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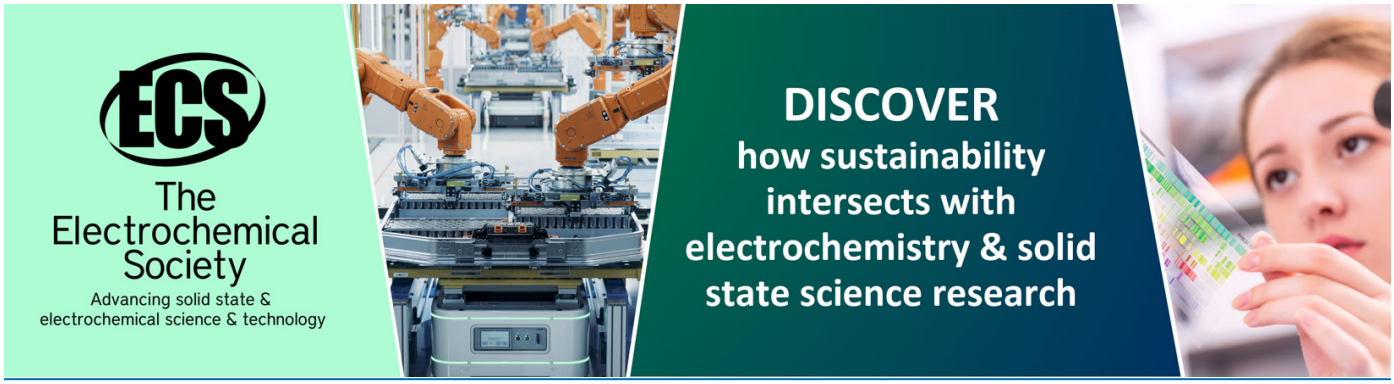

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Study on the relationship between stress intensity factor and J integral for mixed mode crack with arbitrary inclination based on SBFEM

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Abstract. The J integral and the stress intensity factor (SIF) K are both important research objects of fracture mechanics, and are often employed to establish criteria for crack propagation. The relationship between them has always been a research hotspot. In this paper, the SIF can be obtained conveniently by the scaled boundary finite element method (SBFEM) due to the fact that analytical solution can be obtained along the radial direction for stress singularity problems. The J integral can be solved analytically using the formulae between J and K for mixed mode crack with arbitrary inclination in elastic materials. Moreover, the J integral values obtained by this method are more accurate and convenient than by its definition. Factors that affect the accuracy of SIF and J integral, such as the distance between the crack and outer boundary, size of the discretized elements and partition of the domain into super-elements, are examined.

1. Introduction

The great earthquake occurring in Wenchuan has dream more attention to the seismic safety of high concrete dams. Dynamic fracture is one of the main damage modes of concrete dams in earthquakes, and the stability of cracks has become an important factor of safety evaluation of dams, so study on cracks of concrete structure is still important and urgent. Fracture energy serves as the basis for study on crack extension. Fracture energy and energy release rate of linear elastic materials are equivalent. J integral is a common method for calculation of fracture energy，which remains constant along integral paths near the region around the crack tip in small-scale yielding condition. The three parameters mentioned are often used as the criterion for crack extension. G is also used by M.A.Hussain to determine the direction of crack propagation[1]. The stress intensity factors (SIF) K, as proposed by Irwin in 1977, is a parameter only related to the local stress-strain field, and can be more easily determined than G.

Several methods including boundary element method, finite element method, mesh independent method, have been used to calculate SIF. In standard finite element method, the region around the crack tip is discretized, while for the boundary element method, discretization of the crack surface and the material interface through the crack tip is required. These two methods can't obtain the analytical solution at the singular point owing to segmented smooth shape functions. With the rapid development

of computers and computing technology, numerical simulation of fracture problems of engineering structures has become feasible. So it is significant to seek a calculation method which is of high precision and high efficiency. The scaled boundary finite element method [2-3] developed by Wolf and Song has outstanding advantages in solving problems characterized by stress singularity and infinite domain. It involves only discretization of boundaries of the investigated domain, yielding reduction of the spatial dimension by one, which greatly saves computational effort. The solution at the crack tip is analytical, so that accuracy of stress intensity factors is significantly improved. Liu Junyu used SBFEM to calculate the stress intensity factor for crack surface loaded with pressure of arbitrary distribution [4]. So SBFEM is used to calculate of SIFs and integral in the presented paper.

For study on correlation between and , Fan Tianyou derived the relationship between integral and for plane problems [5]. Zhou Longhai derived the relationship between integral and for plane stress problems more detailedly [6]. Their relationship has been discussed in many related papers, most of which focus on problems characterized by mode I crack or mode II crack only and the crack was usually assumed to be in the horizontal direction. Zhu Chaolei utilized the Griffith crack model which has analytical expression for the stress field and displacement field and the integral formula proposed by Rice, to deduce the analytical expression for the relationship between integral and , for mixed mode crack with arbitrary inclination. It was shown that SBFEM has higher precision, less degree of freedom and higher computational efficiency in calculation of integral in contrast to FEM [7]. In the presented paper, t SIFs are calculated by the relationship between integral and , deduced in [7].Then the integral values are more accurate than that obtained by its definition, due to the fact that analytical solution can be obtained along the radial direction for stress singularity problems solved by SBFEM. In addition, factors that affect the accuracy of SIF and J integral, such as distance between the crack and outer boundary, size of the discretized elements and partition of the domain into super-elements, are examined.

2. The relationship between J integral and stress intensity factors for mixed mode crack

2.1. Basic theory of J integral

J integral has two definitions: one is the loop integral definition, while, the other is the deformation power definition, and both were proposed by Rice. The loop integral definition is as follows[5]:

$$J = \int_{\Gamma} (W dx_2 - \vec{T} \frac{\partial \vec{u}}{\partial x_1} ds) \tag{1}$$

$$\text{Or } J = \int_{\Gamma} (W dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds) \tag{2}$$

in which W denotes the strain energy density unit volume; $\vec{T}$ denotes the stress vector which is defined along the outer normal of integral curve Γ. $\vec{u}$ denotes the corresponding displacement vector; ds denotes a section arc length along integral curve Γ; Γ denotes arbitrary integral loop starts from the lower surface to the upper surface of the crack in the anticlockwise direction, as shown in figure 1. A circle with its center located at the crack tip and with radius r is chosen to idealize Γ. The angle between the crack and x-axis of the global coordinate system is α.

2.2. Formula for J integral of mixed mode Griffith crack with arbitrary inclination[7]

J with arbitrary inclination for plane stress problems:

$$J = \frac{1}{4E}[(1-\mu)(K_I^2 - K_{II}^2)\cos\alpha + K_I^2 + 7K_{II}^2 + 2(1-\mu)K_I K_{II}\sin\alpha - \mu(K_I^2 - K_{II}^2) \tag{3}$$
$$+2(1+\mu)(K_I^2 - K_{II}^2)\cos^2\alpha + 4(1+\mu)K_I K_{II}\sin\alpha\cos\alpha]$$

J with arbitrary inclination for plane strain problems:

$$J=\frac{1}{4E}[(1-\mu)(K_I^2-K_{II}^2)\cos\alpha+K_I^2+7K_{II}^2+2(1-\mu)K_IK_{II}\sin\alpha+2(1+\mu)(K_I^2-K_{II}^2)\cos^2\alpha \\ +4(1+\mu)K_IK_{II}\sin\alpha\cos\alpha-2\mu^2(K_I^2-K_{II}^2)\cos\alpha-\mu^2(2K_I^2+6K_{II}^2)-4\mu^2K_IK_{II}\sin\alpha-\mu(K_I^2-K_{II}^2)] \tag{4}$$

in which, E means elastic modulus and α means poisson's ratio. Particularly, when $\alpha=0$,the plane stress case:

$$J=\frac{1}{E}(K_I^2+K_{II}^2)=G \tag{5}$$

and the plane strain case:

$$J=(1-\mu^2)(K_I^2+K_{II}^2)/E=G \tag{6}$$

are yielded

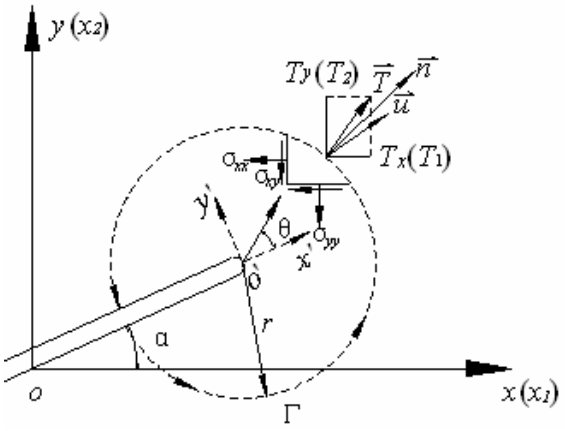


Figure 1. Schematic diagram of the crack

3. Basic theory of SBFEM

The displacement field, stress field and strain field given by SBFEM are as follows, see literature [2,3,8].

$$\{u(\xi,\eta)\}=\sum_{i=1}^{n}c_i\xi^{-\lambda_i}[N^u(\eta)]\{\phi\} \tag{7}$$

$$\{\sigma(\xi,\eta)\}=\sum_{i=1}^{n}c_i\xi^{-\lambda_i-1}\{\psi(\eta)\} \tag{8}$$

$$\{\varepsilon(\xi,\eta)\}=[B^1(\eta)]\{u(\xi)\}_{,\xi}+\frac{1}{\xi}[B^2(\eta)]\{u(\xi)\} \tag{9}$$

where $[N^u(\eta)]$ means shape function for boundary elements; $\{\phi\}_i$ and $\{\psi(\eta)\}_i$ mean the displacement mode and the stress mode solved by SBFEM; c_i means integral constants. The meaning of $[B]$ is similar to that in FEM and is related to geometry of the boundary. ξ means radial ray coordinate starting from the crack tip, η means circumferential direction coordinate. The radial functions of the displacement, stress and strain given by SBFEM are analytical, but the precision in the circumferential direction is equivalent to that of finite element discretization.

The formula for stress intensity factors by SBFEM is shown as follows (see literature [2,4]):

$$\begin{Bmatrix}K_{\mathrm{I}}\\K_{\mathrm{II}}\end{Bmatrix}=\sqrt{2\pi}l_f^{\lambda_s+1}\left(c_{\mathrm{I}}\begin{Bmatrix}\psi_{yy}(\theta=0)\\\psi_{xy}(\theta=0)\end{Bmatrix}_{\mathrm{I}}+c_{\mathrm{II}}\begin{Bmatrix}\psi_{yy}(\theta=0)\\\psi_{xy}(\theta=0)\end{Bmatrix}_{\mathrm{II}}\right) \tag{10}$$

As shown in figure 2, l_f means the distance from the scaling centre to the boundary along the radial line at $\theta = 0$ (in front of the crack tip); $\{\psi(\eta)\}_i$ means the two stress-singular modes in equation (8) with eigenvalue $\lambda_s = -0.5$.

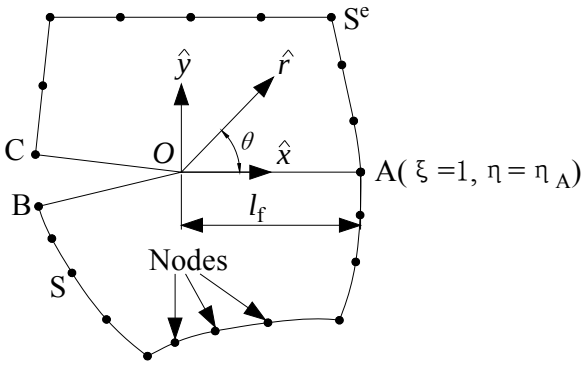


Figure 2. A cracked domain modelled by super-element with scaling centre at the crack tip

4. Numerical validation

For sake of comparison, a plate with Griffith crack with a length of $2a$ in the middle and loaded with tensile stress σ in the far ends [7] is studied. The angle between the direction of σ and crack surface is β and the angle between Ox and crack surface is α (shown in figure 3). Only plane strain problem is considered.

4.1. Idealization and parameters of the model

4.1.1. The formula for SIF is as follows [5] :

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}\sin^2\beta \tag{11}$$

$$K_{\mathrm{II}} = \sigma\sqrt{\pi a}\sin\beta\cos\beta \tag{12}$$

4.1.2. Idealization of the model :

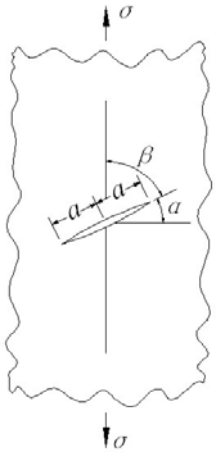


Figure 3. A Griffith crack under tension loading

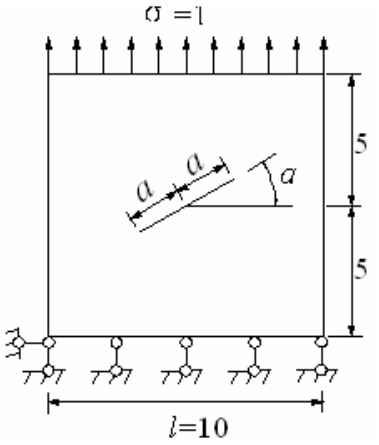


Figure 4. Rectangular plate with inclined crack

The model as shown in figure 4 is employed to idealize the crack in figure 3 approximately. The length and width of the model are $l = 10$ respectively, the length of the crack is $2a$, the angle between

the crack and Ox is α, with the crack center coincides with the scaling center. The upper boundary is loaded with distributed force with $\sigma = 1$, and the lower boundary is constrainted. The material parameters of this model (dimensionless): elastic modulus $E = 1.0$; poisson's ratio $\mu = 0.3$.

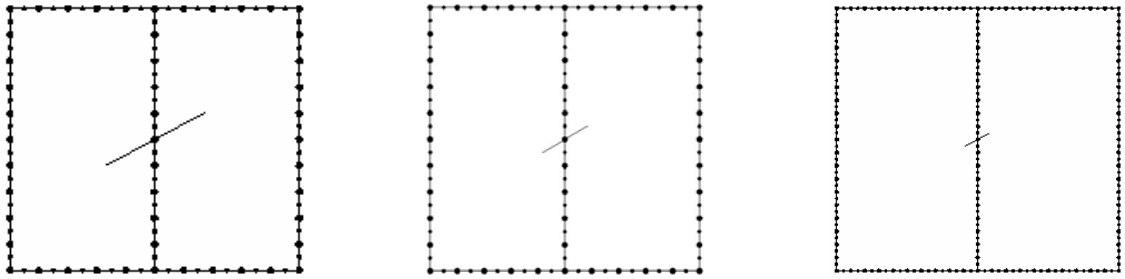
4.1.3. Partition of the model into super element of SBFEM

Three schemes are adopted for verification, and the origin is placed at the center of the model (as shown in figure 5). The structure is divided into two super elements for the three models. For model a, crack length $a = 2$ and $l/a = 5$. Only the boundary is discretized, and the element length l^e is chosen to be 1, which result in a total of 100 nodes (200 degrees of freedom) and 60 elements. For model b, crack length $a = 1$ and $l/a = 10$, the topological relation is same to model a. For model c, crack length $a = 0.5$ and $l/a = 20$, and the element length l^e is chosen to be 0.5 by densifying elements by one time, which result in a total of 200 nodes(400 degrees of freedom) and 120 elements.

Owing to symmetry between α varies from -90° to 0° and from 0° to 90°, only the latter is considered. The scaling centers of the two super elements are $(-a\cos\alpha\ ,\ -a\sin\alpha)$, $(a\cos\alpha\ ,a\sin\alpha)$ respectively. Accordingly, equations (11) and (12) become:

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}\cos^2\alpha \tag{13}$$

$$K_{\mathrm{II}} = \sigma\sqrt{\pi a}\sin\alpha\cos\alpha \tag{14}$$



(a) (b) (c)

Figure 5. Crack element and its discretization, (a) $a = 2, l/a = 5$ (b) $a = 1,\ l/a = 10$ (c) $a = 0.5, l/a = 20$

4.2. Calculation of the stress intensity factors by SBFEM

The stress intensity factors are obtained by SBFEM according to equation (10) and analytically by equation (13) and equation (14) respectively. The results of model a, b, c are shown in figure 6(a), (b) and (c). The outer boundaries should be located far enough from the crack to guarantee satisfactory accuracy which can be seen from figure 6. For model c with $l/a = 20$, the stress intensity factors better tally with theoretical solution. For model a with $l/a = 5$, considerable discrepancy exists between K_I and the theoretical solution. For model b with $l/a = 10$, the error is up to 14%. Because the theoretical solution deduced based on the crack in the infinite domain, this comparison result is imaginable. For model c, when the element length of the boundary discretization $l^e = a = 0.5$ is assumed, the errors of K_I and K_{II} are limited to7% and 4% respectively, as shown in figure 7(a). Particularly, when $\alpha < 45^\circ$, the error of K_I are all lower than 2%. However, when the element lengths are doubled, the maximum errors of K_I and K_{II} are as much as 26% and 18% respectively, as shown in figure 7(b), and the calculated values are shown in figure 6(d). It can be seen that l/a should be large enough to better simulate the real situation for the crack in infinite domain. Simultaneously, the size of l^e affects the

calculation precision greatly for the same size of outer boundary of the calculated domain and the discretized element. $l^e \le a$ is suggested from the comparison above.

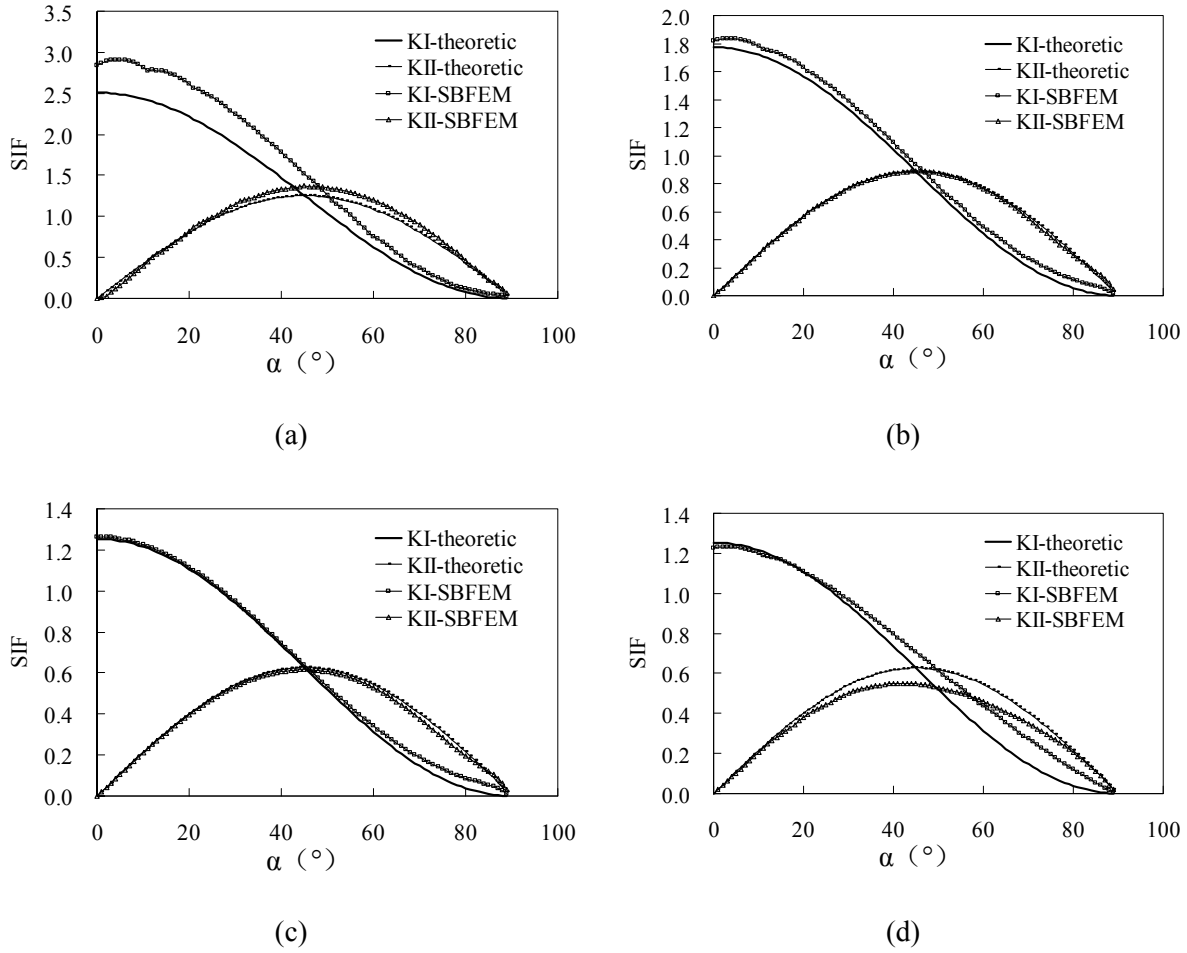


Figure 6. Variation of the SIF with the change of crack angle (a) $a=2$, $l/a=5$ (b) $a=1$, $l/a=10$ (c) $a=0.5$, $l/a=20$, $l^e=a$ (d) $a=0.5$, $l/a=20$, $l^e=2a$

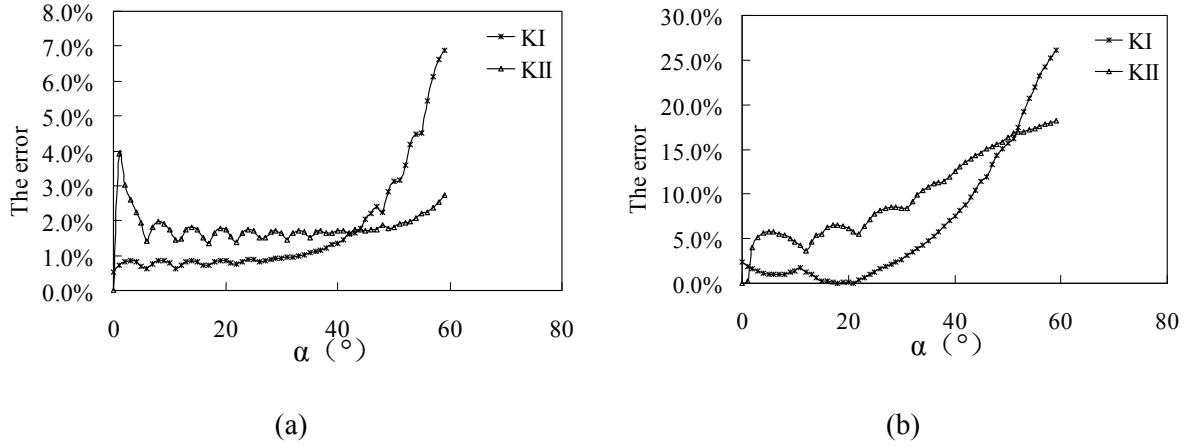


Figure 7. Error analysis for SIF (a) $a=0.5$, $l/a=20$, $l^e=a$ (b) $a=0.5$, $l/a=20$, $l^e=2a$

4.3. Calculation of J integral using Equation (4)

The J integral calculated by substituting the K_I, K_{II} into equation (4) obtained in the precious section for model a, b, c, are shown in figure 8(a), (b) and (c).

The tendency observed is similar to that in section 4.2, as shown in figure 8. For model c with $l/a = 20$, the J integral obtained is identical with the theoretical solution. For model a with $l/a = 5$, considerable discrepancy exists between J integral and the theoretical solution. For model b with $l/a = 10$, the error is also up to 7.68%. For model c, when the element length of the boundary discretization $l^e = a = 0.5$ is assumed, the errors are limited to 1.7%, as shown in figure 9 (a). However, when the element lengths are doubled, the errors can add up to 12.7%, as shown in figure 9 (b), and the calculation results are shown in figure 8 (d).

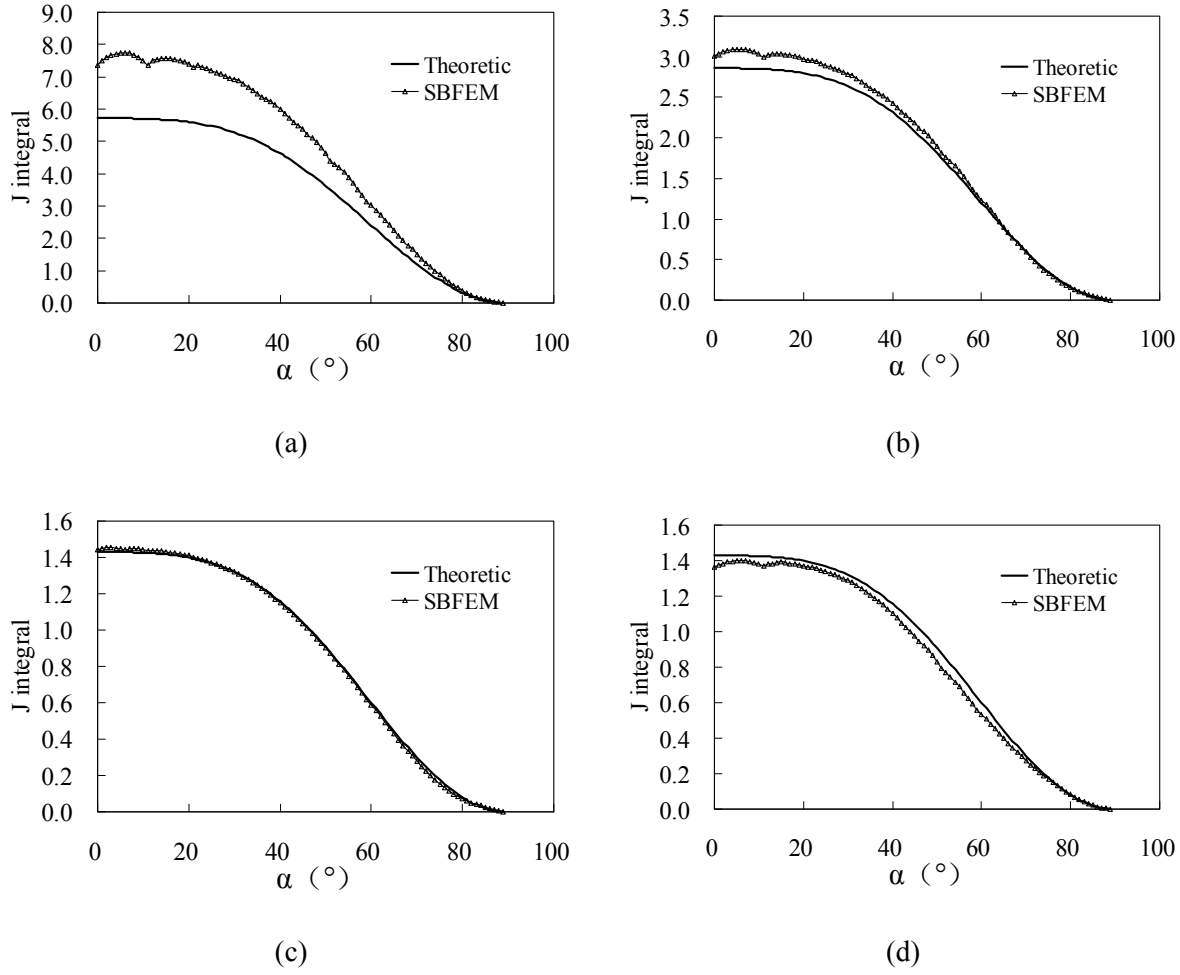
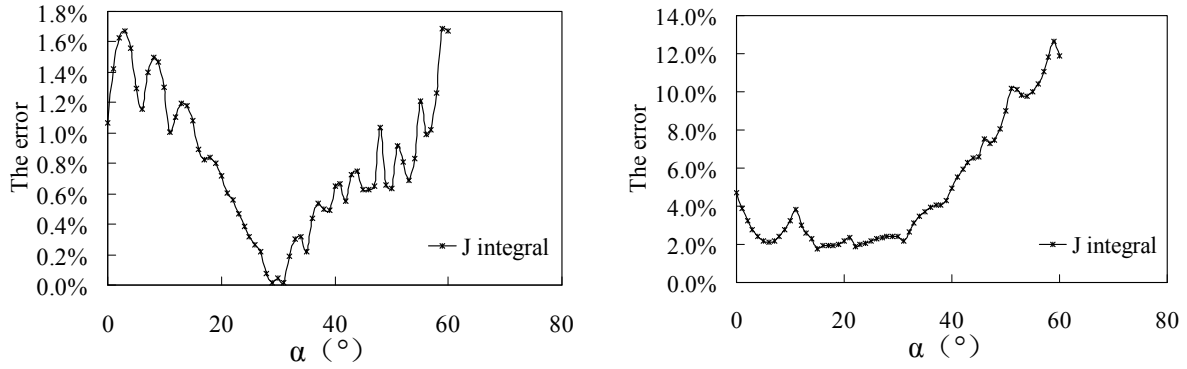


Figure 8. Variation of the J integral with the change of crack angle (a) $a = 2$, $l/a = 5$ (b) $a = 1$, $l/a = 10$ (c) $a = 0.5$, $l/a = 20$, $l^e = a$ (d) $a = 0.5$, $l/a = 20$, $l^e = 2a$



(a) (b)

Figure 9. Error analysis for J integral (a) $a = 0.5$, $l/a = 20$, $l^e = a$ (b) $a = 0.5$, $l/a = 20$, $l^e = 2a$

4.4. Calculation of J integral and SIF for a new super element

The error analysis above are only performed for $\alpha < 60^\circ$, because the scaling center would be so close to the border of the super element in the domain, causing high singularity when $\alpha \geq 60^\circ$. This can be improved by re-division of the super element. Only model c, shown in figure 5(c), is re-evaluated, and the size of discretized element $l^e = a = 0.5$. The new super element is shown in figure 10.

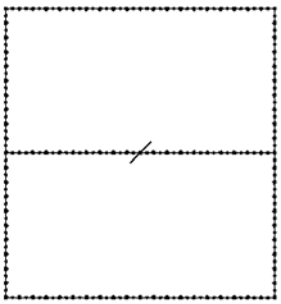


Figure 10. New super elements and its discretized mesh with $a = 0.5$, $l/a = 20$

The J integral and the SIF are calculated for $40^\circ \leq \alpha < 90^\circ$, with the results shown in figure 11 (a) and (b). It can be seen that the two curves are almost coincident. The error between the calculated value and the theoretical solution of K_I, K_{II} calculated by the new super element are shown in figure 11 (c), where the errors of K_I and K_{II} are less than 2.9% and 1.4% respectively. The error between the calculated value and the theoretical solution of J integral is shown in figure 11 (d), with the maximum deviation less than 0.5%. It can be seen from figure 7 (a) and figure 11 (c) that the maximum errors of K_I and K_{II} are less than 2.9% and 3.9% respectively by the rational super element. It can be seen from figure 9 (a) and figure 11 (d) that the error of J integral is less than 1.7% and is less than 2.5% for the best approximation path of model c [7]. So in order to achieve better accuracy, partition of the domain into super elements should be reasonably employed.

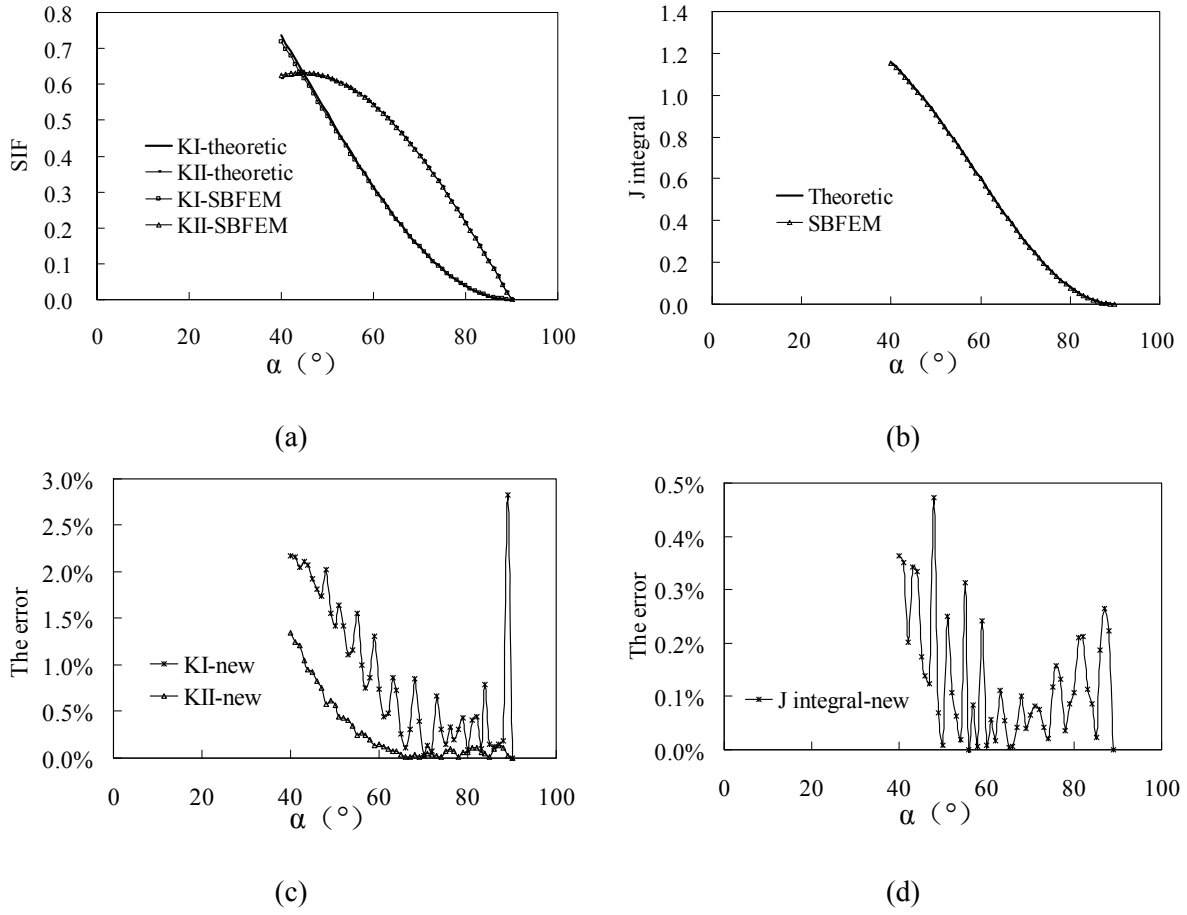


Figure 11. SIF and *J* integral obtained for the new super elements

5. Conclusions

J integral and energy release rate *G* of linear elastic material are equivalent, so it can be used to solve energy release rate by the relationship between the value of *J* integral and the stress intensity factors *K* for the mixed mode Griffith crack in elastic materials, thus *J* criteria and *G* criteria can be applied more conveniently.

For *J* integral solved using the definition formulae by FEM, the model needs to be remeshed as soon as the crack direction changes, furthermore, refinement of elements in the crack tip is needed to improve the accuracy, which is quite computationally expensive. On the contracy, only location of the scaling center is changed in SBFEM, and only the outer boundary is discretized, so the degrees of freedom could be much less than in FEM.

It also can be shown that the *J* integral values obtained by the formulae deduced in paper [7], are more convenient and accurate than by its definition, due to the analytical property of the solution in the radial direction for stress singularity problems solved by SBFEM, and this method needn't to operate integral calculation along the integral paths. Numerical verification show that certain factors, such as size of outer boundary of the calculated domain and the size of the discretized element, affect the accuracy of the results. In addition, reasonable partion sheme of the domain into super-element lead to higher accuracy.

Acknowledgements
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