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# Tomography in Hilbert spaces

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**Abstract.** We present a method of constructing the tomographic probability distributions describing quantum states in parallel with density operators in abstract Hilbert spaces. After the study of an infinite dimensional example, the well known Husimi-Kano quasi-distribution is reconsidered in the new setting and a new tomographic scheme based on coherent states is suggested. Starting from the Sudarshan's diagonal coherent state representation, the associated identity decomposition providing Gram-Schmidt operators is explicitly given.

## 1. Introduction

There are several representations of quantum states providing the possibility to present different, but equivalent, formulations of quantum mechanics [1]. These representations are based on different integral transforms of the density operator [2, 3] taken in the position representation. The density operator in the position representation is mapped by means of the integral transforms either to Wigner quasi-distribution function [4], or Husimi-Kano  $K$ -function [5, 6]. In this paper we have decided to keep up with the original notations of the pioneer papers on the subject. Quasi-distributions are usually referred to as phase space representations of quantum states. Another important phase space representation is related to the Sudarshan's  $\phi$ -diagonal coherent state representation [7, 8].

Recently the tomographic representations of quantum states was suggested [9, 10, 11] using the Radon integral transform of their Wigner functions. The tomographic representation exhibits some specific property in comparison with the other phase space representations. The tomographic probability distributions (tomograms) associated with quantum states are standard positive probability distributions. The mathematical mechanism of constructing the tomographic probabilities in abstract Hilbert spaces was first elucidated in [12] for the finite dimensional case and in [13] for the infinite dimensional case.

As in the usual formulation of quantum mechanics there are several schemes like Schrödinger picture, Heisenberg picture, Dirac picture, even in the tomographic approach exist several different schemes, like symplectic tomography, photon number tomography and so on. Our aim in this paper is to point out the common general mechanism of constructing all these tomographic schemes and in this way to extend the list of the tomographies by including into the construction properties of the coherent states [14] and properties of  $K$ -function and Sudarshan's diagonal coherent state representation [15] in the sense of their relation to distributions (i.e., generalized functions). We call this extension *coherent state tomography*. From this point of view the Husimi-Kano  $K$ -function will be interpreted as a tomogram of a quantum state.

The paper is organized as follows. After a short account of our construction of a tomographic setting in abstract Hilbert spaces in section 2, an infinite dimensional example is discussed in section 3. The coherent state tomography is developed in section 4, where a resolution of the unity is obtained which provides a relation between Sudarshan's diagonal coherent state representation and Husimi-Kano  $K$ -function in the suggested tomographic setting. In section 5, some further developments and perspectives are briefly drawn.

## 2. Tomography in abstract Hilbert spaces

We have given [12, 13] an interpretation of quantum tomography in an abstract Hilbert space  $\mathcal{H}$  by means of complete sets of rank-one projectors  $\{P_\mu\}_{\mu \in M}$ , where  $M$  is a set of (multi-) parameters, discrete or continuous, collectively denoted by  $\mu$ .

In general, a tomogram of a quantum state  $|\psi\rangle$  is a positive real number  $\mathcal{T}_\psi(\mu)$ , depending on the parameter  $\mu$  which labels a set of states  $|\mu\rangle \in \mathcal{H}$ , defined as

$$\mathcal{T}_\psi(\mu) := |\langle \mu | \psi \rangle|^2 . \quad (1)$$

Our main idea was to regard the tomogram  $\mathcal{T}_\psi(\mu)$  as a scalar product on the (Hilbert) space  $\mathbb{H}$  of the rank-one projectors  $|\mu\rangle \langle \mu| = P_\mu \rightarrow |P_\mu\rangle \in \mathbb{H}$ :

$$\mathcal{T}_\psi(\mu) = \text{Tr}(P_\mu \rho_\psi) =: \langle P_\mu | \rho_\psi \rangle . \quad (2)$$

This equation may be generalized to define the tomogram of any density operator  $\hat{\rho}$  or any other (bounded) operator  $\hat{A}$

$$\mathcal{T}_A(\mu) := \text{Tr}(P_\mu \hat{A}) = \langle P_\mu | A \rangle . \quad (3)$$

Equation (3) shows in general that to any operator  $\hat{A}$  a function  $\langle \mu | \hat{A} | \mu \rangle$  of the variables  $\mu$  corresponds in a given functional space. So a tomograph, that is a functional, linear in the second argument:

$$\mathcal{T}_\bullet(\mu) := \text{Tr}(P_\mu \bullet) = \langle P_\mu | \bullet \rangle \quad (4)$$

$$\mathcal{T}_\bullet(\mu) : \hat{A} \rightarrow \langle \mu | \hat{A} | \mu \rangle , \quad (5)$$

may be thought of as a de-quantization, and in fact is an useful tool to study the quantum-classical transition by comparing classical limits of quantum tomograms with the corresponding classical tomograms [16].

In the same sense, the inverse correspondence  $\mathcal{T}_A(\mu) \rightarrow \hat{A}$  may be considered to give a quantization. The reconstruction of the operator  $\hat{A}$  from its tomogram  $\mathcal{T}_A(\mu)$  may be written as

$$\hat{A} = \sum_{\mu \in M} \hat{G}_\mu \text{Tr}(P_\mu \hat{A}) \Leftrightarrow |A\rangle = \sum_{\mu \in M} |G_\mu\rangle \langle P_\mu | A \rangle . \quad (6)$$

Here the  $\hat{G}_\mu$ 's are Gram-Schmidt operators, which take into account that in general the projectors  $P_\mu$ 's are not orthogonal, while the sum may be an integral with a suitable measure.

In other words, the reconstruction of any operator is possible because the tomographic set  $\{P_\mu\}_{\mu \in M}$  provides a resolution of the identity (super-) operator on  $\mathbb{H}$ :

$$\hat{\mathbb{I}} = \sum_{\mu \in M} \hat{G}_\mu \text{Tr}(P_\mu \cdot) = \sum_{\mu \in M} |G_\mu\rangle \langle P_\mu| . \quad (7)$$

We may then view  $|G_\mu\rangle$  and  $\langle P_\mu|$  as dual supervectors.

Thus, for the finite  $n$ -dimensional case,  $\mathbb{H} = \mathcal{H} \otimes \mathcal{H}^*$  is  $n^2$ -dimensional and a *minimal* tomographic set is a basis  $\{P_k\}$ ,  $k \in \{1, \dots, n^2\}$ , of rank-one projectors which may be orthonormalized by a Gram-Schmidt procedure

$$|V_j\rangle = \sum_{k=1}^{n^2} \gamma_{jk} |P_k\rangle \quad , \quad \langle V_i|V_j\rangle = \delta_{ij} \quad . \quad (8)$$

In general, every  $|V_j\rangle$  is a linear combination of projectors, rather than a single projector like  $|P_k\rangle$ . Then a resolution of the super-unity on  $\mathbb{H}$  in terms of the  $P_k$ 's reads as

$$\begin{aligned} \hat{\mathbb{I}}_{n^2} &= \sum_{i=1}^{n^2} |V_i\rangle \langle V_i| = \sum_{i,j,l=1}^{n^2} \gamma_{il}^* \gamma_{ij} P_j \text{Tr}(\hat{P}_l \bullet) \\ &= \sum_{l=1}^{n^2} |G_l\rangle \langle P_l| = \sum_{j=1}^{n^2} |P_j\rangle \langle G_j| \quad , \end{aligned} \quad (9)$$

where the Gram-Schmidt operator  $\hat{G}_l$  has been introduced

$$|G_l\rangle = \sum_{i=1}^{n^2} \gamma_{il}^* |V_i\rangle = \sum_{i,j=1}^{n^2} \gamma_{il}^* \gamma_{ij} |P_j\rangle \quad . \quad (10)$$

We observe that  $\hat{G}_l$  is a nonlinear function of the projectors  $P_k$ , because also the coefficients  $\gamma$ 's depend on the projectors. Moreover

$$\begin{aligned} \langle P_i|G_l\rangle &= \sum_{j=1}^{n^2} \gamma_{jl}^* \langle P_i|V_j\rangle = \sum_{j,k=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ik}^{-1} \langle V_k|V_j\rangle \\ &= \sum_{j=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ij}^{-1} = \delta_{il} \quad . \end{aligned} \quad (11)$$

Similar formulae hold even for any other tomographic, i.e. (over-) complete, set  $\{P_\mu\}_{\mu \in M}$ .

For instance for the spin tomography, in the maximal qu-bit case  $M$  is the Bloch sphere  $S^2$  of all rank-one projectors and we have

$$\hat{\mathbb{I}} = \int_0^{2\pi} \int_0^\pi |G(\theta, \phi)\rangle \text{Tr}(P(\theta, \phi) \bullet) \sin \theta d\theta d\phi \quad , \quad (12)$$

where, in matrix form,

$$P(\theta, \phi) = \frac{1}{2} \begin{bmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 - \cos \theta \end{bmatrix} \quad (13)$$

and

$$\hat{G}(\theta, \phi) = \frac{1}{4\pi} \begin{bmatrix} 1 + 3 \cos \theta & 3e^{-i\phi} \sin \theta \\ 3e^{i\phi} \sin \theta & 1 - 3 \cos \theta \end{bmatrix} \quad , \quad (14)$$

so that, for any operator  $\hat{A}$ , it results [12]:

$$\hat{A} = \int_0^{2\pi} \int_0^\pi \hat{G}(\theta, \phi) \text{Tr}(P(\theta, \phi) A) \sin \theta d\theta d\phi \quad (15)$$

$$= \int_0^{2\pi} \int_0^\pi P(\theta, \phi) \text{Tr}(\hat{G}(\theta, \phi) A) \sin \theta d\theta d\phi \quad . \quad (16)$$

In summary, the set  $\{P_\mu\}_{\mu \in M}$  is tomographic if it is complete in  $\mathbb{H}$ . A tomographic set determines a tomograph

$$(P_\mu, \hat{A}) \rightarrow \mathcal{T}_A(\mu) = \text{Tr}(P_\mu \hat{A}) . \quad (17)$$

This definition is appropriate in the finite  $n$ -dimensional case, where

$$|\mu\rangle \in \mathcal{H}_n \Leftrightarrow P_\mu \in \mathbb{H}_{n^2} = B(\mathcal{H}_n) = \mathcal{H}_n \otimes \mathcal{H}_n^* ,$$

but in the infinite dimensional case the relation  $\mathbb{H} = B(\mathcal{H})$  is no more valid and there are several relevant spaces, as the space of bounded operators  $B(\mathcal{H})$  and that of compact operators  $C(\mathcal{H})$ , the space of Hilbert-Schmidt operators  $\mathfrak{J}_2$  and that of trace-class operators  $\mathfrak{J}_1$ . Their mutual relations are:

$$\mathfrak{J}_1 \subset \mathfrak{J}_2 \subset C(\mathcal{H}) \subset B(\mathcal{H}) .$$

$B(\mathcal{H})$  (and  $C(\mathcal{H})$ ) are Banach spaces, with the norm  $\|A\| = \sup_{(\|\psi\|=1)} \|A\psi\|$ , while  $\mathfrak{J}_2$  is a Hilbert space with scalar product  $\langle A|B \rangle = \text{Tr}(A^\dagger B)$  and corresponding norm  $\|\bullet\|_2$ . Finally,  $\mathfrak{J}_1$  is a Banach space with the norm  $\|A\|_1 = \text{Tr}(|A|)$ . The following inequalities hold true:

$$\|A\| \leq \|A\|_2 \leq \|A\|_1 .$$

So  $\mathfrak{J}_2$ , the only Hilbert space at our disposal to implement our definition of tomographic set, is endowed with a topology which, when restricted to the trace-class operators, is not equivalent to the topology of  $\mathfrak{J}_1$ . This may have serious consequences.

In fact, in the finite dimensional case, the set  $\{P_\mu\}_{\mu \in M}$  is complete iff

$$\text{Tr}(P_\mu A) = 0 \quad \forall \mu \in M \implies A = 0 . \quad (18)$$

Such a condition guarantees the full reconstruction of any observable from its tomograms. Now, in  $\mathfrak{J}_2$ , Eq. (18) reads:

$$\langle P_\mu | A \rangle = 0 \quad \forall \mu \in M \implies A = 0 \quad \& \quad A \in \mathfrak{J}_2 . \quad (19)$$

Then, as  $\mathfrak{J}_2$  is a  $*$ -ideal in  $B(\mathcal{H})$ , there may exist a non-zero operator  $B$ , which is bounded but not Hilbert-Schmidt, such that

$$\text{Tr}(P_\mu B) = 0 \quad \forall \mu \in M .$$

In other words, different observables may be tomographically separated only when their difference is Hilbert-Schmidt.

Nevertheless there is a second case, when the set  $\{P_\mu\}_{\mu \in M}$  of trace-class operators is complete even in  $\mathfrak{J}_1$ . Then, recalling that  $\mathfrak{J}_1$  is a  $*$ -ideal in its dual space  $B(\mathcal{H})$ :

$$\mathfrak{J}_1^* = B(\mathcal{H}) ,$$

the expression  $\text{Tr}(P_\mu A)$  is nothing but the value of the linear functional  $\text{Tr}(\bullet A)$  in  $P_\mu$ . Hence, Eq. (18) holds unconditionally

$$\text{Tr}(P_\mu A) = 0 \quad \forall \mu \in M \implies 0 = \|\text{Tr}(\bullet A)\| = \|A\| \implies A = 0 . \quad (20)$$

Thus, the finest tomographies are those based on sets of rank-one projectors which are complete both in  $\mathfrak{J}_2$  and in  $\mathfrak{J}_1$ . As a matter of fact, this is the case for the main tomographic sets, like the symplectic, the photon number and the coherent state tomographic sets.

After the discussion of some of the topological subtleties of the infinite dimensional case, we are now ready to study an example, which allows for the construction of a minimal tomographic set, i.e., a basis of rank-one projectors.

### 3. A (minimal) tomographic set spanning both $\mathfrak{I}_2$ and $\mathfrak{I}_1$

Let  $\{e_n\}_{n=1}^\infty$  be an orthonormal basis of an Hilbert space  $\mathcal{H}$ . Now switch to the Dirac notation,  $e_n \longleftrightarrow |n\rangle$ , and get an orthonormal basis  $\{E_{nm}\} = \{|n\rangle\langle m|\}_{n,m=1}^\infty$  of  $\mathfrak{I}_2$ .

In the basis  $\{|n\rangle\}$ , we have

$$(E_{nm})_{jk} = \langle j | E_{nm} | k \rangle = \delta_{jn} \delta_{mk} ; \quad (21)$$

$$\begin{aligned} \text{Tr}(E_{qp}^\dagger E_{nm}) &= \sum_{jk} \langle j | E_{qp}^\dagger | k \rangle \langle k | E_{nm} | j \rangle \\ &= \sum_{jk} \delta_{jp} \delta_{qk} \delta_{kn} \delta_{mj} = \delta_{qn} \delta_{pm} . \end{aligned} \quad (22)$$

A Hermitian orthogonal basis may be constructed with the compact operators

$$E_{nm}^+ = \frac{1}{2} (E_{nm} + E_{nm}^\dagger) \quad (n \leq m) ; \quad (23)$$

$$E_{nm}^- = \frac{i}{2} (E_{nm} - E_{nm}^\dagger) \quad (n > m) . \quad (24)$$

The Hermitian basis is readily diagonalizable: for  $n \neq m$  the set  $\{E_{nm}^+, E_{nm}^-\}_{n,m}$  is isospectral, with simple eigenvalues  $\pm 1/2$  and respective eigenvectors

$$|\Psi_{nm}^{+,\pm}\rangle = \frac{1}{\sqrt{2}} (|n\rangle \pm |m\rangle) , \quad |\Psi_{nm}^{-,\pm}\rangle = \frac{1}{\sqrt{2}} (|m\rangle \pm i |n\rangle) , \quad (25)$$

where  $\pm$  label the eigenvalues. Their associated projectors

$$P_{nm}^{+,\pm} = |\Psi_{nm}^{+,\pm}\rangle \langle \Psi_{nm}^{+,\pm}| , \quad P_{nm}^{-,\pm} = |\Psi_{nm}^{-,\pm}\rangle \langle \Psi_{nm}^{-,\pm}| , \quad (26)$$

together with the diagonal ( $n = m$ ) projectors

$$P_{nn} = |n\rangle \langle n| , \quad (27)$$

are a tomographic set. In fact, as

$$P_{nm}^{+,\pm} = \frac{1}{2} (P_{nn} + P_{mm}) \pm E_{nm}^+ , \quad (28)$$

$$P_{nm}^{-,\pm} = \frac{1}{2} (P_{nn} + P_{mm}) \pm E_{nm}^- , \quad (29)$$

the set contains a basis of  $\mathfrak{I}_2$  of rank-one projectors. Moreover, the set is complete in  $\mathfrak{I}_1$ . In fact, assume that the linear functional  $\text{Tr}(A\bullet)$ , with  $A \in B(\mathcal{H})$ , vanishes on the tomographic set. Then the diagonal matrix elements of  $A$  are zero

$$\text{Tr}(AP_{nn}) = \langle n | A | n \rangle = 0 \quad \forall n , \quad (30)$$

and we have

$$\begin{aligned} \text{Tr}(AP_{nm}^{+,\pm}) &= \frac{1}{2} \text{Tr}(A (P_{nn} + P_{mm} \pm 2E_{nm}^+)) \\ &= \pm \frac{1}{2} (\langle m | A | n \rangle + \langle n | A | m \rangle) = 0 , \end{aligned} \quad (31)$$

$$\begin{aligned} \text{Tr}(AP_{nm}^{-,\pm}) &= \frac{1}{2} \text{Tr}(A (P_{nn} + P_{mm} \pm 2E_{nm}^-)) \\ &= \pm \frac{i}{2} (\langle m | A | n \rangle - \langle n | A | m \rangle) = 0 , \end{aligned} \quad (32)$$

which yield

$$\langle m | A | n \rangle = 0 \quad \forall m, n \Leftrightarrow A = 0 , \quad (33)$$

so that  $A$  is the zero operator.

Finally we observe that a *minimal* tomographic set, i.e. a basis of rank-one projectors, may be chosen by taking just one projector from each pair  $P_{nm}^{+,\pm}$  with  $n < m$ , only one projector from each pair  $P_{nm}^{-,\pm}$  with  $n > m$  and all the diagonal  $P_{nn}$ 's. Such a minimal set is obviously complete both in  $\mathfrak{J}_2$  and in  $\mathfrak{J}_1$ .

We can evaluate explicitly the resolution of unity determined by the full (non-minimal) set of projectors. Start from the representation of a (bounded) operator  $B$  as

$$B = \sum_{n,m} \langle n | B | m \rangle | n \rangle \langle m | . \quad (34)$$

In view of the decomposition as a sum of two selfadjoint operators

$$B = \frac{1}{2}(B + B^\dagger) - i\left(\frac{i}{2}(B - B^\dagger)\right) , \quad (35)$$

we may assume  $B$  selfadjoint. Then:

$$\begin{aligned} \langle n | B | m \rangle &= \frac{1}{2} [\langle \Psi_{nm}^{+,+} | B | \Psi_{nm}^{+,+} \rangle - \langle \Psi_{nm}^{+,-} | B | \Psi_{nm}^{+,-} \rangle] \\ &+ \frac{i}{2} [\langle \Psi_{nm}^{-,+} | B | \Psi_{nm}^{-,+} \rangle - \langle \Psi_{nm}^{-,-} | B | \Psi_{nm}^{-,-} \rangle] . \end{aligned} \quad (36)$$

Thus, we get the reconstruction formula

$$\begin{aligned} B &= \sum_n P_{nn} \text{Tr}(BP_{nn}) \\ &+ \sum_{n < m} E_{nm}^+ [\text{Tr}(BP_{nm}^{+,+}) - \text{Tr}(BP_{nm}^{+,-})] \\ &+ \sum_{n < m} E_{nm}^- [\text{Tr}(BP_{nm}^{-,+}) - \text{Tr}(BP_{nm}^{-,-})] . \end{aligned} \quad (37)$$

Upon introducing a third label  $\alpha$  to enumerate the  $P^{\pm,\pm}$ 's, we obtain the resolution of the unity as

$$\hat{\mathbb{I}} = \sum_n |P_{nn}\rangle \langle P_{nn}| + \sum_{n < m, \alpha} |G_{nm}^\alpha\rangle \langle P_{nm}^\alpha| , \quad (38)$$

where

$$|G_{nm}^{+,\pm}\rangle = \pm E_{nm}^+ , \quad |G_{nm}^{-,\pm}\rangle = \pm E_{nm}^- . \quad (39)$$

#### 4. The coherent state tomography

In contrast with the previous example of a countable tomographic set, the coherent state tomographic set is generated by the displacement operators  $\{\mathcal{D}(z)\}$  depending on a complex parameter  $z$

$$\mathcal{D}(z) = \exp\left(z\hat{a}^\dagger - z^*\hat{a}\right) , \quad z \in \mathbb{C} . \quad (40)$$

Upon acting on the projector  $|0\rangle\langle 0|$  of the vacuum Fock state  $\hat{a}|0\rangle = 0$ , they yield the projectors

$$|z\rangle\langle z| = \mathcal{D}(z)|0\rangle\langle 0|\mathcal{D}(z)^\dagger, \quad z \in \mathbb{C}, \quad (41)$$

associated to the usual coherent states

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \exp(z\hat{a}^\dagger) \exp(-z^*\hat{a})|0\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{j=0}^{\infty} \frac{z^j}{j!} \hat{a}^{\dagger j} |0\rangle.$$

We recall that the coherent states are a (over-) complete set in the Hilbert space  $\mathcal{H}$ .

It is possible to interpret the well known Husimi-Kano  $K$ -symbol of a (bounded) operator  $\hat{A}$  as the coherent state (CS) tomogram of  $\hat{A}$ :

$$K_A(z) := \langle z | \hat{A} | z \rangle =: \text{Tr}(|z\rangle\langle z| \hat{A}). \quad (42)$$

In particular, when  $\hat{A}$  is chosen as a density operator  $\hat{\rho}$ , the identity holds

$$\int \frac{d^2z}{\pi} \langle z | \hat{\rho} | z \rangle = \text{Tr}(\hat{\rho}) = 1, \quad (43)$$

which allows for the probabilistic interpretation of the CS tomography. As a matter of fact [14], the  $K$ -symbol exists also for a number of non-bounded operators. The CS tomographic set is complete both in  $\mathfrak{T}_2$ , the space of Hilbert-Schmidt operators, and in  $\mathfrak{T}_1$ , the space of trace class operators acting on the space of states. In fact, the formulae

$$\hat{A} = \int \frac{d^2z}{\pi} \frac{d^2z'}{\pi} \langle z | \hat{A} | z' \rangle |z\rangle\langle z'| \quad (44)$$

and [17]

$$\langle z | \hat{A} | z' \rangle = e^{-\frac{|z|^2+|z'|^2}{2}} \sum_{n,m=0}^{\infty} \frac{(z^*)^n (z')^m}{n!m!} \left[ \frac{\partial^{n+m}}{\partial z^{*n} \partial z^m} \left( e^{|z|^2} \langle z | \hat{A} | z \rangle \right) \right]_{z=0}^{z^*=0}$$

show that if the tomograms  $\langle z | \hat{A} | z \rangle$  of a bounded operator  $\hat{A}$  vanish for any  $z \in \mathbb{C}$ , then  $\hat{A}$  is the zero operator. The previous equation is implicit in Eq. (6) of Sudarshan's paper [7].

So, a resolution of the unity exists, which allows for the full reconstruction of any (bounded) operator from its CS tomograms. We are interested in the explicit determination of such a formula.

Now, the Sudarshan's diagonal coherent state representation  $\phi_A(z)$  of an operator  $\hat{A}$  is defined through the equation

$$\hat{A} = \int \frac{d^2z}{\pi} \phi_A(z) |z\rangle\langle z|. \quad (45)$$

analogous to the (dual) reconstruction formula,

$$|A\rangle = \sum_{\mu \in M} |P_\mu\rangle \langle G_\mu | A \rangle. \quad (46)$$

If our guess is right we can get explicitly the first form of the reconstruction formula,

$$\hat{A} = \sum_{\mu \in M} \hat{G}_\mu \text{Tr}(P_\mu \hat{A}) = \sum_{\mu \in M} |G_\mu\rangle \langle P_\mu | A \rangle. \quad (47)$$

To do that, we have to invert the well-known relation:

$$K_A(z') = \langle z' | \hat{A} | z' \rangle = \int \frac{d^2z}{\pi} \phi_A(z) |\langle z | z' \rangle|^2 = \int \frac{d^2z}{\pi} \phi_A(z) e^{-|z-z'|^2} \quad (48)$$

which follows at once from Eq. (45) defining  $\phi_A(z)$ . This relation shows that  $K_A(z')$  is given by the convolution product of  $\phi_A$  times a gaussian function. Then, denoting with  $K_A(z'_R, z'_I)$  and  $\phi_A(z_R, z_I)$  the  $K$  and  $\phi$  symbols, with  $z' = z'_R + iz'_I$ ;  $z = z_R + iz_I$ , the Fourier transform [18] of Eq. (48) reads:

$$\tilde{K}_A(\xi, \eta) = e^{-(\xi^2 + \eta^2)/4} \tilde{\phi}_A(\xi, \eta) \quad , \quad (49)$$

from which

$$\tilde{\phi}_A(\xi, \eta) = e^{(\xi^2 + \eta^2)/4} \tilde{K}_A(\xi, \eta) \quad , \quad (50)$$

that formally yields

$$\phi_A(z_R, z_I) = \int \frac{d\xi d\eta}{2\pi} e^{(\xi^2 + \eta^2)/4} \tilde{K}_A(\xi, \eta) e^{i(\xi z_R + \eta z_I)} \quad . \quad (51)$$

The presence of the anti-gaussian factor shows that the inverse Fourier transform of  $\tilde{\phi}_A(\xi, \eta)$  exists only when the asymptotic decay of  $\tilde{K}_A(\xi, \eta)$  is faster than the growth of  $e^{(\xi^2 + \eta^2)/4}$ . However, the integral always exists as a generalized function, as proven by Mehta and Sudarshan in ref. [15]. By virtue of this remark, we may go on and substitute the previous expression into Eq. (45) getting

$$\begin{aligned} \hat{A} &= \int \frac{d^2z}{\pi} \left[ \int \frac{d\xi d\eta}{2\pi} e^{(\xi^2 + \eta^2)/4} \tilde{K}_A(\xi, \eta) e^{i(\xi z_R + \eta z_I)} \right] |z\rangle \langle z| \\ &= \int \frac{d^2z}{\pi} \left[ \int \frac{d\xi d\eta}{2\pi} \int \frac{dz'_R dz'_I}{2\pi} K_A(z'_R, z'_I) e^{(\xi^2 + \eta^2)/4} e^{i[\xi(z_R - z'_R) + \eta(z_I - z'_I)]} \right] |z\rangle \langle z| \quad . \quad (52) \end{aligned}$$

Upon interchanging the order of integration, we may write the expected reconstruction formula as

$$\hat{A} = \int \frac{d^2z'}{\pi} \hat{G}(z') K_A(z') \quad , \quad (53)$$

where the Gram-Schmidt operator  $\hat{G}(z')$  reads:

$$\hat{G}(z') := \int \frac{d^2z}{2\pi} \int \frac{d\xi d\eta}{2\pi} e^{(\xi^2 + \eta^2)/4} e^{i[\xi(z_R - z'_R) + \eta(z_I - z'_I)]} |z\rangle \langle z| \quad . \quad (54)$$

So, the resolution of the unity generated by the CS tomographic set is

$$\hat{\mathbb{I}} = \int \frac{d^2z'}{\pi} \hat{G}(z') \text{Tr}(|z'\rangle \langle z'| \bullet) \quad . \quad (55)$$

We note the duality relation

$$\begin{aligned} K_{G(z')}(z) &= \langle z | \hat{G}(z') | z \rangle = \int \frac{d\xi d\eta}{2\pi} e^{i[-\xi z'_R - \eta z'_I]} e^{(\xi^2 + \eta^2)/4} \tilde{K}_{|z\rangle \langle z|}(\xi, \eta) \\ &= \int \frac{d\xi d\eta}{2\pi} e^{i[-\xi z'_R - \eta z'_I]} \tilde{\phi}_{|z\rangle \langle z|}(\xi, \eta) = \phi_{|z\rangle \langle z|}(z') \quad . \quad (56) \end{aligned}$$

This, substituted in the reconstruction formula Eq. (53), gives also the reproducing kernel formula

$$K_{G(z'')}(z) = \int \frac{d^2 z'}{\pi} \langle z | \hat{G}(z') | z \rangle K_{G(z'')}(z') = \int \frac{d^2 z'}{\pi} K_{G(z')}(z) K_{G(z'')}(z') \quad (57)$$

or equivalently

$$\phi_{|z\rangle\langle z|}(z'') = \int \frac{d^2 z'}{\pi} \phi_{|z\rangle\langle z|}(z') \phi_{|z'\rangle\langle z'|}(z'') . \quad (58)$$

Since it results

$$\phi_{|z\rangle\langle z|}(z') = \pi \delta(z - z') , \quad (59)$$

we get the orthonormality relations

$$K_{G(z')}(z) = \text{Tr}(|z\rangle\langle z| \hat{G}(z')) = \pi \delta(z - z') \quad (60)$$

between dual sets of supervectors.

Finally, we can use the previous expression of  $K_G$  to check the reconstruction formula in matrix form as:

$$\langle z | \hat{A} | z \rangle = \int \frac{d^2 z'}{\pi} K_{G(z')}(z) K_A(z') = K_A(z) . \quad (61)$$

## 5. Conclusions

The identity resolution that we have obtained for the CS tomographic set may be recovered even in the context of the generalized phase space distributions associated with the Agarwal-Wolf  $\Omega$ -operator ordering [17]. The coherent states are closely connected with linear vibrations (linear harmonic oscillator). The deformed oscillators, e.g.  $q$ -oscillators [19, 20] and  $f$ -oscillators [21, 22] are related at a classical level with specific non-linear vibrations, so that non-linear coherent states were introduced [23, 24, 25] to describe the corresponding states of a non-linear quantum oscillator, which yields in the linearity limit the standard coherent state counterpart.

We have succeeded [26] in developing the generalized version of the previous CS tomographic approach for generic deformations of coherent states connected with  $f$ -oscillators,  $q$ -oscillators, and  $s$ -deformations associated to operator ordering [27, 28]. In particular we have obtained, for a specific choice of the non-linearity coded by a function  $f = f_q$ , the  $q$ -deformed coherent state tomography.

Also, by using  $(f, s)$ -deformed coherent states, we have developed [26] the  $(f, s)$ -deformed version of the photon number tomography [29, 30, 31] as well as we obtain the identity decomposition for the deformed tomographies.

To conclude, we observe that recently the problem of introducing deformed phase space representations of operators was also addressed in ref. [32] according to some of the ideas exposed above.

We hope to extend the obtained results to the case of multi-mode quantum systems and entangled states in future papers.

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