Periodic Strategies and Rationalizability in Perfect Information 2-Player Strategic Form Games

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Abstract. We define and study periodic strategies in 2-player finite strategic form games. and exploit their connection to non-Nash rationalizable strategies. The result of our findings is that, non-Nash rationalizable strategies are always periodic, but periodic strategies are not necessarily rationalizable.

Non-cooperative game theory [1, 2] has developed to be the most useful tool in strategic decision making and social sciences the last decades, especially when the outcomes of the players are strongly interdependent. This aforementioned attribute renders non-cooperative game theory different from problems of single agent decision theory. The concepts of rationality of the players and also the common belief in rationality of the players that participate in the game are foundational principles of invaluable importance. In the literature rationality is somehow a controversial feature therefore in this paper we assume that rationality means, that each player forms beliefs about his own and his opponents possible strategies. The beliefs, are materialized in the game quantitatively in terms of subjective probabilities and acts according to those probabilities. The word act embodies the fact that a player chooses an optimal strategy based on the beliefs of the player for his opponents strategies. We shall assume that each player acts rationally and additionally that he believes in his opponents rationality. Additionally we shall work within the context of common knowledge [3], which means that either the game and rationality of the players are assumed to be common knowledge. We also confine ourselves to outcomes that are measured by von Neumann-Morgenstern utilities functions [4], so optimality means utility maximization. In this article we shall consider finite strategic form games with complete information that are played simultaneously and only for one time. Strategic form games are very important in game theory since even extensive games can be reduced in strategic form games. A strategic game consists of a finite set of players, a finite set of actions (often so-called pure-strategies), and a utility function for each player. One of the most fundamental concepts in non-cooperative game theory is the Nash equilibrium. A pure-strategy Nash equilibrium is an action profile for which no single player can obtain a higher payoff by deviating unilaterally from this strategy profile. However, as Bernheim notes in his paper [5], Nash equilibrium is neither a necessary consequence of rationality nor a reasonable empirical proposition. A very important refinement of the Nash equilibrium is the rationalizability solution concept and rationalizable strategies [5, 6]. The rationalizability concept appeared independently in Bernheim’s [5] and Pearce’s work [6]. In this article we shall study periodic strategies in two player strategic form
games. With the term periodic is meant that there exist maps between the two players strategy spaces, such that they constitute an automorphism on each strategy space \( d \), which has the property \( d^n = 1 \) for some \( n \in N \). The periodicity concept appeared in Bernheim’s paper [5] in a different context and not under that name. We shall prove that periodic strategies exist in two player strategic games and some of these indeed are rationalizable too. In particular our findings prove that the set of periodic strategies include all the rationalizable strategies that are not Nash equilibria. In addition there are Nash equilibria that can be periodic and also there are periodic strategies that are neither Nash strategies nor rationalizable. Being a superset of non-Nash rationalizable strategies, the periodic strategies could serve as a more general concept than rationalizability. This paper is based on the working paper [7], and due to the constraints of space, we omit the proofs and also the study of mixed strategies. These can be found in [7].

We consider a simultaneous move game with two players A and B, with perfect information and we assume that the game is played only once. The actions are considered to be finite in number, as a first approach. Each player can have a finite number of actions but the actions of the two players can be different in number. Denote with \( M(A) \) the strategy space of all A’s actions and with \( N(B) \) the strategy space of all B’s actions. The strategic form game is then defined as:

- The set of players: \( I = 1, 2 \)
- The strategy spaces of players \( A, B \), namely \( M(A) \) and \( N(B) \), and the total strategy space \( \hat{G} = M(A) \times N(B) \)
- The payoff functions \( u_i(\hat{G}) : \hat{G} \to \mathbb{R}^2, i = 1, 2 \)

We define two continuous maps, \( \phi_1 \) and \( \phi_2 \), between the strategy spaces \( M(A) \) and \( N(B) \), to act as follows:

\[
\phi_1 : M(A) \to N(B) \quad \phi_2 : N(B) \to M(A)
\]  

These two maps are considered to be injective (monomorphisms). Let’s define the action of these maps to each strategy space. With the payoff being defined as:

\[
u_i : M(A) \times N(B) \to \mathbb{R}^2
\]  

the actions of the aforementioned maps \( \phi_{1,2} \) are defined in such a way that the following hold at each level:

\[
u_1(x, \phi_1(x)) \geq \nu_1(x, y_1) \quad \forall y_1 \in N(B)
\]

\[
u_2(\phi_2 \circ \phi_1(x), \phi_1(x)) \geq \nu_2(x_1, \phi_1(x)) \quad \forall x_1 \in M(A)
\]

\[
u_1(\phi_2 \circ \phi_1(x), \phi_1 \circ \phi_2 \circ \phi_1(x)) \geq \nu_1(\phi_2 \circ \phi_1(x), y_1) \quad \forall y_1 \in N(B)
\]

Let us clarify the meaning of the above inequalities. We start with an action \( x \in M(A) \), upon which we act with the map \( \phi_1 \). The map is acting in such a way that the inequality \( u_1(x, \phi_1(x)) \geq u_1(x, y_1) \quad \forall y_1 \in N \) holds true. What this means is that \( \phi_1 \) maps \( x \) to B’s strategy space such that this action of B’s yields the best payoff for player A playing the action \( x \). So we could say that the map yields the actions of player B to which the action \( x \) of A is a best response. At the next step, the map \( \phi_2 \) acts on the strategy space of player B and yields an action \( \phi_2 \circ \phi_1(x) \in M(A) \) such that \( u_2(\phi_2 \circ \phi_1(x), \phi_1(x)) \geq u_2(x_1, \phi_1(x)) \quad x_1 \in M(A) \). So we could say that \( \phi_2 \circ \phi_1(x) \) is an action of player A to which \( \phi_1(x) \) is a best response for player B. If we proceed this way, it is possible to end up to the initial \( x \) action of player A. Let us depict this procedure with a one dimensional chain of actions, as follows:

\[
x \xrightarrow{\phi_1} \phi_1(x) \xrightarrow{\phi_2} \phi_2 \circ \phi_1(x) \xrightarrow{\phi_2} \phi_2 \circ \phi_2 \circ \phi_1(x) \cdots \xrightarrow{\phi_2} x
\]
where with the letter $P$ we denote the procedure described in relation (3) above. Therefore it is possible to construct a chain of continuous action of the maps $\phi_{1,2}$ so that the final action is identical to the action at the beginning of the game. We shall call that action periodic. Notice that for periodic actions, the operator $d = \phi_2 \circ \phi_1$, has the property that $d^n = \text{id}_M$ for some number $n \in N$. This is a very important property of the space of actions of actions of player A. It means that we can find an operator that acts as an automorphism on a subset of $M(A)$ and leaves this subset invariant under this map. It is exactly this subset of the total strategy space of player $A$ that constitutes the set of periodic strategies of players $A$. So we can formally define the periodic strategies and actions as follows.

**Definition 1:** In a 2-player simultaneous move strategic form game with finite actions, we define periodic strategies for a player $A$ to be a subset of his available strategies $M(A)$ such that there exists an operator $d: M(A) \to M(A)$, such that $\exists n \in N$ for which $d^n = \text{id}_M$. It is presumed that the map consists of maps that act in such a way that the inequalities of relation (3) are fulfilled.

There is a clear conceptual distinction between periodic strategies and rationalizable strategies. We will make this clear with some characteristic examples that we analyze in detail. As a comment we have to say that to our knowledge the number $n$ defined previously, depends on the payoff details of the game. So there is no direct connection of this number, to the number of actions and the number of players, at least for finite 2-player strategic form games. There is also some sort of connection between the total number of Nash equilibria and the number $n$, a relation which at the moment we fail to understand from what arguments it originates. We hope to address these problems along with the case of continuous games in the future. According to our findings, all non-Nash rationalizable actions are periodic. Moreover there are Nash actions that are periodic and some Nash strategies that are not periodic. The most important outcome of our work is materialized in the following theorem:

**Theorem:** In a 2-player finite action strategic form game, the rationalizable actions that do not belong to a Nash equilibrium, are periodic. The converse is not true.

The proof can be found in [7]. Denote with $P(A)$ the set of all the periodic actions that are contained in player’s A strategy space, $N(A)$ the set of player A actions that belong to some Nash equilibrium, $R(A)$ be the set of all rationalizable actions and $R'(A)$ the set of non-Nash rationalizable actions. So $R'(A) = R(A) \setminus N(A)$. The theorem states that all actions that belong to $R'(A)$ also belong to $P(A)$. We could say that the following hold true:

$$R'(A) \subseteq P(A) \quad (5)$$

It is obvious that the set of periodic actions contains the set of non-Nash rationalizable actions. Therefore finding the periodic actions in a finite game but with many actions, is a first step towards finding the set of all rationalizable strategies. Before closing we present an example that will enlighten the concepts that we presented in this section and show us that the theorem finds application. Consider game 1. According to the inequalities of relation (3), we can construct the following periodicity cycles, namely:

$$a_1 P_3 b_3 P_3 a_3 P_3 b_1 P_3 a_1$$
$$a_3 P_2 b_1 P_2 a_1 P_2 b_3 P_2 a_2 \quad (6)$$

It is obvious in the above two examples that $n = 2$ for both the actions, $a_1$ and $a_3$. Moreover for the actions that constitute a Nash equilibrium it is not possible to construct such a cycle. Now let us consider the rationalizability cycles. The actions $a_1$ and $a_3$ are both rationalizable and neither of them is a Nash strategy. Therefore the theorem we presented previously applies to them. So these two actions are both rationalizable and periodic, is there exist a rationalizability cycle for these two. By rationalizability cycle is meant a cycle based on rationality and by rationality
is meant acting optimally under some beliefs about the opponents actions. Indeed such a cycle exists and it looks like:

\begin{align}
    a_1 &\rightarrow R \quad b_3 \rightarrow R \quad a_3 \rightarrow R \quad b_1 \rightarrow R \quad a_1 \rightarrow R \\
    a_3 &\rightarrow P \quad b_1 \rightarrow P \quad a_1 \rightarrow P \quad b_3 \rightarrow P \quad a_2 \rightarrow P
\end{align}

The reasoning behind this cycle is based on this system of beliefs: Player A considers action $a_1$ rational if he believes that player B will play $b_3$, which is rational for player B if he believes that player A will play $a_3$. Accordingly, A will consider playing $a_3$ rational if he believes that player B will play $b_1$, which would be rational for player B if he believes that player A will play $a_1$. So we obtain a cycle of rationalizability based on pure utility maximization rationality. For the Nash action $a_2$ it is not easy to construct such a cycle because A will be forced to play $a_2$ since B would never play $b_1$ or $b_3$ as a best response to $a_2$. So the Nash strategy is forced to be rationalizable. In this game the non-Nash rationalizable actions are periodic actions and actually are the only periodic strategies. So the theorem finds direct application in this case.

We have exploited an intrinsic property of 2-player finite simultaneous strategic form games, which we called periodicity of strategies. Additionally we provided a theorem which states that every action that is rationalizable but does not belong to a Nash equilibrium is periodic. Additionally we found that the set of periodic actions is a larger set than these of non Nash rationalizable actions. Clearly there could be some reasoning behind this that resides in some fixed point argument of the appropriate operator $d^n$, for some $n \in \mathbb{N}$, but why this operation excludes some of the Nash equilibria is still a mystery for us. Probably there exists an underlying pattern strongly related to the specific payoff of the games or the combination of actions in a specific way, that we still fail to understand. The periodicity feature for finitely many actions of strategic form games can be very useful. Indeed, all the periodic actions can be found using some simple program code, a fact that is clearly a good step in finding all the rationalizable actions that are non- Nash equilibria. This can be very useful for games that have, as we mentioned, finitely many actions. In [7] we have included into the study the mixed strategies for $2 \times 2$ games and also commented about continuous 2-player games. In addition the reference list is more complete in the working paper [7].

References