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Generating function rationality for anisotropic vicious walk configurations on the directed square lattice

F M Bhatti[1] **and J W Essam**[2]

[1]Department of Mathematics, Lahore University of Management Sciences, Sector U, DHA, Lahore, Pakistan.

[2]Department of Mathematics and Statistics, Royal Holloway College, University of London, Egham, Surrey TW20 0EX, England.

E-mail: fmbhatti@lums.edu.pk, j.essam@rhul.ac.uk

Abstract. Guttmann and Vöge introduced a model of f-friendly walkers and argued that a generating function for the number of n-walker configurations making a total of k left steps is a rational function with denominator $(1 - x^n)^{k+1}$. They also found that for $f = 0, 1$ and 2 the sums of the numerator coefficients for watermelon configurations in which each of 3 walkers made w left steps were 3-dimensional Catalan numbers. Here it is shown that for n vicious walker ($f = 0$) watermelon configurations the m^{th} coefficient of the numerator is the generalised Naryana number $N(w, n, m)$ of Sulanke which is symmetric under interchange of w and n. The sums, $C_{w,n}$, of these coefficients as a sequence indexed by w are n-dimensional Catalan numbers or w-dimensional Catalan numbers if indexed by n. The unexpected symmetry in n and w is seen to follow from duality.

Inui and Katori introduced Fermi walk configurations which are non-crossing subsets of the directed random walks between opposite corners of a rectangular $\ell \times w$ grid. They related these to Bose configurations which biject to vicious walker watermelon configurations. Bose configurations include multisets. Here we consider generating functions for the numbers of configurations in which ℓ and w are fixed. It is found that the maximum number of walks in a Fermi configuration is $\ell w + 1$ and the number of configurations corresponding to this number of walks is $C_{\ell,w}$. This limit on the number of walks in a Fermi configuration leads to the rationality of the Bose generating function and by duality to the rationality of the generating function of Guttmann and Vöge.

1. Background and definitions

Suppose that each of n walkers start at points of the square lattice such that $x+y = 0$ and simultaneously make t steps with step vectors $(1, 0)$ or $(0, 1)$. The walkers therefore follow paths on a directed square lattice. Let $\{x_i(\tau), y_i(\tau)\}$ be the position of the i^{th} walker after the first τ steps and $Y_i(\tau) = y_i(\tau)-x_i(\tau)$. The walks are said to be *non-crossing* if for each $\tau \leq t$

$$Y_{i+1}(\tau) \geq Y_i(\tau) \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}. \tag{1.1}$$

Guttmann and Vöge [1] considered a series of walk models they called f-friendly. These are non-crossing models with the further conditions that not more than two walkers are allowed occupy the same lattice site and two walkers can only stay together for at most f consecutive sites. $f = 0$ corresponds to *vicious* walkers and $f = 1$ to *osculating* walkers. The walkers were assumed [1] to start at the points $\{(-(i-1), i-1), i \in \{1, 2, \ldots, n\}\}$. Vicious walkers were introduced by Fisher in his Boltzmann medal lecture [2] where he considered the probabilities of *reunion* and *survival* of drunken walkers on a one-dimensional lattice who shoot one another if they arrive at the same site. The space time trajectories of these walkers correspond to walks on the directed square lattice. Fisher also described applications including the commensurate-incommensurate phase transition. The walk configurations may also be considered as representing polymer networks studied by Duplantier and Saleur $\{[3],[4],[5]\}$ who introduced the terms *star* and *watermelon* to describe configurations which contribute to survival and reunion probabilities respectively. f-friendly walks with $f = 0, 1$ and 2 may be mapped $\{[1],[6]\}$ to lattice statistical vertex model configurations in which each vertex has five, six and ten states respectively. These may be considered as models of ferroelctric materials. In the polymer context they represent networks with loops. The case $f > 0$ is of interest since it is a lattice path problem the solution of which may *not* be expressed as a Gessel-Viennot determinant $\{[7],[8]\}$.

Suppose that the steps are described as either right or left relative to the $(1, 1)$ direction. The models were made anisotropic by giving different weight to left and right steps which led to the definition of the following "anisotropic generating functions"

$$H_{f,k,n}(x) \equiv \sum_{j=0}^{\infty} h_{f,k,j,n} x^j \tag{1.2}$$

where $h_{f,k,j,n}$ is the number of f-friendly, n-walk configurations which between them make a total of k left steps and j right steps. Two types of endpoint condition were considered and the corresponding generating functions were distinguished by a superscript on H. *Watermelon configurations* were such that each walk makes the same number, w, of left steps so that $k = nw$ and for *star configurations* only the total number of left steps was fixed.

Based on data for $f = 0, 1$ and 2 and a range of values of k and n Guttmann and Vöge [1] observed that $H_{f,k,n}(x)$ is a rational function of x

$$H_{f,k,n}(x) = \frac{P_{f,k,n}(x)}{(1 - x^n)^{k+1}} \tag{1.3}$$

with a denominator which is independent of f. For vicious watermelons the numerator was observed to have degree $w - 1$ as a function of x^2 for $n = 2$ and $2(w - 1)$ as a function of x^3 for $n = 3$. They argued that the rational forms were due to the way in which the configurations grow on increasing the lengths of the walks but keeping a fixed number of left steps. Roughly speaking the left steps partition the configurations into at most $k + 1$ segments the lengths of which may be repeatedly extended independently by adding stages consisting of n parallel right steps. The numerator arises from configurations which cannot be obtained by increasing the length of a smaller configuration in this way. There is an upper limit on the length of such configurations and hence the polynomial form of the numerator.

Guttmann and Vöge also observed that for three walk watermelons the sum of the numerator coefficients is

$$P^{\text{watermelon}}_{f,3w,3}(1) = \frac{2(3w)!}{w!(w+1)!(w+2)!} = C_{3,w}, \tag{1.4}$$

a three-dimensional Catalan number, independently of f. Similarly for stars they found the sum to be a Motzkin number [9]

$$P^{\text{star}}_{f,k,3}(1) = \frac{1}{k+1} \sum_{j=0}^{\lceil (k+1)/2 \rceil} \binom{k+1}{j} \binom{k+1-j}{j-1}. \tag{1.5}$$

Sulanke [10] defined the d-dimensional Naryana number $N(d, n, m)$ to be the number of configurations of a d-dimensional lattice path of length dn from the origin to the point $\mathbf{n} = (n, n, \ldots, n)$

lying in the region $\{(x_1, x_2, \ldots, x_d) : 0 \le x_1 \le x_2 \le \cdots \le x_d\}$ and having m ascents (an *ascent* is a move into a higher dimension). Proposition 1 of [10] states that these numbers are given by the formula

$$N(d,n,m) = \sum_{j=0}^{m} (-1)^{m-j} \binom{dn+1}{m-j} \prod_{i=0}^{d-1} \binom{n+i+j}{n} \binom{n+i}{n}^{-1} \tag{1.6}$$

and they have the following properties

$$N(d,n,m) = 0 \qquad \text{for } m > (d-1)(n-1) \tag{1.7}$$

$$N(d,n,(d-1)(n-1)-m) = N(d,n,m) \qquad \text{for } 0 \le m \le (d-1)(n-1) \tag{1.8}$$

$$\sum_{m=0}^{(d-1)(n-1)} N(d,n,m) = C_{d,n} \equiv (dn)! \prod_{i=0}^{d-1} \frac{i!}{(n+i)!} \tag{1.9}$$

Notes

- The first two are from Corollary 1 of [10].
- $C_{d,n}$ is known as a d-dimensional Catalan number and determines the number of d-dimensional lattice paths from the origin to $\mathbf{n}$ with no constraint on the number of ascents. For $d = 2$ this is the ordinary Catalan number giving the number of Dyck paths [9]. Sulanke attributes (1.9) to MacMahon (see [11], art. 93–103).
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$$N(2,n,m) = N(n,m+1) \tag{1.10}$$

  where

$$N(n,m) \equiv \frac{1}{n}\binom{n}{m-1}\binom{n}{m} = \frac{(n-m+1)_m (n-m+2)_m}{(m-1)!m!} = N_{n,n-m+1} \tag{1.11}$$

  is an ordinary Naryana number [12, 13, 14] introduced by Naryana in 1955 as counting the number of parallelogram polyominoes of perimeter $2(n+1)$ with m columns.

Here we prove the following proposition (see sections 2 and 5) which extends the above results of Guttmann and Vöge in the case of vicious walker watermelon configurations ($f = 0$).

Proposition 1. *The generating function for the number of n-walk watermelon configurations in which each walk makes w left steps is*

$$H^{\textit{watermelon}}_{0,nw,n}(x) = \frac{N_{w,n}(x^n)}{(1-x^n)^{nw+1}}$$

where the numerator is the w-dimensional Naryana polynomial

$$N_{w,n}(z) = \sum_{m=0}^{(w-1)(n-1)} N(w,n,m) z^m$$

and the sum of the numerator coefficients is

$$N_{w,n}(1) = C_{w,n} = (wn)! \prod_{i=0}^{w-1} \frac{i!}{(n+i)!}.$$

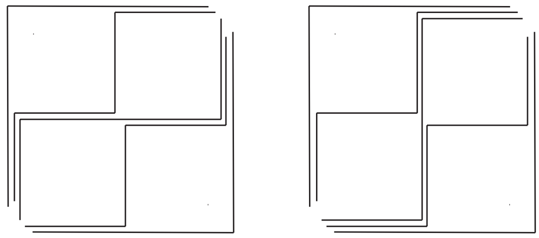

Figure 1. The two maximal Fermi walk configurations on a 2×2 grid.

Inui and Katori [15] considered two sets of $n-$walk non-crossing configurations in which all of the walks start at the lattice origin and end at the point (ℓ, w). The walks are thus confined to an $\ell \times w$ rectangular grid. No further conditions are imposed on *Bose* configurations but *Fermi* configurations are subject to the additional constraint that each of $\binom{\ell+w}{w}$ directed lattice paths between the corners of the rectangle may be used by at most one walker. Two of the Fermi configurations with $\ell = w = 2$ are shown in figure 1.
The numbers of Bose and Fermi configurations will be denoted by $f^{\text{Bose}}_{\ell,w,n}$ and $f^{\text{Fermi}}_{\ell,w,n}$ respectively.

Bose configurations biject [16] to directed integer flows with a source of strength n at the origin and a sink at (ℓ, w) with the flow on a given lattice bond being equal to the number of walkers traversing that bond. The number of flows, and hence the number of Bose configurations, was conjectured by Arrowsmith et al [16] to be

$$f^{\text{Bose}}_{\ell,w,n} = \prod_{j=1}^{w} \frac{(\ell + w - j + 1)_n}{(j)_n} = \prod_{j=1}^{w} \frac{(\ell + j)_n}{(j)_n} \tag{1.12}$$

where $(a)_k \equiv a(a+1)\ldots(a+k-1)$. The first formula above was subsequently derived [17] by enumerating the vicious walker configurations using a Lindstrom-Gessel-Viennot determinant [18, 7, 8]. It is clear from this result that for fixed w and n $f^{\text{Bose}}_{\ell,w,n}$ is a polynomial of degree nw in ℓ. The usual convention $(a)_0 = 1$ implies that $f^{\text{Bose}}_{\ell,w,0} = 1$.

2. Outline of the proof of proposition 1

Vicious walk watermelon configurations biject [16] to Bose configurations by translating the i^{th} walk by the vector $(i-1, -(i-1))$ so that the starting points coincide. Therefore if we define a Bose generating function by

$$H^{\text{Bose}}_{w,n}(z) \equiv \sum_{\ell=0}^{\infty} z^{\ell} f^{\text{Bose}}_{\ell,w,n} \tag{2.1}$$

the vicious watermelon generating function is given by

$$H^{\text{watermelon}}_{0,nw,n}(x) = H^{\text{Bose}}_{w,n}(x^n). \tag{2.2}$$

It turns out to be more useful to work with the alternative generating function

$$G^{\text{Bose}}_{\ell,w}(z) \equiv \sum_{n=0}^{\infty} z^{n} f^{\text{Bose}}_{\ell,w,n} \tag{2.3}$$

However by rearranging the formulae of (1.12) it may be seen (section 4) that $f^{\text{Bose}}_{\ell,w,n}$ is invariant under any permutation of its indices so that

$$H^{\text{Bose}}_{w,n}(z) = G^{\text{Bose}}_{w,n}(z) \tag{2.4}$$

The symmetry of $f^{\text{Bose}}_{\ell,w,n}$ with respect to ℓ and w is clear but the important invariance under interchange of ℓ and n is surprising. In section 4 we show, independently of the specific combinatorial formulae, that this follows from duality. A further proof of this symmetry uses a bijection ([7, 8] and [19, section 3.3]) to plane partitions. Plane partitions may be visualised [19] as stacks of unit cubes pushed into a corner. Plane partitions which biject to Bose configurations correspond to stackings in the corner of the rectangular parallelepiped with edges of lengths ℓ, w and n. The number of such stackings is manifestly invariant under any permutation of the edge lengths.

The following proposition states that $G^{\text{Bose}}_{\ell,w}(z)$ is a rational function whose numerator is a Narayana polynomial and as a result the coefficients are symmetric and sum to a w-dimensional Catalan number.

Proposition 2.

(a)

$$G^{\textit{Bose}}_{\ell,w}(z) = \frac{Q^{\textit{Bose}}_{\ell,w}(z)}{(1-z)^{\ell w+1}}$$

where $Q^{\textit{Bose}}_{\ell,w}(z)$ is a polynomial of degree $(\ell-1)(w-1)$

(b) $q^{\textit{Bose}}_{\ell,w,n} \equiv [z^n]Q^{\textit{Bose}}_{\ell,w}(z) = \sum_{k=0}^{n}(-1)^{n-k}\binom{\ell w+1}{n-k} f^{\textit{Bose}}_{\ell,w,k}$

(c) $q^{\textit{Bose}}_{\ell,w,n} = N(w,\ell,n)$ *and* $Q^{\textit{Bose}}_{\ell,w}(z) = N_{w,\ell}(z)$

(d) $Q^{\textit{Bose}}_{\ell,w}(1) = f^{\textit{Fermi}}_{\ell,w,\ell w+1} = C_{\ell,w}$

(e) *For* $n = 0, 1, \ldots, (\ell-1)(w-1)$, $q^{\textit{Bose}}_{\ell,w,(\ell-1)(w-1)-n} = q^{\textit{Bose}}_{\ell,w,n}$.

(f) $z^{(\ell-1)(w-1)}Q^{\textit{Bose}}_{\ell,w}(1/z) = Q^{\textit{Bose}}_{\ell,w}(z)$

Examples of $Q^{\text{Bose}}_{\ell,w}(z)$ are given in Appendix A.

The proof of proposition 2 in section 5 together with (2.2) and (2.4) establishes proposition 1 since using parts (a) and (c)

$$H^{\text{watermelon}}_{0,nw,n}(x) = \frac{Q^{\text{Bose}}_{n,w}(x^n)}{(1-x^n)^{nw+1}} = \frac{N_{w,n}(x^n)}{(1-x^n)^{nw+1}}. \tag{2.5}$$

and parts (d) and (e) show that the numerator coefficients of $H^{\text{watermelon}}_{0,nw,n}(x)$ are symmetric and sum to $C_{n,w}$.

An important step in our proof of proposition 2 in section 5 requires a relation between the numbers of Fermi and Bose configurations. Fermi walk configurations were first considered by Inui and Katori [15] in the context of directed percolation (DP) theory: they found the following relation.

$$f^{\text{Bose}}_{\ell,w,n} = \sum_{k=1}^{n}\binom{n-1}{n-k} f^{\text{Fermi}}_{\ell,w,k} \tag{2.6}$$

The factor $\binom{n-1}{n-k}$ arises from the number of ways to assign a further $n-k$ walks to the paths used by the k Fermi walks. Möbius inversion [20] of (2.6) gives

$$f^{\text{Fermi}}_{\ell,w,n} = \sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{n-k} f^{\text{Bose}}_{\ell,w,k}. \tag{2.7}$$

which, together with (1.12), is an explicit formula for the number of Fermi configurations

For a given rectangular grid there is clearly a limit to the number of walks in a Fermi configuration. On the 2×2 grid of figure 1 there can be be no more than 5 walks and the figure shows the two maximal configurations. Given ℓ and w the maximal number of Fermi walks is $\ell w+1$ and the number of maximal Fermi configurations is a w-dimensional Catalan number.

$$f^{\text{Fermi}}_{\ell,w,\ell w+1} = C_{\ell,w} \tag{2.8}$$

This result is part of proposition 2 the proof of which uses the above properties of the Naryana numbers. A direct proof based on equations (1.12) and (2.7) will be given in [21] where we also give product formulae for the numbers of Fermi configurations having one, two and three less walks than the maximum. A further proof will be given in [22] using a bijection to standard Young tableaux which are enumerated by a product formula in terms of hook lengths [23].

The following generating function for the numbers of Fermi walks is therefore a polynomial of degree $\ell w+1$.

$$G^{\text{Fermi}}_{\ell,w}(z) \equiv \sum_{n=0}^{\ell w+1} z^n f^{\text{Fermi}}_{\ell,w,n}. \tag{2.9}$$

Equation (2.6) together with the convention $f^{\text{Fermi}}_{\ell,w,0} = 1$ implies the relation

$$G^{\text{Bose}}_{\ell,w}(z) = G^{\text{Fermi}}_{\ell,w}\left(\frac{z}{1-z}\right). \tag{2.10}$$

3. Rationality of a second generating function for Fermi walks

A generating function $H^{\text{Fermi}}_{w,n}(z)$ may be defined in the same way as $H^{\text{Bose}}_{w,n}(z)$ and it follows from (2.7) and proposition 2(a) that

$$H^{\text{Fermi}}_{w,n}(z) \equiv \sum_{\ell=0}^{\infty} z^\ell f^{\text{Fermi}}_{\ell,w,n} = \frac{P^{\text{Fermi}}_{w,n}(z)}{(1-z)^{wn+1}} \tag{3.1}$$

where $P^{\text{Fermi}}_{w,n}(z)$ is the polynomial

$$P^{\text{Fermi}}_{w,n}(z) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} (1-z)^{(n-k)w} Q^{\text{Bose}}_{w,k}(z) \tag{3.2}$$

and we have used the symmetry of $f^{\text{Bose}}_{\ell,w,n}$. Setting $z=1$ shows that the sum of the numerator coefficients $P^{\text{Fermi}}_{w,n}(1) = Q^{\text{Bose}}_{w,n}(1) = C_{w,n}$ and since $Q^{\text{Bose}}_{w,k}(z)$ has degree $(k-1)(w-1)$, $P^{\text{Fermi}}_{w,n}(z)$ has degree $(n-1)w$ arising from the term $k=1$ in (3.2).

Tables of $P^{\text{Bose}}_{w,n}(z)$ and $P^{\text{Fermi}}_{w,n}(z)$ for low values of n and w are given in Appendix A.

4. Duality and the invariance of $f^{\text{Bose}}_{\ell,w,n}$

The second product of (1.12) may be arranged in the forms

$$f^{\text{Bose}}_{\ell,w,n} = \prod_{j=1}^{w} \frac{(\ell+j)_n}{(j)_n} = \prod_{j=1}^{n} \frac{(\ell+j)_w}{(j)_w} = \prod_{j=1}^{n} \frac{(w+j)_\ell}{(j)_\ell}. \tag{4.1}$$

which brings out the invariance of $f^{\text{Bose}}_{\ell,w,n}$ under interchange any two of the indices ℓ, w and n with the third fixed and hence under any permutation of these indices. Invariance under interchange of ℓ and w is required by symmetry but the further invariance involving n is unexpected but follows from the following duality.

Bose configurations biject [16] to Vicious walk watermelon configurations by translating the i^{th} walk by the vector $(-(i-1), i-1)$. Consider a watermelon configuration ω of n vicious walkers which

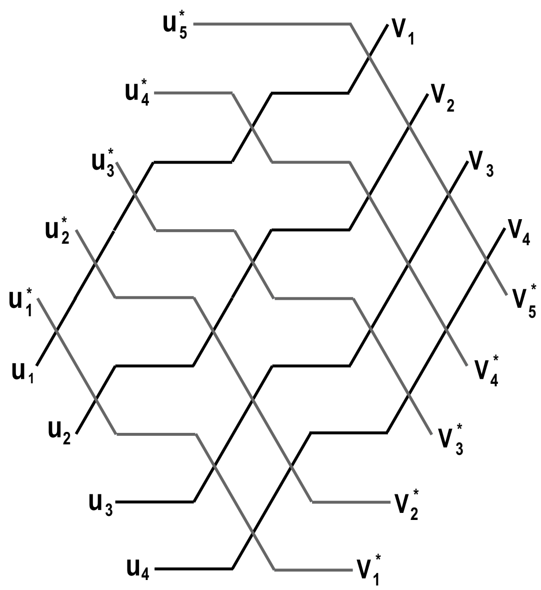

Figure 2. The four primary walks have $\ell = 2$ and $w = 5$. The corresponding five dual walks have $\ell^* = 4$ and $w^* = 2$

corresponds to a Bose configuration on an $\ell \times w$ grid. Each walk has w left steps and ℓ right steps and in order to define the dual configuration ω^* we distort the walks so that they lie on a triangular lattice.

Define step vectors $\mathbf{e}_1 = \{2, 0\}, \mathbf{e}_2 = \{1, \sqrt{3}\}$ and $\mathbf{e}_3 = \{1, -\sqrt{3}\}$ which are the nearest-neighbour vectors of a directed triangular lattice .

Suppose that the right steps of ω have vector $\mathbf{e}_1$ and the left steps have vector $\mathbf{e}_2$ and that the i^{th} primary walk from the left, $i \in \{1, \dots n\}$, starts at $\mathbf{u}_i = (i-1)\mathbf{e}_3$ and ends at $\mathbf{v}_i = \mathbf{u}_i + \ell\mathbf{e}_1 + w\mathbf{e}_2$.

The dual walk configuration ω^* is defined as follows (see fig 2 for example).

- The left step vector of ω^* is $\mathbf{e}_1$, the right vector is $\mathbf{e}_3$ and the j^{th} dual walk from the right, $j \in \{1, \dots n^*\}$, starts at $\mathbf{u}^*_j = \{0, \sqrt{3}\} + (j-1)\mathbf{e}_2$.
- For each left step of the primary walks draw a right step of the dual walks, so that these steps bisect one another.
- Join the right steps of the dual walks by left steps and, if necessary, add further left steps so that the dual walks end at $\mathbf{v}^*_j = \mathbf{u}^*_j + \ell^*\mathbf{e}_3 + w^*\mathbf{e}_1$ where

$$w^* = \ell \qquad \ell^* = n \qquad n^* = w. \tag{4.2}$$

The dual construction is bijective since given a dual configuration the primary configuration may be recovered by interchanging right and left in the dual construction. Moreover the dual configurations are also watermelon configurations with different parameters so

$$f^{\text{Bose}}_{\ell,w,n} = f^{\text{Bose}}_{\ell^*,w^*,n^*} = f^{\text{Bose}}_{n,\ell,w}. \tag{4.3}$$

5. Proof of proposition 2

Proof. Using (2.9) in (2.10) gives

$$(1-z)^{\ell w+1} G^{\text{Bose}}_{\ell,w}(z) = \sum_{n=0}^{\ell w+1} z^n (1-z)^{\ell w+1-n} f^{\text{Fermi}}_{\ell,w,n} = Q^{\text{Bose}}_{\ell,w}(z) \tag{5.1}$$

where $Q^{\text{Bose}}_{\ell,w}(z)$ is a polynomial of degree at most $\ell w+1$. The actual degree is determined below. The rationality of $G^{\text{Bose}}_{\ell,w}(z)$ may thus be seen as a consequence of the upper limit on the number of walks in a Fermi configuration.

The coefficient of z^n on the left of (5.1) is the convolution of part (b).

The second product in (1.12) may be converted to the form

$$f^{\text{Bose}}_{\ell,w,n} = \prod_{j=0}^{w-1} \binom{\ell+j+n}{\ell}\binom{\ell+j}{\ell}^{-1} \tag{5.2}$$

and substituting in part (b) gives the explicit expression for the numerator coefficients

$$q^{\text{Bose}}_{\ell,w,n} = \sum_{k=0}^{n}(-1)^{n-k}\binom{\ell w+1}{n-k}\prod_{j=0}^{w-1}\binom{\ell+j+k}{\ell}\binom{\ell+j}{\ell}^{-1}. \tag{5.3}$$

Comparing this with the definition (1.6) establishes the connection (c) with the w-dimensional Naryana numbers and the degree $(\ell-1)(d-1)$ in (a) follows from property (1.7). $Q^{\text{Bose}}_{\ell,w}(z)$ is therefore the Naryana polynomial $N_{w,\ell}(z)$.

The first equality in (d) is obtained by setting $z=1$ in (5.1)

$$Q^{\text{Bose}}_{\ell,w}(1) = \sum_{n=0}^{\ell w+1} q^{\text{Bose}}_{\ell,w,n} = f^{\text{Fermi}}_{\ell,w,\ell w+1} \tag{5.4}$$

and the second follows from part (c) together with property (1.9) of the Naryana numbers.

Part (e) is Naryana property (1.8) which Sulanke [10] proved using a relation which transcribes to

$$f^{\text{Bose}}_{\ell,w,-\ell-w-n} = (-1)^{\ell w} f^{\text{Bose}}_{\ell,w,n}. \tag{5.5}$$

This follows from (1.12) which may be rewritten in the form

$$f^{\text{Bose}}_{\ell,w,n} = \prod_{j=1}^{w}\frac{(n+j)_\ell}{(j)_\ell}. \tag{5.6}$$

Part (f) follows simply from (e). □

Notes:

- Notice that the degree of $Q^{\text{Bose}}_{\ell,w}(z)$ differs by $\ell+w$ from the limit $\ell w+1$ implied by equations (2.9) and (2.10).
- The inverse of part (b) is

$$f^{\text{Bose}}_{\ell,w,n} = \sum_{k=0}^{n}\binom{\ell w+n-k}{\ell w} q^{\text{Bose}}_{\ell,w,k} \tag{5.7}$$

  which using (c) is a transcription of [10], proposition 4.

6. An inversion relation for $G^{\text{Bose}}_{\ell,w}(z)$

Guttmann and Vöge noted that symmetry of the numerator coefficients (proposition 2(e)) is a property of vicious walker watermelon configurations ($f = 0$) but of no other degree f of friendliness. They used this to derive an inversion relation (their equation (7.71)) for the generating function of vicious walk watermelon configurations with a given number of walkers. The general formula (2.5) is implicit in this relation.

By the invariance of $f^{\text{Bose}}_{\ell,w,n}$ under interchange of w and n this also applies to the generating function

$$W_w(z, y) \equiv \sum_{\ell=0}^{\infty} G^{\text{Bose}}_{\ell,w}(z)y^{\ell}. \tag{6.1}$$

The inversion relation follows from proposition 2(a) and (f) which is only valid for $\ell \geq 1$.

$$W_w(z, y) + \frac{1}{z^w}W_w(\frac{1}{z}, (-1)^w\frac{y}{z}) = \frac{1 - z^{1-w}}{1 - z}. \tag{6.2}$$

where the expression on the right arises from the anomalous term $G^{\text{Bose}}_{0,w} = 1/(1 - z)$.

7. Summary and conclusion

The work of Guttmann and Vöge [1] on anisotropic generating functions for f-friendly walks has been extended in the case $f = 0$ to an arbitrary number of walkers. The rationality of the generating function is shown to follow from the existence of an upper limit to the number of walks in a Fermi walk configuration together with the invariance of $f^{Bose}_{\ell,w,n}$ under the interchange of ℓ and n. This invariance was shown to follow from duality. The binomial form of the denominator arises from a connection between the generating functions for Bose and Fermi configurations and the degree is equal to the number of walks in a maximal Fermi configuration. The same connection allowed us to show that the numerator for $n-$walk configurations having w left steps is a generalised Naryana polynomial $N_{n,w}(z)$ [10] which is symmetric in n and w. Also the coefficients have reflection symmetry and sum to a generalised Catalan number $C_{n,w}$ in agreement with the observations [1] for configurations of three walks. A Fermi walk generating function was shown to be a rational function similar to that for vicious walkers but with a numerator of different degree. It should be possible to extend this work to star configurations. Also it is hoped that the methods used here will throw some light on the more difficult problem of osculating walkers whose generating functions were also found [1] to be rational with the same sum of numerator coefficients.

Acknowledgments

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Appendix A. The polynomials $P^{Bose}_{w,n}(z)$ and $P^{Fermi}_{w,n}(z)$.

(a) $P^{Bose}_{w,n}(z) = Q^{Bose}_{w,n}(z)$

w	1	2	3	4
n				
1	1	1	1	1
2	1	$1+z$	$1+3z+z^2$	$1+6z+6z^2+z^3$
3	1	$1+3z+z^2$	$1+10z+20z^2+10z^3+z^4$	$1+22z+113z^2+190z^3+113z^4+22z^5+z^6$
4	1	$1+6z+6z^2+z^3$	$1+22z+113z^2+190z^3+113z^4+22z^5+z^6$	$1+53z+710z^2+3548z^3+7700x^4+\ldots$

(b) $P^{Fermi}_{w,n}(z)$

w	1	2	3	4
n				
1	1	1	1	1
2	z	$3z-z^2$	$6z-2z^2+z^3$	$10z+5z^3-z^4$
3	z^2	$z+9z^2-6z^3+z^4$	$4z+45z^2-20z^3+16z^4-4z^5+z^6$	$10z+165z^2+116z^3+165z^4-10z^5$ $+25z^6-10z^7+z^8$
4	z^3	$9z^2+24z^3-27z^4$ $+9z^5-z^6$	$z+92z^2+394z^3-166z^4+184z^5$ $-68z^6+30z^7-6z^8+z^9$	$5z+509z^2+4365z^3+7055z^4+7820z^5+2960z^6+$ $+1586z^7-340z^8+125z^9-75z^{10}+15z^{11}-z^{12}$

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