AN ADDENDUM TO THE PAPER “3-FOLD LOG FLIPS”

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The present paper is written in connection with Kollár's counterexample to Proposition 8.3. After a $\mathbb{Z}_2$-covering, this counterexample corresponds to the following small blowup. Consider the three-dimensional quadric $\bar{Z} = (xy + uv = 0)$ on which the generator $\tau \in \mathbb{Z}_2$ acts according to the formula $\tau\colon (x, y, u, v) \mapsto (x, y, -u, -v)$. This quadric contains four planes, viz.

$$V_1 = (x - v = y + u), \qquad V_2 = (x + v = y - u = 0);$$
$$T_1 = (x = v = 0), \qquad T_2 = (y = u = 0).$$

Consider the small resolution $\bar{X} \to \bar{Z}$ corresponding to the fundamental subset of the pencil (V_1, V_2) or, which is the same, the monoidal transformation with center at T_1 or T_2. Denote by V_i' and T_i' the proper transforms of V_i and T_i in $\bar{X}$. The action of τ lifts to $\bar{X}$ and

$$\tau(V_i') = V_{3-i}', \qquad \tau(T_i') = T_i'.$$

Put $X = \bar{X}/\tau$, $S_1 = T_1'/\tau$, $S_2' = T_2'/\tau + (V_1' + V_2')/\tau$. Then $K_X + S_1$ is a log terminal divisor. The intersection $S_1 \cap S_2'$ consists of two curves crossing normally. Furthermore, X has an isolated singularity of index 2 on S_1 and a curve of ordinary double singularities on S_1 (and a similar curve on S_2'). Then all conditions of Proposition 8.3 are satisfied with the exception of irreducibility of S_2'. However, the surface S_2' can be replaced by a general element of the linear system $|S_2'|$ passing through $S_1 \cap S_2'$ (the double points automatically lie in the fundamental subset).

The error in the proof of Proposition 8.3 was almost at the very end and consisted in that in a suitable analytic neighborhood $(\pi \circ q)^{-1} S_2^+$ has two components intersecting in a curve (cf. [1, Russian p. 179, English p. 166]). This cover yields normalization of S_2^+. Furthermore, this is the only possibility. As could be expected, the corresponding flip $X^- \to X^+$ yields the double fold on S_2. As a result, Proposition 8.3 transforms into a statement about properties of the above flip involving the original conditions (i)–(iii) as well as the following new condition:

(iv) $K + S_1 + S_2$ is numerically trivial in a neighborhood of the contracted curve.

Since in the original paper Proposition 8.3 was used to show that many specific cases actually do not occur, the new version of this proposition led to numerous gaps, specifically in the proof of Proposition 8.8. After a minor modification of its statement and considerable changes in the proof, this last proposition has become a proposition-reduction, which means that if the assumptions of this proposition are not satisfied, then there exists a flip. In particular, this is so if $d_i > 1$ for all exceptional divisors E_i with $a_i = 0$.

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See [1].

However in general the sequence of flips is the same as before. Proposition 8.3 is now deduced from general considerations not using Proposition 8.8.

For the reader's convenience the author decided to publish a new version of the entire § 8 rather than a list of corrections.*

§ 8. Special flips of index 2

8.1. **The setup.** In this section $f\colon X \to Z$ is a *special nonexceptional contraction of index* 2. This means that there exist a boundary B and an irreducible surface S on X such that the following conditions hold:

(8.1.1) $K+S+B$ is log canonical and $2(K+S+B)$ is linearly equivalent to 0 in a neighborhood of the contracted curve;
(8.1.2) S is negative on the contracted curve;
(8.1.3) $K+S$ is purely and strictly log terminal;
(8.1.4) $K+S$ is negative on the contracted curve;
(8.1.5) the restriction $(K+S+B)|_S$ is not exceptional in a neighborhood of the contracted fiber.

According to Proposition 6.12, we can assume that

(8.1.6) The locus of log canonical singularities of $K+S+B$ is contained in S.

In particular, shrinking if necessary the neighborhood of the contracted curve, we may assume that the irreducible components of B have multiplicities $\frac{1}{2}$ or 0. By assumption (8.1.3), X is $\mathbb{Q}$-factorial. We also suppose that f is extremal, and the contracted curve is connected. In the analytic case, all of this holds in a neighborhood of the flipping curve, that is, with $W = \mathrm{pt}$ being the image of the flipping curve, and hence the flipping curve is irreducible.

8.2. **Reduction.** *It can be assumed that there is exactly one irreducible curve C not contracted by f, with multiplicity* 1 *in the boundary of the restricted log divisor $(K+S+B)|_S$. Furthermore, each connected component of* Supp$(B|_S)$ *lying outside C and intersecting the locus of log canonical singularities of $(K+S+B)|_S$ is contracted to a point by f.*

Proof. Suppose first that there is at least one irreducible curve C not contracted by f, with multiplicity 1 in the boundary of the log divisor $(K+S+B)|_S$. If B intersects S in a curve $\neq C$ having multiplicity 1 in the boundary of $(K+S+B)|_S$ and not contracted by f, then, by the connectedness of the locus of log canonical singularities of $(K+S+B)|_S$ and (8.1.3), we can change the boundary B_S outside C so that it remains $\geq 0_S$, and K_S+B_S becomes log terminal and numerically negative with respect to $f|_S$. Then by Corollary 5.11, $g^*(K+S)|_S$ will have a 1-complement for any blowing up g. Hence by the proof of Theorem 5.12, $K+S$ has a 1-complement, and the flip of f exists by Proposition 6.8. In a similar way one proves that there exist a 1-complement of $K+S$ and a flip of f in the case when Supp$(B|_S)$ has a connected component that is not contained in C, intersects the locus of log canonical singularities of $(K+S+B)|_S$, and is not exceptional with respect to f. Hence it remains to carry out the reduction in the case when *all curves having multiplicity* 1 *in the boundary of the log divisor $(K+S+B)|_S$ are contracted by f*.

In the analytic setup, the above arguments prove existence of the required complement of $K+S$ in a neighborhood of a flipping ray, since B is positive on the flipping curve, and therefore cuts out on S a connected component which is not contracted

*Editor's note. In the translation of [1], §8 was revised to take account of Kollár's counterexample.

by f and intersects the locus of log canonical singularities of $(K+S+B)|_S$. In the algebraic setup, the required complement B can be found on one of the ends of the given log canonical set, but this complement might behave badly on the other end. In this case we must reduce to other flips.

Consider a strictly log terminal blowup $g\colon Y \to X$ existing in a neighborhood of the flipping curve by Corollary 5.19. Since g is log crepant, by Corollary 3.16 there exists a prime divisor E exceptional with respect to f such that $f \circ g(E) = \mathrm{pt}$. By (8.1.6) the multiplicity d of E in g^*S is positive. Put $H = \varepsilon(g^{-1}S + dE)$ for small $\varepsilon > 0$. Since the exceptional divisors for g together with $g^{-1}S$ form a fiber of g, we can apply Corollary 4.6 to get a new blowup g with a single exceptional divisor E. The required flips are of type III, since $K_Y + g^{-1}S + B_Y$ is log terminal outside the reduced part of the boundary coinciding with $\operatorname{Supp} g^*S$, and this boundary forms a fiber of g. From this and by construction it follows that the supports of modified rays are contained in exceptional divisors other than E. Hence the modifications terminate.

Thus, for the new blowup, $(K_Y + g^{-1}S + E)|_{E^\nu}$ is numerically trivial and contains the intersection curve $C_1 = g^{-1}S \cap E$ in the locus of log canonical singularities. The latter is connected in view of Theorem 6.9 and assumption (8.1.5). Note that by construction $K_Y + g^{-1}S$ is log terminal. But $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2. We now apply the arguments of the proof of Reduction 7.2. Suppose that R_1 corresponds to the contraction g. Obviously R_1 is positive with respect to $g^{-1}B$. Next we need to consider modifications of 0-contractions for $H = \varepsilon g^*B$ corresponding to R_2. Actually it suffices to consider only the flipping rays R_2 nonpositive with respect to $g^{-1}S$.

Assume first that R_2 is negative with respect to $g^{-1}S$. If R_2 is positive with respect to E, then the flip of R_2 exists by Corollary 5.20. Moreover, this flip preserves the log terminality of $K_Y + g^{-1}S$. If E is numerically nonpositive on R_2, then $g^{-1}B$ is positive on R_2 and the flip exists by Lemma 6.10; the log terminality of $K_Y + g^{-1}S$ is again preserved. If the intersection curve C_1 is lost as a result of such flips, then the surfaces $g^{-1}S$ and E no longer intersect each other, and we get a flip by contracting E to a point, as in Reduction 7.2.

Finally, if $g^{-1}S$ is numerically trivial on R_2, then E is negative on R_2 and $g^{-1}B$ is positive on R_2. In particular, the support of R_2 is contained in E. We may assume that R_2 is a flipping ray. If one of the connected components G of $\operatorname{Supp} R_2$ intersects $g^{-1}S$, then it is contained in $g^{-1}S$, and the flip in it exists by virtue of Lemma 6.10 with $S_1 = g^{-1}S$ and $S_2 = E$. Otherwise G does not intersect $g^{-1}S$ and hence it does not intersect C_1. For the remaining connected components E can be replaced by S, and $g^{-1}B$ by B. Then assumptions (8.1.1–2) and (8.1.4) will hold. (In the analytic case, after passing to a neighborhood of the component in question the extremality may not be preserved. Then one must construct a flip corresponding to $-B$.) By Theorem 6.9, on the normalization S^ν there is a (possibly reducible) curve B' such that each connected component of the locus of log canonical singularities of $(K+S+B)|_{S^\nu}$ intersects B', and no component of B' is contracted by f. Also by construction $K+S+B$ is log terminal outside the boundary $S+B$.

Suppose that the flipping component is contracted to a point P. Then on any weakly log canonical model of f the locus of log canonical singularities of $K+S+B$ is connected, even over an analytic neighborhood of P. Of course, it always contains the modified S. Since S is $\mathbb{Q}$-Cartier, connectedness holds for any strictly log terminal model of f. Moreover, any two surfaces with log discrepancy 0 are blown up on some strictly log terminal model of f. Hence the locus of log canonical singularities of $g^*(K+S+B)$ is connected for an arbitrary resolution g over a neighborhood of P, which implies what we want.

Thus it remains to carry out a reduction in the case of a flip of the component under consideration. To this end, we perform a strictly log terminal blowing up $g\colon Y \to X$ for $K+S$. This, as well as all its modifications considered in what follows, is a weakly log canonical model of f. The flip of f can be obtained as a result of modifications of 0-contractions of $f \circ g$ with $H = g^{-1}B$—naturally, with a subsequent contraction of the curves that are numerically trivial with respect to $g^{-1}B$. This is possible, since $K+S+B$ is log terminal outside the boundary $S+B$, which forms a fiber of $f \circ g$. We claim that the flips required for this satisfy the reduction (modulo flips that already exist).

Since S and B form a fiber of $f \circ g$ and the support of a 0-flipping extremal ray R is positive with respect to $g^{-1}B$, it is negative with respect to $S' = g^{-1}S$ or with respect to a surface E exceptional for $f \circ g$. We localize to connected components C' of $\operatorname{Supp} R$ exactly as in the proof of Reductions 6.4–5. If C' intersects another such component S_Y, then the flip exists by Lemma 6.11. Otherwise $C' \subset S'$ and does not intersect any other component S_Y. Hence this is a special flip of index 2, and it exists if it is exceptional. We can thus restrict attention to the case when it is not exceptional. That means that $(K_Y + g^{-1}S + g^{-1}B)|_{S'}$ is not exceptional in a neighborhood of C'. In the case when $S' = g^{-1}S = S_Y$ and $(f \circ g)^{-1}P \subset g^{-1}S = S'$ over a neighborhood of P, $(K_Y + g^{-1}S + g^{-1}B)|_{S^\nu}$ is numerically trivial on $g^{-1}S \to S^\nu$, and therefore the required curve C either joins C' with the proper transform of $(\nu^{-1}f)B'$, or coincides with this proper transform. Otherwise, since the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B$ over P is connected, either Theorem 6.9 yields the required curve C, or the locus of log canonical singularities of $(K_Y + g^{-1}S + g^{-1}B)$ does not coincide with S' in a neighborhood of the flipping curve C'. In the last case the flip exists by Proposition 6.12. □

In the preprint of this paper, the following result was incorrectly stated as nonexistence of flips (or of the corresponding configuration). But, as Kollár pointed out to me, such flipping contractions actually exist.

8.3. **Proposition.** *Let $f\colon X \to Z$ be an extremal contraction of an irreducible curve C, and let $B = S_1 + S_2$, where S_1, S_2 are surfaces and $C \subset S_1 \cap S_2$. Assume that the following conditions are satisfied:*

(i) *S_1 and S_2 cross normally along C;*
(ii) *f is special of type* (6.6.1) *for $S = S_1$;*
(iii) *$(K+S_1+S_2)|_{S_1}\big|_C$ is not purely log terminal on C at a single point P;*
(iv) *$K+S_1+S_2$ is numerically trivial in a neighborhood of the contracted curve.*

Then the flip $X^- \to X^+$ of C exists and has the following properties: $C \subset S_1$ is contracted to a nonsingular point Q of a normal surface S_1^+. The normalization of S_2^+ is nonsingular and unibranched over Q. The flipped curve C^+ is irreducible, and the surface S_2^+ is nonnormal along it. The normalization $S_2^{+\nu} \to S_2^+$ defines a double cover $C^* \to C^+$, and Q is a branch point of this cover. The singularities of X^+ along C^+ are canonical of type A_n, and S_2^+ is a Cartier divisor. S_1^+ and S_2^+ cross normally along C'^+ in a neighborhood of Q and $(K_{X^+} + S_1^+ + S_2^+)|_{S_2^{+\nu}} = K_{S_2^{+\nu}} + C^* + C'^+$; furthermore, the curves C^* and C'^+ cross normally at Q.

Proof. The flip exists by Proposition 2.7. We first describe the properties of f. By (ii) and Corollary 3.8, S_1 is normal and irreducible. Taking an analytic neighborhood of C and replacing f by the contraction of C only, we preserve all the above assumptions except for the $\mathbb{Q}$-factoriality of X. However S_1 and S_2 remain $\mathbb{Q}$-Cartier divisors. From our assumption it follows that S_2 is positive on C. Hence by (iii), possibly after shrinking the neighborhood of C, the intersection $S_1 \cap S_2$ consists of two nonsingular irreducible curves C and C' intersecting at P. Therefore by (iii), Corollary 3.10, Lemma 4.2, and our assumption, $C = \mathbb{P}^1$ and

$$(K + S_1 + S_2)|_{S_1}|_C = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P \tag{8.3.1}$$

whereas

$$S_2|_{S_1} = C + cC',$$

where $0 < c \le 1$ in view of (i) and (3.2.2). Note that, like S_2, the divisor $C + cC'$ is positive on C. Let $g\colon T \to S_1$ be a minimal resolution of singularities in a neighborhood of C. Suppose first that the points P_1 and P_2 are singular on S_1. By Corollary 3.10 they are nodes and

$$g^*(C + cC') = g^{-1}C + cg^{-1}C' + \tfrac{1}{2}E^1 + \tfrac{1}{2}E^2 + \sum e_iE_i,$$

where E^1 and E^2 are exceptional curves over P_1 and P_2 respectively, $E_1, \dots, E_n$ is a chain of exceptional curves over P, and E_1 intersects $g^{-1}C$. By (3.18.6), $0 < e_1 \le 1$, from which it follows that

$$\begin{aligned} 0 < (C + cC' \cdot C) &= (g^*(C + cC') \cdot g^{-1}C) \\ &= (g^{-1}C)^2 + \tfrac{1}{2} + \tfrac{1}{2} + \begin{cases} e_1 & \text{for } n \ge 1, \\ c & \text{for } n = 0 \end{cases} \le (g^{-1}C)^2 + 2, \end{aligned}$$

and $(g^{-1}C)^2 \ge -1$. Since $g^{-1}C$ is contractible, it is an exceptional curve of the first kind, i.e. $(g^{-1}C)^2 = -1$. But then the curve $E^1 \cup E^2 \cup g^{-1}C$ is not contractible. Therefore at least one of the points P_i, say P_1, is nonsingular. Then there is a unique irreducible curve C'' with multiplicity $1/2$ in the boundary $(S_2)_{S_1}$ that crosses normally through P_1. In a similar way one checks that if P_2 is nonsingular, then $(g^{-1}C)^2 \ge 0$, and this again contradicts the contractibility of C. Thus P_2 is singular. Arguing as above, we see that

$$g^*(C + cC') = g^{-1}C + cg^{-1}C' + \tfrac{1}{2}E^2 + \sum e_iE_i,$$

where E^2 is an exceptional curve corresponding to P_2, $E_1, \dots, E_n$ is a chain of exceptional curves corresponding to P, and E_1 intersects $g^{-1}C$, from which it follows that

$$\begin{aligned} 0 < (C + cC' \cdot C) &= (g^*(C + cC') \cdot g^{-1}C) \\ &= (g^{-1}C)^2 + \tfrac{1}{2} + \begin{cases} e_1 & \text{for } n \ge 1, \\ c & \text{for } n = 0 \end{cases} \\ &\le (g^{-1}C)^2 + \tfrac{3}{2}, \end{aligned}$$

and $(g^{-1}C)^2 > -3/2$. Hence by contractibility $g^{-1}C$ is an exceptional curve of the first kind and

$$\tfrac{1}{2} < \begin{cases} e_1 & \text{for } n \ge 1, \\ c & \text{for } n = 0. \end{cases}$$

From this it follows that if $n = 0$, then P is nonsingular on S_1. Furthermore, by Corollary 3.10 we have $c = 1$, and the curves C and C' cross normally at P. If $n \ge 1$, then by construction $(E_i)^2 = -m_i$ with $m_i \ge 2$. Since $E^2 \cup E_1 \cup g^{-1}C$ is not contractible, we have $m_1 \ge 3$. By (3.18.7) and the inequality $e_1 > 1/2$ we get $m_1 = 3$, $m_2 = \cdots = m_n = 2$ and $c > 1/2$, hence by Corollary 3.10 we again have

$c = 1$, that is, in either case S_1 and S_2 cross normally along C'. Furthermore, $f|_{S_1}: S_1 \to S$ contracts C to a nonsingular point $Q \in S$ and $K_S + \frac{1}{2}f|_{S_1}(C'')$ is canonical at Q. This means that all the log discrepancies of $K_S + \frac{1}{2}f|_{S_1}(C'')$ over Q are ≥ 1 (so that their discrepancies are ≥ 0). The corresponding terminal blowup is obtained from T after contracting the curves E^2 and $g^{-1}C$, with C'' mapping to a nonsingular curve. Furthermore, for $n \geq 1$ the curve E_1 maps onto an exceptional curve of the first kind having simple tangency with C''; for $n = 0$ the image of C' has simple tangency with the image of C''.

Now we describe the properties of the flip. I claim first that it defines a contraction of C on S_1. By (8.3.1) there is a curve C'' with multiplicity $1/2$ in the boundary $(S_2)_{S_1}$ crossing normally through P_1. Hence, since S_1 and S_2 cross normally along C and C' in a neighborhood of C,

$$(K + S_1)|_{S_1} = K_{S_1} + \tfrac{1}{2}C''$$

is log terminal and negative on C. Hence, in a neighborhood of the transformed curves $C_1, \dots, C_m$ that land on S_1^+,

$$(K_{X^+} + S_1^+)|_{S_1^+} = K_{S_1^+} + \tfrac{1}{2}C''^+ + \sum c_i C_i$$

and is positive, where by the effectivity (3.2.2) all the $c_i \geq 0$. But by the above the curves C_i are contracted to a nonsingular point Q on S, at which $K_S + (1/2)f|_{S_1}(C'')$ is canonical. This is only possible for $m = 0$. Hence $S_1^+ = S$. From this it follows that there are no finite covers $\pi: V \to U$ of degree $l \geq 2$, where U is an irreducible neighborhood of Q, V is irreducible, and π is ramified only along curves not lying on S and passing through Q. We also assume that all these properties are preserved if we restrict π to irreducible analytic neighborhoods of Q. In fact, by Corollary 2.2, $\pi^*(K_U+S) = K_V + \pi^{-1}S$ is purely log terminal, and hence by Lemma 3.6 $\pi^{-1}S$ is normal and the induced finite cover $\pi|_{\pi^{-1}S}: \bigcup D_i \to S$ is unramified outside Q. Hence, since Q is nonsingular on S, π is unramified over Q, which contradicts the possibility of analytically restricting π while preserving the irreducibility of V (cf. Corollary 3.7).

Now note that S_2^+ is an irreducible surface, and the normalization $\nu: S_2^{+\nu} \to S_2^+$ is one-to-one over Q. In fact, otherwise there would exist an analytic neighborhood of Q in which S_2^+ has components passing through Q. But this is impossible, since a $\mathbb{Q}$-Cartier divisor S intersects each of these components along a curve passing through Q, and these curves are distinct because $K + S + S_2^+$ is log canonical. However $S \cap S_2^+ = C'^+$ is an irreducible curve in a neighborhood of Q. Thus the point Q can be identified with $\nu^{-1}Q$, and C'^+ with $\nu^{-1}C'^+$. The restriction $(K_{X^+} + S + S_2^+)|_{S_2^{+\nu}}$ has at most two irreducible curves passing through Q, with multiplicity 1 in the boundary $S_{S_2^{+\nu}}$. Suppose first that there are exactly two such curves, viz. C'^+ and C^*. Then in a neighborhood of Q

$$(K_{X^+} + S_2^+)|_{S_2^{+\nu}} = K_{S_2^{+\nu}} + C^*$$

and is log terminal. On the other hand, $K_{X^+} + S_2^+$ has index 1 at all points of a punctured neighborhood of Q in S, so that by the above it has index 1 at Q itself. Therefore $K_{S_2^{+\nu}} + C^*$ has index 1 at Q, and by (3.9.2) the surface $S_2^{+\nu}$ is nonsingular at Q. Since S has index 2 along C''^+, in a neighborhood of Q it defines a double cover $\pi: V \to U$ ramified only along C''^+. Hence, after shrinking the analytic neighborhood of Q, $\pi^{-1}S_2^+$ consists of two irreducible components each of which has nonsingular normalization. Since $K_V + \pi^{-1}S$ is purely log terminal, there exists a small strictly log terminal blowing up $q: W \to V$ with connected fiber M over Q;

otherwise arguing as above we get a contradiction with π being ramified over Q. But this is impossible if the two components of $\pi^* S_2^+$ intersect in at most points. Indeed, then $(\pi \circ q)^* S_2^+ = (\pi \circ q)^{-1} S_2^+$ is numerically trivial on M, and, after shrinking the analytic neighborhood of Q, it consists of two irreducible components which do not intersect even along M, because $K_V + \pi^{-1} S + \pi^{-1} S_2^+$ is log canonical. Hence S_2^+ is nonnormal along $\nu(C^*)$, and by the above X^+ has a singularity of the required type along $\nu(C^*)$. The irreducibility of the flipped curve C^+ and the fact that it coincides with $\nu(C^*)$ is easily deduced from the fact that all its components pass through Q and are contained in S_2^+. Indeed, if C' is such a curve and $C' \not\supset \nu(C^*)$, then on the normalization $S_2^{+\nu}$ it is an exceptional curve passing through $Q = C^* \cap C'^+$, and numerically trivial with respect to $(K_{X^+} + S + S_2^+)|_{S_2^{+\nu}} = K_{S_2^{+\nu}} + C^* + C'^+$.

We now suppose that there is no C^*, and derive a contradiction. In fact, in that case in a neighborhood of Q

$$(K_{X^+} + S_2^+)|_{S_2^{+\nu}} = K_{S_2^{+\nu}}$$

is log terminal and has index 1. Therefore $S_2^{+\nu}$ is a normal surface and Q is a canonical singularity of $S_2^{+\nu}$. On the other hand, $(K_{X^+} + S + S_2^+)|_{S_2^{+\nu}}$ is log canonical and equal to $K_{S_2^{+\nu}} + C'^+$ in a neighborhood of Q, but not log terminal at Q. Then by classification Q is a Du Val singularity of type D_n. By (ii) and the original assumption, each flipped curve C^+ is negative with respect to S_2^+; hence it is contained in S_2^+. By (ii) again, C^+ passes through Q, and by the original assumption C^+ is numerically trivial with respect to $(K_{X^+} + S + S_2^+)|_{S_2^{+\nu}}$. Hence C^+ is an exceptional curve of the first kind on the minimal resolution of S_2^+. But then a multiple of C^+ is movable on S_2^+, which contradicts to $f^+ \colon X^+ \to Z$ being small. □

The following standard result is useful in simplifying somewhat the induction in the sequel.

8.4. **Reduction.** *Passing to the analytic case, one can assume that the flipping curve is irreducible.*

Conversely, if there exists an algebraic approximation of the contraction and its polarization, then one can return to the algebraic case, resolving singularities that are not log canonical and not $\mathbb{Q}$-factorial outside the flipped curve; however, it seems that an algebraic approximation need not exist.

Proof. Combine the arguments from the end of proof of Reductions 6.4–5 and Reduction 8.2. By Definition 6.1, one can restrict to an algebraic situation and shrink to an analytic neighborhood. □

8.5. **Classification.** We classify rays according to two tests:

Is $K + S + B$ log terminal along the curve contracted by f? Is the contracted curve contained in B (more precisely, in $\operatorname{Supp} B$)?

By (8.1.3) and (8.1.6), negative answers to both tests are impossible. Hence there are the following possibilities:

(8.5.1) $K + S + B$ is purely log terminal along the curve contracted by f, and B does not contain it;

(8.5.2) $K + S + B$ is not log terminal along the curve contracted by f, and B contains it;

(8.5.3) $K + S + B$ is purely log terminal along the curve contracted by f, and B contains it.

The reason for the chosen order will be clear from the reductions in the sequel. Of course, by Reduction 8.2, in each of the above cases it is assumed that there exists exactly one irreducible curve C not contracted by f, having multiplicity 1 in the boundary of the log divisor $(K+S+B)|_S$, and intersecting the contracted curve; furthermore, each connected component of $\operatorname{Supp}(B|_S)$ outside C meeting the locus of log canonical singularities of $(K+S+B)|_S$ is exceptional. In the cases (8.5.1–2), we also assume that the curve contracted by f is irreducible. In what follows we successively reduce (8.5.1) to (8.5.2–3) and exceptional index 2 flips, (8.5.2) to (8.5.3) and exceptional index 2 flips, and (8.5.3) to exceptional index 2 flips. However, the contracted curve in (8.5.3) is possibly reducible, and log terminality of $K+S+B$ along it means log terminality at a general point of each irreducible component. (It is not hard to verify that the contracted curve in (8.5.3) has at most two irreducible components.) Our general strategy is as follows. First we choose a *good* blowing up $g: Y \to X$ in the sense of the following definition. A *good* blowing up g is an *extremal blowing up* g *having a prime exceptional divisor* E and satisfying the following conditions:

(i) $g^*(K+S+B) = K_Y + g^{-1}S + g^{-1}B + E$, *i.e.* g *is log crepant*;
(ii) $K_Y + g^{-1}S + E$ *is log terminal*;
(iii) $B_1 = g^{-1}S \cap E = \mathbb{P}^1$ *is an irreducible curve, and* $g^{-1}S$ *and* E *cross normally along* B_1;
(iv) $(K_Y + g^{-1}S + g^{-1}B + E)|_{g^{-1}S}|_{B_1} = (K_Y + g^{-1}S + g^{-1}B + E)|_E|_{B_1} = K_{\mathbb{P}^1} + \frac{1}{2}P_1 + \frac{1}{2}P_2 + P$, *where* P *is the unique point on* B_1 *at which* $K_Y + g^{-1}S + g^{-1}B + E$ *is not log terminal.*

Note that by Corollary 3.8, (iii) follows from (ii) and (8.1.3), although they are often proved in the opposite order. As in the second half of the proof of Reduction 7.2, we apply Corollary 4.6 to $f \circ g$ with $H = \varepsilon g^*B$. The induction or reduction step is realized for an extremal ray R_2 that is numerically trivial with respect to the modified $g^{-1}S$. Here the base of the contraction is replaced by a divisorial blowup, a modification of the current good blowup. In the analytic case, which is now the main case of interest for us according to Reduction 8.4, the current W is replaced by its preimage; here the original W is a point, the image of the flipping curve. It is assumed that, for such W, f is extremal and X is strictly $\mathbb{Q}$-factorial. (In the analytic case, extremality of f and $\mathbb{Q}$-factoriality of X are not preserved in general under shrinking of a neighborhood of the contracted fiber.) Otherwise, the speciality assumptions will hold.

Note also that W will always be projective and will be contained in the reduced part of the boundary, since this holds for good blowups and is preserved under subsequent modifications, because of the positivity of the flipping ray with respect to E. Hence in a neighborhood of W there exists a strictly log terminal blowing up of $K+S+B$ as in Corollary 5.19. However, in the boundary of the log divisor $K+S+B$ and its restrictions, we usually write down only the components in a neighborhood of the new flipping curve. In doing so, Lemmas 8.9–10 will allow us to stay within the framework of the cases (8.5.1–3). But we cannot avoid allowing the contracted curve to be reducible in (8.5.3). Overall, this reduction of index 2 special flips to exceptional flips is carried out at the end of §8 and completes the proof of Theorems 1.9–10 and Corollary 1.11.

Since we are not in the exceptional case, in the case (8.5.1) the curve contracted by f meets C in a point Q_1 that is not log terminal for $K+S+B$ or for $(K+S+B)|_S$ in a neighborhood of the contracted curve. By Theorem 6.9, in a neighborhood of the contracted curve Q_1 is the only point at which $(K+S+B)|_S$ fails to be log terminal.

On the other hand, by (iii) and (iv), g cannot be a blowing up of a curve C. Thus g is a blowing up of the point Q_1. In this case, we say that a good blowing up g is an *end blowing up* if P is the only possible point on $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ that is not log terminal on E. In the opposite case, by Theorem 6.9 and the assumption that f is not exceptional, the reduced part of the boundary of $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ has the form B_1+B_2, where B_2 is a curve meeting B_1 only in P and containing a point $Q_2 \neq P$ that is not log terminal. By the extremality of g, the divisors $g^{-1}S$ and B_1 are ample on E. Hence B_2 is irreducible and Q_2 is the only point of E at which $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ is not log terminal, except possibly for P. We say that a good blowing up of Q_2 is a *middle blowing up*; a finite chain of successive blowings up ending in an end blowing up is *stopped*. It is convenient to subdivide case (8.5.2) into two subcases, viz.

(8.5.2) *In the conditions of* (8.5.2) *above, on the curve contracted by* f *there is a point* Q_1 *outside* C *at which* $(K+S+B)|_S$ *is not log terminal.*

(8.5.2)* The opposite case.

It is not hard to verify that in the case (8.5.2) (without star) a good blowing up g blows up Q_1. As above, it is called an *end* blowing up if P is the only possible point at which $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ is not log terminal on E. A *middle* blowing up and a *stopped* chain of blowings up are defined in a similar way.

8.6. **Proposition.** *In the cases* (8.5.1–2) *there exists a stopped chain of good blowings up.*

Proof. By Corollary 5.19, over a neighborhood of W there exists a strictly log terminal blowing up (see the remark after Proposition 8.8). We now apply Corollary 4.6 to g with $H=g^{-1}B$. By (8.1.3), performing modifications of 0-contractions we get the original model X. We claim that the final modification g that yields a neighborhood of the point Q_1 is a good blowing up of Q_1.

Suppose first that this modification was a flip. By construction, $g^{-1}B$ is negative on the flipped curve C' and g^*B is numerically trivial; hence there is a divisor E exceptional with respect to g and positive on C'. Moreover, C' is not contained in any divisor that is exceptional with respect to g, and hence is nonnegative with respect to all these divisors. On the other hand, C' is numerically trivial with respect to g^*S. Hence C' is negative with respect to the divisor $g^{-1}S$ and lies on it. By construction, $K_Y+g^{-1}S$ is log terminal, and hence the surface $g^{-1}S$ is normal. Since C' is positive with respect to E, the final divisor cuts out a curve of log canonical singularities of $(K_Y+g^{-1}S+B_Y)|_{g^{-1}S}$ and Q_1 is also contained in the locus of log canonical singularities of this divisor. Thus by Theorem 6.9 C' is a curve of log canonical singularities of $(K_Y+g^{-1}S+B_Y)|_{g^{-1}S}$. But then, since $K_Y+g^{-1}S+E$ was log terminal before the flip, it follows that $g^{-1}B$, hence also B, cuts out in a neighborhood of Q_1 more than the locus of log canonical singularities of $(K+S+B)|_S$, which is not possible by definition of the cases (8.5.1–2).

Thus the final modification g yielding a neighborhood of Q_1, is the contraction of the divisor E. Conditions (ii) and (iii) are satisfied by construction, and (i) holds locally near Q_1. Hence by (8.1.5) and (3.2.3) we get that E is contracted to the point Q_1. Now it is not difficult to check (iv). By the above, if this is not an end blowing up, the boundary of $g^*(K+S+B)|_E$ has two intersecting irreducible components, viz. $B_1=g^{-1}S\cap E$ and B_2. Now on B_2 there exists a unique point Q_2 that is not log terminal for $g^*(K+S+B)|_E$ outside B_1. Since in a neighborhood of Q_1

the support of B intersects S only along the curve of log canonical singularities of $(K+S+B)|_S$, it follows that $g^{-1}B$ intersects B_1 only in P. Hence in a neighborhood of Q_2 the support of $g^{-1}B$ intersects E only in B_2. Therefore the final modification g yielding a neighborhood of Q_2 is again a good blowing up of Q_2. This process terminates because the number of our modifications is finite. □

In the cases (8.5.2)* and (8.5.3) it is convenient to define a natural invariant δ and then establish the existence of a good blowing up decreasing this invariant. We start with a slightly more general setup. Let Q be a point in S, and suppose that the locus of log canonical singularities of $K+S+B$ is contained in S, and $K+S$ is log terminal in a neighborhood of Q. For each exceptional divisor E_i we define the *multiplicity* d_i of E_i in S by the relation

$$g^*(S) = g^{-1}S + \sum d_i E_i,$$

where E_i is exceptional for the contraction $g\colon Y \to X$. Obviously d_i does not depend on the choice of g.

8.7. **Lemma.** *In a neighborhood of Q the set of exceptional divisors E_i with log discrepancies $a_i = 0$ and multiplicities $d_i \leq 1$ in S is finite.*

Here "in a neighborhood of Q" means that the images of E_i contain Q.

Proof. It follows at once from the definition of log discrepancy that the distinguished exceptional divisors E_i have log discrepancy ≤ 1 for $K+B$. Thus it suffices to prove that the set of exceptional divisors with log discrepancy ≤ 1 (that is, discrepancy ≤ 0) is finite. But by our assumptions $K+B$ is purely log terminal, from which it follows that all log discrepancies are $\geq \varepsilon$ for some positive ε. From then on one can argue as in [25, (1.1)].(1) □

Next we define δ by putting

$$\delta = \#\{E_i \mid a_i = 0 \text{ and } d_i \leq 1\},$$

where we only consider exceptional divisors in a neighborhood of Q, that is over Q or over an irreducible curve of log canonical singularities of $K+S+B$ passing through Q. Returning to our setup, in the case (8.5.2)* we take Q to be a general point of the contracted curve, and in the case (8.5.3) we take Q to be the unique point on the contracted curve that is not log terminal for $K+S+B$.

8.8. **Proposition-Reduction.** *In the cases* (8.5.2)* *and* (8.5.3), *either there exists a good blowing up g for which the image of the exceptional divisor E contains Q, $a=0$, and $d \leq 1$, or there exists a good blowing up g for which the restriction*

$$(K_Y + g^{-1}S + g^{-1}B + E)|_E$$

is purely log terminal outside B_1 over Q (in the case (8.5.2)* *the point Q is generic). More precisely, in the case* (8.5.2)* *the map g blows up the curve contracted by f, and in the case* (8.5.3) *the map g blows up the point Q.*

The heading "Proposition-Reduction" means that in the case when its statement is not true the flip f does exist.

As we already observed, in the analytic case we assume that f is extremal and X is $\mathbb{Q}$-factorial with respect to a projective analytic subspace $W \subset S + \lfloor B \rfloor$; then (8.1.6)

(1)Here and in what follows all references are to the bibliography in [1].

holds in a neighborhood of the flipping curve. Hence by Corollary 5.19 there exists a blowing up of a neighborhood of W which is strictly log terminal for $K+S+B$.

Proof. We start with the case (8.5.2)*. Here we claim first that there exists an exceptional divisor E over a curve contracted by f with $a=0$ and $d \leq 1$. Taking a general hyperplane section, we reduce the problem to the 2-dimensional situation. Let Q be a surface singularity, which is log canonical, but not log terminal for $K+S+B$, where S is a curve and the support of B passes through Q. Then over Q there is an exceptional curve E with $a_i = 0$ and $d_i \leq 1$. Using Lemma 3.6, it is not hard to verify that S is irreducible and nonsingular in a neighborhood of Q.

Consider the log terminal blowing up $g \colon Y \to X$ of a neighborhood of Q for $K+S+B$. The exceptional curves E_i over Q are numerically trivial with respect to $g^*(K+S+B) = K_Y + g^{-1}S + g^{-1}B + \sum E_i$. Moreover, from the fact that $\bigcup E_i$ is connected, it follows that the curves $E_i = \mathbb{P}^1$ are nonsingular, rational, and together with $g^{-1}S$ form a chain $E_1, \dots, E_n$, $g^{-1}S$. If one of the curves E_i for $i \geq 2$ is an exceptional curve of the first kind, then we can contract it to get a new log terminal blowing up of Q. Hence we can suppose that g is minimal, in the sense that $E_i^2 \leq -2$ for $i \geq 2$. Then $a_i = 0$ and $d_i \leq 1$ for each exceptional curve E_i with $i \geq 2$ (cf. Lemma 3.18). Furthermore, the required surface E with $a = 0$ and $d \leq 1$ always exists, except in the case when the surfaces S and $\operatorname{Supp} B$ are nonsingular and have simple tangency at Q. But in this case $n = 1$ and we take $E = E_1$. Then the restriction

$$(K_Y + g^{-1}S + g^{-1}B + E)|_E$$

is purely log terminal outside B_1 over the general point Q. Under the assumption of extremality, it will be shown that this restriction satisfies the conditions in the definition of good blowing up. In the case when there exists a E with $a = 0$ and $d \leq 1$, we can apply Corollary 4.6 with $H = E$ and use the fact that g^*S forms a fiber to transform g into an extremal contraction of E. It remains to verify that it is good. By construction, $K_Y + g^{-1}S$ is log terminal. Since on the curve contracted by f (in its intersection with C), $(K+S+B)|_S$ has a unique singularity that is not log terminal, and E is contracted onto this curve, its proper transform yields a curve $B_1 \subset g^{-1}S \cap E$, $B_1 = \mathbb{P}^1$ such that

$$(K_Y + g^{-1}S + g^{-1}B + E)|_{g^{-1}S}|_{B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P.$$

This last condition is just property (iv) of good blowings up (cf. (8.5)). It is clear that condition (i) holds, and (ii) will hold if we take $E = E_n$. By extremality, $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays. As usual, we denote by R_1 the extremal ray corresponding to the contraction g; it is positive with respect to $g^{-1}B$. If $B_1 \neq g^{-1}S \cap E$, then (8.5.2)* shows that $g^{-1}B$ does not intersect this curve. Hence $g^{-1}B$ is numerically nonpositive with respect to R_2, and therefore E is positive and $g^{-1}S$ is negative with respect to R_2. But then $g^{-1}B$ is numerically trivial on R_2, and so R_2 is a flipping ray whose support contains B_1.

Thus $g^{-1}S \cap E = \bigcup_{i=1}^n B_i$, where $B_1, \dots, B_n$, $g^{-1}C$ is a chain of curves on $g^{-1}S$. Moreover, the curves B_i with $i \geq 2$ are contracted by g to a point. Hence their intersection with $g^{-1}B$ is positive, and therefore $n = 2$. Thus $g^{-1}S \cap E = B_1 \cup B_2$, where B_1 and B_2 are irreducible. Note that $P = B_1 \cap B_2$ is not log terminal for $K_Y + g^{-1}S + E$. Next we check that there is no other such point in a neighborhood of E. Note that the semiampleness of $g^{-1}B$ on E is essential for this: $g^{-1}B$ is numerically trivial with respect to B_1 and positive on all the other curves of E. In fact, by Theorem 6.9 and the fact that

$$(K_Y + g^{-1}S + g^{-1}B + E)|_{E^\nu}$$

is not purely log terminal at $Q = B_2 \cap g^{-1}C$, the locus of log canonical singularities is connected. Furthermore, this locus is made up of a chain of curves $\nu^{-1}B_1, C_1, \ldots, C_n, \nu^{-1}B_2, \ldots, B_m$, with $g^{-1}B|_{E^\nu} = D+\sum_{i\geq 3} b_iB_i$, where $\operatorname{Supp} D$ does not intersect the locus of log canonical singularities and $b_i > 0$. The last assertion follows from the connectedness of $g^{-1}B|_{E^\nu}$ in view of the semiampleness and the fact that D does not intersect C_i and $\nu^{-1}B_2, \ldots, B_{m-1}$ for $m \geq 3$ (here $B_2 = \nu^{-1}B_2$ for $m = 3$), since the restriction in question is log canonical. But $g^{-1}B$ is numerically trivial only on $\nu^{-1}B_1$. Hence $n = 0$ and there are no curves C_i. Thus by Proposition 5.13 the points at which $K_Y + g^{-1}S + g^{-1}B + E$ is not purely log terminal on E are contained in the support of $g^{-1}S + g^{-1}B$, which yields the required assertion. It also follows from this that E is normal. The support of R_2 equals B_1. The flip in B_1 exists and is described in Proposition 8.3. After the flip, $K_Y^+ + g^{-1}S^+ + E^+$ fails to be log terminal only along the flipped curve $B_1^+ = \nu(C^*)$. Now the intersection $g^{-1}S^+ \cap E^+ = B_2^+$ is irreducible, and we can argue as in Reduction 8.2. Then the nontrivial case is that of the flip in a ray that is numerically trivial with respect to $g^{-1}S^+$, negative with respect to E^+ and positive with respect to $g^{-1}B^+$. Thus B_2^+ has positive selfintersection on $E^{+\nu}$. Note that by Proposition 8.3, the normalization map $E^{+\nu} \to E^+$ is one-to-one over B_2^+, and so we identify B_2^+ with its preimage in the normalization. Again by Proposition 8.3, B_2^+ can be singular only at $Q^+ = B_2^+ \cap g^{-1}C^+$. Hence B_2^+ is a curve with selfintersection ≥ 0 on the minimal resolution of singularities of $E^{+\nu}$, and selfintersection is ≥ 1 in the case when B_2^+ is nonsingular. But

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$$

has $C^* \cup B_2^+$ as its locus of log canonical singularities in a neighborhood of C^*. Hence E is obtained from $E^{+\nu}$ by the following procedure. First we perform a minimal strictly log terminal blowing up of

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$$

on $E^{+\nu}$. As a result of this, we get a chain $(B_1 =)C_1, \ldots, C_m, C^*, B_2^+$ in a neighborhood of C^*. Here minimality means that C_i with $i \geq 2$ are not exceptional curves of the first kind. After that we contract the curves C_i with $i \geq 2$ and C^*. Hence B_2 as well as B_2^+ is a curve with selfintersection ≥ 0 on the minimal resolution of E, and selfintersection ≥ 1 if E is nonsingular at Q in B_2. But such a curve cannot lie in a fiber of the ruling determined on E by g, which yields a contradiction.

We now turn to the case (8.5.3). We first assume that there exists an exceptional divisor E_i over Q or over C with $a_i = 0$ and $d_i \leq 1$. Flipping log terminal blowing up for $K+S+B$ and using Corollary 4.6 with $H = \varepsilon(\sum d_iE_i)$, where the sum is taken over $d_i \leq 1$, and the fact that g^*S forms a fiber, we get a simultaneous blowing up $g\colon Y \to X$ of all the E_i with $a_i = 0$ and $d_i \leq 1$ and only such exceptional divisors. All, since by Corollary 3.8 all exceptional divisors with log discrepancy 0 over a log terminal blowup of Y lie over the normal crossings of components of the reduced part of the boundary of $g^{-1}S + \sum E_i$, and using arguments from the proof of Proposition 6.7 it is not hard to perform an additional subblowing up for which the E_i with $a_i = 0$ and $d_i \leq 1$ are not exceptional. By construction

$$g^*(K + S + B) = K_Y + g^{-1}S + g^{-1}B + \sum E_i,$$

$K_Y + g^{-1}S$ is log terminal, and the surface $g^{-1}S$ is normal. By Theorem 6.9, from this it follows that the intersection $g^{-1}S \cap \bigcup E_i$ is a chain of irreducible curves

$B_1, \ldots, B_n$, where B_n is the proper transform of C. For $H = g^{-1}S$, we get the original model without involving any curves outside $\bigcup E_i$. Hence S is obtained from $g^{-1}S$ by contracting $B_i = \mathbb{P}^1$ with $i \leq n-1$. We claim that

$$(8.8.1) \qquad \left(K_Y + g^{-1}S + g^{-1}B + \sum E_i\right)|_{g^{-1}S}|_{B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P_0,$$

where $P_0 = B_1 \cap B_2$. Assume the contrary. Then by the definition of case (8.5.3), B_1 has a point P that lies on $g^{-1}B$ and is not log terminal. Suppose that B_1 (and possibly something else) is cut out by $E = E_1$. Then none of the other E_i pass through P. By construction the following holds:

(8.8.2) *For each exceptional divisor E_i over P or over a curve through P with log discrepancy $a_i = 0$ the multiplicity d_i of E_i in $g^{-1}S + E$ is greater than* 1.

Using Corollary 4.6 we can assume that g is an extremal blowing up of Q preserving (8.8.2). As in the proof of Reduction 8.2, we take $H = \varepsilon(g^{-1}S + dE)$. From (8.8.2) and Lemma 3.18 it follows that in a neighborhood of P the point P is the only point that is not log terminal for $g^*(K + S + B) = K_Y + g^{-1}S + g^{-1}B + E$, or equivalently, $K_Y + g^{-1}S + E$, $K_Y + g^{-1}S + g^{-1}B$, $K_Y + g^{-1}B + E$ are all log terminal in a neighborhood of P. Moreover, P is $\mathbb{Q}$-factorial, since otherwise the log terminal blowing up of $K_Y + g^{-1}S + E$ is automatically small, and the proper transforms of the divisors $g^{-1}S$, E, and $g^{-1}B$ pass through the fiber curves. But by Corollary 3.16, this contradicts the log canonical property of $K_Y + g^{-1}S + g^{-1}B + E$.

We note that from (8.1.3) and the positivity of multiplicity of E in B it follows that $K_Y + g^{-1}S$ is purely log terminal. Hence $g^{-1}S$ is normal. To prove the required log terminal properties one can use Proposition 5.13. In addition to (8.8.2) we prove the following claim.

(8.8.3) *For each exceptional divisor E_i over P with log discrepancy $a_i' \leq 1$ for $K_Y + g^{-1}S + E$ the multiplicity d_i of E_i in $g^{-1}S + E$ is greater than 1, except in the case when there is an exceptional surface E_j with $a_j = 0$ over P such that E_i is obtained by blowing up a curve of ordinary double points on the extremal blowup E_j, $a_i' > a_i = 1/2$ and $d_i = (1/2)d_j > 1/2$.*

By monotonicity $a_i < a_i' \leq 1$. Hence $a_i = 0$ or $1/2$. When $a_i = 0$, the claim immediately follows from (8.8.2). To verify it in the case $a_i = 1/2$ we use a strictly log terminal blowing up $h\colon W \to Y$ for $K_Y + g^{-1}S + g^{-1}B + E$, the exceptional divisors E_j of which lie over P. Such a blowing up exists by Corollary 5.19, since in a neighborhood of P the point P is the only point at which $g^*(K + S + B) = K_Y + g^{-1}S + g^{-1}B + E$ is not log terminal, and the intersection $g^{-1}S \cap E$ is normal along B_1.

This fails only if $\operatorname{Supp} g^{-1}B$ is tangent to E in a neighborhood of P. But then, perturbing $g^{-1}B$ while keeping $P \in g^{-1}B$, we do not change $a_i = 1/2$. (If P becomes log terminal for $K_Y + g^{-1}S + g^{-1}B + E$, then P is a nonsingular point and all $d_i \geq 2$.) As before, the log terminal divisor

$$h^*(K_Y + g^{-1}S + g^{-1}B + E) = K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + \sum E_j$$

has index 2. Since P is $\mathbb{Q}$-factorial, we conclude that $h^{-1}P = \bigcup E_j$ and that E_i lies over one of the exceptional divisors E_j. Suppose first that some E_i with log discrepancy $a_i = 1/2$ is contracted to a point P'. Then in a neighborhood of P' the index of $K_W + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + \sum E_j$ is not equal to 1. By Corollary 3.8, the point P' lies in the intersection of at most two reduced components of the boundary $h^{-1}g^{-1}S + h^{-1}E + \sum E_j$; moreover, if it belongs to the intersection of two components, then P' is $\mathbb{Q}$-factorial and, according to Corollary 3.7, has index 1 if $h^{-1}g^{-1}B$ passes through P' and index 2 in the opposite case. By (8.8.2),

$$h^*(g^{-1}S+E)=h^{-1}g^{-1}S+h^{-1}E+\sum d_jE_j,$$

where $d_j>1$. Hence $d_i>1/2+1/2=1$, since P' lies on at least one of the exceptional components E_j.

Suppose now that P' lies on only one of the reduced boundary components E_j. Then we can modify h into an extremal blowing up of $E'=E_j$ preserving the neighborhood of P' and, in particular, preserving the log terminality of $h^*(K_Y+g^{-1}S+g^{-1}B+E)=K_W+h^{-1}g^{-1}S+h^{-1}g^{-1}B+h^{-1}E+E'$ in a neighborhood of P'. Since h contracts E' to a point, the reduced part of the boundary of $(K_W+h^{-1}g^{-1}S+h^{-1}g^{-1}B+h^{-1}E+E')|_{E'}$ consists of two irreducible curves $C_1=h^{-1}g^{-1}S\cap E'$ and $C_2=h^{-1}E\cap E'$. On the other hand, $K_Y+g^{-1}S+E$ has a 1-complement $\bar{0}$ in a neighborhood of P such that the log discrepancies of E' and E_i for $K_Y+g^{-1}S+E+\bar{0}$ are all equal to 0. To show this, one needs to use the proof of Theorem 5.12 with $S=g^{-1}S$ and $B=(1-\varepsilon)E$ for sufficiently small $\varepsilon>0$. Hence the log canonical divisor

$$h^*(K_Y+g^{-1}S+E+\bar{0})=K_W+h^{-1}g^{-1}S+h^{-1}E+E'+\bar{0}$$

has index 1, and

$$(K_W+h^{-1}g^{-1}S+h^{-1}E+E'+\bar{0})|_{E'}=K_E+C_1+C_2+C_3,$$

where the curve $C_3=h^{-1}\bar{0}\cap E'$ is also irreducible. Note that the curves $C_i=\mathbb{P}^1$ intersect pairwise in one point. Since the log discrepancy of E_i is 0, by construction P' lies on C_3 outside C_1 and C_2. But then P' is a log terminal point of $K_E+C_1+C_2+C_3$, and by Proposition 5.13 the log discrepancy of $K_W+h^{-1}g^{-1}S+h^{-1}E+E'+\bar{0}$ at E_i is ≥ 1. Hence this case is impossible. We also note that the log discrepancy of the exceptional divisor E_i on curves distinct from C_1 and C_2 is equal to 0 only if it lies over C_3. Moreover, if $a_i=1/2$ and $d_i\leq 1$, then C_3 is an ordinary double curve, E_i is its blowup, and $d_i=(1/2)d_j>1/2$. This proves (8.8.3) in the case when E_i with log discrepancy $a_i=1/2$ is contracted to a curve lying on only one exceptional surface E_j. This completes the proof of property (8.8.3), since the index of $K_W+h^{-1}g^{-1}S+h^{-1}g^{-1}B+h^{-1}E+\sum E_j$ on the intersection curves for reduced components of the boundary of $h^{-1}g^{-1}S+h^{-1}E+\sum E_j$ is equal to 1.

Next we verify that Y is nonsingular outside $g^{-1}S\cup E$ in a neighborhood of P. It is clear that it suffices to consider the singularities along curves C_i not lying on $g^{-1}S\cup E$ and passing through P. As we already know, $K_Y+g^{-1}S+E$ has a 1-complement in a neighborhood of P, and the curves of noncanonical singularities C_i lie in the boundary of the complement. As above, we can construct an extremal blowing up $h\colon W\to Y$ over one of such curves of an exceptional surface E' with log discrepancy $0<a'<1$ for $K_Y+g^{-1}S+E$, so that

$$h^*(K_Y+g^{-1}S+E)=K_W+h^{-1}g^{-1}S+h^{-1}E+(1-a')E'.$$

But then the surfaces $h^{-1}g^{-1}S$, $h^{-1}E$, and E' all pass through the fiber curve $h^{-1}P$, which contradicts the log terminality of $K_Y+g^{-1}S+E$ in a neighborhood of P. In the case when the singularities along the curves C_i are canonical we can use the arguments of Proposition 4.3 and the log terminality of $K_Y+g^{-1}S+E$ to construct a blowing up h of the exceptional divisors over C_i with log discrepancy 1 (that is, discrepancy 0), and no others. By monotonicity (1.3.3), there are no exceptional surfaces over P (cf. (1.5.7)), and $h^{-1}P$ is again a curve, so that the above arguments yield a contradiction.

We proceed with proving the following claim.

(8.8.4) *If P is an isolated singularity, then the index of $K_Y + g^{-1}S + E$ is odd, and equal to $2m+1$, where $m \geq 2$ is a natural number; moreover, there exists an extremal blowing up $h\colon W \to Y$ of the point P with exceptional divisor E' having multiplicity $d' = 1 + 1/(2m+1)$ in $g^{-1}S + E$, log discrepancy $a' = 0$ for $K_Y + g^{-1}S + g^{-1}B + E$, and log discrepancy $a'' = 1/(2m+1)$ for $K_Y + g^{-1}S + E$. Moreover, E' has a singular point P' locally satisfying the same conditions as P, but with index of $K_W + h^{-1}g^{-1}S + h^{-1}E + E'$ even and equal to $2m$.*

First of all, we claim that P is a singular point of $g^{-1}S$. To see this, we observe that there exists an extremal 0-contraction $h\colon Y \to W$ of a curve $C_1 \subset g^{-1}S \cap \operatorname{Supp} g^{-1}B$. This is a 0-contraction for $H = \varepsilon g^*B$ over Z. In fact, $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/Z)$ has two extremal rays R_1 and R_2. Let R_1 correspond to the contraction g. It is clear that R_1 is nonnegative with respect to $g^{-1}B$. On the other hand, by Reduction 8.2 the curve $g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is exceptional. Hence there is a curve over Z that is negative with respect to $g^{-1}B$. Thus R_2 is negative with respect to $g^{-1}B$. But g^*B is numerically trivial on R_1 and positive on $g^{-1}S \cap \operatorname{Supp} g^{-1}B$. Hence R_2 is positive on E and negative on $g^{-1}S$, which yields the desired assertion.

Moreover, the contracted curve B_0 passes through P. It is not hard to verify that in the case when P is nonsingular the curve B_0 is irreducible, nonsingular, crosses B_1 normally at a single point P, and is an exceptional curve of the first kind. Moreover, there is a curve B_{-1} passing through P such that $B_{-1} \subset g^{-1}S$ but $B_{-1} \not\subset \operatorname{Supp} g^{-1}B$; this curve has multiplicity $1/2$ in the boundary of $(K_Y + g^{-1}S + g^{-1}B + E)|_{g^{-1}S}$, and in a neighborhood of B_0 this restriction has the form $K_{g^{-1}S} + B_1 + (1/2)B_0 + (1/2)B_{-1}$. But then Y has ordinary double points along B_{-1}, which contradicts our assumption that P is an isolated singularity. In particular, from this it follows that P is actually singular and the index of $K_Y + g^{-1}S + E$ is greater than 1. We claim that P is a terminal singularity.

To see this it suffices to verify that $a_i' + d_i > 1$ for the exceptional divisors E_i over P. This follows immediately from (8.8.3). By the above and [7, (5.2)], the index r of the point P is greater than 1, hence by Kawamata's theorem in the Appendix there exists an exceptional divisor E_i over P with log discrepancy $1 + 1/r$ (that is, discrepancy $1/r$). Hence $a_i' + d_i = 1 + 1/r$. On the other hand, ra_i' and rd_i are positive natural numbers. Again by (8.8.3), this is only possible when $r = 2m+1$ is odd, $a_i = 1/2$, $a_i' = (m+1)/(2m+1)$, $d_i = (m+1)(2m+1)$, and the exceptional divisor E_i is obtained by blowing up a curve C_3 of quadratic singularities on the exceptional divisor E' over P with log discrepancy 0 for $K_Y + g^{-1}S + g^{-1}B + E$ and $d' = 2d_i = 1 + 1/(2m+1)$. We may assume that E' is exceptional for the extremal blowing up $h\colon W \to Y$. Then by construction, and since $g^{-1}B$ passes through P, the restriction of $h^*(K_Y + g^{-1}S + g^{-1}B + E)$ to E' is numerically trivial and has the form

$$K_{E'} + C_1 + C_2 + \tfrac{1}{2}C_3 + \tfrac{1}{2}C_4,$$

where the curves $C_1 = h^{-1}g^{-1}S \cap E' = \mathbb{P}^1$, $C_2 = h^{-1}E \cap E' = \mathbb{P}^1$, $C_3 = \mathbb{P}^1$, and $C_4 = \operatorname{Supp} h^{-1}g^{-1}B \cap E'$ are irreducible. Since W has ordinary double points along C_3, the log discrepancy of E' for $K_Y + g^{-1}S + E$ is of the form

$$a' = 1 - 2(1 - a_i') = \frac{1}{2m+1}.$$

Hence

$$h^*(K_Y + g^{-1}S + E) = K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'.$$

However, by [7, (5.2)], the index of this divisor divides $2m+1$. Hence, arguing as in the proof of Lemma 4.2 and Corollary 3.10, we see that this index is equal to $2m+1$, W is nonsingular along C_1 and C_2, and the crossings along $C_1 = h^{-1}g^{-1}S \cap E'$ and $C_2 = h^{-1}E \cap E'$ are normal at generic points. Then by the same arguments the unique intersection point $Q' = C_1 \cap C_2$ is nonsingular on $h^{-1}g^{-1}S$ and $h^{-1}E$. Hence by Corollaries 3.7–8 W is nonsingular, and $h^{-1}g^{-1}S$, $h^{-1}E$, and E' cross normally at Q'. In particular, E' is nonsingular in a neighborhood of Q'. Again by [7, (5.2)], the index of $g^{-1}S$ and E divides $2m+1$. Hence the multiplicities of E' in $g^{-1}S$ and in E do not exceed 1. From this and the fact that the boundary $h^{-1}g^{-1}S + h^{-1}E + E'$ has normal crossings at the point Q' we deduce that C_1 and C_2 cannot be exceptional curves of the first kind on the minimal resolution of singularities of $h^{-1}g^{-1}S$ and $h^{-1}E$, respectively.

Now we turn to the surface E' and show that there is a singular point of E' on C_2. Indeed, if it were not so, then $C_2 = \mathbb{P}^1$ and all the singularities of W in a neighborhood of the point $P' = C_2 \cap C_3$ would lie on the curve C_3, which crosses C_2 normally, since C_2 is ample and hence also meets C_1 and C_4. Then from the fact that E' is nonsingular in a neighborhood of P' it follows that the divisor $K_W + h^{-1}E + E'$ has index 2; hence $(K_W + h^{-1}E + E')|_{h^{-1}E}$, which in a neighborhood of P' is of the form $K_{h^{-1}E} + C_2$, also has index 2. Since P' is log terminal, it is an ordinary double point on $h^{-1}E$. On the other hand, the restriction

$$\begin{aligned} h^*(K_Y + g^{-1}S + E)|_{h^{-1}E} &= \left(K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'\right)\Big|_{h^{-1}E} \\ &= K_{h^{-1}E} + \frac{2m}{2m+1}C_2 + h^{-1}B_1 \end{aligned}$$

is numerically trivial on C_2, from which it follows that C_2 is a nonsingular rational curve with selfintersection $-(m+1)$ on the minimal resolution of $h^{-1}E$, that is, the blowup of the ordinary double point P'. In the same way, since the divisor

$$h^*g^{-1}S|_{h^{-1}E} = (h^{-1}g^{-1}S + d_S E')|_{h^{-1}E} = d_S C_2 + h^{-1}B_1$$

is numerically trivial on C_2, we can compute the multiplicity d_S of E' in $g^{-1}S$; it turns out that $d_S = 2/(2m+1)$. But then the multiplicity of E' in E is equal to $\frac{2m}{2m+1}$. Therefore $\frac{2m}{2m+1}C_1 + h^{-1}B_1$ is numerically trivial on C_1. But the restriction

$$\begin{aligned} h^*(K_Y + g^{-1}S + E)|_{h^{-1}g^{-1}S} &= \left(K_W + h^{-1}g^{-1}S + h^{-1}E + \frac{2m}{2m+1}E'\right)\Big|_{h^{-1}g^{-1}S} \\ &= K_{h^{-1}g^{-1}S} + \frac{2m}{2m+1}C_1 + h^{-1}B_1 \end{aligned}$$

is also numerically trivial on C_1, and hence the canonical divisor $K_{h^{-1}g^{-1}S}$ is numerically trivial on C_1. Here C_1 is not an exceptional curve of the first kind on the minimal resolution of $h^{-1}g^{-1}S$. From this it follows that C_1 is a nonsingular rational curve with selfintersection -2 on the minimal resolution of $h^{-1}g^{-1}S$, and on C_1 there is at most one singular point, which is resolved by a chain of nonsingular rational (-2)-curves. Thus P is a Du Val singularity of type A_{2m} on $g^{-1}S$. But by the above, on $g^{-1}S$ there is an exceptional curve lying in $g^{-1}S \cap \operatorname{Supp} g^{-1}B$ and numerically trivial on the restriction $(K_Y + g^{-1}S + g^{-1}B + E)|_{g^{-1}S}$, whose boundary in a neighborhood of P is $B_1 + \frac{1}{2}(g^{-1}S \cap \operatorname{Supp} g^{-1}B)$. From this we deduce that $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is an irreducible curve, and on the minimal resolution of singularities of $g^{-1}S$ it is a nonsingular rational (-2)- or (-1)-curve passing through a unique singular point P of the surface $g^{-1}S$.

In the first case the contraction of B_0 transforms P into a Du Val singularity of type D_{2m+1}. From this it follows that on the minimal resolution of the point Q of the surface S the curve $g(B_0) = S \cap \operatorname{Supp} B$ is not an exceptional curve of the first kind. But this is impossible since B is positive on the flipping curve $S \cap \operatorname{Supp} B$. In the second case $m = 1$, the preimage $h^{-1}B_0$ does not pass through the singularity of $h^{-1}g^{-1}S$ on C_1 and crosses C_1 normally at a single point. Hence by the numerical triviality of $K_Y + g^{-1}S + g^{-1}B + E$ on B_0 its restriction to $g^{-1}S$ in a neighborhood of B_0 is of the form $K_{g^{-1}S} + B_1 + (1/2)B_0 + (1/2)B_{-1}$, where B_{-1} is a nonsingular curve crossing B_0 normally at a single point $\neq P$; Y has ordinary double points along B_{-1}. Hence B_0 is contracted to a nonsingular point terminal for the image of $K_{g^{-1}S} + (1/2)B_{-1}$. But then (8.8.4) holds except that $m = 1$ and a subsequent point P' need not exist. This case will be excluded later, so that for the moment we assume that $m \geq 2$.

Therefore C_2 has a singular point P', again coinciding with $C_2 \cap C_3$. Since C_2 is ample on the surface E' and has a unique singularity on this surface, it becomes a nonsingular rational curve with nonnegative selfintersection on the minimal resolution of singularities of E'. Now applying Theorem 6.9 to the minimal resolution of the singularity P' of the surface E' one can show that the selfintersection index is equal to 0 and P' is a unique singularity of E' of the simplest possible type, that is, E' is a cone with vertex P' over the nonsingular rational curve C_1.

Note also that the curves C_3 and C_4 intersect C_1 in distinct points P_1 and P_2 respectively. Hence, arguing as above, we see that P_1 is an ordinary double point of $h^{-1}g^{-1}S$, the selfintersection of the curve C_1 on the minimal resolution of $h^{-1}g^{-1}S$ is $-(m+1)$, the selfintersection of C_2 on the minimal resolution of $h^{-1}E$ is -2, and P' is a Du Val singularity of type A_{2m-1}. It follows that the index of $K_W + h^{-1}E + E'$ in a neighborhood of P' coincides with that of the restriction $(K_W + h^{-1}E + E')|_{h^{-1}E} = K_{h^{-1}E} + h^{-1}B_1$ and is equal to $2m$. (Furthermore, one can show that P is a quotient singularity of type $\frac{1}{2m+1}(2, -2, 1)$.)

We check that P' satisfies (8.8.2). Indeed, otherwise there exists an exceptional divisor E_i over P' with $a_i = 0$ for $K_W + h^{-1}g^{-1}B + h^{-1}E + E'$ and multiplicity d_i in $h^{-1}E + E'$ at most 1. Then E_i is exceptional over P with $a_i = 0$ for $K_Y + g^{-1}S + g^{-1}B + E$ and multiplicity $d_i < 1 + 1/(2m+1)$ in

$$h^*(g^{-1}S + E) = h^{-1}g^{-1}S + h^{-1}E + E' + \frac{1}{2m+1}E'$$

and in $g^{-1}S + E$. But since $g^{-1}S + E$ has index $2m+1$ at P, it follows that $d_i \leq 1$, which contradicts (8.8.2). However, we may lose the existence of a contracted curve in the intersection $g^{-1}S \cap \operatorname{Supp} g^{-1}B$, which is important for choosing the component on which the singularity P' appears when $m \geq 2$.

(8.8.5) *If P is a nonisolated singularity, then the index of $K_Y + g^{-1}S + E$ is even and is equal to $4m+2$ for a natural number $m \geq 1$. Furthermore, there exists an extremal blowing up $h: W \to Y$ of the point P with exceptional divisor E' having multiplicity $d' = 1 + \frac{1}{4m+2}$ in $g^{-1}S + E$ and log discrepancy $a' = 0$ for $K_Y + g^{-1}S + g^{-1}B + E$. Moreover, on E' there is a singular point P' locally satisfying the same conditions as P, but with the index of $K_W + h^{-1}g^{-1}S + h^{-1}E + E'$ odd and equal to $4m+1$.*

The only possibility for a curve of singularities through P is the curve of ordinary double points on $g^{-1}S + E$; then $K_Y + g^{-1}S + E$ is log terminal and has index 2 at a generic point of such a curve. Thus the index of $K_Y + g^{-1}S + E$ is even, and there is a double cover $\pi: \tilde{Y} \to Y$ in a neighborhood of P ramified only in such curves.

Next we check that the proper transforms $\pi^{-1}g^{-1}S$, $\pi^{-1}g^{-1}B$ and $\pi^{-1}E$ preserve the previous properties in a neighborhood of the point $\pi^{-1}P$. Under lifting by π the log terminality of $K_Y + g^{-1}S + E$ is preserved outside P by construction, and at P by Corollary 2.2. Thus, as above, by (1.3.3) and Corollary 2.2 applied to $g^{-1}S+g^{-1}B+E$, we get that $\pi^{-1}P$ is $\mathbb{Q}$-factorial, and $\pi^{-1}g^{-1}S$ and $\pi^{-1}E$ are irreducible and normal in a neighborhood of $\pi^{-1}P$. By the proof of Corollary 2.2, the log discrepancy $\tilde{a}_i$ of the exceptional divisor $\tilde{E}_i$ over E_i for Y and over $\pi^{-1}P$ in $K_{\tilde{Y}}+\pi^{-1}g^{-1}S+\pi^{-1}g^{-1}B+\pi^{-1}E$ is equal to 0 only if the log discrepancy of E_i for $K_Y+g^{-1}S+g^{-1}B+E$ vanishes. This implies that property (8.8.2) is preserved. Hence, by (8.8.4), $\pi^{-1}P$ is a terminal point of odd index $2m+1$ and the index of $K_Y+g^{-1}S+E$ is of the form $4m+2$. If $m=0$, then $K_Y+g^{-1}S+E$ and its restriction $(K_Y+g^{-1}S+E)|_{g^{-1}S}$ both have index 2. More precisely, if $g^{-1}S$ does not contain a curve of singularities of Y, then P is an ordinary double point on $g^{-1}S$ and, as above, we see that in a neighborhood of P

$$(K_Y+g^{-1}S+g^{-1}B+E)|_{g^{-1}S}=K_{g^{-1}S}+B_1+\tfrac{1}{2}B_0,$$

where the curve $B_0\subset g^{-1}S\cap\operatorname{Supp}g^{-1}B$ generates a flipping extremal ray (which was denoted by R_2 in the above). The (-2)-curve resolving P on $g^{-1}S$ has log discrepancy 1 for $K_{g^{-1}S}+B_1+\frac{1}{2}B_0$. Thus, upon resolving P, we arrive at a contradiction in the same way as in proving that P is singular in (8.8.4). Therefore the point $P\in g^{-1}S$ is nonsingular, it is contained in a curve B_{-1} of ordinary double points, and in a neighborhood of P we have

$$\big(K_Y+g^{-1}S+g^{-1}B+E\big)|_{g^{-1}S}=K_{g^{-1}S}+B_1+\tfrac{1}{2}B_0+\tfrac{1}{2}B_{-1},$$

where $B_0=g^{-1}S\cap\operatorname{Supp}g^{-1}B=|R_2|$ is an irreducible curve. Note that $g^{-1}S$ is nonsingular in a neighborhood of B_0, B_0 is an exceptional curve of the first kind, and the restriction $(K_Y+g^{-1}S)|_{g^{-1}S}=K_{g^{-1}S}+\frac{1}{2}B_{-1}$ is numerically negative on B_0, since B_0 and B_{-1} cross normally at a unique point P.

We proceed with proving that the flip exists in this case. First we verify that the intersection $g^{-1}S\cap E=B_1\cup\cdots\cup B_n$ is irreducible. All the B_i with $i<n$ are contracted to Q, and are therefore positive with respect to $g^{-1}B$. Hence $n\le 2$. Suppose that $n=2$. The curve B_0 is the support of the next extremal ray R_2. Moreover, the surfaces $g^{-1}S$ and $g^{-1}B$ are negative, and E is positive on B_0. Hence the flip in B_0 exists by Corollary 5.20. One can show that it has all the properties described in Proposition 8.3. To see this it suffices to recall that the image of $K_{g^{-1}S}+\frac{1}{2}B_{-1}$ under the contraction of B_0 is log terminal. In particular, the only point of $g^{-1}S^+$ at which B_1^+ can be singular is $Q'=B_1^+\cap B_2^+$. But now $g^{-1}B^+$ is numerically trivial on B_1^+, from which it follows that it is extremal. This means that B_1^+ is the support of the next extremal ray. Hence E is positive on it. The intersection of $g^{-1}S^+$ and E^+ along B_1^+ is normal by Proposition 8.3. As in Proposition 8.3, we deduce from this using Lemma 3.18 that B_1^+ is movable, which yields a contradiction. Thus the intersection $B_1=g^{-1}S\cap E$ is irreducible.

Suppose now that $g(E)=C$. Then g identifies $g^{-1}S$ with S. Furthermore, by Proposition 3.9 and (8.1.4) we have

$$(K+S\cdot B_0)=\left(K_S+\frac{n-1}{n}C+\frac{1}{2}B_{-1}\cdot B_0\right)=\frac{n-1}{n}-\frac{1}{2}<0,$$

where n is the index of $K+S$ along C. Hence $n=1$ and, in a neighborhood of B_0, X has only ordinary double points along $g(B_{-1})$. Furthermore, the index of $K+S$ is equal to 2. Therefore there is a purely log terminal complement of $K+S$ of index 2 in a neighborhood of B_0, and the flip of f exists by Proposition 2.9.

The case $g(E)=Q$ is similar. Arguing as above, we get

$$(g^*(K+S)\cdot B_0)=\left(K_{g^{-1}S}+\frac{n-1}{n}C+\frac{1}{2}B_{-1}+a'B_1\cdot B_0\right)=a'-\frac{1}{2}<0,$$

from which it follows that $a' < 1/2$. But B_1 is not an exceptional curve of the first kind on the minimal blowup E, and

$$K_{g^{-1}S} + \frac{n-1}{n}C + \frac{1}{2}B_{-1} + a'B_1 = g^* \left(K_S + \frac{n-1}{n}C + \frac{1}{2}B_{-1}\right).$$

From this it follows that B_1 is a (-2)-curve on the minimal blowup E, that X is nonsingular along C, and $P_0 = B_1 \cap g^{-1}C$ is a canonical singularity of type A_n. Hence in a neighborhood of $g(B_0)$ on S there is a purely log terminal complement of $K_S+(1/2)g(B_{-1})$ of index 2. According to the proof of Theorem 5.12, to extend this complement to X for $K+S$, it suffices to have a resolution $Y' \to X$ with normal crossings $S_{Y'}$ which is minimal over S. For this we need to use a partial resolution of g and extend it. Since Y has ordinary double singularity along B_{-1}, resolving it does not change $g^{-1}S$. Thus it suffices to find such a resolution for P_0. Now P_0, just as P, is a $\mathbb{Q}$-factorial point. Furthermore, by Corollary 3.7, it is a quotient singularity of index n.

If P_0 is not an isolated singularity, then the curve of singularities C' lies on E. Moreover, X has a canonical singularity of type $A_{n'}$ with $n'|n$. Performing resolutions of C' as described in Proposition 4.3, we again preserve the minimality assumption and reduce the resolution to isolated singularities of the same type; the blown up surfaces arising under these resolutions are irreducible. In the case when P_0 is isolated it is a terminal singularity of type $\frac{1}{n}(k, -k, 1)$, an economical blowing up of which yields the required resolution. This can also be deduced by induction on n from the theorem in the Appendix. Thus $m \geq 1$. Hence by (8.8.4) there exists an extremal blowing up $\tilde{h}\colon \tilde{W} \to \tilde{Y}$ with exceptional divisor $\tilde{E}'$ such that the multiplicity of $\tilde{E}'$ in $\pi^{-1}g^{-1}S+\pi^{-1}E$ is given by $\tilde{d}' = 1+\frac{1}{2m+1}$, the log discrepancy $\tilde{a}'$ for $K_{\tilde{Y}} + \pi^{-1}g^{-1}S + \pi^{-1}g^{-1}B + \pi^{-1}E$ vanishes, and the log discrepancy for $K_{\tilde{Y}} + \pi^{-1}g^{-1}S + \pi^{-1}E$ is given by $\tilde{a}'' = \frac{1}{2m+1}$. By (8.8.2), from this it follows that the ramification index of π at $\tilde{E}'$ is equal to 1.

Suppose that $\tilde{E}' \subset \tilde{Y}$ lies over $E' \subset Y$, which is an irreducible exceptional surface over P. Then the log discrepancy of E' for $K_Y + g^{-1}S + g^{-1}B + E$ is equal to 0. Let $h\colon W \to Y$ be the extremal blowing up of E'. Using Theorem 6.9, it is not hard to verify that $K_W + E'$ is purely log terminal. Hence, arguing as in the proof of Proposition 8.3, from Corollary 2.2 and Corollary 3.8 we deduce that π is unramified everywhere over E', and hence also over P provided that $\pi^{-1}(E')$ is reducible. Therefore the surface $\tilde{E}' = \pi^{-1}E'$ is irreducible, i.e. the covering involution corresponding to the double cover π acts regularly on the extremal blowing up $\tilde{h}$. Since h and $\tilde{h}$ are extremal, the curves $\tilde{C}_1 = \pi^{-1}h^{-1}g^{-1}S \cap \tilde{E}'$ and $\tilde{C}_2 = \pi^{-1}h^{-1}E \cap \tilde{E}'$ are irreducible and lie over $C_1 = h^{-1}g^{-1}S \cap E'$ and $C_2 = h^{-1}E \cap E'$ respectively. On the other hand, by the proof of (8.8.4) there exists a curve $\tilde{C}_i$ for which

$$(K_{\tilde{Y}} + \pi^{-1}h^{-1}g^{-1}S + \pi^{-1}h^{-1}g^{-1}B + \pi^{-1}h^{-1}E + \tilde{E}')|_{\tilde{E}'}|_{\tilde{C}_i} = K_{\tilde{C}_i} + \tilde{Q} + \tfrac{1}{2}\tilde{P}_1 + \tfrac{1}{2}\tilde{P}_2,$$

where

$$\tilde{Q} = \pi^{-1}h^{-1}g^{-1}S \cap \pi^{-1}h^{-1}E \cap \tilde{E}',$$
$$\tilde{P}_2 = \pi^{-1}h^{-1}g^{-1}S \cap \operatorname{Supp} \pi^{-1}h^{-1}g^{-1}B \cap \tilde{E}'$$

or

$$\tilde{P}_2 = \pi^{-1}h^{-1}E \cap \operatorname{Supp} \pi^{-1}h^{-1}g^{-1}B \cap \tilde{E}'.$$

From this it follows that π is ramified along the curve $\tilde{C}_i$, and by the purity theorem W is singular along the corresponding curve C_i. Since π is unramified along $\tilde{E}'$, the log discrepancy of E' for $K_Y + g^{-1}S + E$ is equal to $\frac{1}{2m+1}$. By construction and [7, (5.2)], the index of $K_Y+g^{-1}S+E$ divides $4m+2$, and hence, arguing as in the proof of Lemma 4.2, we conclude that W has an ordinary double point along C_i. The exceptional divisor corresponding to this singularity has log discrepancy 0 for $K_Y+g^{-1}S+g^{-1}B+E$, log discrepancy $\frac{1}{4m+2}$ for $K_Y+g^{-1}S+E$, and multiplicity $1+\frac{1}{4m+2}$ in $g^{-1}S+E$.

Denote by h the extremal blowing up of this divisor. Arguing as in (8.8.4), we conclude that the curves $C_1 = h^{-1}g^{-1}S\cap E'$ and $C_2 = h^{-1}E\cap E'$ are irreducible and that the above crossings are normal at their generic points. Furthermore, the curves C_1 and C_2 are not exceptional curves of the first kind on the surfaces $h^{-1}g^{-1}S$ and $h^{-1}E$ respectively. Suppose first that $i=2$ above. Then by the proof of (8.8.4) the multiplicity of the previous E' in E is equal to $\frac{2m}{2m+1}$, hence the multiplicity of the current E' in E is equal to $\frac{4m+1}{4m+2}$. Then, arguing as in (8.8.4), one can show that C_1 will be a (-2)-curve on the minimal blowing up, $g^{-1}S$ does not contain a curve of double points of Y, and C_1 passes through a unique singularity, viz. a Du Val singularity of type A_{4m} on $h^{-1}g^{-1}S$. This contradicts the fact that, according to the proof of (8.8.4), for $m\ge 2$ the surface $g^{-1}S$ contains a curve of double points B_{-1}.

Thus $i=1$. By the previous arguments the multiplicity of the current E' in $g^{-1}S$ is equal to $\frac{4m+1}{4m+2}$, E does not contain a curve of double points of Y, P is a Du Val singularity of type A_{4m+1} on E, C_2 is a (-2)-curve on the minimal blowup of $h^{-1}E$, and C_2 passes through a unique singularity P' of the surface $h^{-1}E$, viz. a Du Val singularity of type A_{4m}. On the other hand, by construction we have

$$(K_Y + h^{-1}g^{-1}S + h^{-1}g^{-1}B + h^{-1}E + E')|_{h^{-1}S}|_{C_1} = K_{C_1} + P + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2,$$

where $\operatorname{Supp}\pi^{-1}h^{-1}g^{-1}B$ passes through P_2, and P_2 is nonsingular on $h^{-1}g^{-1}S$. But $h^{-1}g^{-1}S$ must contain the double point curve $h^{-1}B_{-1}$. It is clear that $P_1 = C_1\cap h^{-1}B_{-1}$ is a nonsingular point of $h^{-1}g^{-1}S$. Hence $h^{-1}g^{-1}S$ is nonsingular in a neighborhood of C_1 and $C_1=\mathbb{P}^1$ is a curve with selfintersection $-(2m+1)$. On the other hand, by nonsingularity of $h^{-1}g^{-1}S$, in a neighborhood of C_1 the surface E' has P_1 as an ordinary double point and E' does not have double points along curves on W. Hence P' is an isolated singularity. But P' is a Du Val singularity of type A_{4m}. From this it follows that the index of $K_W+h^{-1}E+E'$ in a neighborhood of P' is the same as that of the restriction $(K_W+h^{-1}E+E')|_{h^{-1}E} = K_{h^{-1}E}+h^{-1}B_1$ and is equal to $4m+1$. As in the proof of (8.8.4), in what follows we check that P' satisfies (8.8.2).

Now it follows from (8.8.4–5) that in (8.8.4) the index of P' is of the form $4m'+2$, hence the index of P is of the form $4m'+3$ with $m'\ge 1$. Hence the case (8.8.5) is impossible altogether, from which it follows that the case (8.8.4) is also impossible. All this holds with a possible exception of one case that we have not yet considered, viz. $m=1$ in (8.8.4). We will show that in this case the flip exists, or reduces to the same type (8.5.3) with $d_i>1$ for $a_i=0$. For this we need the following two lemmas, also used in the proof of main results for preserving the type of flips at subsequent inductive steps.

8.9. **Lemma.** *Let S be a normal projective surface with boundary B, and let C_1, C_2 be* (possibly reducible) *contracted curves such that*:

(i) $2(K+B)\sim 0$;

(ii) $\lfloor B \rfloor = B_1 + B_2$, *where* B_1 *and* B_2 *are irreducible*;
(iii) $B_1^2 > 0$;
(iv) *the curve* B_2 *becomes ample after contracting the curve* C_2;
(v) C_1 *does not intersect* B_1;
(vi) *the point* $P = B_1 \cap B_2$ *is the only point of* B_1 *at which* $K + B$ *is not purely log terminal*;
(vii) *the components of* C_2 *intersect* B_1 *and* B_2 *at* P.

Then either C_1 *does not intersect* B_2 *or* P *is the only point of* B_2 *at which* $K+B$ *is not purely log terminal.*

By Theorem 6.9 the singularities of S are rational, and remain so after contracting C_2. Hence ampleness in (iv) coincides with numerical positivity by the Nakai-Moishezon criterion (cf. [8, 6-1-15 (2)]).

Proof. Suppose that C_1 intersects B_2. Then, combining standard arguments of the theory of extremal rays for the contraction of C_1 with (i), one can find an irreducible contractible curve $C' \subseteq C_1$ intersecting B_2. Hence without changing the assumptions we can restrict to the case when C_1 is irreducible and intersects B_2. By (i), (ii), and Theorem 6.9, the locus of log canonical singularities of $K+B$ coincides with $B_1 \cup B_2$. Suppose now that there exists an irreducible component $C'' \subseteq C_2$ intersecting C_1. Then by Corollary 3.16 and the log canonical assumption on $K+B$ the curve C'' has multiplicity 0 in B. Hence by (i) C'' is an exceptional curve of the first kind on the minimal resolution of S. As in Proposition 8.3, it is easy to verify that $B_1 = \mathbb{P}^1$ and

$$(K + B_1 + B_2)|_{B_1} = K_{\mathbb{P}^1} + \tfrac{1}{2}P_1 + \tfrac{1}{2}P_2 + P.$$

Now let $g\colon T \to S$ be a strictly log terminal blowing up of $K+B$ which is minimal over P. Then $g^{-1}C''$ does not intersect $g^{-1}B_1$, but crosses normally at Q a component that is exceptional over P and has multiplicity 1 in the boundary B_T. Therefore $g^*(K+B)$ has only canonical singularities on $g^{-1}C''$ outside the point Q. It follows that C_1 also has multiplicity 0 in B and is an exceptional curve of the first kind on the minimal resolution of S, since it intersects B_2. Then on the minimal resolution T a suitable multiple of the total preimage of the curve $g^{-1}(C_1 \cup C'')$ is movable. But $g^{-1}(C_1 \cup C'')$ is disjoint from $g^{-1}B_1$, and its intersection with B_T is not mapped to P. Hence $g^{-1}B_1$ is exceptional. Arguing as in Proposition 8.3, we see that from (iii) it follows that exactly one of the points P_i is nonsingular.

Suppose that P_1 is the nonsingular point. Then there is an irreducible curve B_3 crossing normally through P_1 and having multiplicity $1/2$ in the boundary B. By the above C'' does not intersect B_3. Also each irreducible component of C_2 does not intersect B_3, since it passes through P. Hence by (iv) B_3 meets B_2. Moreover, it is not hard to verify that $g^{-1}(C_1 \cup C'')$ is disjoint from $g^{-1}(B_1 \cup B_3)$. Arguing as above, we derive from this that $g^{-1}(B_3)$ is exceptional. But $g^{-1}(B_3)$ intersects the locus of log canonical singularities of $(B_1 + B_2)_T$ at two points, which contradicts Lemma 5.7. Thus we have proved that all irreducible components of C_2 are disjoint from C_1. Contracting C_2 one can assume that $C_2 = \varnothing$; then assumption (iv) means that B_2 is ample. By (iii) $\overline{\mathrm{NE}}(S)$ has an extremal ray R that is positive with respect to B_1. If the corresponding contraction contracts a curve, then by ampleness of B_2 and Lemma 5.7 it intersects B_1 and B_2 at P. Hence one can take this last curve as C_2, and then contract it. The contractions decrease the Picard number of S. Hence after a finite number of such contractions we may assume that the extremal contraction Cont_R is not birational. Since C_1 is disjoint from B_1, Cont_R must be a contraction onto a curve, and the curves B_1 and B_2 are not contained in its fibers.

Now from Theorem 6.9 it follows that P is the only point of B_2 at which $K+B$ is not purely log terminal. □

8.10. **Lemma.** *Let $f\colon S \dashrightarrow T$ be a birational map of normal projective surfaces, and D an effective ample divisor on S such that D_T is irreducible. Then D_T is numerically positive.*

Proof. Consider a resolution of indeterminacies of f, for example a Hironaka hut

$$\begin{tikzcd} & U \arrow[dl, "h"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "f"] & & T. \end{tikzcd}$$

Since D is ample, its support is connected, and since S is normal, its preimage $\bigcup C_i$ is connected on U. We claim that $g(\bigcup C_i) \neq \mathrm{pt}$. Indeed, otherwise there exist $a_i > 0$ such that

$$\left(\sum a_i C_i \cdot C_j\right) < 0$$

for all irreducible components C_j. In particular

$$0 \leq \left(\sum a_i h(C_i) \cdot D\right) = \left(\sum a_i C_i \cdot h^* D\right) = \left(\sum a_i C_i \cdot \sum b_j C_j\right) < 0,$$

since $D = \sum b_j C_j$, where $b_j > 0$ for at least one j; this is a contradiction. From the claim and the irreducibility of D_T (that is, irreducibility of $\operatorname{Supp} D_T$), we infer that $D_T = g(\bigcup C_i)$ and all the curves that are exceptional with respect to h and do not intersect $\bigcup C_i$ are exceptional for g. Thus

$$g^* D_T = \sum c_i C_i,$$

where all $c_i > 0$. If B is a curve on T disjoint from D_T, then $g^{-1}B$ is disjoint from $\bigcup C_i$, and $h \circ g^{-1}B$ is disjoint from the support of D. Therefore $h \circ g^{-1} B = \mathrm{pt}$, and by the above this is impossible. It remains to verify that D_T^2 is positive. Indeed, otherwise $g^* D_T$ is numerically nonpositive on all the curves C_j, which yields a contradiction:

$$0 < \left(\sum c_i h(C_i) \cdot D\right) = \left(\sum c_i C_i \cdot h^* D\right) = \left(\sum c_i C_i \cdot \sum b_j C_j\right) \leq 0. \quad \square$$

We proceed with the proof of Proposition 8.8. Thus we return to the case $m = 1$ in (8.8.4). By what we have already proved, there exists an extremal blowing up of a surface E' over P with $a = 0$ for which (8.8.1) holds. However E' has multiplicity $2/3$ in $g^{-1}S$ and in E, and hence multiplicity $(2/3)(1+d)$ in S, where $d \leq 1$ is the multiplicity of E in S. By assumption we have $1 < (2/3)(1+d) \leq 4/3$, so that $d > 1/2$.

Consider now an extremal blowing up $g\colon Y \to X$ of the new surface $E = E'$. We check that it is good. As above, first we show that $g^{-1}S \cap E$ consists of at most two irreducible curves B_i and that if $g^{-1}S \cap E = B_1 \cup B_2$, then f has a flip. In view of the equality $H = g^* B$, to do this one should first perform a flip in the proper transforms of the flipping curves of f. By definition of the current type, these coincide with the intersection $g^{-1}S \cap g^{-1}B$, and $g^{-1}B$ is negative on them. Hence E is positive on them and $g^{-1}S$ is negative. A flip in them does not interfere with the log terminal property of $K_Y + g^{-1}S + E$ outside $P_0 = B_1 \cap B_2$, which is established as above. From this it follows that E and its modifications are normal. (There are at most two such flips, and they modify at most two curves.)

The flipped curves do not intersect B_2. After this one must perform the flip in B_1 described in Proposition 8.3, since E is positive on B_1. The modified surface E^+ is nonnormal along the flipped curve $B_1^+ = \nu(C^*)$. Now the intersection $B_2^+ = g^{-1}S^+ \cap E^+$ is irreducible. As before we are interested in the subsequent extremal and flipping ray R_2, which is numerically trivial with respect to $g^{-1}S^+$, positive with respect to $g^{-1}B^+$, and negative with respect to E^+. According to Theorem 6.9, if some connected component of the support of R_2 intersects the locus of log canonical singularities of

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$$

outside C^*, then it intersects an (irreducible) curve C^{**} such that the above locus of log canonical singularities is represented in the form $C^* \cup B_2^+ \cup C^{**}$. However, arguing as in the treatment of the case $(8.5.2)^*$, we see that by Proposition 8.3 the curve B_2^+ has at most one singular point, which can only be at $Q^+ = B_2^+ \cap g^{-1}C^+$. It has selfintersection ≥ 0 on the minimal resolution, and even ≥ 1 in the singular case. Performing partial resolutions at Q^+ that are log crepant for

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$$

until B_2^+ becomes a 0-curve, we arrive at a contradiction with Theorem 6.9 for the contraction along the modified B_2^+. Hence, again by Theorem 6.9, the components of the support of R_2 can have log canonical singularities of

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$$

only in C^*. Moreover, if $(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$ is purely log terminal outside B_2^+ in a neighborhood of C^*, then its divisors with log discrepancy zero for $K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+$ lie over the generic point of B_1^+. Hence by Proposition 8.3, after perturbing the surface $g^{-1}S$ with base locus in the support of R_2 we arrive at a purely log terminal divisor in a neighborhood of C^*. By the above, the restricted log divisor on $E^{+\nu}$ does not have log canonical singularities outside C^*. Hence after perturbing we get a flip of type IV. Thus in this case a flip of f exists. Hence from now on we may assume that $(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$ has a point $Q' \in C^*$ outside B_2^+ that is not purely log terminal. We claim that this is impossible. By Theorem 6.9 again, $(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}}$ does not have nontrivial log crepant resolutions with modified 0-curve B_2^+. But such a resolution is trivial only when Q^+ is nonsingular on $E^{+\nu}$, and in a neighborhood of Q^+

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}} = K_{E^{+\nu}} + B_2^+ + \tfrac{1}{2}D,$$

where D is an irreducible curve that is simply tangent to B_2^+ at Q^+, and B_2^+ has selfintersection 1 on $E^{+\nu}$. We observe that in a neighborhood of Q^+ we have $E = E^+ = E^{+\nu}$. The curve D is cut out normally by $g^{-1}B$, and B_2 by $g^{-1}S$. Therefore by Corollary 3.7, $Q = B_2 \cap g^{-1}C$ is nonsingular on Y.

Thus the surface $E^{+\nu}$ is nonsingular on B_2^+, the selfintersection of B_2^+ is equal to 1, and

$$(K_{Y^+} + g^{-1}S^+ + g^{-1}B^+ + E^+)|_{E^{+\nu}} = K_{E^{+\nu}} + C^* + B_2^+ + D'$$

has a point $Q' \in C^*$ that is not purely log terminal; here $D' \geq \frac{1}{2}D$ and D is an irreducible curve tangent to B_2^+ at Q^+. Thus B_2^+ determines a contraction $h\colon E^{+\nu} \to \mathbb{P}^2$ such that $h(B_1^+)$ and $h(B_2^+)$ are lines and $h(D') = h(D)$ is a conic tangent to these lines. The mapping h contracts all curves that do not intersect B_2^+. In particular, all the flipped curves are contracted, since the last flip modifies E into $E^{+\nu}$ in divisors with log discrepancy zero over C^* for $K_{E^{+\nu}} + C^* + B_2^+$, and the flipped curves only intersect the final component B_1 of the log terminal blowup of Q'. In the above conditions and notations, we contract all the curves flipped before B_1. Then the original E is obtained as a result of the procedure described in (8.5.2)* above. We have to perform a minimal log terminal blowing up of Q' for $K_{E^{+\nu}} + C^* + B_2^+ + D'$ and then contract C^* along with all the blown up curves B_i apart from the end one B_1. We claim that they are preserved at the point of tangency $Q'' = h(Q') = h(B_1^+) \cap h(D)$.

Indeed, all the curves C_i contracted by h intersect B_1 on the log terminal blowup, without touching the other components of the blowup. Otherwise C_i would be an exceptional curve of the first kind on a subsequent minimal resolution of $E^{+\nu}$. Since $K_{E^{+\nu}} + C^* + B_2^+ + D'$ is numerically trivial and the minimal blowing up is log crepant, this curve would not intersect B_1 and the modified D'. Hence its modification on E passes through P_0 and does not intersect the modified $D' \geq g^{-1}B|_E$. But this contradicts the ampleness of $g^{-1}B$ on E. Thus we have established what we needed, and we see that a minimal log terminal blowing up of Q' consists of a single curve. By the same arguments, C^* must be an exceptional curve of the first kind on such a blowing up. Hence P_0 is nonsingular on E, and by Corollary 3.7, also on X. Thus P_0 is a canonical singularity of $g^{-1}S$. Its type on $g^{-1}S$ is known from the proof of Proposition 8.3, from which it follows that P_0 is also nonsingular on $g^{-1}S$. From this and the fact that the multiplicity of E in S is greater than 1 it follows that the same holds for the multiplicity in S of all divisors with $a_i = 0$ over a neighborhood of $Q \in X$, which contradicts the construction of E.

Thus the intersection $B_1 = g^{-1}S \cap E$ is irreducible. Then, as in the case (8.5.2)*, the log terminality of $K_Y + g^{-1}S + E$ follows from the ampleness of $g^{-1}B$ on E. Thus g is a good blowing up. In the case when

$$(K_Y + g^{-1}S + g^{-1}B + E)|_E$$

is purely log terminal outside B_1, we get what we want. Incidentally, then the flip of f exists for the following reasons. By Theorem 6.9 and the ampleness of $g^{-1}S$ on E, it remains to consider the case when the locus of log canonical singularities of the above restriction coincides with $B_1 \cup C'$, where C' is an irreducible curve on E intersecting B_1 at $Q = B_1 \cap g^{-1}C$. We reduce this case to flips of type (8.5.3) with $d_i > 1$ for $a_i = 0$. Since $H = g^*B$, to do this we should first perform a flip in the proper transforms of the curves of the flip of f. As before, a flip in them does not change $K_Y + g^{-1}S + E$ being log terminal. From this it follows that E and its modification are normal. (In the case under consideration there exists exactly one such flip.) The flipped curves do not intersect C'.

As usual, we are interested in the subsequent extremal flipping ray R_2, which is numerically trivial with respect to $g^{-1}S$, positive with respect to $g^{-1}B$, and negative with respect to E. Now $g^{-1}B$ intersects B_1 only in Q. If the restriction $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ is purely log terminal outside B_1, then, up to connectedness of the flipping curves, the required flip is exceptional and of index 2, and therefore exists. Hence we can assume that this restriction has a point $Q' \in C'$ outside B_1 that is not purely log terminal. Since the modified $g^{-1}B$ is positive on R_1 and R_2, it is ample on E. By Lemma 8.10, after contracting all the components of $\operatorname{Supp} g^{-1}B|_E$ other than C', it transforms C' into an ample curve. Hence by Lemma 8.9 the support of R_2 coincides with the contracted curves, and so the flip of R_2 is again of type (8.5.3). It remains to verify that $d_i > 1$ for all i such that $a_i = 0$ over a neighborhood of Q'. Suppose that this is not so, and reduce it to (8.8.1). To this end, we observe that by construction on E there is an irreducible curve C_3 of double points of Y passing through Q' and not touched by the flips. Hence by the above there exists a flip of R_2, and hence of f, or an extremal blowing up $h\colon W \to Y$ of a surface E' with multiplicity $d' \le 1$ in E and $a = 0$ for $K + S + B$. Moreover, it lies over Q' or over the generic point of C' and satisfies (8.8.1). The flips do not touch this blowing up, and hence it can be constructed for the original g. However in the case under consideration

$$(K_Y + g^{-1}S + g^{-1}B + E)|_E = K_E + B_1 + C' + \tfrac{1}{2}C_3 + \tfrac{1}{2}C_4,$$

where $C_4 = E \cap \operatorname{Supp} g^{-1}B$, as well as C_3, passes through Q'. Furthermore, $h^{-1}C_4$ is irreducible and movable on $h^{-1}E$. This can be deduced from the existence of a 1-complement of $K_Y + g^{-1}S + E$ in a neighborhood of E with log discrepancy 0 for E' (compare with the proof of (8.8.3)). Hence $h^{-1}C_4$ and the curve $h^{-1}C_3$ that does not meet it define a ruling of $h^{-1}E$, since $h^{-1}C_3$ is not exceptional by Theorem 6.9. This ruling is induced by a contraction on W mapping the surface $h^{-1}E$ onto a curve, possibly after a flip in $h^{-1}C'$. The last flip is involved only when E' lies over Q' and intersects $h^{-1}E$ in two curves, one of which lies in a fiber of the ruling. This yields the relation $d = (1 + dd')/n$, where d is the multiplicity of E in S, dd' is the multiplicity of E' in S, and the integer $n = -(h^{-1}E) \cdot (h^{-1}C_4)$ is positive. For $n = 1$ we have $dd' = d - 1 \le 4/3 - 1 < 1$, which yields (8.8.1) after contracting $h^{-1}E$ to a curve as before. But if $n \ge 2$, then $d = (1 + dd')/n \le (1 + d)/2$, since $d' \le 1$. Therefore $d \le 1$. This final contradiction completes our treatment of the case (8.8.4) for $m = 1$; more precisely, it reduces this case to flips of type (8.5.3) with $d_i > 1$ for $a_i = 0$ over a neighborhood of Q whose existence is discussed below.

Thus (8.8.2) does not hold if the index of $K_Y + g^{-1}S + E$ is not less than 4, and the existence of flipping curves then does not play any role if we do not care about the choice of component on which the new singularity P' appears. They are only required in treating cases with indices ≤ 3.

Thus, returning to the beginning of the proof, we have checked (8.8.1) modulo the existence of flips of type (8.5.3) with $d_i > 1$, which yields an extremal blowing up g satisfying the conditions (i) and (iv) in the definition of a good blowing up. To verify the other properties of a good blowing up, we first restrict ourselves to the case $g(E) = Q$. Recall that, as before, the multiplicity d of E in S does not exceed 1. As above, the intersection $g^{-1}S \cap E$ consists of at most two irreducible curves B_i. We show that if $g^{-1}S \cap E = B_1 \cup B_2$, then either f has a flip or it reduces to a flip of type (8.5.3) with $d_i > 1$ for all E_i over Q with $a_i = 0$. For this, as in the treatment of the case (8.8.4) with $m = 1$, we make the following reduction. The divisor $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$, Y is nonsingular at P_0, $Q = B_2 \cap g^{-1}C$, and the surface E is normal and nonsingular along the curve B_2, which has selfintersection 2. The last assertion follows from the fact that the 1-curve B_2^+ is obtained from B_2 by performing a single blowing up at P_0. The surface $g^{-1}S$ is also nonsingular along B_2.

Suppose that the selfintersection of B_2 on $g^{-1}S$ is equal to $-n$. Then the multiplicity d of the surface E can be computed as follows:

$$\begin{aligned} 0 = (g^{-1}S + dE \cdot B_2) &= (g^{-1}S \cdot B_2) + d(E \cdot B_2) \\ &= (B_1 + B_2 \cdot B_2)_E + d(B_1 + B_2 \cdot B_2)_{g^{-1}S} = 3 - d(n-1), \end{aligned}$$

so that $d = 3/(n-1)$ and $n \ge 4$, since in the case under consideration $d \le 1$. Furthermore, by the above E is the only surface over $g(Q) = Q$ with multiplicity ≤ 1 in S and $a = 0$. However the surface E' obtained by performing the standard monoidal transformation with center at B_1 has multiplicity $1 + d$ in S and $a = 0$.

The same is true for the similarly defined surface E'' over B_2. All other surfaces over $Q \in X$ with $a = 0$ have multiplicity $> 1 + d$ in S. We observe that the monoidal transformation with center at P_0 (resp. at Q) gives a surface with multiplicity $1+d$, but with $a = 1$ (resp. $a = 1/2$).

Now consider the extremal blowing up $g'\colon Y' \to X$ of the divisor E'. This divisor intersects $g'^{-1}S$ along a single curve $B_1 \subset Y'$, the transform of the corresponding curve in Y. This can be proved just as the corresponding assertion in the case (8.8.4) with $m = 1$. Thus the intersection $B_1 = g'^{-1}S \cap E'$ is irreducible. Moreover, the point $Q = B_1 \cap g'^{-1}C$ is a point of type A_1 on $g'^{-1}S$ resolved by the curve B_2 with selfintersection $-n \leq -4$. By construction the crossing is normal along B_1. Then as before we check that $K_{Y'} + g'^{-1}S + E'$ is log terminal, and verify the other properties of good blowing up. If the locus of log terminal singularities of

$$(K_{Y'} + g'^{-1}S + g'^{-1}B + E')|_{E'}$$

is reduced to B_1, then we are done.

Thus it remains to deal with the case when the locus of log canonical singularities of $(K_{Y'} + g'^{-1}S + g'^{-1}B + E')|_{E'}$ contains another curve C'. Furthermore, arguing as before, we can assume that there exists a point $Q' \in C'$ outside B_1 that is not purely log terminal. We claim that all $d_i > 1$ for components over Q' with $a_i = 0$. In fact, if Y' is nonsingular along C', then over $Q = C' \cap g'^{-1}C$ there is a surface E with $a = 0$ for $K + S + B$ and multiplicity ≤ 1 in $g'^{-1}S + E'$. To see this we note that $K_{Y'} + g'^{-1}S + E$ has index $n \geq 4$ at $Q \in Y'$, and one can perturb $g'^{-1}B$ preserving Q as the only point that is not log terminal in a neighborhood of Q. The multiplicity of E in S equals its multiplicity in $g'^{-1}S + (1+d)E'$, which equals $a + b(1+d) < 1 + d$, where $0 < a < 1$ is the multiplicity of E in $g'^{-1}S$, and $0 < b < 1$ is that of E in E', so that $0 < a + b \leq 1$ is the multiplicity of E in $g'^{-1}S + E'$. Thus the surface in question is bimeromorphic to E, and its multiplicity in S is equal to d. Blowing up this surface, we obtain a curve B_2 in the intersection of this blowup with the blowup of $g'^{-1}S$, and over this curve there is a surface $\neq E'$ also lying over $Q \in X$ with $a = 0$ and multiplicity $1 + d$, hence bimeromorphic to E''. Therefore all the multiplicities for $(1+d)E'$ over Q' are greater than $1 + d$, from which it follows that $d_i > 1$ for all i such that $a_i = 0$ over Q'.

Now assume that Y' is singular along C'. By the same arguments this is a singularity of type A_1, and it is resolved by E. Furthermore, E'' lies over $Q \in Y'$, or more precisely over a curve in the preimage of Q for an extremal resolution of E. This completes the treatment of the cases when the intersection $g^{-1}S \cap E$ is reducible. Otherwise g satisfies (iii), which implies (ii) since $g^{-1}B$ is ample on E. Thus g is a good blowing up, modulo reduction to those cases in Proposition 8.8 for which $d_i > 1$ for $a_i = 0$ over Q for each exceptional divisor E_i over Q.

The case when $g(E) = C$, (8.8.1) holds, and the multiplicity d of the divisor E in S does not exceed 1 is also of the above type. In this case we carry out the reduction to flips of type (8.5.3) with $d_i > 1$ for $a_i = 0$ over a neighborhood of Q. By (8.8.1) we have $g(B_1) = Q \in X$. Furthermore, from Theorem 6.9 and the fact that $g^{-1}B$ is ample with respect to g we deduce that the intersection $g^{-1}S \cap E$ consists of two irreducible curves B_1 and B_2 over a neighborhood of $Q \in X$. Moreover, we have $g(B_2) = C$. Then we check that $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$. This implies that E is normal. Note that the curves $D \subset Y$ over Q lie on E and intersect $g^{-1}B$, and thus do not pass through P_0 except for $D = B_1$. The contraction of a curve $D \neq B_1$ does not violate (8.8.1) and does not change the singularity of P_0 on E. Hence P_0 either is nonsingular on E or is an ordinary double point of E.

In the first case Y is also nonsingular at P_0. We check that the same holds in the second case. To this end we observe that for our choice of $H = g^*B$, first come flips in the proper transforms of curves of the flip of f. Flips in these do not touch B_2, and in particular P_0. After these comes the flip in B_1 described in Proposition 8.3. By the proof of Proposition 8.3, P_0 either is nonsingular on $g^{-1}S$ or has index ≥ 3; in the last case the index of $K_Y + g^{-1}S$ is also ≥ 3. But the degree of a cover of E ramified only in P_0 does not exceed 2. This yields the required nonsingularity. Thus E is the only surface over a neighborhood of $g(Q) = Q$ with multiplicity $d \leq 1$ and $a = 0$. Moreover, the multiplicity of other divisors with $a = 0$ is not less than $1 + d$, and this value over $Q \in X$ is attained only for E', the monoidal transform of B_1. As above, we consider the blowing up $g'\colon Y' \to X$ of the divisor E' and verify that it intersects $g'^{-1}S$ along a single curve B_1, a modification of the curve with the same name. By construction this crossing is normal along B_1. Next we verify the log terminal property of $K_{Y'} + g'^{-1}S + E'$ and the other properties of a good blowing up. If the locus of log terminal singularities of

$$(K_{Y'} + g'^{-1}S + g'^{-1}B + E')|_{E'}$$

coincides with B_1, then we are done. As before, it remains to consider the case when the locus of log canonical singularities of $(K_{Y'} + g'^{-1}S + g'^{-1}B + E')|_{E'}$ contains another curve C'. Furthermore, as before, we can assume that there exists a point $Q' \in C'$ outside B_1 that is not purely log terminal. But then all $d_i > 0$ for $a_i = 0$ over a neighborhood of Q'. For otherwise there would exist a surface over Q' or over the general point of C' with $d_i \leq 1$ and $a_i = 0$. Hence it lies over Q and its multiplicity in S does not exceed $1 + d$ if $a = 0$, which is impossible.

This completes the reduction of Proposition 8.8 to flips of type (8.5.3) with $d_i > 1$ for $a_i = 0$ over a neighborhood of Q in the cases when the required good blowing up does not exist. It remains to establish the existence of the flip of f in these exceptional cases. For them there exists a minimal multiplicity d in S for $a = 0$ over a neighborhood of Q. By assumption, $d > 1$. On the other hand, $d \leq 2$, since $\operatorname{Supp} B$ touches S along C, and there is a surface over the general point of C with $d = 2$ and $a = 0$. There exist only a finite number of surfaces E over a neighborhood of Q with $a = 0$ and a given multiplicity d. They are all blown up by a log terminal blowing up of $K + S + B$. Hence, as above, we can choose an extremal blowing up g of such a surface E satisfying (8.8.1) or (8.8.2) for $P \in B_1 \subseteq g^{-1}S \cap E$. To this end we observe that $g^{-1}S$ and E cross normally along B_1 and the other components of $g^{-1}S \cap E$, for otherwise by (3.18.6) over a general point of B_1 there would exist a surface E' with $a = 0$ having multiplicity $a + b \leq 1$ in $g^{-1}S + E$, where $0 < a$, b are the multiplicities of E' in $g^{-1}S$ and E respectively. Hence the multiplicity of E' in $g^{-1}S + dE$, equal to the multiplicity in S, is $a + bd < (a + b)d \leq d$, which contradicts the choice of d.

Next we show that the case (8.8.2) is only possible if P is an isolated singularity of Y from (8.8.4) with $m = 1$. But this case again reduces to flips of type (8.5.3) with $d_i > 1$ for $a_i = 0$ over a neighborhood of Q. Indeed, the multiplicity $d' := \frac{2}{3}(1 + d)$ of the new surface E', like d itself, does not exceed 2, from which it follows that for $n = 1$ we have $dd' = d - 1 \leq 2 - 1 = 1$, which again contradicts the choice of d. However now Q is contained in the curve of double singularities C_3, which reduces the existence of the required flips to the case when g satisfies (8.8.1).

Assume first that $g(E) = Q$. As before, the intersection $g^{-1}S \cap E$ consists of at most two irreducible curves B_i. We show that if $g^{-1}S \cap E = B_1 \cup B_2$, then the flip of f exists. To this end, as in the similar case with $d \le 1$, we reduce to the following setup. The divisor $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$, Y is nonsingular at P_0 and $Q = B_2 \cap g^{-1}C$, the surfaces E and $g^{-1}S$ are normal and nonsingular on the curve B_2 having selfintersection 2 on E and $-3 = -n$ on $g^{-1}S$. The last assertion follows from $d = 3/(n-1)$ and $n = 3$, since $1 < d \le 2$. From this it also follows that $d = 3/2$. We also note that the selfintersection index of the curve B_1 on the minimal resolution of E does not exceed 0, since B_2 crosses B_1 normally at a single point P_0. On the other hand, by the ampleness of $g^{-1}S$ on E we have

$$0 < (g^{-1}S \cdot B_1) = (B_1 + B \cdot B_1)_E = (B_1 \cdot B_1)_E + 1.$$

Hence $(B_1 \cdot B_1)_E = 0$ and E is nonsingular on B_1. Otherwise E would have a unique ordinary double point, say P_1, and the selfintersection of B_1 on the minimal resolution of E would be equal to 0 or -1. Hence, depending on the case, $(g^{-1}S \cdot B_1) = 1$, $3/2$, or $1/2$. Furthermore,

$$0 = \left(g^{-1}S + \tfrac{3}{2}E \cdot B_1\right) = (g^{-1}S \cdot B_1) + \tfrac{3}{2}(E \cdot B_1),$$

and thus $(E \cdot B_1) = -2/3$, -1, and $-1/3$ respectively. The fractional cases are impossible, since P_0 is a nonsingular point of $g^{-1}S$, and $g^{-1}S$ has at most one ordinary double point on B_1 (viz. P_1). Therefore B_1 has an ordinary double point P_1 on E, B_1 has selfintersection 0 on the minimal resolution of E, and $g^{-1}S$ is nonsingular on $B_1 \cup B_2 = g^{-1}S \cap E$. Moreover, B_1 is a (-2)-curve and B_2 is a (-3)-curve.

Through the point P_1 on $g^{-1}S$ there passes the curve of double points B_{-1}. From this it follows that the curve $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is irreducible, has a unique singularity Q' (not over $Q \in X$) of type A_1, resolved by a (-3)-curve, and is an exceptional curve of the first kind on the minimal resolution of $g^{-1}S$. But in this case $K + S$ has a purely log terminal complement of index 2, and hence the flip of f exists. To see this we observe that, by Proposition 5.13 and Corollary 5.19, $K + S + 2B$ is strictly log terminal at Q' and has index 3; one half of its 1-complement at Q' gives the required index 2 complement. Furthermore, $K + S$ has index 2 at $Q \in X$, since E is a quadratic cone with vertex P_1 and $g^*(K+S) = K_Y + g^{-1}S + (1/2)E$.

Next we consider the case when $g(E) = Q$ and $B_1 = g^{-1}S \cap E$ is irreducible. Then, arguing as above, we check that g is good. We can also assume that the locus of log canonical singularities of

$$(K_Y + g^{-1}S + g^{-1}B + E)|_E$$

contains another curve C'. But then $Q = B_1 \cap g^{-1}C$ is at worst an isolated singularity of Y. In view of the choice of d and (3.18.4), neither $g^{-1}S$ nor E contain curves of singularities of Y through Q. The fact that there are no other curves of singularities through Q follows as before from the log canonical property of $K_Y + g^{-1}S + g^{-1}B + E$. Moreover, if we perturb $g^{-1}B$ in a neighborhood of Q (while fixing $Q \in g^{-1}B$), then Q will satisfy (8.8.2). Otherwise, arguing as in the proof of the fact that $g^{-1}S$ crosses E normally along B_1, we can find a surface over Q with $a = 0$ and multiplicity $< d$ in S, which contradicts the choice of d. Thus (8.8.2) holds, so that Q is a singularity of type (8.8.4) with $m = 1$ or $m = 0$. If $m = 1$, then, choosing $g^{-1}B$ as above, we get a log canonical singularity of $(g \circ h)^*(K + S + B)$ on a curve of double points $C_3 \subset E'$. Hence the surface resolving C_3 has $a = 0$, and its multiplicity in $g^{-1}S + E$ is equal to $2/3 < 1$ and $< d$ for S. Therefore $m = 0$ and Q is nonsingular. In this case, blowing up the points P_i if necessary, we can construct an exceptional 2-complement.

We now turn to the final case $g(E) = C$. Then $d = 2$. As before, g satisfies (8.8.1). Furthermore, as in the similar case above with $d \le 1$, we check that the intersection $g^{-1}S \cap E$ consists of two irreducible curves B_1 and B_2 over a neighborhood of $Q \in X$. Moreover, $g(B_1) = Q$ and $g(B_2) = C$. Then we check that $K_Y + g^{-1}S + E$ is log terminal outside $P_0 = B_1 \cap B_2$ and E is normal. Again P_0 is nonsingular on X and on $g^{-1}S$, and is either nonsingular or an ordinary double point of E. This time the latter case is impossible since $(g^{-1}B \cdot D) = 1/2$, where D is a general fiber of the surface E over C, and $(g^{-1}B \cdot B_1) \ge 1/2$. Hence $(g^{-1}B \cdot B_1) = 1/2$ and the curve B_1 is numerically equivalent to D. In particular

$$-\frac{1}{2} = (E \cdot D) = (E \cdot B_1) = (B_1 \cdot B_1)_{g^{-1}S} + (B_2 \cdot B_1)_{g^{-1}S} = (B_1 \cdot B_1)_{g^{-1}S} + 1,$$

and thus $(B_1 \cdot B_1)_{g^{-1}S} = -3/2$. Therefore $g^{-1}S$ has a unique ordinary double point on B_1, say P_1. Then B_1 is a (-2)-curve on the minimal resolution of $g^{-1}S$. By the same arguments $(B_1 \cdot B_1)_E = 0$, from which it follows that E is nonsingular on B_1 and B_1 is a complete fiber of E over C. Moreover, P_1 is contained in the curve of double points of Y. Hence $K + S$ has index 1 at $Q \in X$, since $g^*(K+S) = K_Y + g^{-1}S$ has index 1 on B_1. On the other hand, the curve $B_0 = g^{-1}S \cap \operatorname{Supp} g^{-1}B$ is irreducible, does not meet the singularities of $g^{-1}S$, and is an exceptional curve of the first kind on $g^{-1}S$ crossing normally the curve B_{-1} of double points. Thus the index of $K + S$ in a neighborhood of the flipping curve $g(B_0)$ is equal to 2, and $(K + S \cdot g(B_0)) = -1/2$. Hence one half of the general hyperplane section of B_1 gives a purely log terminal complement of index 2 and the flip exists by Proposition 2.9. □

Proof of Theorems 1.9–10 *and Corollary* 1.11. According to Reductions 6.4–5, Reduction 7.6, and Propositions 6.7–8, it suffices to establish the existence of nonexceptional flips of index 2. By Reduction 8.2, Proposition 8.3, and Reduction 8.4, we can restrict ourselves to flips of type (8.5.1–3). In what follows we denote by $h: Y \to X$ the good blowing up of Propositions 8.6 and 8.8, E the unique exceptional divisor of h, and $B_1 = g^{-1}S \cap E = \mathbb{P}^1$ the irreducible curve of property (iii) in 8.5. Since h is extremal, we have $\rho(Y/Z) = 2$ and $\overline{\mathrm{NE}}(Y/X)$ has two extremal rays R_1 and R_2. From now on we proceed as in Reductions 7.2 and 8.2. In particular, we assume that R_1 corresponds to the contraction g. The flips of R_2 are considered separately depending on their type.

We start with type (8.5.1). Suppose first that $B_0 \subset g^{-1}S$, the preimage of the flipping curve, does not pass through $P = B_1 \cap g^{-1}C$. Then by construction B_0 is irreducible and is not contained in $g^{-1}B$. Hence R_2 is nonnegative with respect to $g^{-1}B$, positive with respect to E, and negative with respect to $g^{-1}S$. Therefore the support of R_2 coincides with B_0, since B_1 is contained in R_1.

Note that the flip in B_0 exists by Corollary 5.20. Since $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal in a neighborhood of B_0, the flip transforms the curve B_0 into a curve B_0^+ on the modified surface E preserving the index 2 or 1 and the log terminal property of the divisor on E in a neighborhood of B_0^+. It is not hard to check that the transformed curve B_0^+ intersects the modified B_1 and is irreducible. The subsequent ray R_2 can be negative with respect to $g^{-1}S$ only when it is generated by the modified $B_1 = g^{-1}S \cap E$ and thus is negative with respect to E. As in Reduction 7.2, in this case the flip of f exists. Thus, except for the case of a divisorial contraction, it remains to consider the case when the next flipping curve C_1 lies on E and does not intersect B_1. Furthermore, in this case $B_1^2 > 0$. Since

C_1 is numerically trivial with respect to $g^{-1}S$, it must be negative with respect to E and positive with respect to $g^{-1}B$.

If the restriction $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ is log terminal in a neighborhood of the support of R_2, then the flip of R_2 exists by Corollary 7.3, since it is an exceptional flip of index 2 for each connected component of the flipping curve. (In the analytic case, passing to connected components while preserving the assumptions that the contraction is extremal and the space is $\mathbb{Q}$-factorial can be carried out either by blowing up the base outside a fixed fiber or by localizing as in the proof of Reductions 6.4–5.) Otherwise by Theorem 6.9 E contains a curve B_2 intersecting the support of R_2 in a unique point Q at which $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ is not log terminal, and the reduced part of the boundary of the most recent restriction has the form $B_1 + B_2$. The curves B_1 and B_2 intersect in a unique point P, and since B_1 is ample on the original E, it follows that B_2 is irreducible. But after flipping, the curve B_2 is numerically effective and numerically trivial only on B_0^+. Since B_1 is ample on the original E, after contracting any of the irreducible components of C_1 we get that B_2 is ample. By Lemma 8.9 the flipped curve is contracted as a whole, hence it is irreducible. Consequently a flip in C_1 again has type (8.5.1), and exists by Proposition 8.6 because the number of good blowings up has decreased.

We now proceed to the case when B_0 passes through P. Then the extremal ray R_2 generated by B_0 is positive with respect to E and $g^{-1}B$, but negative with respect to $g^{-1}S$. The flip in B_0 exists by Corollary 5.20, the flipped curve B_0^+ lies in the intersection of the modified $E \cap \operatorname{Supp} g^{-1}B$, and $B_1 = E \cap g^{-1}S$. Again it suffices to consider the case when the flipping curve C_1 is on E and does not intersect $g^{-1}S$. If the locus of log canonical singularities of the modified restriction $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ is disjoint from C_1, then the flip exists and is of type IV by Proposition 5.13. Otherwise, by Theorem 6.9, B_0^+ is irreducible and is contained in the reduced part of the boundary of the most recent restriction.

On the other hand, before the flip the divisor $g^{-1}B|_E$ was ample, and its support intersected B_1 only in P. Hence the support of the modified $g^{-1}B|_E$ is contained in C_1 and is a contractible curve. After its contraction, by Lemma 8.10 the curve B_0^+ becomes numerically ample and by Lemma 8.9 the image of C_1 must be trivial, that is, the support of the modified $g^{-1}B|_E$ must coincide with C_1. If the divisor $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal along all components of C_1, we arrive at a flip of type (8.5.3); otherwise C_1 is irreducible and defines a flip of type (8.5.2). This completes the reduction in the case (8.5.1).

Consider now the case (8.5.2) proper, when the good blowing up g has an exceptional divisor E over a point. By construction, the proper transform of B_0, the curve contracted by f, generates R_2. Thus R_2 is positive with respect to E and negative with respect to $g^{-1}S$. Hence the flip of B_0 exists by Corollary 5.20.

After the flip the curve $g^{-1}S \cap E$ may become reducible. However this is only possible when $g^{-1}B$ is numerically negative on B_0. By our choice of $H = g^*B$, $K_Y + g^{-1}S$ remains log terminal. Moreover, the flipped curves on $g^{-1}S$ lie in the intersection with E. As in the proof of Proposition 8.8, the intersection $g^{-1}S \cap E$ contains at most two curves, viz. B_1 and a flipped curve B_2. In particular, B_2 is exceptional on E^ν. Now B_1 becomes the support of the subsequent extremal ray, which is numerically trivial with respect to $g^{-1}B$, positive with respect to E, and negative with respect to $g^{-1}S$. From this it follows that $g^{-1}B$ is positive on all the remaining curves of E. As in the proof of Proposition 8.8, using this one can verify the log terminality of $K_Y + g^{-1}S + E$ in a neighborhood of E, except at the point $P_0 = B_1 \cap B_2$. Hence E is normal. The flip in B_1 is described in Proposition 8.3.

Arguments from the proof of Proposition 8.8 in the case (8.5.2)* allow us to show either that the flip of f exists, or that B_2 is numerically effective on the minimal resolution of E. But the last case is impossible, since B_2 is exceptional on E. Thus we can assume that the intersection $g^{-1}S \cap E = B_1$ is again irreducible. The log terminal property of $K_Y + g^{-1}S + E$ is preserved if $g^{-1}B$ is numerically effective on B_0; otherwise it can be deduced from the ampleness of $g^{-1}B$ on the modified E.

Thus again the new flipping curve C_1 is contained in E and does not intersect $g^{-1}S$. If the singularities of the restriction $(K_Y+g^{-1}S+g^{-1}B+E)|_E$ are log terminal on C_1, then, as before, the flip exists by Corollary 7.3. Otherwise by Theorem 6.9 there exists an irreducible curve B_2 contained together with B_1 in the boundary of $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ after modification and intersecting B_1 in P. We claim that B_2 is contained in the support of the new ray R_1, that is, B_2 is obtained after flipping B_0. Indeed, otherwise all the components of the flipped curve B_0^+ would intersect B_1 at a single point P. After contracting B_0^+ we return to the situation before flipping, when the curve $B_2 = \operatorname{Supp} g^{-1}S \cap E$ is ample on E. Therefore by Lemma 8.9 there is no C_1. Thus B_2 is contained in B_0^+, the remainder of B_0^+ is contracted to a point P, and its components intersect B_1 and B_2 only at P.

Since $K_Y + E$ is log terminal, the surface E is normal. By Theorem 6.9 and Lemma 8.9, after blowing down the support of $g^{-1}B|_E$ outside B_2 we see that C_1 coincides with the given contracted curve. By Lemma 8.10, B_2 becomes ample after blowing down C_1 and the components of B_0^+ other than B_2. (The components of $g^{-1}B|_E$ other than B_2 are contained in C_1, since they do not intersect B_1 and are numerically trivial on $g^{-1}S$.) If the support of $g^{-1}B|_E$ outside B_2 contains a curve along which $K_Y + g^{-1}S + g^{-1}B + E$ has singularities that are not log terminal, then it coincides with it, and the contraction of the curve in question has type (8.5.2). Furthermore, in the case (8.5.2) proper, Proposition 8.6 shows that the number of good blowings up is decreased and the flip exists by induction. In the opposite case we get a reduction to type (8.5.2)*. Type (8.5.3) arises if $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal along C_1.

In the case (8.5.2)*, the ray R_2 that is negative with respect to $g^{-1}S$ at the first step leads to a flip in B_1 and separates the surfaces E and $g^{-1}S$. After this the contraction of E to a point gives a flip of f. Thus the case that is essential for us is when the flipping curve C_1 lies in E and does not intersect $g^{-1}S$. As above, we need only consider the case when C_1 passes through a point at which the restriction $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ is not log terminal. Then the fiber B_2 of the ruled surface E over $P = B_1 \cap g^{-1}C$ is irreducible and is contained in the boundary of $(K_Y+g^{-1}S+g^{-1}B+E)|_E$. Since $g^{-1}B$ is positive on R_1 and R_2, it is positive on E, and after contracting all components of $\operatorname{Supp} g^{-1}B|_E$ except B_2 the curve B_2 becomes ample. Thus again by Lemma 8.9, C_1 coincides with the given contracted curve. If C_1 contains a curve from the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B + E$, then it coincides with it, and the contraction of the given curve is of type (8.5.2)*. Here by our choice of good blowing up in Proposition 8.8, δ decreases. Indeed, the exceptional divisors of E_i over C_1 have log discrepancy 0 for $K_Y + g^{-1}S + g^{-1}B + E$ precisely when $a_i = 0$, and the multiplicity of E_i in E is equal to its multiplicity in $g^{-1}S + E$, and is not less than its multiplicity in $g^{-1}S + dE = g^*S$. This yields strict monotonicity for δ. In the remaining case we get a reduction to type (8.5.3).

In case (8.5.3), we first perform flips in curves of the intersection $E \cap \operatorname{Supp} g^{-1}B$. These curves intersect B_1 in points P_1 and P_2 where $K_Y + g^{-1}S + g^{-1}B + E$ is log terminal. Hence after such flips the intersection $B_1 = g^{-1}S \cap E$ remains irreducible. However now $g^{-1}B$ intersects B_1 only in a point $P = B_1 \cap g^{-1}C$ which is possibly not log terminal, and the curve B_1 becomes the only curve on $g^{-1}S$ over Z in a neighborhood of the flipping fiber. Again it remains to consider the case when the subsequent flipping curve C_1 lies on E and does not intersect $g^{-1}S$. As before, we need only consider the case when C_1 passes through a point Q' at which the restriction $(K_Y + g^{-1}S + g^{-1}B + E)|_E$ is not log terminal. Then there exists an irreducible curve $B_2 \neq B_1$ such that $B_1 + B_2$ is the reduced part of the boundary of the restriction and $Q' \in B_2$. Since $g^{-1}B$ is positive on R_1 and R_2, it is ample on E. By Lemma 8.10, after blowing down the components of $\operatorname{Supp} g^{-1}B|_E$ other than B_2 we transform B_2 into an ample curve. Thus, again by Lemma 8.9, C_1 coincides with the given contracted curve. But by construction C_1 is not contained in the locus of log canonical singularities of $K_Y + g^{-1}S + g^{-1}B + E$. Furthermore, by our choice of good blowing up in Proposition 8.8, δ decreases. More precisely, δ' for Q' is less than δ. □

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