn}(A,c+\ell+(k_j-\ell_j)e_j)\bigr) \to S\bigl(\mathrm{Horn}(A,c+\ell+(k_j-\ell_j+1)e_j)\bigr)$$

has a non-trivial kernel. This implies that the vector space

$$\prod_{k=1}^{-k_j} (\langle \mathbf{A}_j,\theta\rangle + c_j - k) S\bigl(\mathrm{Horn}(A,c+\ell+(k_j-\ell_j)e_j)\bigr)$$

is an invariant subspace of $S(\mathrm{Horn}(A,c+\ell-\ell_j e_j))$ under the monodromy action. Thus $S(\mathrm{Horn}(A,c+\ell-\sum_{j\in J,\ell_j<0} \ell_j e_j))$ has an invariant subspace of codimension greater than 1. We now consider the vector space

$$\prod_{i\notin J,\,\ell_i<0} \prod_{\lambda_i=0}^{-\ell_i-1} \bigl(\langle \mathbf{A}_i,\theta\rangle + c_i + \ell_i + \lambda_i\bigr) S\Biggl(\mathrm{Horn}\Biggl(A, c+\ell-\sum_{j\in J,\,\ell_j<0} \ell_j e_j\Biggr)\Biggr).$$

It contains a non-trivial monodromy-invariant subspace of $S\bigl(\mathrm{Horn}\bigl(A, c+\ell-\sum_{\ell_j<0}\ell_j e_j\bigr)\bigr)$. Thus it suffices to prove the proposition in the case when $\ell \in \mathbb{Z}^n_{\geqslant 0}$. A straightforward calculation shows that

$$\prod_{j=1}^{n} \prod_{\lambda_j=0}^{\ell_j-1} \bigl(\langle \mathbf{A}_j,\theta\rangle + c_j + \lambda_j\bigr)^{-1} \bigl(S(\mathrm{Horn}(A,c+\ell))/\Psi_0\bigr)$$

is an invariant subspace of the space of all holomorphic solutions of the system $\mathrm{Horn}(A,c)$. We remark here that none of the operators $\langle \mathbf{A}_j,\theta\rangle + c_j + \lambda_j$ for $j = 1,\ldots,n$ and $\lambda_j = 0,\ldots,\ell_j - 1$ occurs as a factor in the operators $P_j(\theta)$, $Q_j(\theta)$, $i = 1,\ldots,n$, that define the system of equations $\mathrm{Horn}(A,c+\ell)$. $\square$

Corollary 3.13. *In the case of two variables, suppose that*

$$\sum_{\mathbf{A}_j,\mathbf{A}_k \text{ lin. indep.}} \nu(\mathbf{A}_j,\mathbf{A}_k) = 0,$$

where the sum is taken over all pairs of linearly independent rows of the matrix defining the Horn system. Then, for almost all $c \in \mathbb{C}^m$, *the monodromy representations of the Horn systems* $\mathrm{Horn}(A,c)$ *and* $\mathrm{Horn}(A,c-e_j)$ *are equivalent for any* $j = 1,\dots,m$.

Proof. By Theorem 3.7, 3), the condition on the indices of the rows of the defining matrix means precisely that the corresponding Horn system has no persistent polynomial solutions. Thus for generic parameters all solutions are fully supported (that is, the convex hull of the support of each solution has dimension 2). No such series is annihilated by a differential operator of the form (3.3) and hence the intertwining operators have trivial kernels. This means that the monodromy representations are equivalent. □

Example 3.14. Consider the hypergeometric system defined by the matrix

$$\begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{3.4}$$

and an arbitrary vector of parameters $(c_1,c_2,c_3) \in \mathbb{C}^3$. This system is generated by the differential operators

$$\begin{gathered} x_1(\theta_1+2\theta_2+c_1)+(\theta_1+\theta_2-c_2), \\ x_2(\theta_1+2\theta_2+c_1)(\theta_1+2\theta_2+c_1+1)-(\theta_1+\theta_2-c_2)(\theta_2-c_3). \end{gathered} \tag{3.5}$$

It is holonomic for all (c_1,c_2,c_3) and has rank 2. The following *universal basis* in the space of holomorphic solutions of (3.5) consists of functions that are linearly independent for all $(c_1,c_2,c_3) \in \mathbb{C}^3$:

$$f_1(x;c) = x_1^{c_1+2c_2} x_2^{-c_1-c_2},$$
$$f_2(x;c) = x_1^{c_1+2c_2}\left(x_2^{-c_1-c_2} - x_2^{c_3}(x_1+x_1^2+x_2)^{-c_1-c_2-c_3}\right)/(c_1+c_2+c_3).$$

When $c_1+c_2+c_3=0$ this basis degenerates into the pair of functions

$$x_1^{c_1+2c_2}x_2^{-c_1-c_2}, \quad x_1^{c_1+2c_2}x_2^{-c_1-c_2}\log\frac{x_1+x_1^2+x_2}{x_2}.$$

Observe that the system (3.5) is resonant if and only if $c_1+c_2+c_3 \in \mathbb{Z}$. The notion of maximal resonance coincides in this example with that of ordinary resonance since there is only one (up to scaling) linear relation between the rows of the matrix (3.4). Let $\mathrm{Sol}(c)$ be the vector space of local solutions of (3.5) at a non-singular point. The intertwining operators for this Horn system are given by

$$\begin{aligned} I_1 &= \theta_1+2\theta_2+c_1-1\colon\ \mathrm{Sol}(c_1-1,c_2,c_3) \to \mathrm{Sol}(c), \\ I_2 &= -\theta_1-\theta_2+c_2-1\colon\ \mathrm{Sol}(c_1,c_2-1,c_3) \to \mathrm{Sol}(c), \\ I_3 &= -\theta_2+c_3-1\colon\ \mathrm{Sol}(c_1,c_2,c_3-1) \to \mathrm{Sol}(c). \end{aligned}$$

Observe that

$$\begin{aligned} I_1(f_1(x;c)) &= I_2(f_1(x;c)) = -f_1(x;c), \\ I_3(f_1(x;c_1,c_2,c_3-1)) &= (c_1+c_2+c_3-1)f_1(x;c), \\ I_1(f_2(x;c_1-1,c_2,c_3)) &= I_2(f_2(x;c_1,c_2-1,c_3)) \\ &= (c_1+c_2+c_3)f_2(x;c) - f_1(x;c), \\ I_3(f_2(x;c_1,c_2,c_3-1)) &= (c_1+c_2+c_3)f_2(x;c). \end{aligned}$$

This example shows that the intertwining operators constructed above may have non-trivial kernels despite the fact that the monodromy of (3.5) depends only on the values of c_1, c_2, c_3 modulo the integer lattice $\mathbb{Z}$.

§ 4. Explicit monodromy calculation for simplicial and parallelepipedal hypergeometric systems

4.1. Atomic hypergeometric systems. In this section we investigate the monodromy representations of certain important families of hypergeometric systems. They will generate two classes of polygons corresponding to Horn systems with maximally reducible monodromy representations to be described in § 6.

Recall that by Definition 3.4 an atomic hypergeometric system of equations is a confluent Horn system defined by a non-singular square matrix. An atomic system can be transformed into a system of differential equations with constant coefficients by means of the isomorphism in [8], Lemma 5.1, Corollary 5.2. By the Malgrange–Ehrenpreis–Palamodov fundamental principle [16], a basis of the space of holomorphic solutions of an atomic system is given by products of Puiseux polynomials and exponential functions whose arguments are also Puiseux polynomials. Observe that an atomic system is confluent by definition since the non-confluence condition (2.4) is a linear relation between the rows of the defining matrix. Also by definition, an atomic system is never resonant. Every solution of a holonomic atomic system is either a persistent Puiseux polynomial or a fully supported Puiseux series. In the case of two variables one can tell exactly how many Puiseux polynomial solutions an atomic system can have and what their initial exponents are (see Definition 2.7).

Theorem 4.1. 1) *For every non-singular* 2×2 *integer matrix* $M=\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ *with* G. C. D.$(\det(M), a_1, b_1, a_2, b_2) = 1$ *and every* $\tilde{c}\in\mathbb{C}^2$, *the holonomic rank of the associated atomic system is given by*

$$\operatorname{rank}(\operatorname{Horn}(M,\tilde{c})) = |\det(M)| + \nu(M).$$

Furthermore, the system $\operatorname{Horn}(M,\tilde{c})$ *has* $|\det(M)|$ *fully supported series solutions while the remaining* $\nu(M)$ *solutions are persistent Puiseux polynomials.*

2) *When* $\nu(M)>0$ *the initial exponents of the Puiseux polynomial solutions of* $\operatorname{Horn}(M,\tilde{c})$ *are of the form* $-M^{-1}(\mathcal{R}_M+\tilde{c})$, *where*

$$\mathcal{R}_M = \begin{cases} \{(u,v)\in\mathbb{N}^2 \colon u<|b_1|,\ v<|a_2|\}, & \text{if } |a_1b_2|>|b_1a_2|, \\ \{(u,v)\in\mathbb{N}^2 \colon u<|a_1|,\ v<|b_2|\}, & \text{if } |a_1b_2|<|b_1a_2|. \end{cases}$$

Proof. 1) By Proposition 4 in [2], the system $\mathrm{Horn}(M,\tilde{c})$ admits a solution of the following form for a suitable cycle $\mathcal{C}$:

$$\begin{aligned}
&\frac{|\det(M)|}{(2\pi i)^2}\int_{\mathcal{C}}\Gamma(a_1s_1+b_1s_2+\tilde{c}_1)\Gamma(a_2s_1+b_2s_2+\tilde{c}_2)x_1^{s_1}x_2^{s_2}\,ds_1\,ds_2\\
&\quad=\sum_{k\in\mathbb{Z}^2_{\geqslant 0}}\frac{(-1)^{|k|}}{k!}x^{-M^{-1}(k+\tilde{c})}=x^{-M^{-1}\tilde{c}}\sum_{k\in\mathbb{Z}^2_{\geqslant 0}}\frac{1}{k!}\prod_{j=1}^{2}(-x^{-M^{-1}e_j})^{k_j}\\
&\quad=x^{-M^{-1}\tilde{c}}\exp\biggl(-\sum_{j=1}^{2}x^{-M^{-1}e_j}\biggr).
\end{aligned}\tag{4.1}$$

The dimension of the linear span of the set of all analytic continuations of (4.1) (that is, the space of fully supported solutions) equals $|\det(M)|$ since

$$\mathrm{G.C.D.}(\det(M),a_1,b_1,a_2,b_2)=1.$$

By Lemma 6.5 in [8], the dimension of the space of persistent Puiseux polynomial solutions is equal to $\nu(M)$. We conclude that $\mathrm{rank}(\mathrm{Horn}(M,\tilde{c}))=|\det(M)|+\nu(M)$.

2) This follows from the construction of persistent Puiseux polynomial solutions in [8], Lemma 6.5. □

The supports of persistent polynomial solutions of a bivariate Horn system can be characterized as follows. By Theorems 3.7, 3) and 4.1, only submatrices $\mathbf{A}_I=(\mathbf{A}_i,\mathbf{A}_j)$ with $\nu(\mathbf{A}_i,\mathbf{A}_j)>0$ contribute to the space of persistent solutions of $\mathrm{Horn}(A,\tilde{c})$. By making the change of variables $x_1\to\frac{1}{x_1}$ if necessary, we can assume without loss of generality that $\mathbf{A}_i=(a_1,b_1)\in\mathbb{N}^2$ and $\mathbf{A}_j=(a_2,b_2)\in-\mathbb{N}^2$. Interchanging x_1 and x_2 if necessary, we can also assume without loss of generality that $|a_1b_2|>|a_2b_1|$. In this case $\mathcal{R}_{\mathbf{A}_I}=\{(u,v)\in\mathbb{N}^2\colon u<b_1,\ v<|a_2|\}$.

Corollary 4.2. *Under the normalization above, we define an index set*

$$\tilde{\mathcal{R}}_{\mathbf{A}_I}=\bigl\{(u,v)\in\mathbb{N}^2\colon 0\leqslant u<\min(a_1,b_1),\ 0\leqslant v<\min(|a_2|,|b_2|)\bigr\}$$

contained in $\mathcal{R}_{\mathbf{A}_I}$.

1) *The support of a persistent monomial solution of the atomic system* $\mathrm{Horn}(\mathbf{A}_I,\tilde{c}_I)$ *is given by* $\alpha\in-{\mathbf{A}_I}^{-1}(\tilde{\mathcal{R}}_{\mathbf{A}_I}+\tilde{c}_I)$.

2) *With every* $\alpha_0\in-{\mathbf{A}_I}^{-1}((\mathcal{R}_{\mathbf{A}_I}\setminus\tilde{\mathcal{R}}_{\mathbf{A}_I})+\tilde{c}_I)$ *we associate a tuple of indices* $S_{\alpha_0}:=\bigcup_{k=0}^{K}\{\alpha_k\}$ *that will be defined later in the proof. Then the support of every persistent polynomial solution of* $\mathrm{Horn}(\mathbf{A}_I,\tilde{c}_I)$ *is the union of* S_{α_0} *and the supports of the persistent monomial solutions.*

Proof. We first remark that under the normalization above, the condition $\alpha\in-{\mathbf{A}_I}^{-1}(\mathcal{R}_{\mathbf{A}_I}+\tilde{c})$ means that $P_2(\alpha)=0$ and $Q_1(\alpha)=0$. The cardinality of the set of lattice points satisfying this condition is equal to $|a_2b_1|$.

1) If $\alpha\in-{\mathbf{A}_I}^{-1}(\tilde{\mathcal{R}}_{\mathbf{A}_I}+\tilde{c}_I)$, then

$$\alpha\in\ker\bigl(\langle\mathbf{A}_i,\theta\rangle+\tilde{c}_i+u_i\bigr)\cap\ker\bigl(\langle\mathbf{A}_j,\theta\rangle+\tilde{c}_j+v_j\bigr)$$

for $(u_i,v_j)\in\tilde{\mathcal{R}}_{\mathbf{A}_I}$. Hence the operator $\langle\mathbf{A}_i,\theta\rangle+\tilde{c}_i+u_i$ with $u_i<\min(a_1,b_1)$ is a factor of both $P_1(\theta)$ and $P_2(\theta)$. In a similar way, the operator $\langle\mathbf{A}_j,\theta\rangle+\tilde{c}_j+v_j$ with $v_j<\min(|a_2|,|b_2|)$ is a factor of both $Q_1(\theta)$ and $Q_2(\theta)$.

We put $i = 1$ when $(\mathcal{R}_{\mathbf{A}_I} \setminus \tilde{\mathcal{R}}_{\mathbf{A}_I}) \cap \mathbb{N} \times \{0\} \neq \varnothing$ and adopt the usual notation $e_1 = (1, 0)$. We similarly put $i = 2$ when $(\mathcal{R}_{\mathbf{A}_I} \setminus \tilde{\mathcal{R}}_{\mathbf{A}_I}) \cap \{0\} \times \mathbb{N} \neq \varnothing$, and $e_2 = (0, 1)$.

2) When $|b_2| < |a_2|$, the case $i = 2$ occurs. Hence there is an α_0 such that $P_2(\alpha_0) = Q_1(\alpha_0) = 0$ but $Q_2(\alpha_0) \neq 0$. The following equalities hold:

$$H_2(\mathbf{A}_I, \tilde{c}_I)x^{\alpha_0} = (x_2 P_2(\theta) - Q_2(\theta))x^{\alpha_0} = -Q_2(\alpha_0)x^{\alpha_0},$$
$$H_2(\mathbf{A}_I, \tilde{c}_I)x^{\alpha_0 - e_2} = P_2(\alpha_0 - e_2)x^{\alpha_0} - Q_2(\alpha_0 - e_2)x^{\alpha_0 - e_2}.$$

We now consider a sequence of integer lattice points $\alpha_0, \alpha_1 = \alpha_0 - e_2, \ldots$ such that $\alpha_k - \alpha_{k+1} = -e_1$ or e_2. The points α_k lie inside the cone $C(i,j) := \{s \colon \langle \mathbf{A}_j, s\rangle + \tilde{c}_j \leqslant 0\} \cap \{s \colon \langle \mathbf{A}_i, s\rangle + \tilde{c}_i \leqslant 0\}$. Since the sequence must terminate at a certain step, the union of all points $\{\alpha_k\}_{k \geqslant 0}$ is a finite subset of $C(i,j)$. Thus there is a finite set of integer points S_{α_0} such that a linear combination of the polynomials $H_2(\mathbf{A}_I, \tilde{c}_I)x^{\alpha_k}$ (resp. $H_1(\mathbf{A}_I, \tilde{c}_I)x^{\alpha_k}$), $k = 1, \ldots, K$, is identically equal to zero (see [8], Lemma 6.5, Fig. 2, depicting a process equivalent to the construction of S_{α_0}). If $|a_2| \leqslant |b_2|$ and $a_1 \geqslant b_1$, then $\tilde{\mathcal{R}}_{\mathbf{A}_I} = \mathcal{R}_{\mathbf{A}_I}$. Thus all persistent polynomial solutions are actually monomials.

If $|a_2| \leqslant |b_2|$ and $a_1 < b_1$, then the case $i = 1$ occurs. As in the case $i = 2$, we obtain a polynomial solution supported on the set of integer points $S_{\alpha_0} = \bigcup_{k \geqslant 0}\{\alpha_k\}$, $\alpha_1 = \alpha_0 - e_1, \ldots$, such that $\alpha_k - \alpha_{k+1} = -e_2$ or $\alpha_k - \alpha_{k+1} = e_1$. $\square$

Example 4.3. Consider the atomic hypergeometric system defined by the matrix

$$M = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$$

and the zero parameter vector. It is generated by the operators

$$\begin{aligned} &x_1(3\theta_1 + 2\theta_2)(3\theta_1 + 2\theta_2 + 1)(3\theta_1 + 2\theta_2 + 2) \\ &\qquad - (-4\theta_1 - 3\theta_2)(-4\theta_1 - 3\theta_2 + 1)(-4\theta_1 - 3\theta_2 + 2)(-4\theta_1 - 3\theta_2 + 3), \\ &x_2(3\theta_1 + 2\theta_2)(3\theta_1 + 2\theta_2 + 1) - (-4\theta_1 - 3\theta_2)(-4\theta_1 - 3\theta_2 + 1)(-4\theta_1 - 3\theta_2 + 2). \end{aligned} \tag{4.2}$$

By Theorem 4.1, 1), the dimension of the space of persistent polynomial solutions is equal to 8.

The persistent monomial solutions are given by

$$1,\ x_1^{-2}x_2^3,\ x_1^{-4}x_2^6,\ x_1^{-3}x_2^4,\ x_1^{-5}x_2^7,\ x_1^{-7}x_2^{10}.$$

The polynomials

$$x_1^{-6}x_2^8 - \frac{1}{3}x_1^{-6}x_2^9, \quad x_1^{-9}x_2^{13} - 4x_1^{-9}x_2^{12} + x_1^{-8}x_2^{13} + 12x_1^{-8}x_2^{11}$$

are the essentially polynomial persistent solutions.

Observe that all Puiseux polynomial solutions of an atomic system are necessarily persistent. This is of course not the case for an arbitrary hypergeometric system.

4.2. Simplicial hypergeometric configurations. An important particular case of a general non-confluent Horn system is the system defined by a matrix whose rows are the vertices of an n-dimensional integer simplex. More precisely, let M be a non-singular $n \times n$ integer matrix and $\alpha \in \mathbb{C}^n$ a parameter vector. Put $\tilde{\alpha} = (\alpha, \alpha_{n+1}) \in \mathbb{C}^{n+1}$. Let $M_1, \ldots, M_n$ be the rows of M. We put $M_{n+1} = -M_1 - \cdots - M_n$ and define $\widetilde{M}$ as the $(n+1) \times n$ matrix with rows $M_1, \ldots, M_{n+1}$. The non-confluent Horn system $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ associated with this data is said to be *simplicial*.

Proposition 4.4 [1]. *For generic $\tilde{\alpha}$, a holonomic simplicial hypergeometric system* $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *admits the solution*

$$x^{-M^{-1}\alpha}\left(1 + \sum_{j=1}^{n} x^{-M^{-1}e_j}\right)^{-|\tilde{\alpha}|}, \tag{4.3}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ (1 in the jth place). Every solution of $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *either lies in the linear span of analytic continuations of* (4.3) *or is a persistent Puiseux polynomial. When $-|\tilde{\alpha}| \in \mathbb{Z}_{\geqslant 0} \setminus \{0\}$ the monodromy representation of* $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *is maximally reducible.*

Example 4.5. The Horn system

$$\begin{gathered} x_1(\theta_1 + \theta_2 - 3)(\theta_1 - 2\theta_2 - 1) - (-2\theta_1 + \theta_2)(-2\theta_1 + \theta_2 - 1), \\ x_2(\theta_1 + \theta_2 - 3)(-2\theta_1 + \theta_2 - 1) - (\theta_1 - 2\theta_2)(\theta_1 - 2\theta_2 - 1) \end{gathered} \tag{4.4}$$

is holonomic of rank 4. A pure basis of its solution space is given by the Puiseux polynomials

$$\begin{gathered} \frac{1}{x_1 x_2}, \qquad 4 + 2x_1 + 2x_2 + 6x_1x_2 + x_1^2x_2 + x_1x_2^2, \\ x_1^{-2/3}x_2^{-1/3}(5 + 10x_1 + 30x_1x_2 + 20x_1^2x_2 + x_1^3x_2 + 5x_1x_2^2 + 10x_1^2x_2^2), \\ x_1^{-1/3}x_2^{-2/3}(5 + 10x_2 + 30x_1x_2 + 20x_1x_2^2 + x_1x_2^3 + 5x_1^2x_2 + 10x^2x_2^2). \end{gathered}$$

The Newton polygons of these polynomials are shown in Fig. 1. Consider the Mellin–Barnes integral with weight given by the Ore–Sato coefficient

$$\varphi(s) = \frac{\Gamma(-c + s_1 - 2s_2 - 1)\Gamma(-2s_1 + s_2 - 1)e^{\sqrt{-1}\,\pi(s_1+s_2)}}{\Gamma(-s_1 - s_2 + 4)}$$

that defines the system (4.4). We assume that $c \in \mathbb{R}$ is generic and the contour $\mathcal{C}$ is invariant under unit shifts in real directions. Computing the residues, we arrive at a fully supported Puiseux series solution of the Horn system obtained by perturbing (4.4), that is, by replacing $\theta_1 - 2\theta_2$ by $\theta_1 - 2\theta_2 - c$:

$$f_c = x_1^{-c/3-1}x_2^{-2c/3-1}(x_1^{2/3}x_2^{1/3} + x_1^{1/3}x_2^{2/3} + 1)^{5-c}.$$

Observe that when $c = 0$ we get a Puiseux polynomial solution

$$f_0 = \frac{(x_1^{2/3} x_2^{1/3} + x_1^{1/3} x_2^{2/3} + 1)^5}{x_1 x_2}.$$

The presence of a polynomial solution when $c = 0$ is explained by the fact that the poles of the numerator $\Gamma(-c + s_1 - 2s_2 - 1)$ in the Ore–Sato coefficient $\varphi(s)$ are not cancelled by those of the denominator $\Gamma(-s_1 - s_2 + 4)$ for generic c. But when $c = 0$ we have a half-space cancellation of poles (see Definition 6.2) and the non-zero residues lie only in the strip $\{s\colon -2 \leqslant s_1 + s_2 \leqslant 3\}$.

Linear combinations of branches of f_0 produce the last three Puiseux polynomial solutions of (4.4). The only persistent solution in this example is the Laurent monomial $\frac{1}{x_1 x_2} \in \ker(\theta_1 - 2\theta_2 - 1) \cap \ker(-2\theta_1 + \theta_2 - 1)$. This solution generates a one-dimensional invariant subspace of the space of holomorphic solutions of (4.4).

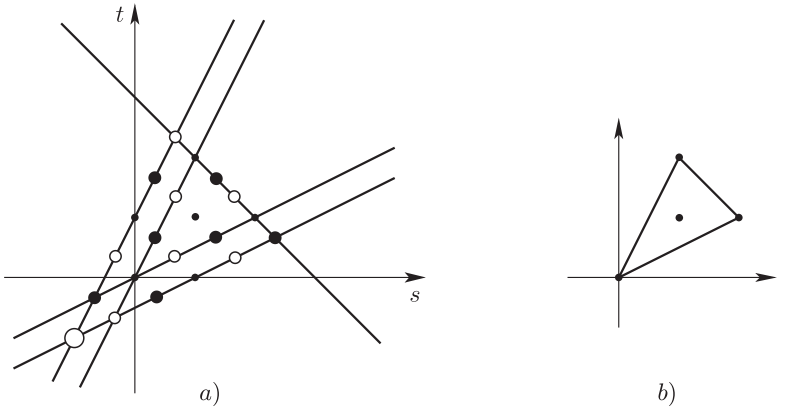

Figure 1. *a*) the supports of solutions of (4.4); *b*) the polygon of the Ore–Sato coefficient defining (4.4)

Example 4.6. Consider the bivariate ($n = 2$) simplicial hypergeometric system generated by the matrix

$$M = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and the vector of parameters $\tilde{\alpha} = (0, 0, c)$. There is no loss of generality in making this choice of parameters since changing the first two coordinates of $\tilde{\alpha}$ only results in a shift of the exponent space. The system is generated by the differential operators

$$\begin{aligned} &x_1(2\theta_1 + 2\theta_2 + c)(2\theta_1 + 2\theta_2 + c + 1) - 2\theta_1(2\theta_1 - 1), \\ &x_2(2\theta_1 + 2\theta_2 + c)(2\theta_1 + 2\theta_2 + c + 1) - 2\theta_2(2\theta_2 - 1). \end{aligned} \tag{4.5}$$

By Theorem 3.8 the holonomic rank of (4.5) equals 4 for generic $c \in \mathbb{C}$. Moreover, Theorem 6.10 in [13] shows that this holds for all $c \in \mathbb{C}$ since $\mathcal{Z}_{\mathrm{Andean}}(I) = \varnothing$ for this system. By Proposition 4.4, $(1 + \sqrt{x_1} + \sqrt{x_2})^{-c}$ is a generating solution of (4.5). It follows from Theorem 3.7 that (4.5) has no persistent Puiseux polynomial solutions and, therefore, a basis of the space of analytic solutions of (4.5) for generic $c \in \mathbb{C}$ is given by

$$\begin{aligned} f_1(c) &= (1+\sqrt{x_1}+\sqrt{x_2}\,)^{-c}, & f_2(c) &= (1+\sqrt{x_1}-\sqrt{x_2}\,)^{-c}, \\ f_3(c) &= (1-\sqrt{x_1}+\sqrt{x_2}\,)^{-c}, & f_4(c) &= (1-\sqrt{x_1}-\sqrt{x_2}\,)^{-c}. \end{aligned} \tag{4.6}$$

However, this basis degenerates for two special values of c, namely, $c = 0$ (when all the basis elements (4.6) are identically equal to 1) and $c = -1$ (when $f_1(-1) + f_4(-1) - f_2(-1) - f_3(-1) = 0$). We now construct bases in the solution space of (4.5) for both of these resonant values of c.

When $c = -1$, the corresponding resonant basis is given by $f_1(-1), f_2(-1), f_3(-1)$ and the function

$$\tilde{\tilde{f}}_4 = (f_1 \log f_1 - f_2 \log f_2 - f_3 \log f_3 + f_4 \log f_4)|_{c=-1}.$$

When $c = 0$, a resonant basis of the solution space of (4.5) is given by $f_1(0)$ and the three additional resonant solutions

$$\begin{aligned} \tilde{f}_2 &= \log\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr) - \log\bigl(1+\sqrt{x_1}-\sqrt{x_2}\,\bigr), \\ \tilde{f}_3 &= \log\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr) - \log\bigl(1-\sqrt{x_1}+\sqrt{x_2}\,\bigr), \\ \tilde{f}_4 &= \log\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr) - \log\bigl(1-\sqrt{x_1}-\sqrt{x_2}\,\bigr). \end{aligned}$$

It turns out to be possible to endow the space of analytic solutions of (4.5) with a single universal basis whose elements remain linearly independent after passing to the limit as $c \to 0$ or $c \to -1$. This basis is given by

$$\begin{aligned} \hat{f}_1(c) &= \bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c}, \\ \hat{f}_2(c) &= \Bigl(\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c} - \bigl(1+\sqrt{x_1}-\sqrt{x_2}\,\bigr)^{-c}\Bigr)/c, \\ \hat{f}_3(c) &= \Bigl(\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c} - \bigl(1-\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c}\Bigr)/c, \\ \hat{f}_4(c) &= \Bigl(\bigl(1+\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c} - \bigl(1+\sqrt{x_1}-\sqrt{x_2}\,\bigr)^{-c} \\ &\qquad - \bigl(1-\sqrt{x_1}+\sqrt{x_2}\,\bigr)^{-c} + \bigl(1-\sqrt{x_1}-\sqrt{x_2}\,\bigr)^{-c}\Bigr)/(c+c^2). \end{aligned} \tag{4.7}$$

It is easy to check that the functions $\hat{f}_1(c), \ldots, \hat{f}_4(c)$ are linearly independent for all $c \in \mathbb{C}$.

Given the basis (4.7), it is straightforward to find the monodromy representation of the fundamental group of the complement of the singularities of solutions of (4.5). It is generated by three matrices corresponding to loops around the coordinate axes $\{x_1 = 0\}$, $\{x_2 = 0\}$ and the essential singularity $\{\mathcal{S}(x) := 1 - 2x_1 + x_1^2 - 2x_2 - 2x_1x_2 + x_2^2 = 0\}$. These matrices are given by

$$M_{x_1} = \begin{pmatrix} 1 & 0 & -c & 0 \\ 0 & 1 & 0 & -1-c \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad M_{x_2} = \begin{pmatrix} 1 & -c & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1-c \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$M_{\mathcal{S}} = \operatorname{diag}(e^{-2\pi\sqrt{-1}\,c}).$$

4.3. Parallelepipedal hypergeometric configurations. Now let M be a non-singular $n \times n$ integer matrix and let $\alpha, \beta \in \mathbb{C}^n$ be two parameter vectors. We write $\widetilde{M}$ for the $2n \times n$ matrix obtained by concatenating the rows of M and $-M$. The rows of $\widetilde{M}$ are the vertices of a parallelepiped of non-zero n-dimensional volume. Let $\tilde{\alpha}$ be the vector with components $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$. As in the simplicial case, it turns out to be possible to construct a generating solution of the corresponding hypergeometric system (4.5) by computing multidimensional residues.

Proposition 4.7 [2]. *For generic $\tilde{\alpha} \in \mathbb{C}^n$, the holonomic hypergeometric system* $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *admits a solution of the form*

$$x^{-M^{-1}\alpha} \prod_{j=1}^{n} (1 + x^{-M^{-1}e_j})^{-\alpha_j - \beta_j}, \tag{4.8}$$

where $e_j = (0, \dots, 1, \dots, 0)$ (1 in the jth place). Every solution of $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *either lies in the linear span of analytic continuations of* (4.8) *or is a persistent Puiseux polynomial. If $-\alpha_j - \beta_j \in \mathbb{Z}_{\geqslant 0} \setminus \{0\}$ for all $j = 1, \dots, n$, then the monodromy representation of* $\mathrm{Horn}(\widetilde{M}, \tilde{\alpha})$ *is maximally reducible.*

§ 5. Bases in the solution space of the hypergeometric Horn system

Let us denote by q the number of vertices of the Newton polytope of the polynomial which defines the singular hypersurface of the hypergeometric system under consideration. In this section we construct a family of q bases in the space of fully supported solutions of that system. This will be used in § 6 to deduce our main result.

Definition 5.1. The *amoeba* $\mathcal{A}_f$ of a Laurent polynomial $f(x)$ (or of the algebraic hypersurface $f(x) = 0$) is defined to be the image of the set $f^{-1}(0)$ under the map $\mathrm{Log}\colon (x_1, \dots, x_n) \mapsto (\log|x_1|, \dots, \log|x_n|)$.

Let $\mathcal{A}(\varphi)$ be the amoeba of the singularity of the hypergeometric system $\mathrm{Horn}(\varphi)$.

Definition 5.2. The *recession cone* C_B of a convex set $B \subset \mathbb{R}^n$ is defined as

$$C_B = \bigl\{s \in \mathbb{R}^n \colon u + \lambda s \in B \ \ \forall\, u \in B,\ \lambda \geqslant 0\bigr\}$$

(see [9], § 4). Hence the recession cone of a convex set is the maximal element (with respect to inclusion) in the family of cones whose shifts are contained in this set.

The following theorem (compare with the results in [7] for the GKZ system) shows that for every vertex of the Newton polygon of the singularity of a bivariate hypergeometric function there is a basis of the solution space of the corresponding Horn system. This basis consists of hypergeometric series which converge in the pre-image of the connected component of the complement of the amoeba corresponding to that vertex.

Theorem 5.3. 1) *For every bivariate non-confluent Ore–Sato coefficient φ with generic parameters and every connected component M of ${}^c\mathcal{A}(\varphi)$ there is a pure Puiseux series basis $f_{M,i}$, $i = 1, \dots, \mathrm{rank}(\mathrm{Horn}(\varphi))$, of the solution space of* $\mathrm{Horn}(\varphi)$ *such that the recession cone of the support of $f_{M,i}$ is contained in $-C_M^{\vee}$.*

2) *The domain of convergence of the series* $f_{M,i}$ *contains* $\mathrm{Log}^{-1}(M)$ *for any* $i = 1, \ldots, \mathrm{rank}(\mathrm{Horn}(\varphi))$.

Proof. The Ore–Sato coefficient $\varphi(s)$ defining the Horn system can be represented in the form

$$\varphi(s) = \prod_{i=1}^{m} \Gamma(a_i s_1 + b_i s_2 + c_i),$$

where $(a_i, b_i) \in \mathbb{Z}^2$, $\sum_{i=1}^{m}(a_i, b_i) = (0, 0)$ and $(c_1, \ldots, c_m) \in \mathbb{C}^m$ is a generic parameter vector. By Definition 2.5, the vectors $\{(a_i, b_i)\}_{i=1}^{m}$ are the outer normals to the sides of the polygon $\mathcal{P}(\varphi)$ of the Ore–Sato coefficient φ (observe that some of them may coincide) ([17], Theorem 2). This theorem also implies that the number of distinct vectors in this set equals q. To simplify the notation, we denote the distinct elements in this set by $(\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_q, \bar{b}_q)$. There is no loss of generality in assuming that these normals are ordered anticlockwise (in accordance with the principal branch of the argument of the complex number $\bar{a}_k + \bar{b}_k\sqrt{-1}$) from $(\bar{a}_1, \bar{b}_1)$ to $(\bar{a}_q, \bar{b}_q)$. Let v_i be the vertex of $\mathcal{P}(\varphi)$ lying on the sides with normals $(\bar{a}_i, \bar{b}_i)$ and $(\bar{a}_{i+1}, \bar{b}_{i+1})$ (we also define v_q as the common vertex of the first and last sides).

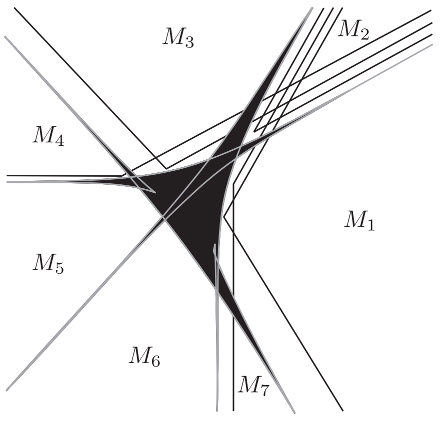

Figure 2. The amoeba of the singularity of a Horn system and the recession cones of the supports of its solutions

By Theorem 7 in [9], there is a one-to-one correspondence between the vertices $v_1, \ldots, v_q$ of the polygon $\mathcal{P}(\varphi)$ and the connected components of the complement of the amoeba $\mathcal{A}(\varphi)$. We denote these connected components by $M_1, \ldots, M_q$.

Fig. 2 shows the particular case of the amoeba of the singularity of the Horn system defined by the Ore–Sato coefficient $\Gamma(s_1 + 2s_2)\Gamma(s_1 - 2s_2)\Gamma(-s_1 + 3s_2) \times \Gamma(-s_1 - 3s_2)\Gamma(s_1)\Gamma(-s_1 - s_2)\Gamma(s_2)$. In this case $q = 7$. The continuous curve that bounds the amoeba and goes somewhere inside is its contour (that is, the set of critical values of the logarithmic Gauss map on the hypersurface that defines the amoeba [18]). The shape of the amoeba was found by means of the Horn–Kapranov parametrization [19] using the computer algebra system Mathematica 9.0. Fig. 2 also shows the recession cones of the convex hulls of the connected components of the complement of the amoeba that are strongly convex (that is, they are convex and contain no lines [9]) and whose shifts contain M_2. The duals of these cones support hypergeometric series that satisfy the Horn system and whose domains of convergence contain $\mathrm{Log}^{-1} M_2$. To prove the theorem, we must show that the number of such series is independent of the connected component of the complement of the amoeba.

We claim that for every $i = 1, \ldots, q$ the system $\mathrm{Horn}(\varphi)$ has the same number of fully supported Puiseux series solutions which converge on $\mathrm{Log}^{-1}(M_i)$. To prove this, we will show that the number of such series whose domain of convergence is $\mathrm{Log}^{-1}(M_1)$, coincides with the number of Puiseux series solutions that converge on $\mathrm{Log}^{-1}(M_2)$. Repeating this argument, one can prove that for any two adjacent components in the complement of $\mathcal{A}(\varphi)$ the number of Puiseux series solutions that converge on the pre-images of these components under the map Log is independent of $i = 1, \ldots, q$. This will prove that any such connected component carries the same number of fully supported Puiseux series solutions. We remark that the desired assertion can also be proved using the approach and results in [20].

We define a single-valued branch arg of the argument Arg of a complex number by setting $\arg(-a - b\sqrt{-1}\,) = 0$ and $\lim_{\varepsilon\to 0^-} \arg e^{\sqrt{-1}\,\varepsilon}(-a - b\sqrt{-1}\,) = 2\pi$, and introduce a partial order $\prec$ on the lattice $\mathbb{Z}^2$ by saying that $(a, b) \prec (c, d)$ if $\arg(a + b\sqrt{-1}\,) < \arg(c + d\sqrt{-1}\,)$. We say that $(a, b) \preccurlyeq (c, d)$ if $\arg(a + b\sqrt{-1}\,) \leqslant \arg(c + d\sqrt{-1}\,)$.

By Lemma 11 in [9] and Theorem 3.7, 2), the number of fully supported Puiseux series solutions of the hypergeometric system $\mathrm{Horn}(\varphi)$ that converge in the domain $\mathrm{Log}^{-1}(M_i)$ equals

$$S_i = \sum_{\substack{j:\ -(\bar a_{i+1}, \bar b_{i+1}) \prec (\bar a_j, \bar b_j) \preccurlyeq (\bar a_i, \bar b_i) \\ \ell:\ (\bar a_{i+1}, \bar b_{i+1}) \preccurlyeq (\bar a_\ell, \bar b_\ell) \prec -(\bar a_j, \bar b_j)}} k_j k_\ell \begin{vmatrix} \bar a_\ell & \bar b_\ell \\ \bar a_j & \bar b_j \end{vmatrix},$$

where k_j is the number of those vectors in the set $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ which coincide with $(\bar a_j, \bar b_j)$. Observe that all determinants in this formula are positive by our choice of the indices of summation. To prove that $S_1 = S_2$, we make use of the fact that these two sums have many common terms. Indeed, the sum of terms in S_1 that are not present in S_2 is given by

$$\sum_{j:\ -(\bar a_2, \bar b_2) \prec (\bar a_j, \bar b_j) \preccurlyeq (\bar a_1, \bar b_1)} k_2 k_j \begin{vmatrix} \bar a_2 & \bar b_2 \\ \bar a_j & \bar b_j \end{vmatrix} = \det\biggl(k_2(\bar a_2, \bar b_2), \sum_{j:\ -(\bar a_2, \bar b_2) \prec (\bar a_j, \bar b_j) \preccurlyeq (\bar a_1, \bar b_1)} k_j (\bar a_j, \bar b_j) \biggr). \tag{5.1}$$

Similarly, the sum of terms in S_2 that are not present in S_1 is given by

$$\sum_{\ell:\ (\bar a_3, \bar b_3) \preccurlyeq (\bar a_\ell, \bar b_\ell) \prec -(\bar a_2, \bar b_2)} k_2 k_\ell \begin{vmatrix} \bar a_\ell & \bar b_\ell \\ \bar a_2 & \bar b_2 \end{vmatrix} = \det\biggl(\sum_{\ell:\ (\bar a_3, \bar b_3) \preccurlyeq (\bar a_\ell, \bar b_\ell) \prec -(\bar a_2, \bar b_2)} k_\ell (\bar a_\ell, \bar b_\ell), k_2(\bar a_2, \bar b_2) \biggr). \tag{5.2}$$

The non-confluence condition $\sum_{i=1}^{q} k_i (\bar a_i, \bar b_i) = \sum_{j=1}^{m} (a_j, b_j) = (0, 0)$ implies that the determinant on the right-hand side of (5.1) equals the determinant on the right-hand side of (5.2). This proves that any connected component of the complement of the amoeba carries equally many fully supported solutions of the Horn system that are convergent on its pre-image under the map Log (and, possibly, on a larger set).

Recall that any solution of a hypergeometric system with generic parameters can be expanded into a Puiseux series centred at the origin. (This series may in particular be a Puiseux polynomial.) Since a Puiseux polynomial solution of a Horn system is defined everywhere except (possibly) the coordinate hyperplanes, it works for every connected component in the complement of the amoeba of the singularity. Thus for every such component M there is a Puiseux series basis of the space of those solutions of the Horn system all of whose elements converge (at least) in the domain $\operatorname{Log}^{-1}(M)$.

We remark that this assertion can also be proved using the approach and results in [20].

It remains to show that we can take pure Puiseux series as a basis. To do this, we show that suitable linear combinations of the analytic continuations of a solution

$$P(x)=\sum_{k=1}^{\mu} x_1^{\frac{v_{1k}}{N_1}} x_2^{\frac{v_{2k}}{N_2}} p_k(x_1,x_2)$$

(here $p_k(x)$, $k=1,\dots,\mu$, are Laurent series) admit power-series expansions that converge in the domain $\operatorname{Log}^{-1}(M_i)$ for fixed values of i, $N_1,N_2\in\mathbb{N}$, $v_{1,k},v_{2,k}\in\mathbb{Z}$. Notice that $\mu\leqslant N_1N_2$. The result of analytic continuation along a loop turning ℓ_1 times around $x_1=0$ and ℓ_2 times around $x_2=0$ (in the anticlockwise direction) is given by the formula

$$(M^{\ell_1}_{x_1=0}M^{\ell_2}_{x_2=0})_*P(x)=\sum_{k=1}^{\mu} e^{\left(\frac{\ell_1 v_{1k}}{N_1}+\frac{\ell_2 v_{2k}}{N_2}\right)2\pi\sqrt{-1}} x_1^{\frac{v_{1k}}{N_1}} x_2^{\frac{v_{2k}}{N_2}} p_k(x_1,x_2).$$

To write $x_1^{\frac{v_{1k}}{N_1}} x_2^{\frac{v_{2k}}{N_2}} p_k(x_1,x_2)$ as a linear combination of $(M^{\ell_1}_{x_1=0}M^{\ell_2}_{x_2=0})_*P(x)$, $0\leqslant \ell_1\leqslant N_1-1$, $0\leqslant \ell_2\leqslant N_2-1$, it suffices to consider the inverse of a Vandermonde matrix of order μ. This completes the proof of the theorem. □

§ 6. Maximally reducible monodromy

In this section we continue our study of bivariate Horn systems. Let A be an $m\times 2$ integer matrix whose rows add up to the zero vector. Such a matrix, together with a vector of parameters, defines a bivariate non-confluent hypergeometric system of equations. It is convenient to associate with A a convex polygon $\mathcal{P}$ with integer vertices such that the outer normals to the sides of $\mathcal{P}$ are the rows of A. We also require that the relative length of each side of $\mathcal{P}$ in the integer lattice equals the number of occurrences of the corresponding normal as a row of A. (Observe that the normals to a polygon whose lengths are adjusted in this way add up to zero.) The polygon $\mathcal{P}$ satisfying these conditions is uniquely determined (up to a shift by an integer vector) by the matrix A. Conversely, every convex integer planar polygon $\mathcal{P}$ determines a matrix $A(\mathcal{P})$ whose rows are the outer normals to the sides of $\mathcal{P}$ (possibly with some of them repeated). The order of the rows of this matrix is irrelevant since they all lead to the same hypergeometric system of equations. Thus, together with a vector of parameters c, such a polygon determines a non-confluent hypergeometric system of equations. We denote this system by $\operatorname{Horn}(A(\mathcal{P}),c)$. This was illustrated by Example 4.5.

The results in § 4 yield that any Horn system defined by a matrix whose rows are the vertices of a simplex or a parallelepiped admits a basis of Puiseux polynomials for suitable values of its parameters. In particular, the monodromy representation of such a Horn system (with this very particular choice of parameters) is maximally reducible.

The following problem was posed in [6]: describe those GKZ hypergeometric systems whose solution space contains a one-dimensional subspace on which the monodromy acts trivially (this corresponds to the existence of a rational solution). In this section we solve the closely related problem of describing the class of Horn hypergeometric systems with maximally reducible monodromy representations. Apart from systems with rational bases of solutions, such systems have the simplest possible monodromy representation since the corresponding monodromy groups are generated by diagonal matrices.

Recall that a *zonotope* is the Minkowski sum of segments. The main result in this section is the following theorem.

Theorem 6.1. *The monodromy representation of a bivariate non-confluent hypergeometric system* $\mathrm{Horn}(A(\mathcal{P}), c)$ *is maximally reducible for some* $c \in \mathbb{C}^n$ *if and only if the polygon* $\mathcal{P}$ *is one of the following*:

1) *a zonotope*;

2) *the Minkowski sum of a triangle* $\triangle$ *and an arbitrary number of segments each of which is parallel to a side of* $\triangle$.

For example, the zonotope in Fig. 6 (see below) corresponds to the matrix (6.9) whose rows are the outer normals to its sides.

Theorem 6.1 implies that every triangle determines a hypergeometric system with maximally reducible monodromy for a suitable choice of the vector of parameters. A quadrilateral defines a system with maximally reducible monodromy if and only if it is a trapezium.

We divide the proof of Theorem 6.1 into three steps.

We first give a detailed description of a key technical notion named 'half-space cancellation of poles' (Definition 6.2, Lemma 6.3). Then we prove that the conditions 1), 2) are necessary and sufficient for the conclusion of the theorem to hold (Propositions 6.5, 6.6). Finally we use Proposition 6.6 to establish that the maximal reducibility of monodromy is equivalent to the existence of a Puiseux polynomial basis for a proper choice of parameters (Corollary 6.7).

To prove the necessity and sufficiency of the conditions in Theorem 6.1, we need the following auxiliary technical notion.

Definition 6.2. We say that the Ore–Sato coefficient $\varphi(s) = \frac{\prod_{j=1}^{a}\Gamma(\alpha_j)}{\prod_{i=1}^{b}\Gamma(\beta_i)}$ admits a *half-space cancellation of poles* if the poles of $\varphi(s)$ lie in the set $\{s\colon \alpha_j(s) = \sigma,\ \sigma \in \mathbb{Z}_{\leqslant 0},\ \gamma_j \leqslant \sigma \leqslant 0\}$ for some $\gamma_j < 0$, $j \in [1, a]$.

Lemma 6.3. *The half-space cancellation of poles in the Ore–Sato coefficient* $\varphi(s) = \frac{\prod_{j=1}^{a}\Gamma(\alpha_j)}{\prod_{i=1}^{b}\Gamma(\beta_i)}$ *is a necessary condition for the Mellin–Barnes integral* $\mathrm{MB}(\varphi, \mathcal{C})$ *to represent a set of Puiseux polynomial solutions for every contour* $\mathcal{C}$, *satisfying the conditions of Theorem* 3.3.

Example 6.4. Consider the function

$$\varphi(s) = \frac{\Gamma(s_1+s_2-3)\Gamma(-s_2)}{\Gamma(s_1+1)\Gamma(s_2+2)\Gamma(-s_2+2)}.$$

Its poles lie on the lines $\{s\colon -s_2=\sigma,\ \sigma=-1,0,\ s_1\neq -1,-2,\dots\}$. In this case $\mathrm{MB}(\varphi,\mathcal{C}) = \mathrm{const}\cdot(x_1+1)^2(2x_1-3x_2+2)$, where the contour $\mathcal{C}$ is located near the integer lattice points inside $\{s\colon s_1+s_2\leqslant 3,\ 0\leqslant s_1,\ 0\leqslant s_2\}$.

We now use Definition 6.2 and Lemma 6.3 to prove the sufficiency of either or the conditions 1), 2).

Proposition 6.5. *For a polygon $\mathcal{P}$ of type* 1) *or* 2), *the space of holomorphic solutions of* $\mathrm{Horn}(A(\mathcal{P}),c)$ *admits a Puiseux polynomial basis for some parameter $c\in\mathbb{C}^n$ and hence admits a maximally reducible monodromy representation.*

Proof. Let A be the $m\times 2$ matrix whose rows are the outer normals to the sides of the zonotope which is the polygon of an Ore–Sato coefficient; see Definition 2.5. We recall that the number of occurrences of a vector as a row of A equals the relative length of the corresponding side of the zonotope in the integer lattice. We will first show that there is a $c\in\mathbb{C}^m$ such that the space of holomorphic solutions of the hypergeometric system $\mathrm{Horn}(A,c)$ at a generic point has a basis consisting of functions of the form $x^\alpha p(x)$, where $\alpha\in\mathbb{C}^n$ and $p(x)$ is a Taylor polynomial (that is, a Puiseux polynomial with integer non-negative exponents). Since the analytic continuation of such a function along any path is proportional to itself, this will prove that the monodromy representation of $\mathrm{Horn}(A,c)$ is maximally reducible.

Since the matrix A is determined by a zonotope, there is no loss of generality in assuming (possibly after interchanging some of its rows) that it consists of blocks of the form $B_i = \left(\begin{smallmatrix} a_i & b_i \\ -a_i & -b_i\end{smallmatrix}\right)$. Let k_i be the number of occurrences of the block B_i in A, and let l be the number of different blocks. There is no loss of generality in assuming that the numbers a_i and b_i are relatively prime. We will also assume for simplicity that all entries of A are different from zero. The case when some of them are equal to zero can be treated in a similar way.

By Theorem 3.8 the holonomic rank of the system $\mathrm{Horn}(A,c)$ equals

$$r(A) = \left(\sum_{i=1}^{l} k_i|a_i|\right)\left(\sum_{j=1}^{l} k_j|b_j|\right) - \sum_{i=1}^{l} k_i^2|a_ib_i| = \sum_{\substack{i,j=1\\ i\neq j}}^{l} k_ik_j|a_ib_j|.$$

Induction on l shows that the vector space of analytic solutions of the hypergeometric system $\mathrm{Horn}(A,c)$ admits a Puiseux polynomial basis. Indeed, when $l=2$ we have a parallelogram, which by Proposition 4.7 (with $-\alpha_j-\beta_j\in\mathbb{N}$ in (4.8)) determines a system with a Puiseux polynomial basis in its solution space. We define a matrix B_{l+1} by the formula $B_{l+1} = \left(\begin{smallmatrix} a_{l+1} & b_{l+1} \\ -a_{l+1} & -b_{l+1}\end{smallmatrix}\right)$. Let A' be the matrix obtained by appending k_{l+1} copies of the block B_{l+1} to the matrix A, and let $r(A')$ be the holonomic rank of the associated Horn system. As above, there is no loss of generality in assuming that $a_{l+1}\neq 0$, $b_{l+1}\neq 0$. We can also assume that the vector (a_{l+1},b_{l+1}) is not proportional to (a_i,b_i) for any $i=1,\dots,l$. Indeed, if these two vectors were proportional, then adding the block B_{l+1} would be equivalent to increasing the number k_i of occurrences of the block B_i in the matrix A.

Observe that appending the block B_{l+1} to the matrix A corresponds to adding (in the Minkowski sense) the segment $(-b_{l+1}, a_{l+1})$ to the polygon defined by A. To obtain a polygon of different combinatorial structure, we must add a segment which is not parallel to the sides of the original polygon. Then the amoeba of the singularity of the corresponding hypergeometric systems sprouts two new tentacles in opposite directions. By Theorem 5.3, the number of Puiseux series solutions of the Horn system defined by A' is the same for all connected components of the complement to the amoeba of its singularity. We will show that for a suitable (very specific) choice of the parameters of the system, these series actually turn out to be polynomials.

Under the assumptions above, the holonomic rank $r(A')$ of the hypergeometric system defined by the matrix A' and a generic vector of parameters is given by

$$\begin{aligned} r(A') &= \sum_{\substack{i,j=1\\ i\neq j}}^{l+1} k_i k_j |a_i b_j| = r(A) + \sum_{i=1}^{l} k_i k_{l+1} |a_i b_{l+1}| + \sum_{j=1}^{l} k_{l+1} k_j |a_{l+1} b_j| \\ &= r(A) + \sum_{i=1}^{l} \big((k_i|a_i| + k_{l+1}|a_{l+1}|)(k_i|b_i| + k_{l+1}|b_{l+1}|) - k_i^2|a_i b_i| - k_{l+1}^2 |a_{l+1} b_{l+1}| \big) \\ &= r(A) + \sum_{i=1}^{l} r(k_i B_i, k_{l+1} B_{l+1}), \end{aligned}$$

where $r(k_i B_i, k_{l+1} B_{l+1})$ stands for the holonomic rank of the parallelepipedal hypergeometric system defined by the matrix obtained by joining together k_i copies of the block B_i and k_{l+1} copies of the block B_{l+1}.

We will now show that adding (in the Minkowski sense) a segment to a planar zonotope preserves the property of the corresponding hypergeometric system of having a Puiseux polynomial basis in its space of holomorphic solutions for a suitable choice of the vector of parameters.

We first observe that for any positive integer m_{l+1} the poles of the meromorphic function

$$\frac{\Gamma(a_{l+1}s_1 + b_{l+1}s_2 + c_{l+1})}{\Gamma(a_{l+1}s_1 + b_{l+1}s_2 + c_{l+1} + m_{l+1} + 1)}$$

lie on the lines $\bigcup_{h=0}^{m_{l+1}} \{s \colon a_{l+1}s_1 + b_{l+1}s_2 + c_{l+1} + h = 0\}$. The poles of the function

$$\prod_{i=1}^{l} \prod_{j=1}^{k_i} \frac{\Gamma(a_i s_1 + b_i s_2 + c_{i,j})}{\Gamma(a_i s_1 + b_i s_2 + c_{i,j} + m_{i,j} + 1)}$$

also lie on the finite family of lines $\bigcup_{i=1}^{l} \bigcup_{j=1}^{k_i} \bigcup_{h=0}^{m_{i,j}} \{s \colon a_i s_1 + b_i s_2 + c_{i,j} + h = 0\}$. We conclude that for a suitable choice of the vector of parameters c the number of double poles of the meromorphic function

$$\prod_{i=1}^{l+1} \prod_{j=1}^{k_i} \frac{\Gamma(a_i s_1 + b_i s_2 + c_{i,j})}{\Gamma(a_i s_1 + b_i s_2 + c_{i,j} + m_{i,j} + 1)}$$

is finite. To prove this fact, it suffices to choose the vector of parameters c in such a way that the parallelogram

$$\Pi(i,j;k,\ell) = \bigcup_{t=0}^{1}\bigcup_{u=0}^{1}\{s\colon a_is_1+b_is_2+c_{i,j}+tm_{i,j}=0,\ a_ks_1+b_ks_2+c_{k,\ell}+um_{k,\ell}=0\}$$

is disjoint from any similar parallelogram $\Pi(i',j';k',\ell')$ whenever $|i-i'|+|j-j'|+|k-k'|+|\ell-\ell'|\neq 0$. Note that all double poles of the meromorphic function

$$\frac{\Gamma(a_is_1+b_is_2+c_{i,j})\Gamma(a_ks_1+b_ks_2+c_{k,\ell})}{\Gamma(a_is_1+b_is_2+c_{i,j}+m_{i,j}+1)\Gamma(a_ks_1+b_ks_2+c_{k,\ell}+m_{k,\ell}+1)}$$

that contribute to the solutions of $\mathrm{Horn}(A,c)$, are contained in the parallelogram $\Pi(i,j;k,\ell)$ because of the cancellation of poles (compare Definition 6.2) of the two factors $\Gamma(a_is_1+b_is_2+c_{i,j})$ and $\Gamma(a_ks_1+b_ks_2+c_{k,\ell})$. Since a parallelogram is the image of the square $\{(t,u)\colon 0\leqslant t\leqslant 1,\ 0\leqslant u\leqslant 1\}$ under a linear map, one can choose the values of the parameters $c_{i,j}$, $c_{k,\ell}$, $c_{i',j'}$, $c_{k',\ell'}$ in such a way that $\Pi(i,j;k,\ell)$ is disjoint from $\Pi(i',j';k',\ell')$ when $(i,j;k,\ell)\neq(i',j';k',\ell')$. The set of such pairs is finite and, therefore, the desired choice of parameters can always be made.

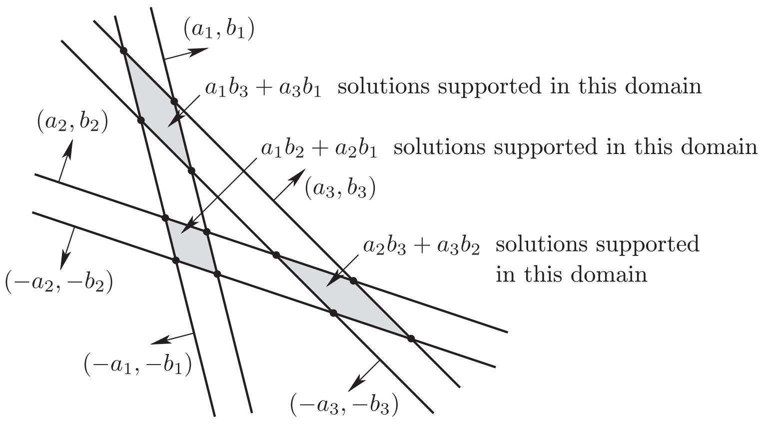

Figure 3. Adding a segment to the zonotope that defines a Horn system

The inductive step described above is illustrated by Fig. 3 under the assumption that $a_i,b_i>0$ for $i=1,2,3$. The shaded regions contain the supports of the Puiseux polynomial solutions of the Horn system obtained by adding the block $B_3=\begin{pmatrix} a_3 & b_3\\ -a_3 & -b_3\end{pmatrix}$ to a matrix composed of the blocks B_1 and B_2. The computation of the rank (see above) shows that the number of Puiseux polynomial solutions whose supports lie on the intersections of the new (third) pair of divisors with the initial divisors is exactly sufficient to compensate for the growth in rank. In fact, by Theorem 3.8, the rank of the system defined by all three pairs of divisors equals $(a_1+a_2+a_3)(b_1+b_2+b_3)-a_1b_1-a_2b_2-a_3b_3$. This is exactly how many Puiseux polynomials are supported by the three parallelograms depicted in Fig. 3.

Similar arguments show that the second class of polygons in Theorem 6.1 (the Minkowski sums of triangles and multiples of their sides) also define hypergeometric systems with Puiseux polynomial bases in their solution spaces for suitable choices of the parameters.

Since every pure Puiseux polynomial spans a one-dimensional invariant subspace, it follows that the monodromy representation of a hypergeometric system satisfying the conditions of Theorem 6.1 is maximally reducible. □

We now prove the necessity of one of the conditions 1), 2) of Theorem 6.1.

Proposition 6.6. *If a bivariate hypergeometric system* $\operatorname{Horn}(A, c)$ *has a maximally reducible monodromy representation, then its Ore–Sato polygon is one of the following*:

1) *a zonotope*;
2) *the Minkowski sum of a triangle and segments parallel to its sides.*

Proof. For simplicity we consider the case when the matrix A is of the form

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & b_1 \\ \dots & \dots \\ a_r & b_r \end{pmatrix}, \tag{6.1}$$

where $1 + \sum_{j=1}^{r} a_j = 1 + \sum_{j=1}^{r} b_j = 0,\ m = r + 2$. The proof for general A can be achieved in a completely parallel way.

If the Ore–Sato polygon is a triangle, then condition 2) holds automatically. Therefore in what follows we assume that $r \geqslant 2$ and hence $m \geqslant 4$. We shall use the notation $\alpha_j(s) = a_j s_1 + b_j s_2$. Consider two groups of linear functions $\alpha_j(s)$, which are indexed by I_+, I_- in such a way that $j+ \in I_+$ (resp. $k- \in I_-$) if and only if $a_{j+} > 0$ (resp. $a_{k-} < 0$). The poles of the function $\Gamma(\alpha_{j+}(s) + \gamma_{j+})$ with $\alpha_{j+}(s) = -m - \gamma_{j+}$, $m \in \mathbb{Z}_{\geqslant 0}$ (resp. $\Gamma(\alpha_{k-}(s) + \gamma_{k-})$ with $\alpha_{k-}(s) = -m - \gamma_{k-}$, $m \in \mathbb{Z}_{\geqslant 0}$) restricted to the complex plane $\{s \in \mathbb{C}^2 \colon s_2 + \delta_2 + n = 0,\ n \in \mathbb{Z}_{\geqslant 0}\}$ exhibit the asymptotic behaviour as $s_1 \to -\infty$ (resp. $s_1 \to +\infty$).

For the Ore–Sato coefficient

$$\varphi_{2,j+,k-}(s) = \frac{\Gamma(s_2 + \delta_2)\Gamma(\alpha_{j+}(s) + \gamma_{j+})\Gamma(\alpha_{k-}(s) + \gamma_{k-})}{\Gamma(1 - s_1 - \delta_1)\prod_{\ell \neq j+,k-}^{r} \Gamma(1 - \alpha_\ell(s) - \gamma_\ell)} \tag{6.2}$$

we examine the subspace of solutions of $S(\operatorname{Horn}(A', c'))$ spanned by the integrals of the form

$$u_{2,j+}(x) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\mathcal{C}_{2,j+}} \varphi_{2,j+,k-}(s) x^s \, ds$$

and their analytic continuations. Here $c' = (\delta_1, \delta_2, \gamma_1, \dots, \gamma_r)$ and

$$\mathcal{C}_{2,j+} = \{s \in \mathbb{C}^2 \colon |s_2 + \delta_2 + n| = |\alpha_{j+}(s) + \gamma_{j+} + m| = \varepsilon,\ (n, m) \in \mathbb{Z}^2_{\geqslant 0}\},$$

where the radius ε of the circle is chosen so small that each disc inside the circle contains one isolated double pole of $\varphi_{2,j+,k-}(s)$.

The system Horn(A', c'), whose solution space has non-diagonalizable local monodromy is a resonant system (see Definition 2.13). That is, for such a system at least one of the monodromy representation matrices has a non-trivial Jordan block of order at least 2. Thus it is not maximally reducible. Therefore we may assume that the system of equations Horn(A', c') is non-resonant. This means in particular that its solution $u_{2,j+}(x)$ can be expanded into the Puiseux series

$$\sum_{(n,m)\in\mathbb{Z}^2_{\geqslant 0}} c_{n,m} \left(\frac{x_1^{\frac{b_{j+}}{a_{j+}}}}{x_2} \right)^{n+\delta_2} x_1^{\frac{-m-\gamma_{j+}}{a_{j+}}} \tag{6.3}$$

in a neighbourhood of the point $(\frac{1}{x_1}, \frac{1}{x_2}) = (0,0)$. Repeated application of the monodromy action

$$\frac{1}{x_1} \to \frac{1}{e^{2\pi\sqrt{-1}}x_1}$$

to this series representation of $u_{2,j+}(x)$ produces an (a_{j+})-dimensional subspace $S_{2,j+} \subset S(\text{Horn}(A', c'))$ because the Vandermonde matrix is non-singular.

We now consider an analytic continuation of the Puiseux series solution $u_{2,j+}(x)$ (6.3) that transforms it into the integral

$$u_{2,k-}(x) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\mathcal{C}_{2,k-}} \varphi_{2,j+,k-}(s) x^s \, ds \tag{6.4}$$

by means of an operation to be called 'the Mellin–Barnes contour throw' throughout the rest of the paper (Fig. 4).

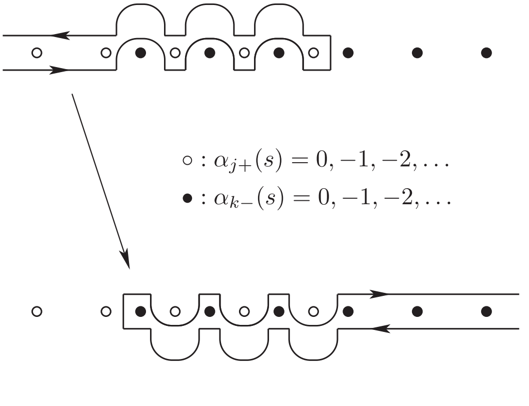

Figure 4. Mellin–Barnes contour throw

The integral (6.4) is calculated as the sum of residues inside the contours

$$\mathcal{C}_{2,k-} = \{s \in \mathbb{C}^2 \colon |s_2 + \delta_2 + n| = |\alpha_{k-}(s) + \gamma_{k-} + m| = \varepsilon,\ n, m \in \mathbb{Z}_{\geqslant 0}\},$$

that encircle the poles for which $s_1 \to +\infty$ on the complex line $\{s \in \mathbb{C}^2 \colon s_2 + \delta_2 + n = 0,\ n \in \mathbb{Z}_{\geqslant 0}\}$. The Puiseux expansion of $u_{2,k-}(x)$ in a neighbourhood of $\left(x_1, \frac{1}{x_2}\right) = (0,0)$ takes the form

$$\sum_{(n,m)\in\mathbb{Z}^2_{\geqslant 0}} d_{n,m} \left(\frac{x_1^{\frac{b_{k-}}{a_{k-}}}}{x_2}\right)^{n+\delta_2} x_1^{\frac{-m-\gamma_{k-}}{a_{k-}}},$$

where $a_{k-} < 0$. Repeated application of the monodromy action $x_1 \to e^{2\pi\sqrt{-1}}x_1$ to the series representation for $u_{2,k-}(x)$ produces a $|a_{k-}|$-dimensional subspace $S_{2,k-} \subset S(\mathrm{Horn}(A', c'))$ in the solution space of the Horn system because of the non-singularity of a Vandermonde matrix.

We now analyze the following steps in the analytic continuation of the solutions of the hypergeometric system in question:

a) the analytic continuation of $u_{2,j+}$ that transforms it into $S_{2,k-}$ by means of the contour throw in the Mellin–Barnes integral,

b) the monodromy action on the subspace of solutions $S_{2,k-}$ induced by the map $x_1 \mapsto e^{2\pi h\sqrt{-1}}x_1$, that is,

$$\varphi_{2,j+,k-}(s)x^s \mapsto \varphi_{2,j+,k-}(s)e^{2\pi h s_1\sqrt{-1}}x^s, \qquad h \in \mathbb{Z},$$

c) the inverse analytic continuation of $S_{2,k-}$ to $S_{2,j+}$.

Under the condition of maximal reducibility of the monodromy, if the above procedures a)–c) give rise to a well-defined non-trivial monodromy around $x_1 = \infty$, then the image of $S_{2,j+}$ under this monodromy action has dimension $|a_{k-}|$ and, therefore, $|a_{j+}| = |a_{k-}|$. This means that for every $j+ \in I_+$ there is a $k- \in I_-$ such that $a_{j+} + a_{k-} = 0$.

Interchanging s_2 and s_1 (as well as x_2 and x_1 in (6.3) and (6.4)) and using the same argument, we conclude that for every $b_{p+} > 0$ there is a $b_{q-} < 0$ such that $b_{p+} + b_{q-} = 0$.

We now prove a stronger assertion: for every $j+ \in I_+$ there is a $k- \in I_-$ such that

$$a_{j+} + a_{k-} = 0, \qquad b_{j+} + b_{k-} = 0. \tag{6.5}$$

To prove the existence of such an index, we study the domains of convergence of all series obtained as residues of $\varphi_{i,j+,k-}(s)x^s$.

Let $D_{j+,k-}$ be the domain of convergence of the series

$$u_{j+,k-}(x) = \sum_{n,m\geqslant 0} \operatorname{Res}_{\substack{\alpha_{j+}(s)+\gamma_{j+}=-n \\ \alpha_{k-}(s)+\gamma_{k-}=-m}} \varphi_{i,j+,k-}(s)x^s$$

for $i = 1, 2,\ j+ \in I_+,\ k- \in I_-$. Here we use the notation

$$\varphi_{2,j+,k-}(s) = \frac{\Gamma(s_2+\delta_2)\Gamma(\alpha_{j+}(s)+\gamma_{j+})\Gamma(\alpha_{k-}(s)+\gamma_{k-})}{\Gamma(1-s_1-\delta_1)\prod_{\ell\neq j+,k-}^{r}\Gamma(1-\alpha_\ell(s)-\gamma_\ell)}.$$

We similarly define a function $\varphi_{1,j+,k-}(s)$.

In a similar way, we look at the domains $D_{i,j+}$ of convergence of the series

$$u_{i,j+}(x) = \sum_{n,m\geqslant 0} \operatorname{Res}_{\substack{\alpha_{j+}(s)+\gamma_{j+}=-m \\ s_i+\delta_i=-n}} \varphi_{i,j+,k-}(s)x^s$$

as well as the domains $D_{i,k-}$ of convergence of the series

$$u_{i,k-}(x) = \sum_{n,m\geqslant 0} \operatorname{Res}_{\substack{\alpha_{k-}(s)+\gamma_{k-}=-m \\ s_i+\delta_i=-n}} \varphi_{i,j+,k-}(s)x^s$$

for $i = 1, 2$.

We claim that the domain $D_{j+,k-}$ has non-empty intersection with at least one of the four domains $D_{1,j+}$, $D_{2,j+}$, $D_{1,k-}$, $D_{2,k-}$. To prove this, we consider the supporting cones $C_{j+,k-}$, $C_{i,j+}$ and $C_{i,k-}$ of the solutions $u_{j+,k-}(x)$, $u_{i,j+}(x)$ and $u_{i,k-}(x)$ respectively. It follows from Abel's lemma (see [7], Proposition 2, and [9], Lemma 1) that

$$\operatorname{Log} x^{(a,b)} - C^{\vee}_{a,b} \subset \operatorname{Log}(D_{a,b})$$

for some $x^{(a,b)} \in D_{a,b}$ and some multi-index (a,b) that coincides with one of the multi-indices $(j+,k-)$, $(i,j+)$, $(i,k-)$. After an easy case-by-case study we see that $C^{\vee}_{j+,k-}$ has non-empty two-dimensional intersection with one of the four dual cones $C^{\vee}_{1,j+}$, $C^{\vee}_{2,j+}$, $C^{\vee}_{1,k-}$, $C^{\vee}_{2,k-}$. This proves the claim (see Fig. 5).

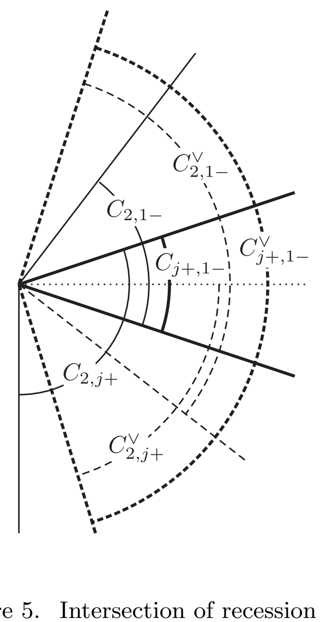

Figure 5. Intersection of recession cones

Let us assume, for example, that $D_{j+,k-} \cap D_{2,j+} \neq \varnothing$. The analytic continuation of $S_{2,j+}$ induced by the Mellin–Barnes contour throw $\mathcal{C}_{2,j+} \to \mathcal{C}_{j+,k-}$ on the complex lines $\{s \in \mathbb{C}^2 \colon \alpha_{j+}(s) + \gamma_{j+} \in \mathbb{Z}_{\leqslant 0}\}$ yields a $|a_{j+}(b_{j+} + b_{k-})|$-dimensional subspace of Puiseux series solutions in $S(\operatorname{Horn}(A', c'))$ that consists of Puiseux series converging in the domain $D_{j+,k-}$ by Theorem 3.7, 2). This dimension is calculated from the following equality:

$$\left|\det\begin{pmatrix} a_{j+} & b_{j+} \\ a_{k-} & b_{k-} \end{pmatrix}\right| = |a_{j+}(b_{j+} + b_{k-})|, \tag{6.6}$$

where $a_{j+} = -a_{k-}$. On the other hand, we have already noticed that the analytic continuation $S_{2,k-}$ of $S_{2,j+}$ induced by the Mellin–Barnes contour throw $\mathcal{C}_{2,j+} \to \mathcal{C}_{2,k-}$ on the complex lines $\{s \in \mathbb{C}^2 \colon s_2 + \delta_2 \in \mathbb{Z}_{\leqslant 0}\}$ has dimension $|a_{k-}| = a_{j+}$. Thus we obtain an analytic continuation of the elements of $S_{2,j+}$ to the domain $D_{j+,k-} \cap D_{2,j+} \neq \varnothing$, whose dimension is $a_{j+} + |a_{j+}(b_{j+} + b_{k-})|$ by Theorem 3.7, 2). If the monodromy representation of the hypergeometric system is maximally reducible, then the analytic continuation of an element of $S_{2,j+}$ along any path must have dimension a_{j+}. In particular, this applies to the monodromy action. This means that $b_{j+} + b_{k-} = 0$ and, therefore, (6.5) follows.

The same argument works when $D_{j+,k-} \cap D_{2,k-} \neq \varnothing$.

When $D_{j+,k-} \cap D_{1,j+} \neq \varnothing$ or $D_{j+,k-} \cap D_{1,k-} \neq \varnothing$, we interchange the roles of x_1 and x_2 and arrive at the equality $|b_{j+}| = |b_{j+}| + |a_{j+}(b_{j+} + b_{k-})|$, whence $b_{j+} + b_{k-} = 0$. Thus we again obtain (6.5).

Using the condition $1 + \sum_{j=1}^{r} a_j = 1 + \sum_{j=1}^{r} b_j = 0$, where $m = r + 2$, we see that the matrix A' defining a hypergeometric system $\operatorname{Horn}(A', c')$ with a maximally reducible monodromy representation must be either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ a_1 & b_1 \\ -a_1 & -b_1 \\ \dots & \dots \\ a_{r/2-1} & b_{r/2-1} \\ -a_{r/2-1} & -b_{r/2-1} \end{pmatrix} \tag{6.7}$$

for even r, or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ a_1 & b_1 \\ -a_1 & -b_1 \\ \dots & \dots \\ a_{(r-1)/2} & b_{(r-1)/2} \\ -a_{(r-1)/2} & -b_{(r-1)/2} \end{pmatrix} \tag{6.8}$$

for odd r.

Elementary plane geometry shows that the matrix A' of the form (6.7) corresponds to a hypergeometric system defined by a zonotope.

To examine the case (6.8), we use the notation $\mathbf{A}_{1-} = (-1, -1)$, $1- \in I_-$. When $j+ \in I_+$ we see that either $D_{j+,1-} \cap D_{2,j+} \neq \varnothing$, or $D_{j+,1-} \cap D_{2,1-} \neq \varnothing$.

When $D_{j+,1-} \cap D_{2,j+} \neq \varnothing$, the analytic continuation by means of the Mellin–Barnes contour throw in the complex line $\{s \in \mathbb{C}^2 \colon \alpha_{j+}(s) + \gamma_{j+} = -m,\ m \in \mathbb{Z}_{\geqslant 0}\}$ of the solution

$$u_{2,j+}(x) = \sum_{n,m \geqslant 0} \mathrm{Res}_{\substack{\alpha_{j+}(s)+\gamma_{j+}=-m \\ s_2+\delta_2=-n}} \varphi_{2,1-,j+}(s) x^s$$

of the Horn system yields the solution

$$u_{j+,1-}(x) = \sum_{n,m \geqslant 0} \mathrm{Res}_{\substack{\alpha_{j+}(s)+\gamma_{j+}=-m \\ -s_1-s_2+\gamma_{1-}=-n}} \varphi_{2,1-,j+}(s) x^s.$$

Using Theorem 3.7, 2), we arrive at the equality $a_{j+} = a_{j+} + |a_{j+} - b_{j+}|$. This means that $a_{j+} - b_{j+} = 0$.

When $D_{2,1-} \cap D_{j+,1-} \neq \varnothing$, we apply the same argument to the analytic continuation $u_{2,1-}(x) \to u_{j+,1-}(x)$ and arrive at the equality $1 = 1 + |a_{j+} - b_{j+}|$. Hence we again obtain that $a_{j+} - b_{j+} = 0$, that is, the vector $\mathbf{A}_{j+}$ is collinear with $(-1,-1)$.

In an analogous way we conclude that the analytic continuation by means of the Mellin–Barnes contour throw along the complex lines $\{s \in \mathbb{C}^2 \colon s_2 + \delta_2 \in \mathbb{Z}_{\leqslant 0}\}$ transforms the function

$$u_{2,1-}(x) = \sum_{n,m \geqslant 0} \mathrm{Res}_{\substack{-s_1-s_2+\gamma_{1-}=-m \\ s_2+\delta_2=-n}} \varphi_{2,1-,j+}(s) x^s$$

into the function

$$u_{2,k-}(x) = \sum_{n,m \geqslant 0} \mathrm{Res}_{\substack{\alpha_{k-}(s)+\gamma_{k-}=-m \\ s_2+\delta_2=-n}} \varphi_{2,1-,j+}(s) x^s.$$

In view of the relation $C^\vee_{2,1-} \subset C^\vee_{2,k-}$, we see that $1 + |a_{k-}| = 1$, that is, $|a_{k-}| = 0$ and, therefore, the vector $\mathbf{A}_{k-}$ is collinear with $(0,1)$.

We now apply the same argument to the residues of the functions $\varphi_{1,1-,j+}(s)x^s$ and $\varphi_{1,1-,k-}(s)x^s$ and conclude that every row vector of the matrix (6.8) is collinear with one of the three vectors $(1,0)$, $(0,1)$, $(-1,-1)$. This means that the Ore–Sato polygon of the Horn system $\mathrm{Horn}(A',c')$, with A' defined in (6.8), must be the Minkowski sum of a triangle and segments parallel to its sides. □

Corollary 6.7. *A bivariate hypergeometric system* $\mathrm{Horn}(A,c)$ *has a maximally reducible monodromy representation if and only if the space of solutions of* $\mathrm{Horn}(A,\tilde{c})$ *is spanned by Puiseux polynomials for some choice of the vector of parameters* $\tilde{c}$.

Proof. If the space of solutions of $\mathrm{Horn}(A,\tilde{c})$ is spanned by Puiseux polynomials, then the monodromy representation is maximally reducible. Indeed, each Puiseux monomial spans a one-dimensional subspace which is invariant under the analytic continuation along any path and, therefore, $\mathrm{Horn}(A,\tilde{c})$ is a direct sum of such subspaces.

Proposition 6.6 shows that the polygon of the Ore–Sato coefficient defining the hypergeometric system $\mathrm{Horn}(A,c)$ with a maximally reducible monodromy representation must be either a zonotope or the Minkowski sum of a triangle and segments parallel to its sides. By Proposition 6.5, the solution space of $\mathrm{Horn}(A,\tilde{c})$ admits a Puiseux polynomial basis for a suitably chosen parameter $\tilde{c}$. □

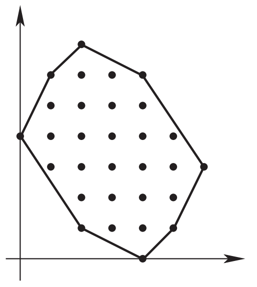

Figure 6. The zonotope which defines the matrix (6.9)

Example 6.8 (a randomly chosen zonotope). Consider the Minkowski sum of the four segments shown in Fig. 6. The matrix of the outer normals to its sides takes the form

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ -1 & 1 \\ 1 & -1 \\ -3 & -2 \\ 3 & 2 \\ 2 & -1 \\ -2 & 1 \end{pmatrix} \tag{6.9}$$

Choose the vector of parameters to be $c = (3, -5, -2, 1, -2, -1, -1, -1)^T$. The corresponding hypergeometric system $\mathrm{Horn}(A, c)$ is holonomic of rank 31 by Theorem 3.8. Here is a pure Puiseux polynomial basis of its solution space (which was computed with Mathematica 9.0): the persistent solutions are

$$x_2, \; x_1^3 x_2^5, \; \frac{\sqrt{x_1}}{x_2^{7/4}}, \; \frac{x_1}{x_2^2}, \; \frac{x_1^{5/2}}{x_2^{15/4}}, \; \frac{x_1^3}{x_2^4}$$

while the non-persistent Puiseux polynomial solutions are

$$\frac{x_1^2}{x_2^3}, \; \frac{x_1^{3/2}}{x_2^{11/4}}, \; \frac{1}{x_1^{4/5} x_2^{8/5}}, \; \frac{x_1^{2/7}}{x_2^{3/7}}, \; \frac{\sqrt[7]{x_2}}{x_1^{3/7}}, \; \frac{x_2^{3/5}}{x_1^{2/5}}, \; \frac{x_2}{x_1},$$

$$13068 x_1^2 x_2^4 + 18900 x_1^2 x_2^3 + 74529 x_1 x_2^3 + 715715 x_1 x_2^2, \; \frac{54}{x_1^{4/7} \sqrt[7]{x_2}} + \frac{5 x_1^{3/7}}{\sqrt[7]{x_2}},$$

$$\frac{99 x_2^{3/7}}{x_1^{2/7}} - \frac{52}{x_1^{2/7} x_2^{4/7}}, \; \frac{230 x_2^{5/7}}{\sqrt[7]{x_1}} - \frac{407}{\sqrt[7]{x_1} x_2^{2/7}}, \; \frac{5}{x_2} - 9,38 \sqrt[7]{x_1}\, x_2^{2/7} - \frac{99 \sqrt[7]{x_1}}{x_2^{5/7}},$$

$$\frac{234 x_2^{6/5}}{x_1^{4/5}} - \frac{1463 \sqrt[5]{x_2}}{x_1^{4/5}}, \; \frac{14 x_2^{7/5}}{x_1^{3/5}} - \frac{837 x_2^{2/5}}{x_1^{3/5}}, \; \frac{119 x_2^{4/5}}{\sqrt[5]{x_1}} - \frac{4 x_2^{4/5}}{x_1^{6/5}}, \; \frac{275}{x_1^2 x_2} - \frac{7}{x_1^3 x_2},$$

$$\frac{129115}{x_1^{5/3}x_2^{2/3}} - \frac{7904}{x_1^{8/3}x_2^{2/3}},\ \frac{203}{x_1^{7/3}\sqrt[3]{x_2}} - \frac{170}{x_1^{7/3}x_2^{4/3}},\ \frac{22869x_1^2}{x_2^{7/2}} + \frac{16065x_1^2}{x_2^{5/2}} - \frac{143650x_1}{x_2^{5/2}},$$

$$-\frac{2600150x_1^{5/2}}{x_2^{13/4}} + \frac{29637333x_1^{3/2}}{x_2^{9/4}} + \frac{4075291x_1^{3/2}}{x_2^{13/4}},\ \frac{1}{x_1x_2^2} - \frac{7}{x_1x_2},$$

$$\frac{19}{x_1^{7/5}x_2^{9/5}} + \frac{143}{x_1^{2/5}x_2^{9/5}},\ \frac{238}{x_1^{6/5}x_2^{7/5}} + \frac{999}{\sqrt[5]{x_1}x_2^{7/5}},\ \frac{2511}{x_1^{3/5}x_2^{6/5}} - \frac{88}{x_1^{3/5}x_2^{11/5}}.$$

Fig. 7 depicts the supports of these solutions of $\mathrm{Horn}(A, c)$. The big bullets correspond to monomials (persistent or not), and the small bullets to all other solutions. The parallelograms that carry the supports arise as intersections of the divisors of the defining Ore–Sato coefficient.

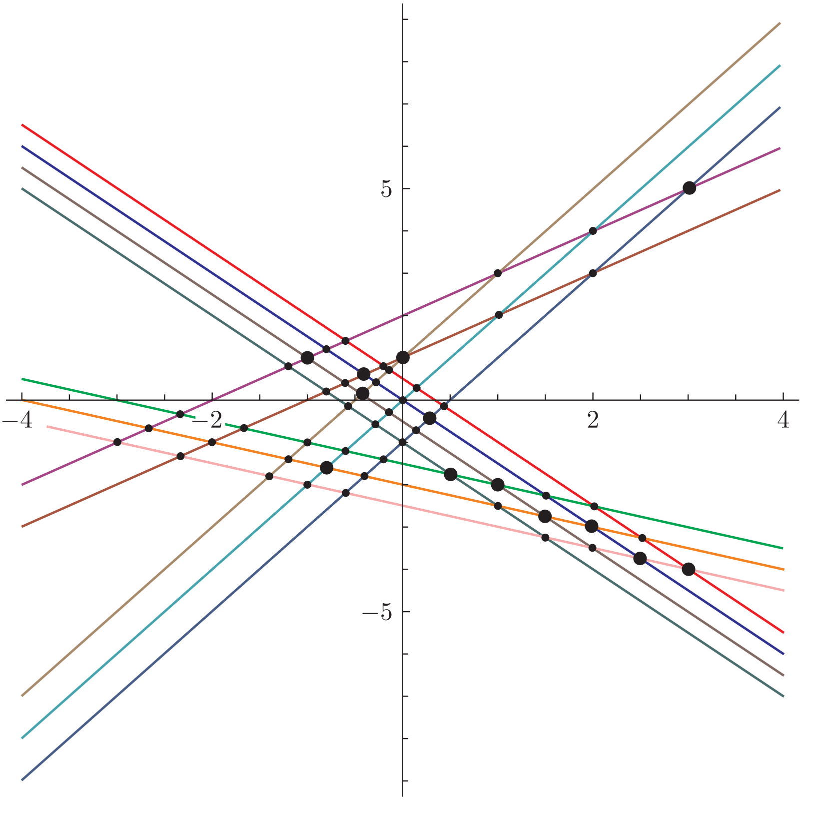

Figure 7. The supports of the solutions of the system $\mathrm{Horn}(A, c)$ defined by the matrix (6.9)

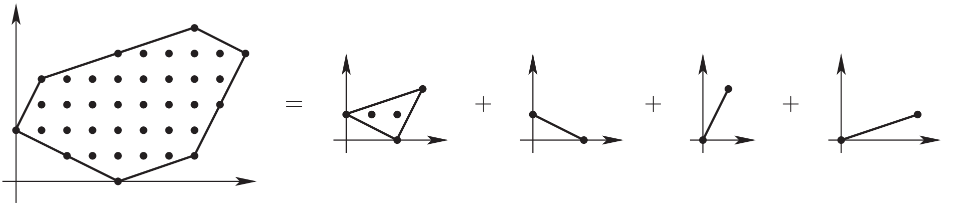

Figure 8. The polygon defining the matrix (6.10), and its Minkowski decomposition

Example 6.9 (the Minkowski sum of a triangle and its sides). The following matrix comes from the Minkowski sum of a triangle and all its sides (Fig. 8):

$$A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \\ -2 & 1 \\ -1 & 3 \\ -1 & 3 \\ 1 & -3 \\ 1 & 2 \\ -1 & -2 \\ -1 & -2 \end{pmatrix} \tag{6.10}$$

Choose the vector of parameters to be $c = (-1, -6, 3, -2, -10, 5, 3, -1, -6)^T$. By Theorem 3.8 the corresponding hypergeometric system is holonomic of rank 40 and is defined by the differential operators

$$\begin{aligned} &x_1(\theta_1 - 3\theta_2 + 5)(2\theta_1 - \theta_2 - 6)(2\theta_1 - \theta_2 - 5)(2\theta_1 - \theta_2 - 1)(2\theta_1 - \theta_2)(\theta_1 + 2\theta_2 + 3) \\ &\quad - (\theta_1 + 2\theta_2 + 6)(\theta_1 + 2\theta_2 + 1)(2\theta_1 - \theta_2 - 4) \\ &\qquad\qquad \times (2\theta_1 - \theta_2 - 3)(\theta_1 - 3\theta_2 + 10)(\theta_1 - 3\theta_2 + 2), \end{aligned}$$

$$\begin{aligned} &x_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 + 1)(\theta_1 - 3\theta_2 + 2)(\theta_1 - 3\theta_2 + 8)(\theta_1 - 3\theta_2 + 9) \\ &\qquad \times (\theta_1 - 3\theta_2 + 10)(2\theta_1 - \theta_2 - 3)(\theta_1 + 2\theta_2 + 3)(\theta_1 + 2\theta_2 + 4) \\ &\quad - (\theta_1 - 3\theta_2 + 5)(\theta_1 - 3\theta_2 + 6)(\theta_1 - 3\theta_2 + 7)(2\theta_1 - \theta_2 - 6)(2\theta_1 - \theta_2 - 1) \\ &\qquad \times (\theta_1 + 2\theta_2)(\theta_1 + 2\theta_2 + 1)(\theta_1 + 2\theta_2 + 5)(\theta_1 + 2\theta_2 + 6). \end{aligned}$$

This system has the following five persistent Puiseux polynomial solutions (which actually turn out to be monomials): x_1x_2, $x_1^4x_2^2$, $x_1^{14/5}x_2^{13/5}$, $x_1^{13/5}x_2^{21/5}$, $x_1^{28/5}x_2^{26/5}$. The following 30 pure Puiseux polynomial solutions of $\mathrm{Horn}(A, c)$ were computed with Mathematica 9.0:

$$28 + 15/x_1,\;\; x_1^{-4/5}x_2^{-3/5}(7x_1 + 22x_2 + 44x_1x_2),$$

$$x_1^{-1/5}x_2^{-2/5}(196 + 297x_2 + 231x_1x_2),$$

$$x_1^{-3/5}x_2^{-1/5}(198 + 140x_1 + 165x_1x_2),\;\; x_1^{-7/5}x_2^{1/5}(25 + 120x_1 + 72x_1^2),$$

$$x_1^{4/5}x_2^{-17/5}(3+1254x_2+52x_1x_2),$$

$$x_1^{17/5}x_2^{14/5}(298452+129675x_2+27930x_1x_2+588x_1x_2^2+85x_1^2x_2^2),$$

$$x_1^{2/5}x_2^{-16/5}(91+15x_1+15675x_2+3135x_1x_2),\ \ x_2^{-3}(1040+819x_1+62700x_1x_2),$$

$$x_1^{3/5}x_2^{-14/5}(2340+182x_1+72675x_2),\ \ x_1^{19/5}x_2^{18/5}(8892+266x_1+105x_2+72x_1x_2),$$

$$x_1^3x_2^3(426360+34884x_1+26600x_1x_2+1200x_1^2x_2+51x_1^2x_2^2),$$

$$x_1^{18/5}x_2^{16/5}(43605+741x_1+3325x_2+1125x_1x_2),$$

$$x_1^{16/5}x_2^{17/5}(46512+6669x_1+900x_1x_2+64x_1^2x_2),$$

$$2660x_1+34884x_1^2+51x_2+4500x_1x_2+74100x_1^2x_2/x_1^7,$$

$$x_1^{-38/5}x_2^{-1/5}(8151x_1^2+9x_2+1980x_1x_2+73150x_1^2x_2+639540x_1^3x_2),$$

$$x_1^{-32/5}x_2^{1/5}(1200+33345x_1+170544x_1^2+336x_2+13300x_1x_2),$$

$$x_1^{-34/5}x_2^{2/5}(32+1596x_1+17442x_1^2+38760x_1^3+105x_1x_2),$$

$$x_1^{-36/5}x_2^{3/5}(17+1575x_1+31122x_1^2+149226x_1^3),$$

$$x_1^{1/5}x_2^{-18/5}(16x_1+48279x_2+18018x_1x_2),$$

$$x_1^{6/5}x_2^{-8/5}(33x_1+9996x_2+3672x_1x_2+22100x_1x_2^2+1326x_1^2x_2^2 \\ +4641x_1x_2^3+2652x_1^2x_2^3),$$

$$x_1^{9/5}x_2^{-7/5}(81+3024x_2+192x_1x_2+5720x_2^2+1872x_1x_2^2+624x_1x_2^3+72x_1^2x_2^3),$$

$$x_1x_2^{-1}(420+216x_1+2925x_1x_2+175x_1^2x_2+2145x_1x_2^2+819x_1^2x_2^2),$$

$$x_1^{8/5}x_2^{-4/5}(23520+1728x_1+109200x_2+34125x_1x_2+38220x_1x_2^2+2912x_1^2x_2^2),$$

$$x_1^{7/5}x_2^{-6/5}(9504+990x_1+128700x_2+41580x_1x_2+113256x_1x_2^2 \\ +7280x_1^2x_2^2+4455x_1^2x_2^3),$$

$$x_1^{-22/5}x_2^{-9/5}(1225x_1^2+3780x_1^3+1512x_1^4+75x_2+2730x_1x_2 \\ +18018x_1^2x_2+27300x_1^3x_2),$$

$$x_1^{-4}x_2^{-2}(120x_1^2+216x_1^3+45x_2+819x_1x_2+3250x_1^2x_2+2925x_1^3x_2),$$

$$x_1^{-18/5}x_2^{-11/5}(3456x_1^2+2835x_1^3+5824x_2+65520x_1x_2 \\ +163800x_1^2x_2+82320x_1^3x_2+38220x_1x_2^2),$$

$$x_1^{-19/5}x_2^{-13/5}(66x_1^3+2652x_1x_2+12852x_1^2x_2+11424x_1^3x_2 \\ +1377x_2^2+18564x_1x_2^2+48620x_1^2x_2^2),$$

$$x_1^{-16/5}x_2^{-12/5}(198x_1^2+1456x_2+10725x_1x_2+16632x_1^2x_2 \\ +3696x_1^3x_2+3432x_2^2+18876x_1x_2^2).$$

We omit the other five solutions since they are too cumbersome to display. Their initial exponents are

$$\left(-\frac{23}{5},\frac{9}{5}\right),\ \left(-\frac{21}{5},\frac{8}{5}\right),\ \left(-\frac{19}{5},\frac{7}{5}\right),\ \left(-\frac{17}{5},\frac{6}{5}\right),\ (-3,1).$$

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Timur M. Sadykov
Siberian Federal University, Krasnoyarsk
Department of Mathematics and Computer Science, Plekhanov Russian University of Economics, Moscow
E-mail: Sadykov.TM@rea.ru

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Susumu Tanabé
Department of Mathematics,
Galatasaray University, Istanbul, Turkey
E-mail: tanabe@gsu.edu.tr