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FAST TRACK COMMUNICATION

The n-level, $n-1$-mode Jaynes–Cummings model: spectrum and eigenvectors

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Abstract

We explicitly find eigenvectors and eigenvalues of the general n-level, $n-1$-mode Jaynes–Cummings model in Λ-configuration possessing arbitrary detuning parameters Δ_i, $i \in \overline{1, n}$ and arbitrary interaction strengths g_i, $i \in \overline{2, n}$. In the case of equal interaction strengths $g_i = g_j$, $i, j \in \overline{2, n}$, we compare the obtained answers with those obtained by the algebraic Bethe ansatz method (Skrypnyk 2008 *J. Phys. A: Math. Theor.* **41** 475202; 2009 *J. Math. Phys.* **50** 103523).

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1. Introduction

An important quantum mechanical problem is the description of the interaction of charged matter with radiation [1, 2]. The standard Jaynes–Cummings model [3] is the simplest non-trivial model describing the interaction of a two-level atom with an electromagnetic field at resonance. More complicated is the so-called n-level, many-mode Jaynes–Cummings model describing the interaction of an n-level atom with an electromagnetic field. Mathematically, the corresponding model is a generalized spin-boson model, where the role of the 'generalized spin operators' is played by the elements of the Lie algebra $gl(n)$ in an n-dimensional representation.

Usually for physical applications, the most important role is played by the so-called n-level, $n-1$-mode model describing the interaction of an n-level atom with $n-1$ modes of the electromagnetic field when transitions between certain energy levels are forbidden. In this communication, we are interested in the so-called Λ-configuration when only transitions from a fixed energy level of the atom to all other energy levels are allowed. The corresponding models were introduced and considered (for the cases $n = 3$ and for arbitrary n) in the papers [4–9], under the additional requirement that the so-called detuning is scalar, i.e. $\Delta_i = 0$, $i \in \overline{2, n}$. In this paper, we consider the n-level, $n-1$-mode Jaynes–Cummings model in 'Λ-configuration' with arbitrary Δ_i, $i \in \overline{1, n}$ and arbitrary interaction strengths g_i, $i \in \overline{2, n}$.

It is necessary to mention that in the papers [10, 11], this and more general Jaynes–Cummings models were shown to be quantum integrable under the requirement of the equal interaction strengths $g_i = g_j$, $i, j \in \overline{2, n}$ in an arbitrary representation of a generalized spin algebra. The corresponding eigenvectors and eigenvalues were constructed with the help of the 'nested' algebraic Bethe ansatz. Nevertheless, the obtained answers are not completely explicit: they require solutions of the 'nested' Bethe equations.

For physical applications, more explicit answers are desirable. That is why in this communication, we find them using the direct method. We show that in the case most important for applications—the n-dimensional representation of the generalized spin algebra—the Jaynes–Cummings model is exactly solvable even in the case of non-equal interaction strengths. We show that the corresponding eigenvalues satisfy one algebraic equation of order n and explicitly construct the corresponding eigenvectors. The mentioned algebraic equation coincides with the 'n-level generalization' of the Rabi equation. We show that in the case $\Delta_i = 0$, $i \in \overline{2, n}$, the constructed generalization of the Rabi equation passes to the usual quadratic Rabi equation. Nevertheless, this case is in a sense degenerated—when for $n > 2$, the corresponding eigenvalues do not constitute the complete basis in the space of quantum states.

We compare the obtained answers with those obtained by the Bethe ansatz method in the case of equal interaction strengths. We show that in the case of $n > 2$ and $\Delta_i = 0$, $i \in \overline{2, n}$ in order to obtain the quadratic Rabi equations, one has to impose an additional condition on the solutions of the Bethe equations, meaning that the Bethe ansatz method provides a wider set of eigenvectors than the direct method. In the case when $n = 2$, there is no additional condition and the Bethe equations provide the same quadratic Rabi equations for the energy spectrum as the direct method.

The structure of this communication is as follows: in section 2, we describe the n-level, $n - 1$-mode Jaynes–Cummings Hamiltonian and its abelian symmetry algebra; in section 3, we construct its eigenvectors and eigenvalues. Finally in section 4, we compare the obtained answers with those obtained by the nested Bethe ansatz.

2. The n-level, $n - 1$-mode Jaynes–Cummings model

The so-called n-level, $n - 1$-mode Jaynes–Cummings Hamiltonian in the Λ-configuration and rotating wave approximation is given by the following formula (see, e.g., [4–9]):

$$\hat{H}_{n\mathrm{JC}} = \sum_{i=2}^{n} w_i \hat{b}_i \hat{b}_i^{\dagger} + \sum_{i=2}^{n} g_i \left(\hat{b}_i \hat{X}_{1i} + \hat{b}_i^{\dagger} \hat{X}_{i1} \right) + \sum_{i=1}^{n} \epsilon_i \hat{X}_{ii}, \tag{1}$$

where g_i are interaction strengths, ϵ_i are energy levels of the atom, w_i are frequencies of free modes of the electromagnetic field, operators X_{ij} constitute a fundamental representation of the Lie algebra $gl(n)$ in the n-dimensional space, i.e. $(\hat{X}_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$,

$$[\hat{X}_{ij}, \hat{X}_{kl}] = \delta_{kj}\hat{X}_{il} - \delta_{il}\hat{X}_{kj},$$

and $\hat{b}_i^{\dagger}$, $\hat{b}_i$, $i \in \overline{2, n}$ are standard Bose creation–annihilation operators:

$$\left[\hat{b}_i^{\dagger}, \hat{b}_j\right] = \delta_{ij}, \left[\hat{b}_i^{\dagger}, \hat{b}_j^{\dagger}\right] = [\hat{b}_i, \hat{b}_j] = 0.$$

This Hamiltonian describes (in the dipole and rotating wave approximation) the physical problem of the interaction of an n-level atom with $n - 1$ modes of the radiation field in the case when the transitions between the first and other $n - 1$ levels are possible and all other atomic transitions are forbidden.

It is easy to show that the Hamiltonian (1) possesses $n-1$ commuting integrals of motion of the form

$$M_i = \hat{X}_{ii} + \hat{b}_i\hat{b}_i^\dagger, \tag{2}$$

where $i \in \overline{2,n}$, in terms of which the Hamiltonian of the model is written as follows:

$$\hat{H}_{n\mathrm{JC}} = H_0 + H_I = \sum_{i=2}^{n} w_i\hat{M}_i + \sum_{i=1}^{n}\Delta_i\hat{X}_{ii} + \sum_{i=2}^{n} g_i(\hat{b}_i\hat{X}_{1i} + \hat{b}_i^\dagger\hat{X}_{i1}),$$

$$\text{where } H_0 = \sum_{i=2}^{n} w_i\hat{M}_i,\ H_I = \sum_{i=1}^{n}\Delta_i\hat{X}_{ii} + \sum_{i=2}^{n} g_i(\hat{b}_i\hat{X}_{1i} + \hat{b}_i^\dagger\hat{X}_{i1}).$$

Here $\Delta_1 = \epsilon_1$, $\Delta_i = \epsilon_i - w_i$, $i \in \overline{2,n}$ are the so-called detuning parameters.

Due to the fact that $[\hat{H}_{n\mathrm{JC}}, \hat{M}_i] = 0$, $[\hat{M}_j, \hat{M}_i] = 0$, it follows that $[\hat{M}_i, \hat{H}_I] = 0$ and the problem of the diagonalization of $\hat{H}_{n\mathrm{JC}}$ is reduced to the problem of the simultaneous diagonalization of $\hat{M}_i$ and $\hat{H}_I$.

3. Spectrum and eigenvectors

Let us obtain the spectrum and eigenvalues of the Hamiltonian $\hat{H}_{n\mathrm{JC}}$, integrals $\hat{M}_i$ and $\hat{H}_I$.

The following theorem holds true.

Theorem 3.1. *The energy spectrum of the Hamiltonian (1) is given by the following formula:*

$$E = \sum_{i=2}^{n} w_i m_i + E_I,$$

where m_i are arbitrary non-negative integers coinciding with the eigenvalues of the operators $\hat{M}_i$ and the energy E_I satisfies the following polynomial algebraic equation of the order n:

$$\prod_{i=1}^{n}(E_I - \Delta_i) = \sum_{i=2}^{n} g_i^2 m_i \prod_{k=2,k\neq i}^{n}(E_I - \Delta_k). \tag{3}$$

Proof. In order to prove this theorem, it will be convenient to use the representation of the Heisenberg algebra in the space of holomorphic functions of variables z_i, $i \in \overline{2,n}$:

$$\hat{b}_i^\dagger = \partial_{z_i},\ \hat{b}_i = z_i.$$

Let us write the operators $\hat{H}_I$ and $\hat{M}_i$, $i \in \overline{2,n}$ in the matrix form. We have

$$\hat{M}_i = z_i\partial_{z_i}\hat{1}_n + \hat{X}_{ii} = \mathrm{diag}(z_i\partial_{z_i}, \ldots, z_i\partial_{z_i} + 1, \ldots, z_i\partial_{z_i}),$$

$$\hat{H}_I = \begin{pmatrix} \Delta_1 & g_2z_2 & \ldots & g_nz_n \\ g_2\partial_{z_2} & \Delta_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ g_n\partial_{z_n} & 0 & \ldots & \Delta_n \end{pmatrix}.$$

From the explicit form of the operators $\hat{M}_i$, it follows that the following vectors

$$\overrightarrow{v}_{m_2,\ldots,m_n} = \begin{pmatrix} z_2^{m_2}z_3^{m_3}\ldots\ldots z_n^{m_n} \\ k_2z_2^{m_2-1}z_3^{m_3}\ldots\ldots z_n^{m_n} \\ k_3z_2^{m_2}z_3^{m_3-1}\ldots\ldots z_n^{m_n} \\ \ldots\ldots \\ k_nz_2^{m_2}z_3^{m_3}\ldots\ldots z_n^{m_n-1} \end{pmatrix} \tag{4}$$

are their eigenvectors with the eigenvalues m_i: $\hat{M}_i \vec{v}_{m_2,\ldots,m_n} = m_i \vec{v}_{m_2,\ldots,m_n}$, where m_i are arbitrary non-negative integers and the coefficients k_i in the definition of $\vec{v}_{m_2,\ldots,m_n}$ are arbitrary. We will use the freedom in their definition in order to diagonalize the Hamiltonian $\hat{H}_I$. The direct calculation gives

$$\hat{H}_I \vec{v}_{m_2,\ldots,m_n} = \begin{pmatrix} \left(\Delta_1 + \sum_{i=2}^{n} g_i k_i\right) z_2^{m_2} z_3^{m_3} \ldots .. z_n^{m_n} \\ (k_2\Delta_2 + g_2 m_2) z_2^{m_2-1} z_3^{m_3} \ldots .. z_n^{m_n} \\ (k_3\Delta_3 + g_3 m_3) z_2^{m_2} z_3^{m_3-1} \ldots .. z_n^{m_n} \\ \ldots . \\ (k_n\Delta_n + g_n m_n) z_2^{m_2} z_3^{m_3} \ldots .. z_n^{m_n-1} \end{pmatrix}. \tag{5}$$

Comparing expressions (4) and (5), we obtain that (4) is the eigenvector of $\hat{H}_I$ with the eigenvalue E_I if and only if

$$g_i m_i + k_i \Delta_i = E_I k_i, \, i \in \overline{2, n}, \quad \Delta_1 + \sum_{i=2}^{n} k_i g_i = E_I. \tag{6}$$

Using this, we obtain the answer for the coefficients of the eigenvectors $k_i = \frac{g_i m_i}{E_I - \Delta_i}$, where the energy E_I satisfies the polynomial algebraic equation (3). Now, from the explicit form of the Hamiltonian $\hat{H}_{n\mathrm{JC}}$, it easily follows that the full energy is given by the formula $E = \sum_{i=2}^{n} w_i m_i + E_I$. The theorem is proven. □

Remark 1. Observe that the n-level Jaynes–Cummings model in Λ-configuration is exactly solvable for any 'vector' of detuning $(\Delta_1, \Delta_2, \ldots, \Delta_n)$. The equation for the spectrum (3) has order n. It may be called the 'n-level Rabi equation'. There is a special case when this equation is substantially simplified.

Example 1. Let us consider the case when $\Delta_i = \Delta, \forall i \in \overline{2, n}$. In this case, we may put without loss of generality $\Delta = 0$ and obtain from the system of equations (6) the following quadratic Rabi-type equation for the spectrum of H_I:

$$E_I^2 - \Delta_1 E_I - \sum_{i=2}^{n} g_i^2 m_i = 0 \tag{7}$$

with its following Rabi-type solutions:

$$E_I = \frac{1}{2}\left(\Delta_1 \pm \sqrt{\Delta_1^2 + 4\sum_{i=2}^{n} g_i^2 m_i}\right)$$

and the coefficients k_i of the eigenvectors are given by the following formula:

$$k_i = \frac{2 g_i m_i}{\Delta_1 \pm \sqrt{\Delta_1^2 + 4\sum_{i=2}^{n} g_i^2 m_i}}. \tag{8}$$

Remark 2. Observe that when $n > 2$, the basis of the eigenvectors constructed in example 1 is not complete (the same thing happens in the other 'degenerate' cases when some of the coefficients Δ_i coincide). Indeed, the subspaces of the eigenvectors of the operators $\hat{M}_i$ with the eigenvalues m_i, $i \in \overline{2, n}$ have the same dimension n. While the space of the eigenvectors (5) of $\hat{H}_I$ with the coefficients k_i given by formula (8) has a dimension 2, there exist only two different solutions for k_i in this case.

4. Comparison with the Bethe ansatz

4.1. Bethe ansatz solutions

Let us recall [10, 11] that in the case of the uniform coupling constants $g_1 = g_2 = .. = g_n = g$, the JC model with the Hamiltonian (1) is exactly solvable by means of the algebraic Bethe ansatz. Let us specify the corresponding 'Bethe ansatz' answers in the case at hand.

The following theorem holds true.

Theorem 4.1.

(i) The spectrum of the Hamiltonian and $\hat{H}_I$ on the 'nested' Bethe vectors has the following form:

$$E_I = \Delta_1 + \sum_{i=1}^{M_1} \frac{1}{v_i^{(1)}}, \tag{9}$$

where rapidities $v_i^{(k)}$, $i \in \overline{1, M_k}$, $k \in \overline{1, n-1}$ satisfy the following Bethe-type equations:

$$-v_i^{(1)}g^2 + (\Delta_1 - \Delta_2) + \frac{1}{v_i^{(1)}} = \sum_{j=1;\, j\neq i}^{M_1} \frac{2}{\left(v_i^{(1)} - v_j^{(1)}\right)} - \sum_{j=1}^{M_2} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)}, \tag{10a}$$

$$(\Delta_{m+1} - \Delta_{m+2}) = \sum_{j=1;\, j\neq i}^{M_{m+1}} \frac{2}{\left(v_i^{(m+1)} - v_j^{(m+1)}\right)} - \sum_{j=1}^{M_m} \frac{1}{\left(v_i^{(m+1)} - v_j^{(m)}\right)}$$
$$- \sum_{j=1}^{M_{m+2}} \frac{1}{\left(v_i^{(m+1)} - v_j^{(m+2)}\right)}, \quad m \in \overline{1, n-3}, \tag{10b}$$

$$(\Delta_{n-1} - \Delta_n) = \sum_{j=1;\, j\neq i}^{M_{n-1}} \frac{2}{\left(v_i^{(n-1)} - v_j^{(n-1)}\right)} - \sum_{j=1}^{M_{n-2}} \frac{1}{\left(v_i^{(n-1)} - v_j^{(n-2)}\right)}. \tag{10c}$$

(ii) The spectrum of the additional integrals $\hat{M}_i$, $i \in \overline{2, n}$ on the 'nested' Bethe vectors has the following explicit form:

$$m_i = (M_{i-1} - M_i), \quad \text{where} \;\; M_n = 0. \tag{11}$$

Proof. This theorem is a corollary of the more general theorem from the papers [10, 11]. Its statement is obtained by putting the number of generalized spin operators in the corresponding theorems of [10, 11] equal to 1 ($N = 1$) and by the specification of the form of the highest weight vector $\lambda^{(1)} = (1, 0, \ldots, 0)$ of the finite-dimensional representation of $gl(n)$ and auxiliary diagonal matrices $K = \mathrm{diag}(k_1, k_2, k_2, \ldots, k_2)$ and $C = \mathrm{diag}(c_1, c_2, \ldots, c_n) = \mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$, where the interaction strength is connected with the elements of the matrix K as follows: $g^2 = k_2 - k_1$. After such an identification, the integrable Hamiltonian from [10, 11] describing the n-level, $n - 1$-mode subcase of the Jaynes–Cummings model ($N = 1$, $\nu_1 = 0$ in the terminology of [10, 11]) acquires the required form:

$$\hat{H}_I = \sum_{i=1}^{n} \Delta_i \hat{S}_{ii} + g \sum_{i=2}^{n} \left(\hat{b}_i^+ \hat{S}_{i1} + \hat{b}_i^- \hat{S}_{1i}\right).$$

Observe that for the chosen representation of $gl(n)$, we have $\hat{S}_{ij} \equiv \hat{X}_{ij}$.

The formula for the spectrum of the Hamiltonian $\hat{H}_I$

$$h_I = \sum_{i=1}^{n} c_i\lambda_i^{(1)} + \lambda_1^{(1)} \sum_{i=1}^{M_1} \frac{1}{v_i^{(1)}} - \sum_{k=2}^{n-1} \lambda_k^{(1)} \left(\sum_{i=1}^{M_{k-1}} \frac{1}{v_i^{(k-1)}} + \sum_{i=1}^{M_k} \frac{1}{v_i^{(k)}} \right) - \lambda_n^{(1)} \sum_{i=1}^{M_{n-1}} \frac{1}{v_i^{(n-1)}}.$$

acquires in this case the simple form (9).

In a similar way, substituting the above data into the Bethe equations for the n-level, $n-1$-mode subcase of the Jaynes–Cummings model from [11], we obtain the Bethe equations (10) and the formula for the spectrum of the additional integrals (11). The theorem is proven. □

4.2. *Derivation of Rabi equations from the Bethe ansatz equations*

In the case of arbitrary n, a derivation of equations (3) from the nested Bethe ansatz equations is a difficult combinatorial problem. We will solve it only in the ‘degenerated’ case $\Delta_2 = \Delta_3 = \ldots = \Delta_n = 0$.

The following proposition holds true.

Proposition 4.1. *Let* $\Delta_2 = \Delta_3 = \ldots = \Delta_n = 0$. *Then the ‘nested’ Bethe equations (10) on the subset of their solutions given by*

$$\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{v_i^{(1)}}\frac{1}{v_j^{(2)}} + \sum_{i=1}^{M_2}\sum_{j=1}^{M_3} \frac{1}{v_i^{(2)}}\frac{1}{v_j^{(3)}} + \cdots + \sum_{i=1}^{M_{n-2}}\sum_{j=1}^{M_{n-1}} \frac{1}{v_i^{(n-2)}}\frac{1}{v_j^{(n-1)}}$$
$$= \sum_{i,j=1, i\neq j}^{M_2} \frac{1}{v_i^{(2)}}\frac{1}{v_j^{(2)}} + \sum_{i,j=1, i\neq j}^{M_3} \frac{1}{v_i^{(3)}}\frac{1}{v_j^{(3)}} + \cdots + \sum_{i,j=1, i\neq j}^{M_{n-1}} \frac{1}{v_i^{(n-1)}}\frac{1}{v_j^{(n-1)}} \qquad (12)$$

yield equation (7), i.e. the following equation:

$$E_I^2 - \Delta_1 E_I - g^2 \sum_{i=2}^{n} m_i = 0.$$

Proof. In order to prove this proposition, it is necessary to consider in more detail the nested Bethe equations (10). The first of these equations in the case at hand has the following form:

$$-g^2 v_i^{(1)} + \Delta_1 + \frac{1}{v_i^{(1)}} = \sum_{j=1; j\neq i}^{M_1} \frac{2}{\left(v_i^{(1)} - v_j^{(1)}\right)} - \sum_{j=1}^{M_2} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)}.$$

Multiplying these equations by $\frac{1}{v_i^{(1)}}$ and summing them with respect to the index i, we obtain the following equation:

$$-M_1 g^2 + \Delta_1 \sum_{i=1}^{M_1} \frac{1}{v_i^{(1)}} + \left(\sum_{i=1}^{M_1} \frac{1}{v_i^{(1)}} \right)^2 = -\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{v_i^{(1)}} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)}.$$

Let us transform the expression $\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{v_i^{(1)}} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)}$. We have

$$\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{v_i^{(1)}} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)} = -\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{v_i^{(1)}}\frac{1}{v_j^{(2)}} - \sum_{j=1}^{M_1}\sum_{i=1}^{M_2} \frac{1}{v_i^{(2)}} \frac{1}{\left(v_i^{(2)} - v_j^{(1)}\right)}.$$

Now using the next set of Bethe equations, namely the equations

$$0 = \sum_{j=1; j\neq i}^{M_2} \frac{2}{\left(v_i^{(2)} - v_j^{(2)}\right)} - \sum_{j=1}^{M_1} \frac{1}{\left(v_i^{(2)} - v_j^{(1)}\right)} - \sum_{j=1}^{M_3} \frac{1}{\left(v_i^{(2)} - v_j^{(3)}\right)},$$

multiplying them by $\frac{1}{v_i^{(2)}}$ and summing them with respect to the index i, we obtain that

$$\sum_{i=1}^{M_2}\sum_{j=1}^{M_1}\frac{1}{v_i^{(2)}}\frac{1}{\left(v_i^{(2)}-v_j^{(1)}\right)} = -\sum_{i,j=1}^{M_2}\frac{1}{v_i^{(2)}}\frac{1}{v_j^{(2)}}+\sum_{i=1}^{M_3}\sum_{j=1}^{M_2}\frac{1}{v_i^{(3)}}\frac{1}{v_j^{(2)}}+\sum_{j=1}^{M_2}\sum_{i=1}^{M_3}\frac{1}{v_i^{(3)}}\frac{1}{\left(v_i^{(3)}-v_j^{(2)}\right)}.$$

Proceeding further in a similar way, using the next Bethe equations, we obtain

$$-\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\frac{1}{v_i^{(1)}}\frac{1}{(v_i^{(1)}-v_j^{(2)})} = \sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\frac{1}{v_i^{(1)}}\frac{1}{v_j^{(2)}}+\sum_{i=1}^{M_2}\sum_{j=1}^{M_3}\frac{1}{v_i^{(2)}}\frac{1}{v_j^{(3)}}+\cdots$$
$$+\sum_{i=1}^{M_{n-2}}\sum_{j=1}^{M_{n-1}}\frac{1}{v_i^{(n-2)}}\frac{1}{v_j^{(n-1)}}-\left(\sum_{i,j=1,i\neq j}^{M_2}\frac{1}{v_i^{(2)}}\frac{1}{v_j^{(2)}}\right.$$
$$\left.+\sum_{i,j=1,i\neq j}^{M_3}\frac{1}{v_i^{(3)}}\frac{1}{v_j^{(3)}}+\cdots+\sum_{i,j=1,i\neq j}^{M_{n-1}}\frac{1}{v_i^{(n-1)}}\frac{1}{v_j^{(n-1)}}\right).$$

Taking into account that $m_i = M_{i-1} - M_i$, we obtain that $\sum_{i=2}^{n} m_i = M_1$. Taking further into account that by the very definition $E_I = \Delta_1 + \sum_{i=1}^{M_1}\frac{1}{v_i^{(1)}}$ and additional condition (12), we derive equation (7). The proposition is proven. □

Remark 3. Observe that equation (12) imposes an additional restriction on the rapidities $v_i^{(k)}$ and onto the corresponding Bethe vectors. This is in good agreement with the above-mentioned fact that for $n > 2$, the constructed eigenvectors, corresponding to the eigenvalues satisfying (7), do not form a complete family, i.e. in this case the family of eigenvectors provided by the nested Bethe ansatz is wider than the set of the eigenvectors obtained by the direct method.

Remark 4. Observe that when $n = 2$, one can put $\Delta_2 = 0$ without loss of generality. Moreover, in this case there is only one set of rapidities $v_i^{(1)}$, $i \in \overline{1, M_1}$. That is why there is no additional condition (12) in this case. In other words, the algebraic Bethe ansatz method and the direct method produce the same results in the case $n = 2$. Let us also remark that the $n = 2$ Bethe equations for the Jaynes–Cummings and Dicke models were first derived in [12] and re-derived later in [13].

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