

## Skyrmion multi-walls

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2009 J. Phys. A: Math. Theor. 42 482001

(<http://iopscience.iop.org/1751-8121/42/48/482001>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 38.107.179.212

The article was downloaded on 22/02/2012 at 10:15

Please note that [terms and conditions apply](#).

## FAST TRACK COMMUNICATION

# Skyrmion multi-walls

**J Silva Lobo and R S Ward**

Department of Mathematical Sciences, Durham University, Durham DH1 3LE, UK

E-mail: [j.i.silva-lobo@durham.ac.uk](mailto:j.i.silva-lobo@durham.ac.uk) and [richard.ward@durham.ac.uk](mailto:richard.ward@durham.ac.uk)

Received 23 October 2009, in final form 28 October 2009

Published 17 November 2009

Online at [stacks.iop.org/JPhysA/42/482001](http://stacks.iop.org/JPhysA/42/482001)**Abstract**

Skyrmion walls are topologically nontrivial solutions of the Skyrme system which are periodic in two spatial directions. We report numerical investigations which show that solutions representing parallel multi-walls exist. The most stable configuration is that of the square  $N$ -wall, which in the  $N \rightarrow \infty$  limit becomes the cubically symmetric Skyrme crystal. There is also a solution resembling parallel hexagonal walls, but this is less stable.

PACS numbers: 11.27.+d, 11.10.Lm, 11.15.-q

The Skyrme system, originally introduced as a model of nucleons, is an archetypal (3+1)-dimensional classical field theory admitting topological soliton solutions. Much is known about various types of skyrmion solutions, for example: isolated skyrmions in  $\mathbb{R}^3$ , up to relatively high charge [1–3]; a triply-periodic ‘Skyrme crystal’ [2, 4–6]; a doubly-periodic ‘Skyrme domain wall’ [7] and various types of singly-periodic ‘Skyrme chains’ [8].

The purpose of this communication is to investigate static  $N$ -wall solutions, i.e. the  $N > 1$  generalization of the single-wall fields discussed in [7]. If one has two (or indeed  $N$ ) well-separated parallel walls, then the force between them can be made attractive by a suitable relative orientation of the fields. So one expects there to be solutions representing  $N$  walls bound together, although *a priori* the walls might merge together to form a single wall.

We investigate this by numerical minimization of the energy, and our main findings are as follows. There are two obvious shapes for a single wall, namely square and hexagonal, and it is known [7] that the latter has slightly lower energy than the former. If walls are allowed to attract, then they do not merge but remain identifiable as separate parallel walls. There is a stable bound configuration representing two parallel hexagonal walls, but this is not the lowest energy 2-wall state. For  $N \geq 2$ , the lowest energy state consists of  $N$  parallel square walls (each one being a square array of half-skyrmions), and as  $N \rightarrow \infty$  this approaches the Skyrme crystal.

The energy density of a static  $SU(2)$ -valued Skyrme field  $U(x^j)$  on  $\mathbb{R}^3$  is defined to be

$$\mathcal{E} := -\frac{1}{2} \text{tr}(L_i L_i) - \frac{1}{16} \text{tr}([L_i, L_j][L_i, L_j]), \quad (1)$$

where  $L_i = U^{-1} \partial U / \partial x^i$ , and  $x^j = (x, y, z)$  are the spatial coordinates. In what follows, let us write  $U = \Phi_4 + i\Phi_j \sigma_j$ , where  $\sigma_j$  are the Pauli matrices, and the real 4-vector field  $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$  satisfies  $\Phi \cdot \Phi = 1$ .

In this communication, we deal with configurations which resemble  $N$  walls or sheets, each parallel to the  $xy$ -plane: so the field is periodic in  $x$  and  $y$  (with periods  $L_x$  and  $L_y$ , respectively) and satisfies the boundary condition

$$\Phi_4 \rightarrow \begin{cases} 1 & \text{as } z \rightarrow -\infty, \\ (-1)^N & \text{as } z \rightarrow \infty. \end{cases} \quad (2)$$

For  $N = 1$ , and more generally for  $N$  odd, one has a domain wall which separates two vacuum regions, where  $\Phi_4 = 1$  and  $\Phi_4 = -1$ , respectively; for  $N$  even, one has the same vacuum on both sides of the multi-layered sheet. In the asymptotic region  $|z| \gg 1$ , the three fields  $\Phi_j$  are small and they satisfy the Laplace equation, since the energy density reduces to  $\mathcal{E} \approx (\partial_i \Phi_j)^2$ . Assuming (without loss of generality) that  $L_y \geq L_x$ , we see by separating variables that the leading behaviour as  $|z| \rightarrow \infty$  is typically  $\Phi_j \approx C \sin(\mu y) \exp(-\mu|z|)$ , where  $\mu = 2\pi/L_y$ . In particular, the fields approach their asymptotic values exponentially quickly, with a scale determined by the larger of  $L_x$  and  $L_y$ .

The topological charge  $Q$  (over a single cell) is

$$Q = \int_{T^2 \times \mathbb{R}} \mathcal{Q} \, dx \, dy \, dz, \quad (3)$$

where

$$\mathcal{Q} = \frac{1}{24\pi^2} \varepsilon_{ijk} \text{tr}(L_i L_j L_k) \quad (4)$$

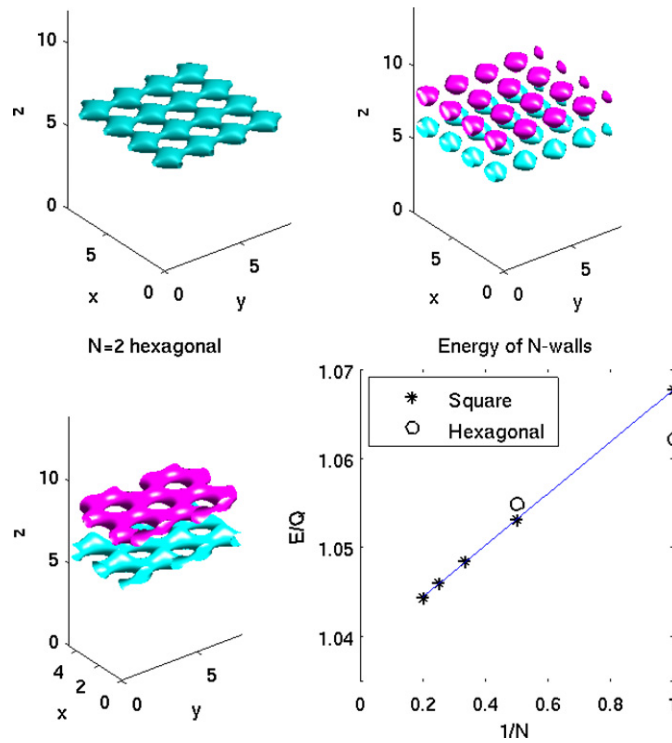
is the topological charge density. We claim that  $Q$  is an integer. If  $N$  is even, then (2) allows us to regard  $\Phi$  as being defined, for topological purposes, on  $T^2 \times S^1$ , and then  $Q$  equals the degree of  $\Phi$ . If  $N$  is odd, then it is not quite so obvious why  $Q$  is an integer, but it follows from the theorem in the appendix of [8]. The energy  $E$  is defined to be

$$E := \frac{1}{12\pi^2} \int_{T^2 \times \mathbb{R}} \mathcal{E} \, dx \, dy \, dz, \quad (5)$$

and it satisfies the usual Faddeev bound  $E \geq Q$ .

In what follows, we describe  $N$ -wall configurations which were found by numerical minimization of the energy functional  $E$ . We used a first-order finite-difference scheme for  $E$ , with the spatial points  $(x, y, z)$  being represented by a rectangular lattice having lattice spacing  $h$ , and we applied conjugate-gradient minimization. The lattice error in  $E$  goes like  $h^2$ , and we extrapolated the finite- $h$  results for both  $E$  and  $Q$  to  $h = 0$ . The extrapolated value of  $Q$  then gives a measure of the remaining error, which for the situations described below turns out to be less than 0.2%. The boundary condition (2) was modelled by imposing  $\Phi_4 = 1$  at  $z = -L_z/2$  and  $\Phi_4 = (-1)^N$  at  $z = L_z/2$ . As remarked above, the walls are exponentially localized in  $z$ , and so as long as  $L_z$  is taken to be large enough, there is no discernable finite-size effect; a value of  $L_z = 10 + 2N$  turns out to be sufficient for this. In each case, we adjusted the periods  $L_x$  and  $L_y$  to their optimal size, meaning that the energy-per-cell is made as small as possible. Numerical minima were randomly perturbed and then re-minimized, as a test of their stability. As initial configurations we used the same sort of ‘rational map ansatz’ as in [7], involving a Weierstrass elliptic function of  $x + iy$  (the lemniscatic form to get square symmetry, and the equianharmonic form to get hexagonal symmetry), together with a suitable profile function  $f(z)$  satisfying  $f(-L_z/2) = 0$  and  $f(L_z/2) = N\pi$ .

The results are consistent with the anticipated general principle that the lowest-energy configurations are arrays of half-skyrmions. For an  $N$ -wall, we expect that each fundamental



**Figure 1.** Energy densities of the square 1-wall, square 2-wall and hexagonal 2-wall, and plot of the energy  $\hat{E}$  for the square  $N$ -wall ( $1 \leq N \leq 5$ ) and hexagonal  $N$ -wall ( $1 \leq N \leq 2$ ). (This figure is in colour only in the electronic version)

**Table 1.** Energy  $\hat{E}$  and cell size  $L$  for the square  $N$ -wall.

$N$	$\hat{E}$	$L$
1	1.068	4.25
2	1.053	4.47
3	1.048	4.54
4	1.046	4.58
5	1.044	4.61

cell will contain a multiple of  $4N$  half-skyrmions, and therefore its topological charge  $Q$  will be a multiple of  $2N$ ; this indeed turns out to be the case. As mentioned above, the walls do not merge, but retain their identity; the location of each wall can be determined by looking at the locus where  $\Phi_4 = 0$ .

The simplest case to describe is the square one, with  $L_y = L_x = L$ ; our results for  $1 \leq N \leq 5$  are summarized in table 1, which gives the energy-per-charge  $\hat{E}$ , and the optimal value of  $L$ , for each  $N$ . Pictures of the  $N = 1$  and  $N = 2$  cases are presented in figure 1, together with a plot of the energy data in table 1. Let us first comment on the data. The normalized energy  $\hat{E}$  of the square  $N$ -wall is surprisingly close to having a  $1/N$ -dependence (although there is no obvious reason why this should be so), and extrapolating on this basis gives  $\hat{E} \approx 1.039$  in the  $N \rightarrow \infty$  limit. This is very close to the energy of the (triply-periodic) Skyrme crystal, a cubic array in which each fundamental cube contains eight half-skyrmions:

its energy-per-charge, computed using the method described above, is  $\widehat{E} = 1.038$ . Further support for the claim that the square  $N$ -wall tends to the Skyrme crystal as  $N \rightarrow \infty$  comes from looking at the symmetries of the field  $\Phi$ . These include, for example, the translations

$$\begin{aligned} x \mapsto x + \frac{1}{2}L_x &\Rightarrow (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (-\Phi_1, -\Phi_2, \Phi_3, \Phi_4), \\ y \mapsto y + \frac{1}{2}L_y &\Rightarrow (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (\Phi_1, -\Phi_2, -\Phi_3, \Phi_4), \\ z_p \mapsto z_{p+1} &\Rightarrow (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (\Phi_1, -\Phi_2, \Phi_3, -\Phi_4), \end{aligned}$$

where the third translation (in  $z$ ) denotes moving from the  $p$ th wall to the  $(p+1)$ st wall. These are exactly the same as the translation symmetries of the Skyrme crystal [2]. The values for the optimal cell length  $L = L_x = L_y$  are consistent with their approaching  $L = 4.7$  as  $N \rightarrow \infty$ , this being the cell size of the Skyrme crystal (and similarly the distance between each parallel pair of walls is approximately  $4.7/2$ , as one would expect).

Each three-dimensional plot in figure 1 is an isosurface of the energy density  $\mathcal{E}$ , namely where  $\mathcal{E}$  equals 0.6 times its maximum value. For the square case, the plots are over four fundamental cells. One clearly sees square arrays of half-skyrmions. Observe that, for  $N = 2$ , the half-skyrmions are aligned in the  $z$ -direction; the same is true for  $N > 2$ .

Let us turn now to the case of hexagonal symmetry. For ease of computation, we follow the same scheme as in [7], namely taking  $L_y = \sqrt{3}L_x$  and fitting two fundamental parallelograms into the corresponding rectangle. Each such rectangle, of each wall, contains eight half-skyrmions, as is seen in the hexagonal 2-wall picture of figure 1. For  $N = 1$ , the energy of the hexagonal arrangement is  $\widehat{E} \approx 1.062$ , less than that of the corresponding square case [7], but for  $N \geq 2$ , the hexagonal arrangement is less efficient than the square one, and (depending on the values of  $L_x$  and  $L_y$ ) it is either a local minimum of the energy functional or it is unstable. There is a local minimum hexagonal 2-wall solution with energy  $\widehat{E} \approx 1.055$ , which is only very slightly (less than 0.2%) higher than that of the square 2-wall. Its energy density is depicted in figure 1; one feature to note is that the two walls are not aligned in the  $z$ -direction, but are offset. If  $L_x$  and  $L_y$  are allowed to change so that the relation  $L_y = \sqrt{3}L_x$  no longer holds, then this solution becomes unstable and changes into the square 2-wall.

An isolated skyrmion of charge  $Q \geq 3$  typically has a polyhedral shell structure, analogous to carbon fullerenes, and it may be viewed as constructed from a section of the hexagonal 1-wall (graphene), with the insertion of defects to create a spherical shell [2, 7]. There has also been an investigation [9] of the possibility of constructing skyrmions as multi-walled spherical shells, with the ‘shell material’ consisting of a double or triple wall. For the cases that were examined in [9], either the walls coalesced, or one obtained a structure which resembled a shell-like part of the Skyrme crystal. The findings reported above are consistent with this; in particular, multiple hexagonal walls appear to be rather unstable, and therefore unsuitable for constructing shells. But it does raise the possibility of stable high-charge skyrmions constructed as shells of square multi-wall material, or equivalently as hollow chunks of Skyrme crystal, and this would be worth investigating further.

## Acknowledgment

Support from the UK Science and Technology Facilities Council, through the Rolling Grant ‘Particles, Fields and Spacetime’, is gratefully acknowledged.

## References

- [1] Batty R and Sutcliffe P M 2002 Skyrmions, fullerenes and rational maps. *Rev. Math. Phys.* **14** 29–85
- [2] Manton N S and Sutcliffe P M 2004 *Topological Solitons* (Cambridge: Cambridge University Press)

- [3] Battye R, Manton N S and Sutcliffe P M 2007 Skyrmions and the  $\alpha$ -particle model of nuclei *Proc. R. Soc. A* **463** 261–79
- [4] Klebanov I 1985 Nuclear matter in the Skyrme model *Nucl. Phys. B* **262** 133–43
- [5] Kugler M and Shtrikman S 1988 A new Skyrmion crystal *Phys. Lett. B* **208** 491–4
- [6] Castillejo L, Jones P S J, Jackson A D, Verbaarschot J J M and Jackson A 1989 Dense Skyrmion systems *Nucl. Phys. A* **501** 801–12
- [7] Battye R and Sutcliffe P M 1998 A Skyrme lattice with hexagonal symmetry *Phys. Lett. B* **416** 385–91
- [8] Harland D and Ward R S 2008 Chains of skyrmions *J. High Energy Phys.* JHEP12(2008)093
- [9] Manton N S and Piette B M A G 2001 Understanding skyrmions using rational maps *Prog. Math.* **201** 469–80