INTERSECTION THEORY OF DIVISORS ON AN ARITHMETIC SURFACE

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INTERSECTION THEORY OF DIVISORS ON AN ARITHMETIC SURFACE

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Abstract. In this article it is explained how to construct for a nonsingular model of a curve defined over a number field a theory analogous to the theory of divisors, and the intersection numbers of divisors, on a compact algebraic surface.

Introduction

In this article we define the notion of a divisor class on the nonsingular model of an algebraic curve defined over an algebraic number field $K$. This notion replaces the usual one of the class of divisors on an ordinary algebraic variety with respect to linear equivalence.

By a method analogous to that used by Néron we can construct an intersection theory for such classes which is fairly geometric in spirit, and which is reminiscent of intersection theory on an algebraic surface.

§0. The definition

Let $K$ be a number field, $\Lambda \subset K$ the ring of integers. The symbol $\infty$ will denote any embedding of $K$ in the field of complex numbers; there are thus $n$ infinities, where $n = [K: \mathbb{Q}]$, and $\sum_{\infty}$ will denote summation over all of these infinities. Consider a smooth complete curve $X$ over $K$.

$X_{\infty}$ will denote the Riemann surface of the curve $X_{\infty} \otimes \mathbb{C}$. Let $V$ be any smooth and complete model of $X$ over $\Lambda$.

By a finite divisor on $V$ we will mean a divisor in the usual sense

$$D_{\text{lin}} = \sum k_i C_i,$$

where $k_i$ are integers, and $C_i \subset V$ are irreducible closed subsets of codimension 1.

The curves $X_{\infty}$ are the fibers of the morphism $f : V \rightarrow \text{Spec} \Lambda$ over the places of $K$ at infinity. We will call these the components at infinity; by definition these components can occur in a divisor with real coefficients.

with $k_i \in \mathbb{Z}$, $\lambda_\infty \in \mathbb{R}$. Divisors of this type form a group, which we denote by $\text{Div}^{-}(V)$. We can also consider the group $\text{Div}^{-}(K)$—this will be the group of divisors of the form

$$d = \sum p \cdot \mathfrak{m} + \sum \lambda_\infty \cdot \infty,$$

where the $\mathfrak{p}$ are the prime ideals of the field $K$, $\mathfrak{m}_\mathfrak{p} \in \mathbb{Z}$ and $\lambda_\infty \in \mathbb{R}$. In this group the principal divisors are, by definition, divisors of the form

$$\sum p(a) \cdot \mathfrak{m} + \sum (-\log |a|_\infty),$$

with $a \in K$, and $|a_\infty| = |\infty(a)|$.

The degree of the divisor $d$ is the real number

$$\deg d = \sum m_p \log N_p + \sum \lambda_\infty;$$

the degree of the principal divisor is

$$\sum p(-\log |a|_p) + \sum (-\log |a|_\infty) = 0$$

by the product formula.

We now define the principal divisor of an element $f \in K(X)$. For every $\infty$, $f$ is a meromorphic function on the Riemann surface $X_\infty$, and we want to define $\nu_X(f)$ as the mean values of $(-\log |f|)$ on $X_\infty$. For this, consider and fix on each $X_\infty$ a Hermitian metric $ds_\infty^2$; we will denote the corresponding volume element by $d\mu_\infty$. We will suppose that $d\mu_\infty$ is normalized in such a way that $\int_{X_\infty} d\mu_\infty = 1$. Define $\nu_\infty(f)$ by the formula

$$\nu_\infty(f) = \int_{X_\infty} -\log |f| d\mu_\infty.$$

By definition, a principal divisor is

$$(f) = (f)_{\text{fin}} + \sum \nu_\infty(f) \cdot X_\infty,$$

where $(f)_{\text{fin}}$ is the usual divisor of a rational function on the two-dimensional scheme $V$. Let $P^{-}(V)$ be the group of principal divisors. We define

$$\text{Cl}(K) = \text{Div}^{-}(V)/P^{-}(V).$$

We can fix up the same sort of group for Spec $K$:

$$\text{Cl}(K) = \text{Div}^{-}(K)/P^{-}(K).$$

We have homomorphisms $\text{Cl}^{-}V \to \text{Cl} V$ and $\text{Cl}^{-}K \to \text{Cl} K$ of the groups just defined into the usual divisor class groups. Both of these homomorphisms are surjective, and
have the same kernel, which we denote by $G$. $G$ is the quotient group of real $n$-dimensional space by the image of the group of units of $K$ under the logarithm map. Of course, it might be natural to identify complex conjugate embeddings of $K$ and consider conjugate metrics $d\mu_\infty$, but it seems to us that this is necessary only in more detailed study of the situation, and for the time being there is no particular need for it. This means that the group $G$ turns out a bit bigger than one might have expected. Finally, note that we have the inverse image homomorphism $f^*: \text{Cl}^{-1}K \to \text{Cl}^{-1}(V)$. Thus the group $G$ comes from below.

§1. Intersection numbers

Firstly we recall the definition of the usual intersection numbers of two finite divisors $D_1$ and $D_2$. Let $f_1$ and $f_2$ be local equations for these, and let $P \in V$ be a 0-dimensional point of $V$; let $f(P) = \gamma \in \text{Spec } \Lambda$. Then the local intersection number $(D_1 \cdot D_2)_P$ is defined by

$$(D_1 \cdot D_2)_P = \log N(P),$$

where $N(P)$ is the number of elements of the field $k(P) = O_{P,V}/m_{P,V}$, and $l(M)$ denotes the length of the $O_{P,V}$-module $M$. If one of the divisors, say $D_2$, is irreducible and horizontal (that is, $f(D_2) = \text{Spec } \Lambda$), then it can be written in the form $D_2 = \epsilon(\text{Spec } \Gamma)$, where $\Gamma$ is the ring of integers of a field $L$ finite over $K$, and $\epsilon: \text{Spec } \Gamma \to V$ is a morphism over $\text{Spec } \Lambda$.

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are the prime ideals of $\Gamma$ lying over the point $P$, then

$$(D_1 \cdot D_2)_P = \prod_{i=1}^{k} -\log \| \epsilon^* f_i \|_{\mathfrak{p}_i}.$$ 

Set

$$(D_1 \cdot D_2)_P = \sum_{l(P) = P} (D_1 \cdot D_2)_P,$$

with $P \in \text{Spec } \Lambda$. We now proceed to the definition of $(D_1 \cdot D_2)_\infty$ for two finite divisors $D_1$ and $D_2$. After this the global index $(D_1 \cdot D_2)$ will be defined to be equal to

$$\sum_{P \in \text{Spec } \Lambda} (D_1 \cdot D_2)_P + \sum_{\infty} (D_1 \cdot D_2)_\infty.$$ 

Suppose that both of the divisors are irreducible. If one of them is a component of a fiber, set $(D_1 \cdot D_2)_\infty = 0$. Suppose now that $D_1$ and $D_2$ are horizontal and are obtained by taking the closure in $V$ of two points $P_1$ and $P_2$ on the curve $X$ with fields of definition $L_1$ and $L_2$. Denote by $\{\infty_1 \}$ and $\{\infty_2 \}$ the embeddings of the fields $L_1$ and $L_2$ which extend the fixed embedding of $K$ into the complex field. To these there correspond points $P_{\infty_1}$ and $P_{\infty_2}$ on the Riemann surface $X_\infty$.

We will define $(P_{\infty_1} \cdot P_{\infty_2})$ for every $\alpha$ and $\beta$, and then set
The definition of \((P \cdot Q)_X\) will make sense for any Riemann surface with a given Hermitian metric, and for any pair of distinct points on it. Hence we now let \(X\) be any Riemann surface, and \(P\) and \(Q\) two points of \(X\). Let \(d\mu\) be the volume element defined by the metric on \(X\).

We define the real index \((P \cdot Q)_X\) by analogy with the finite intersection number. This can be obtained by restricting the local equation of one point on the other, and then taking the absolute value (valuation).

Let us now associate to every point \(P \in X\) a smooth real-valued function \(\phi_P(z)\) on \(X\), which is to serve as the global equation of \(P\). After this we will define the index \((P \cdot Q)_X\) by the formula

\[
(P \cdot Q) = -\log \phi_P(Q).
\]

We will insist that \(\phi\) shall be a nonnegative function with a unique zero at \(P\), and there is a first order zero; that is, near \(P\) we have

\[
\phi_P(z) = |t_P(z)| \cdot u(z),
\]

with \(t_P(z)\) a local parameter near \(P\), and \(u(z)\) a smooth function with \(u(P) \neq 0\).

A way of picking out a single preferred function \(\phi\) from this class of functions was proposed by Paršin. This consists of imposing on the function \(\phi_P\) the Poisson differential equation

\[
\frac{1}{2\pi} \Delta \log \phi_P dxdy = -d\mu.
\]

Any two solutions \(\phi_P\) and \(\tilde{\phi}_P\) of this equation are proportional, since the log of the ratio between them is a smooth harmonic function everywhere on \(X\), which is therefore constant. Hence the solution, normalized by the condition

\[
\int_X \log \phi_P d\mu = 0,
\]

is uniquely determined; we will take this as the global equation of the point \(P\). The existence of the solution will be proved in §2, but first we prove

**Proposition 1.1.** The intersection number \((P \cdot Q)_X\) is symmetrical.

For the proof note that if \(U_P\) and \(U_Q\) are small discs centered on \(P\) and \(Q\) respectively, it follows from (*) and (**) that the integral

\[
\int_{X \setminus U_P \cup U_Q} (\log \phi_P \Delta \log \phi_Q - \log \phi_Q \Delta \log \phi_P) dxdy
\]

tends to zero as the radii of the discs \(U_P\) and \(U_Q\) tend to zero. By the Green equation this integral is
The function \( f \in K(X) \) defines a function \( f^\infty \) on \( X_\infty \); let
\[
(f^\infty) = \sum Q(f) Q.
\]
Let us write out a decomposition of the function \( f^\infty(z) \) over its zeros and poles:
\[
|f^\infty(z)| = \left( \prod_{Q \in X} \varphi_Q(z)^{v_Q(f)} \right) \cdot u(z).
\]
Here the function \( u(z) \) never vanishes. It follows from the properties of \( \varphi_Q \) that \( u \) satisfies the equation \( \Delta \log u = 0 \), so that \( u \equiv c \in \mathbb{R} \). Taking the log of (1) and integrating, we get
\[
c = \int_{X_\infty} \log |f^\infty| d\mu_\infty = -v_\infty(f).
\]
Now let us write out

\[(f) \cdot D = (f)_{\text{lin}} \cdot D + \sum_{\infty} v_\infty(f) \cdot (X_\infty \cdot D),\]

\[(f)_{\text{lin}} \cdot D = \sum_{p \in \text{Spec} A} (f)_{\text{lin}} \cdot D_p + \sum_{\infty} (f)_{\text{lin}} \cdot D_\infty,\]

where

\[
\sum_{p \in \text{Spec} A} (f)_{\text{lin}} \cdot D = \sum_{\mathcal{O} \in \text{Spec} A} -\log \| e^f \|_{\mathcal{O}},
\]

\[
(f)_{\text{lin}} \cdot D_\infty = -\sum_{Q \in X \setminus \infty} v_Q(f) \cdot \log \varphi_Q(P_\infty)
\]

\[
= -\sum_{P_\infty \in X_\infty} \log |f(P_\infty)| - m\nu_\infty(f).
\]

Collecting everything together, we get

\[(f) \cdot D = \sum_{\mathcal{O} \in \text{Spec} A} -\log \| e^f \|_{\mathcal{O}} + \sum_{\infty} -\log \| e^f \|_{\infty},\]

and this is zero by the product formula.

\[\S 2. \text{Hermitian line bundles}\]

We will now give a convenient reformulation of the definition of divisor class. First let \(X\) be a Riemann surface with the volume element \(d\mu\), and let \(L\) be a one-dimensional Hermitian vector bundle on \(X\). The metric on \(L\) will be denoted by \(\| \|\). If \(s\) is a rational section of \(L\), then the form of type \((1, 1)\) given by

\[
\frac{1}{\pi i} d'd^* \log \| s \| = -\frac{1}{2\pi} \Delta \log \| s \| dx dy
\]

is independent of \(s\). It is called the curvature form of the connection, and satisfies the condition

\[
\int_X -\frac{1}{2\pi} \Delta \log \| \| = \deg L,
\]

which can be obtained by applying Green's formula to the functions \(\| s \| \) and 1 on \(X\).

We will only consider those metrics on \(L\) which satisfy

\[
-\frac{1}{2\pi} \Delta \log \| \| = (\deg L) d\mu. \tag{*}
\]

All such metrics are proportional.

Now let us consider an invertible sheaf \(\mathcal{E}\) on the scheme \(V\), and suppose that on every one of the line bundles \(\mathcal{O}_{\infty}\) defined by restriction of \(\mathcal{E}\) to the fiber \(X_\infty\), we are given some Hermitian metric \(\| \|_\infty\), satisfying \((*)\) with \(d\mu = d\mu_\infty\). We call such an object a metrized line bundle on \(V\).
If \( s \) is a rational section of the sheaf \( \mathcal{O} \) on \( V \), then we define the divisor of this section by the equation
\[
s = (s)_{\text{fin}} + \sum_{\infty} v_{\infty}(s) X_{\infty},
\]
where
\[
v_{\infty}(s) = \int_{X_{\infty}} - \log \| s \| d\mu_{\infty},
\]
and \( (s)_{\text{fin}} \) is the finite divisor of the section \( s \) on the scheme \( V \).

If \( s' \) is another rational section, then \( s' = f \cdot s \), with \( f \in K(X) \), and obviously \( (s') = (s) + (f) \). Thus a metrized line bundle defines a divisor class on \( V \). To prove the converse, we use

**Proposition 2.1.** Let \( X \) be a Riemann surface with the volume element \( d\mu \), and let \( L \) be a line bundle over \( X \). Then there is a Hermitian metric \( \| \| \) on \( L \) satisfying (*).

Before giving the proof of this proposition, let us show how to use it to find, for a given class \( \alpha \in \text{Cl}^\wedge V \), a metrized line bundle which determines this class. Consider any representative \( D \) of the class \( \alpha \):
\[
D = D_{\text{fin}} + \sum_{\infty} v_{\infty}(D) X_{\infty}.
\]

Let us take the invertible sheaf \( \mathcal{O} = \mathcal{O}_V(D_{\text{fin}}) \), and define on it an arbitrary metrization, to start with, which satisfies (*). Let \( s \in \mathcal{O} \) be that section of \( \mathcal{O} \) for which \( (s)_{\text{fin}} = D_{\text{fin}} \). Adjusting the metrics by suitable scalar multiples, we achieve the equation \( v_{\infty}(s) = v_{\infty}(D) \), as required. We have obtained

**Proposition 2.2.** The group \( \text{Cl}^\wedge V \) can be interpreted as the group of classes of metrized line bundles on \( V \) under tensor products.

The same interpretation also holds for the group \( \text{Cl}^\wedge K \), where a metrized line bundle over \( \text{Spec} \Lambda \) is now by definition an invertible sheaf (or a projective module of rank 1) \( L \) over \( \text{Spec} \Lambda \), in which every one of the one-dimensional vector spaces \( L \otimes_{\Lambda} \mathbb{C} \) is given a Hermitian metric \( \| \|_\infty \). Just as above, to every metrized line bundle \( L \) over \( \text{Spec} \Lambda \) there corresponds a divisor class \([L] \in \text{Cl}^\wedge K \). We will not write out the corresponding formulas, which are obvious. It follows from this that any metrized line bundle \( L \) has a degree \( \text{deg} L \), which is a real number.

**Proposition 2.3.** Let \( L \) be a finite extension of the field \( K \), and let \( \Gamma \subseteq L \) be the ring of integers of \( L \); let \( \epsilon: \text{Spec} \Gamma \to V \) be a morphism over \( \text{Spec} \Lambda \), and let \( D = \epsilon^*(\text{Spec} \Gamma) \). Then, for a metrized line bundle \( \mathcal{O} \) over \( V \),
\[
(D \cdot \mathcal{O}) = \text{deg} \epsilon^* \mathcal{O},
\]
where \( \epsilon^* \mathcal{O} \) is the metrized line bundle on \( \text{Spec} \Gamma \) obtained by restricting \( \mathcal{O} \).
The proof of this proposition is purely checking, which we omit.

Finally, let us introduce one more proposition. Let $D$ be any finite divisor on $V$, so that $v_\infty(D) = 0$ for all $\infty$. Consider the invertible sheaf $O_V(D)$. Let $O_V \rightarrow O_V(D)$ be the natural embedding, and let $s_0$ be the section of $O_V(D)$ into which the section 1 of $O_V$ is taken under this embedding. It is obvious that $O_V(D)$ can be made into a metrized bundle in a unique way such that $v_\infty(s_0) = 0$; by definition, this will from now on be called $O_V(D)$. Obviously in this definition $(s_0) = D$.

From now on we will be using the language of metrized line bundles as being the most convenient.

Now let us return to the proof of Proposition 2.1.

a) Lemma 2.1. Let $L$ be a line bundle of degree 0 on a Riemann surface $X$.

Then there exists a harmonic metric on $L$.

A metric $\|\|$ on $L$ is said to be harmonic if its curvature form is zero. The existence of such a metric follows from the fact that the transition functions of $L$ can be taken to be constant functions, equal modulo 1, since the line bundle $L$ comes from a one-dimensional representation of the fundamental group of $X$.

b) Lemma 2.2. There exists at least one metric $d\mu_0$ for which Proposition 2.1 holds.

For the proof of this lemma we find a line bundle $L_0$ with a metric $\|\|$ for which $\pi^{-1}d'd''\log \| > 0$. Then we define $d\mu_0$ by the equation

$$d\mu_0 = (\deg L_0)^{-1} \cdot \frac{1}{\pi i} d'd'' \log \|. $$

Now if $L$ is any other line bundle on $X$, then for suitable $m$ and $n$ we have $L^n \cong L_0^m \otimes N$, $\deg N = 0$. Hence the metric on $L^n$ which we want can be obtained by multiplying the metric on $L_0^m$ by a harmonic metric on $N$; taking $n$th roots, we obtain the required metric on $L$.

Now let us find the line bundle $L_0$. Let $i : X \rightarrow \mathbb{P}^N$ be any projective embedding of the curve $X$.

Let $\mathbb{P}^N = \mathbb{P}(V)$, with $V$ an $(N + 1)$-dimensional vector space. Consider any Hermitian metric on $V$. This defines a Hermitian metric on the trivial vector bundle $V \times \mathbb{P}^N$. There is a standard one-dimensional subbundle $\mathcal{O} \subset V \times \mathbb{P}^N$, with $\mathcal{O} \cong O_{\mathbb{P}^N}(-1)$. Let $\|\|$ be that metric on $\mathcal{O}$ which is gotten by restricting the metric on $V \times \mathbb{P}^N$. Then the Hermitian form corresponding to the exterior form $-\pi^{-1}d'd''\log \|$ of type $(1, 1)$ is strictly positive and defines a Kähler metric on $\mathbb{P}^N$. This property is preserved on restricting to the curve $X$, so that we can take for $L_0$ the line bundle $i^*\mathcal{O}_{\mathbb{P}^N}^{-1} = i^*O_{\mathbb{P}^N}(1)$.

c) Lemma 2.3. (On transferring to a different metric). Proposition 2.1 holds for any metric.
Let \( d\mu_1 = \rho(z)d\mu_0 \) be any other metric, and let \( \| \|_0 \) be a Hermitian metric on \( L \) for which
\[
-\frac{1}{2\pi} \Delta \log \| \|_0 dx \wedge dy = \deg L \cdot d\mu_0.
\]
We will look for a metric \( \| \|_1 = \phi(z)\|_0 \) for which
\[
-\frac{1}{2\pi} \Delta \log \| \|_1 dx \wedge dy = \deg L \cdot d\mu_1.
\]
Then the equation for \( \phi \) has the form
\[
\frac{1}{2\pi} \Delta \log \phi = (1 - \rho) d\mu_0.
\]
This is the Poisson equation. The condition that it can be solved is well known: it is just that
\[
\int_X (1 - \rho) d\mu_0 = 0,
\]
and this holds since
\[
\int_X d\mu_0 = \int_X d\mu_1 = 1.
\]
Solving the above equation, we find the metric \( \| \|_1 \) on \( L \). This completes the proof of Proposition 2.1.

Remark 1. There is no obligation to assume that the form \( d\mu_1 \) is strictly positive; one could, for instance, allow it to have a finite number of zeros.

Remark 2. The existence of the global equation of the point \( P \) - the function \( \phi_P(z) \) - can now be proved as follows.

Consider a Hermitian metric \( \| \| \) on the line bundle \( L = O_X(P) \) for which we have
\[
-\frac{1}{2\pi} \Delta \log \| \| = d\mu.
\]
Let \( s \) be the section of \( L \) which has its unique zero at \( P \). Then \( \phi_P(z) = c \cdot \| s \| \), where \( c \) is a suitable constant.

§3. The Green’s function of a Riemann surface

Definition 3.1. Let \( X \) be a Riemann surface with volume element \( d\mu \), and let \( \phi_P(z) \) be the global equation of the point \( P \) with respect to \( d\mu \). We set \( G(P, z) = \phi_P(z) \). The function \( G(P, z) = \log G(P, z) \) is called the Green’s function of the Riemann surface \( X \) with respect to \( d\mu \).

This definition differs slightly from the accepted one. The function \( G(P, Q) \) is the intersection number \( (P \cdot Q)_X \), regarded as a function of \( P \) and of \( Q \). We list here some properties of \( G \):

1) \( G(P, z) \) is a nonnegative smooth function, defined on \( X \times X \setminus \Delta_X \), \( \Delta_X \) being the diagonal.
2) \( G(P, z) \) has a first-order zero on the diagonal.
3) \(-2\pi i \Delta_z G(P, z) = d\mu(z)\).
4) \( G(P, z) = G(z, P) \).
5) \( \int_X \log G(P, z) d\mu(z) = 0 \).

\( G \) is uniquely determined by these properties. The smoothness of the function \( G(P, z) \) as a function of two variables requires proof, of course, although we will not give it here.

Example. Let \( X = \mathbb{P}^1 \), and let \( z \) be an affine coordinate on \( \mathbb{P}^1 \), and

\[
d\mu = \frac{1}{2\pi} \frac{|dz|^2}{(1+|z|^4)}.
\]

Then

\[
G^2(a, z) = e^{\frac{|a - z|^2}{(1 + |a|^4)(1 + |z|^4)}}.
\]

The function \( G(P, z) \) defines a Hermitian metric on the line bundle \( \mathcal{O}(\Delta_X) \). This metric is given by the equation \( \|s_a\|^2 = G(P, z) \), where \( s_\Delta \) is the section of \( \mathcal{O}(\Delta_X) \) that is the image of the unit section of \( \mathcal{O}(\Delta_X) \) under the natural embedding \( \mathcal{O}(\Delta_X) \to \mathcal{O}(\Delta_X) \).

Let us denote this Hermitian metric by \( \| \| \), and state the classical result on its curvature form. For this consider the space \( \Omega^1(X) \) of holomorphic differentials on \( X \), with the Hermitian metric

\[
\langle \xi, \eta \rangle = i \int_X \xi \wedge \overline{\eta}.
\]

Let \( \omega_1, \ldots, \omega_g \) be an orthonormal basis with respect to this metric.

**Proposition 3.1.**

\[
\frac{1}{n!} d'd'' \log \| = p_1^* d\mu + p_2^* d\mu - i \sum_{k=1}^g \left( p_1^* \omega_k \wedge p_2^* \overline{\omega_k} + p_1^* \overline{\omega_k} \wedge p_2^* \omega_k \right).
\]

For the proof, see [1], Chapter 4, §10. Let us give a short sketch of it anyway. A more usable statement of Proposition 3.1 consists of the assertion that the bilinear differential form

\[
\frac{1}{n!} d'd'' \log \| = \omega_1 \wedge \overline{\omega_1} + \cdots + \omega_g \wedge \overline{\omega_g}
\]

is a reproducing kernel for holomorphic differentials; if \( \eta \) is such a differential, then

\[
\left( \int_X \frac{1}{n!} d'd'' \log \| dz \wedge \overline{\eta}(z) \right) d\overline{P} = \overline{\eta(P)}.
\]

From (\( \ast \)), and from the fact that the form

\[
\frac{\partial^2 \log \|}{\partial z \partial \overline{P}}
\]

is holomorphic in \( z \), it is easy to deduce Proposition 1.3. For the proof of (\( \ast \)), we have to rewrite the integral which appears on the left-hand side as
after which the required equation follows by applying Stokes' formula to the surface $X$ minus a small disc centered at the point $P$.

We conclude by noting the transformation law for the function $G(P, z)$ under a change of metric.

**Proposition 3.2.** Let $d\mu_1$ and $d\mu_0$ be two volume elements. Let $\phi$ be a smooth function which satisfies

$$\Delta \log \phi dx dy = d\mu_0 - d\mu_1,$$

and which is normalized so that

$$\int_X \log \phi (d\mu_0 + d\mu_1) = 0. \quad (**)$$

Then, if $G_0$ and $G_1$ are the functions corresponding to the metrics $d\mu_0$ and $d\mu_1$,

$$G_1(P, z) = \phi(P) \cdot \phi(z) \cdot G_0(P, z).$$

For the proof, let us check that the function

$$G_Q(P, z) \cdot \phi(P) \cdot \phi(z)$$

satisfies all the requirements 1)—5) for $G$. 1)—4) are obviously satisfied, and it remains to check that

$$\int_X \log \phi(P) \cdot \phi(z) \cdot G_0(P, z) d\mu_0(z) = 0.$$

For this, using a property of the Green's function $G_0$ and the property (**) we can rewrite the integral in the form

$$\log \phi(P) = \int_X \log \phi(z) d\mu_0(z) + \int_X \log G_0(P, z) [d\mu_1(z) - d\mu_0(z)],$$

and we then get what we want by using Green's formula.

§4. The canonical class and the adjunction formula

**Theorem 4.1.** For some choice of the metrics $ds^2$ there exists a metrized line bundle $K$ on $V$ which has the following properties: Let $C \subset V$ be a smooth horizontal divisor on $V$, not passing through any points of $V$ which are multiple on the fibers, and let $\delta$ denote the absolute discriminant of the divisor $C$; that is, the discriminant over $Q$ of the field $L$ of rational functions on $C$. Then the following adjunction formula holds:

$$C^2 + C \cdot K = \log |\delta|. \quad (*)$$

It is natural to refer to $K$ as the canonical class of the surface $V$. To prove the theorem, let us introduce the notation $B = \text{Spec } \Lambda$ and consider the invertible sheaf
\( \omega_{V/B} \) on the surface \( V \), defined by the fact that it agrees, outside the set of points which are multiple on the fibers, with the sheaf \( \Omega^1_{V/B} \). We will look for suitable Hermitian metrics on the line bundles \( \omega_{V/B}^\infty = \Omega^1_{X_\infty} \), regarding them for the time being as indeterminate. Let us denote by \( \partial_{C/B} \) the sheaf of \( O_C \)-modules corresponding to the relative different of \( C \) over \( B \). We have the two exact sequences

\[
0 \to \partial_{C/B} \to O_C \to \Omega^1_{C/B} \to 0, \quad (1)
\]

\[
0 \to O_V(-C) \big|_C \xrightarrow{\alpha} \Omega_{V/B} \big|_C \to \Omega^1_{C/B} \to 0. \tag{2}
\]

We consider the first two terms of the sequence (2) as metrized line bundles over \( C \). Since \( C = \text{Spec } \Gamma \), with \( \Gamma \) the integers of the field \( L \), every such line bundle over \( C \) has a degree. Suppose that the following requirement is satisfied:

\[ \{ \alpha_\infty \text{ is an isomorphism of Hermitian spaces for every } \infty. \} \tag{3} \]

Then, comparing the degrees, we get

\[
\omega_{V/B} \cdot C + C^a = \log |\delta_{C/B}|,
\]

where \( \delta_{C/B} \) is the relative discriminant of \( C \) over \( B \).

Now let \( \partial \) be the absolute different for \( \text{Spec } \Lambda \), which we regard as a metrized line bundle over \( \text{Spec } \Lambda \). We define \( K \) by the equation \( K = \omega_{V/B} \otimes \partial^{-1}. \) Then

\[
K \cdot C + C^a = (\partial^a \partial^{-1}) \cdot C + \log |\delta_{C/B}| = \log |\delta_C|,
\]

as required.

Now let us consider what is needed in order to satisfy (3). The problem reduces to that of a single Riemann surface \( X \), a point \( P \in X \) and the isomorphism

\[
O_X(-P) \big|_P \xrightarrow{\alpha_P} \Omega^1_X \big|_P
\]

of the two one-dimensional Hermitian vector spaces—the fibers of the line bundles \( O_X(-P) \) and \( \Omega^1_X \) at the point \( P \).

The Hermitian metric on the line bundle \( O_X(-P) \) is obtained from the definition of \( O_V(-C) \) as a metrized line bundle (see \( \S \) 2). The metric is defined by the fact that \( \|s_0\| = \phi^{-1}_P \), where \( s_0 \) is the canonical section with pole at \( P \). Now note that the requirement that the isomorphism \( \alpha_P \) should be an isometry itself defines some Hermitian metric on the line bundle \( \Omega^1_X \); if its curvature form happens to be proportional to the form \( \partial \mu \), then we will be able to take this metric as a definition of \( \omega_{V/B} \) as a metrized line bundle. Let us compute the curvature form. Note that the isomorphisms \( \alpha_P \) can be glued together to form a global isomorphism

\[
\alpha : O_{XX}(-\Delta_X) \big|_{\Delta_X} \to \Omega^1_X,
\]

where \( \Delta_X \subset X \times X \) is the diagonal. We see that the metric that we have constructed on \( \Omega^1_X \) is obtained by restricting onto the diagonal the metric given by the form \( G(P, z)^{-1} \) on the line bundle \( O_{XX}(-\Delta_X). \) Hence the curvature form can be obtained by restricting the corresponding curvature form on \( X \times X \); but this we have computed.
in \( \S 3 \). Restricting onto the diagonal, we get

\[
i \left[ \sum_{k=1}^{g} \left( \omega_k \wedge \bar{\omega}_k + \bar{\omega}_k \wedge \omega_k \right) \right] - 2d\mu.
\]

The condition that this form is proportional to \( d\mu \) is satisfied in one case only, namely, when

\[
d\mu = \frac{1}{2g} \cdot i \cdot \sum_{k=1}^{g} \left( \omega_k \wedge \bar{\omega}_k + \bar{\omega}_k \wedge \omega_k \right).
\]

If we define the metric \( d\mu_\infty \) by this equation, then the adjunction formula will hold.

\( \S 5 \). Change of metric

In the preceding section we made explicit the metric to which one should show preference. However, it would be even more preferable to get rid of the metric altogether. It turns out that in a certain sense this can be done.

Let \( d\mu_0 \) and \( d\mu_1 \) be two volume elements, and let \( \phi(z) \) be the function of Proposition 3.2, so that

\[
\Delta \log \phi \cdot dxdy = d\mu_0 - d\mu_1, \quad \int_X \log \phi \cdot \left( d\mu_0 + d\mu_1 \right) = 0.
\]

Suppose that the line bundle \( L \) has a metric \( \| \|_0 \) for which

\[
- \frac{1}{2\pi} \Delta \log \| \|_0 = \deg L \cdot d\mu_0.
\]

Then for the metric \( \| \|_1 = \phi^{\deg L} \cdot \| \|_0 \) we will have

\[
- \frac{1}{2\pi} \Delta \log \| \|_1 = \deg L \cdot d\mu_1.
\]

In this way we obtain a method by which we can associate to every metrized line bundle \( L \) (metrized with respect to the collection of metrics \( d\mu_{0,\infty} \)) a metrized line bundle having the same underlying sheaf, but metrized with respect to the collection of metrics \( d\mu_{1,\infty} \). Unfortunately, this changeover is not geometrical: under it the multiplicities of the components at infinity of the divisor of a section are altered. However, we do have

**Proposition 5.1.** The intersection numbers are preserved under the changeover from one metric to another defined above.

We do not give the proof. It is just a simple check, which needs to be carried out on a Riemann surface using Green's formula.

Since we now have the possibility of changing over from one metric to another, we can define the inverse image of a metrized line bundle. We will carry out the argument for a pair of Riemann surfaces, \( X \) and \( Y \). Let \( f: X \to Y \) be a map of degree \( d \), and suppose that we have a line bundle \( L \) on \( Y \), metrized with respect to a metric \( d\mu_Y \). Suppose that we are given a metric \( d\mu_0 \) on \( X \). Set \( d\mu_1 = d^{-1} f^* d\mu_Y \). The line bundle
$f^*L$ is naturally metrized with respect to $d\mu_1$. Let us go over from $d\mu_1$ to $d\mu_0$ by the method given above. This defines the inverse image of a metrized line bundle. It is easy to check the following assertion:

**Proposition 5.2.** Under the inverse image homomorphism the intersection number is multiplied by the degree of the map.

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**BIBLIOGRAPHY**


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