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FAST TRACK COMMUNICATION

Multi-operator brackets acting thrice

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Abstract
We generalize an identity first found by Bremner for Nambu 3-brackets. For odd N-brackets built from associative operator products, we show that

$$[[A[B_1 \cdots B_N]B_{N+1} \cdots B_{2N-2}]B_{2N-1} \cdots B_{3N-3}]$$
$$= [[AB_1 \cdots B_{N-1}][B_N \cdots B_{2N-1}]B_{2N} \cdots B_{3N-3}]$$

for any fixed A, when totally antisymmetrized over all the Bs.

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1. Introduction

Nambu introduced a multilinear operator bracket in the context of a novel formulation of mechanics [13]. His N-bracket is defined by

$$[A_1 A_2 \cdots A_N] = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) A_{\sigma_1} \cdots A_{\sigma_N}, \tag{1}$$

where the sum is over all $N!$ permutations of the operators. For example, $[ABC] = ABC - ACB + BCA - BAC + CAB - CBA$. The operator product is assumed to be associative. To avoid ambiguities when some of the entries within a bracket are themselves products, commas are often used to separate the entries. Parentheses also suffice in such cases. For example, $[AD, B, C] = [(AD)BC] = ADBC - ADCB + BCAD - BADC + CADB - CBAD$.

The same construction independently appeared in the mathematical literature [10, 11]. The theory of such multi-operator products, as well as their 'classical limits' in terms of multivariable Jacobians, has been studied extensively [1–4, 6–9, 12, 14–18].

From an algebraic point of view, it is natural to seek the analogue of the Jacobi identity for N-brackets. For the case of even N-brackets, the obvious generalization where one N-bracket acts on another leads to a true identity *if* all entries are totally antisymmetrized (see (5) below). But for odd N-brackets this procedure does not work [1, 4]—the total antisymmetrization over all entries of one odd N-bracket acting on another does not vanish, but rather yields a higher order $(2N-1)$-bracket.

Nevertheless, an interesting generalization of the Jacobi identity was discovered by Bremner for 3-brackets acting three times [1]. He showed

$$[[A[bcd]e]fg] = [[Abc][def]g], \tag{2}$$

where A is fixed, but it is implicitly understood that lowercase entries are totally antisymmetrized by summing over all 6! signed permutations of them. The point of this short communication is to show that Bremner's identity generalizes to all odd N-brackets.

Before discussing the general case, we anticipate and indicate a proof for the case of 3-brackets. The Bremner identity can be proved through a resolution of both left- and right-hand sides as a series of canonically ordered words. By direct calculation we find

$$\begin{aligned}[[A[bcd]e]fg] &= 24Abcdefg - 36bAcdefg + 36bcAdefg - 24bcdAefg \\ &\quad + 36bcdeAfg - 36bcdefAg + 24bcdefgA,\end{aligned} \tag{3}$$

where all lowercase entries are implicitly totally antisymmetrized. Precisely the same expansion holds for $[[Abc][def]g]$, again by direct calculation. Hence, the identity is established.

That is to say, both $[[A[bcd]e]fg]$ and $[[Abc][def]g]$ can be rendered as a 7-bracket plus another 3-bracket containing 3-brackets, when antisymmetrized over lowercase entries:

$$[[A[bcd]e]fg] = \tfrac{1}{20}[Abcdefg] - \tfrac{1}{6}[A[bcd][efg]] = [[Abc][def]g]. \tag{4}$$

Thus, the Bremner identity amounts to the combinatorial statement, as written, that there are two distinct ways to present a 7-bracket in terms of nested 3-brackets.

2. Results for any N

As known, and previously mentioned, even brackets need only act twice to yield an identity. Namely [4, 9]

$$[B_1 \cdots B_{N-1}[B_N \cdots B_{2N-1}]] = 0 \qquad \text{for } N \text{ even}. \tag{5}$$

Total antisymmetrization of all the Bs is understood[1]. When $N = 2$ this is the familiar Jacobi identity. The proof is by direct calculation and follows as a consequence of associativity.

However, for odd N, $[B_1 \cdots B_{N-1}[B_N \cdots B_{2N-1}]] \neq 0$, but instead produces the $(2N-1)$-bracket $[B_1 \cdots B_{2N-1}]$ upon total antisymmetrization [3, 4]. Apparently, the simplest identity obeyed by odd brackets of only one type, that does *not* introduce higher order brackets, requires that they act at least three times. For any odd $N = 2L + 1$, a valid relation is the immediate generalization of that found by Bremner for the case of 3-brackets. To show this, we present two easily established lemmata, followed by our main theorem and its proof. Firstly,

Lemma 1.

$$[\mathcal{A}B_1 \cdots B_J] = J!\sum_{j=0}^{J}(-1)^j B_1 \cdots B_j \mathcal{A} B_{j+1} \cdots B_J. \tag{6}$$

Total antisymmetrization of the Bs is understood. Here we have also used the convention that an empty product equals 1. Explicitly, $B_1 \cdots B_0 = 1 = B_{J+1} \cdots B_J$, so that the first and last terms in the sum are $\mathcal{A}B_1 \cdots B_J$ and $(-1)^J B_1 \cdots B_J \mathcal{A}$, respectively. It is a simple exercise to use this first lemma to prove (5). Similarly,

[1] We will not discuss *all* possible symmetrizations of the bracket entries, but just particular choices that work to give identities for general N. For a thorough study of other symmetrizations, particularly in the 3-bracket case, see [1], as well as the more recent work of Nuyts *et al* [5].

Lemma 2.

$$[\mathcal{A}B_1\cdots B_J\mathcal{Z}] = J!\sum_{j=0}^{J}(-1)^j\sum_{k=0}^{J-j}(-1)^k B_1\cdots B_k\mathcal{A}B_{k+1}\cdots B_{J-j}\mathcal{Z}B_{J-j+1}\cdots B_J$$
$$- J!\sum_{j=0}^{J}(-1)^j\sum_{k=0}^{J-j}(-1)^k B_1\cdots B_k\mathcal{Z}B_{k+1}\cdots B_{J-j}\mathcal{A}B_{J-j+1}\cdots B_J. \tag{7}$$

Finally, it is rather tedious but fairly straightforward to use both lemmata to prove the following.

Theorem 1. *For associative products, with implicit total antisymmetrization of the Bs,*

$$[[A[B_1\cdots B_{2L+1}]B_{2L+2}\cdots B_{4L}]B_{4L+1}\cdots B_{6L}]$$
$$= [[AB_1\cdots B_{2L}][B_{2L+1}\cdots B_{4L+1}]B_{4L+2}\cdots B_{6L}]. \tag{8}$$

Proof. The result (8) follows from resolving the left- and right-hand sides into sums of canonically ordered words, as illustrated above for the case of 3-brackets. We have

$$[[AB_1\cdots B_{2L}][B_{2L+1}\cdots B_{4L+1}]B_{4L+2}\cdots B_{6L}] = \sum_{n=0}^{6L}(-1)^n m_n^{(1)} B_1\cdots B_n A B_{n+1}\cdots B_{6L},$$
$$[[A[B_1\cdots B_{2L+1}]B_{2L+2}\cdots B_{4L}]B_{4L+1}\cdots B_{6L}] = \sum_{n=0}^{6L}(-1)^n m_n^{(2)} B_1\cdots B_n A B_{n+1}\cdots B_{6L}. \tag{9}$$

All the coefficients $m_n^{(1,2)}$ in these two resolutions are manifestly positive integers. The theorem is established by showing that $m_n^{(1)} = m_n^{(2)}$ for all n.

By direct calculation, through the use of the two lemmata, we find

$$m_n^{(1)} = m_n^{(2)} = (2L+1)!(2L)!(2L-1)! \times c_n, \tag{10}$$

$$c_n = \begin{cases} (n+1)(4L-n)/2 & \text{for} \quad 0 \leqslant n \leqslant 2L \\ 10L^2 - 6Ln + L + n^2 & \text{for} \quad 2L+1 \leqslant n \leqslant 3L \\ c_{6L-n} & \text{for} \quad 3L+1 \leqslant n \leqslant 6L \end{cases}. \tag{11}$$

The determination of the m_ns is just a matter of enumerating the ways to obtain a particular intercalation of A among the Bs.

Consider in more detail some of the calculations involved. As a first step, with the implicit antisymmetrization[2], the internal brackets $[B_1\cdots B_{2L+1}]$ or $[B_{2L+1}\cdots B_{4L+1}]$ may be supplanted by products: $[B_1\cdots B_{2L+1}] = (2L+1)!(B_1\cdots B_{2L+1})$ or $[B_{2L+1}\cdots B_{4L+1}] = (2L+1)!(B_{2L+1}\cdots B_{4L+1})$. Then we may write, on the one hand,

$$[[AB_1\cdots B_{2L}][B_{2L+1}\cdots B_{4L+1}]B_{4L+2}\cdots B_{6L}]$$
$$= -(2L+1)![[AB_1\cdots B_{2L}]B_{4L+2}\cdots B_{6L}(B_{2L+1}\cdots B_{4L+1})] \tag{12}$$

[2] To avoid any misunderstanding, by *implicit total antisymmetrization* of the Bs in the expression $[[A\,[B_1\cdots B_{2L+1}]\,B_{2L+2}\cdots B_{4L}]\,B_{4L+1}\cdots B_{6L}]$, we mean

$$\sum_{\sigma\in S_{6N}} \mathrm{sgn}(\sigma)\left[\left[A\left[B_{\sigma_1}\cdots B_{\sigma_{2L+1}}\right]B_{\sigma_{2L+2}}\cdots B_{\sigma_{4L}}\right]B_{\sigma_{4L+1}}\cdots B_{\sigma_{6L}}\right],$$

where the sum is over all $(6N)!$ permutations of the indices, $1,\ldots,6N$. Similar meanings apply to the other implicitly antisymmetrized expressions in the communication.

In this expression, we may now rename indices, bearing in mind the antisymmetrization:

$$[[AB_1\cdots B_{2L}][B_{2L+1}\cdots B_{4L+1}]B_{4L+2}\cdots B_{6L}] \\ = (2L+1)![[AB_{2L}\cdots B_{4L-1}]B_1\cdots B_{2L-1}(B_{4L}\cdots B_{6L})]. \tag{13}$$

Next, we apply lemma 2 for $J = 2L-1$, and identify $[AB_{2L}\cdots B_{4L-1}]$ with $\mathcal{A}$, and $(B_{4L}\cdots B_{6L})$ with $\mathcal{Z}$:

$$[\mathcal{A}B_1\cdots B_{2L-1}\mathcal{Z}] = (2L-1)!\sum_{j=0}^{2L-1}(-1)^j \\ \times \sum_{k=0}^{2L-1-j}(-1)^k B_1\cdots B_k\mathcal{A}B_{k+1}\cdots B_{2L-1-j}\mathcal{Z}B_{2L-j}\cdots B_{2L-1} \tag{14a}$$

$$-(2L-1)!\sum_{j=0}^{2L-1}(-1)^j\sum_{k=0}^{2L-1-j}(-1)^k B_1\cdots B_k\mathcal{Z}B_{k+1}\cdots B_{2L-1-j}\mathcal{A}B_{2L-j}\cdots B_{2L-1}. \tag{14b}$$

To continue, consider first the coefficients $m_n^{(1)}$ where $n\leqslant 2L$.

For the determination of $m^{(1)}_{n\leqslant 2L}$, since $\mathcal{Z}$ consists of $(2L+1)$ Bs, it *must* be placed to the *right* of $\mathcal{A}$ in the application of lemma 2. Otherwise there would be too many Bs to the left of A. Thus, for $m^{(1)}_{n\leqslant 2L}$ we need keep only the first line in the last relation, (14*a*). To place a total of n Bs to the left of the A contained in $\mathcal{A} = [AB_{2L}\cdots B_{4L-1}]$, with k Bs already to the left as in (14*a*), we then need only the terms in $\mathcal{A}$ with an additional $(n-k)$ Bs to the left of A. That is to say, from lemma 1, with $J = 2L$ and all B indices shifted up by $2L-1$,

$$\mathcal{A} = [AB_{2L}\cdots B_{4L-1}] = (2L)!\sum_{l=0}^{2L}(-1)^l B_{2L}\cdots B_{l+2L-1}AB_{l+2L}\cdots B_{4L-1}, \tag{15}$$

and from this we need only the term with $l = n-k$. The net result for $m^{(1)}_{n\leqslant 2L}$ is

$$m^{(1)}_{n\leqslant 2L} = (2L+1)!(2L)!(2L-1)!\times c_{n\leqslant 2L}, \tag{16}$$

$$c_{n\leqslant 2L} = \sum_{j=0}^{2L-1}\sum_{k=0}^{2L-1-j}\sum_{l=0}^{2L}\delta_{l,n-k}\Bigg|_{n\leqslant 2L} = \sum_{j=0}^{2L-1}\sum_{k=0}^{\min(n,2L-1-j)}1 = \frac{(n+1)(4L-n)}{2}. \tag{17}$$

On the other hand, with similar steps, we have

$$[[A\,[B_1\cdots B_{2L+1}]\,B_{2L+2}\cdots B_{4L}]\,B_{4L+1}\cdots B_{6L}] \\ = (2L+1)!\,[[AB_1\cdots B_{2L-1}(B_{2L}\cdots B_{4L})]\,B_{4L+1}\cdots B_{6L}]\,. \tag{18}$$

We again apply lemma 2 for $J = 2L-1$, but to $[AB_1\cdots B_{2L-1}(B_{2L}\cdots B_{4L})]$, so now we identify A with $\mathcal{A}$, and $(B_{2L}\cdots B_{4L})$ with $\mathcal{Z}$. As before, consider first only $m_n^{(2)}$ coefficients where $n\leqslant 2L$. For the determination of $m^{(2)}_{n\leqslant 2L}$, $\mathcal{Z}$ *must* once again be placed to the *right* of $\mathcal{A}$, so we need keep only the line (14*a*). We pick up an additional $(n-k)$ Bs by applying again lemma 1, only this time to the remaining *outside* bracket in (18). The net result for $m^{(2)}_{n\leqslant 2L}$ is

$$m^{(2)}_{n\leqslant 2L} = (2L+1)!(2L)!(2L-1)!\times c_{n\leqslant 2L}, \tag{19}$$

with exactly the same expression for $c_{n\leqslant 2L}$ as before, (17). Thus, we have shown $m^{(1)}_{n\leqslant 2L} = m^{(2)}_{n\leqslant 2L}$.

Next, consider the coefficients where $2L+1 \leqslant n \leqslant 3L$. There are still contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ from the line (14*a*), as above, of the form $(2L+1)!(2L)!(2L-1)! \times c_n$, and these contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ still turn out to be the same. But in this case the sums contributing to c_n give

$$\sum_{j=0}^{2L-1} \sum_{k=0}^{2L-1-j} \sum_{l=0}^{2L} \delta_{l,n-k} \Bigg|_{2L+1 \leqslant n \leqslant 3L} = \sum_{j=0}^{2L-1-(n-2L)} \sum_{k=(n-2L)}^{2L-1-j} 1 = \frac{1}{2}(4L+1-n)(4L-n). \tag{20}$$

Moreover, from applying lemma 2, there are now contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ from the second line, (14*b*), where the respective $\mathcal{Z}$s are placed to the *left* of the $\mathcal{A}$s. Following steps similar to those above, it is not difficult to see that these other terms contribute the *same amount* to either $m_n^{(1)}$ or $m_n^{(2)}$, for $2L+1 \leqslant n \leqslant 3L$. Namely, $(2L+1)!(2L)!(2L-1)! \times$

$$\sum_{j=0}^{2L-1} \sum_{k=0}^{2L-1-j} \sum_{l=0}^{2L} \delta_{l,n+j-4L} \Bigg|_{2L+1 \leqslant n \leqslant 3L} = \sum_{j=4L-n}^{2L-1} \sum_{k=0}^{2L-1-j} 1 = \frac{1}{2}(n-2L+1)(n-2L). \tag{21}$$

Thus, the net result is $m^{(1)}_{2L+1 \leqslant n \leqslant 3L} = m^{(2)}_{2L+1 \leqslant n \leqslant 3L} = (2L+1)!(2L)!(2L-1)! \times c_{2L+1 \leqslant n \leqslant 3L}$ with

$$c_{2L+1 \leqslant n \leqslant 3L} = \tfrac{1}{2}(4L+1-n)(4L-n) + \tfrac{1}{2}(n-2L+1)(n-2L) = 10L^2 - 6Ln + L + n^2. \tag{22}$$

Finally, consider the coefficients for $3L+1 \leqslant n \leqslant 6L$. These are given by an elementary reflection symmetry: $m_n^{(1)} = m_{6L-n}^{(1)}$ and $m_n^{(2)} = m_{6L-n}^{(2)}$. Thus, $m_n^{(1)} = m_n^{(2)} = (2L+1)!(2L)!(2L-1)! \times c_{6L-n}$ for $3L+1 \leqslant n \leqslant 6L$. □

As a check, the coefficients must sum to give the number of generic terms that appear in three nested $(2L+1)$-brackets (i.e. in either $[[[\cdots]\cdots]\cdots]$ or $[[\cdots]\cdots[\cdots]]$). That is $\sum_{n=0}^{6L} m_n = ((2L+1)!)^3$. Equivalently,

$$\sum_{n=0}^{6L} c_n = 2L(2L+1)^2. \tag{23}$$

This condition is indeed satisfied by the c_n given in (11).

3. Conclusion

Perhaps *N*-brackets and algebras have an important role to play in physics, as originally suggested by Nambu. Recently there has been considerable interest in *N*-brackets, especially 3-brackets, as expressed in the physics literature (see [2] and references therein). These ideas await further development.

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