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Quantum gauge theories from geometry

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Abstract. Geometrical theories have been developed to describe quantum interacting particles with full mathematical covariance. They possess a sophisticated gauge structure that derives from the fundamental properties of the geometry. These theories are all implicitly quantized and come in three known types: Weyl, non-compactified Kaluza-Klein, and, as presented here, Dirac. The spin one-half particle is a conformal wave in an eight dimensional Riemannian space. The coordinates transform locally as spinors and project into space time to give the known gravitational and electromagnetic forces. The gauge structure of the weak interactions appears as well, as in this space the electron transforms into a neutrino under hyper-rotations. The possibility of including the strong interactions and the corresponding gauge system is discussed.

1. Introduction

Gauge structures have an ubiquitous presence in modern particle theory. The original concept of gauge was derived by Weyl early in his attempt to unify electromagnetism with general relativity. Modern quantum versions of his theory as well as of a non-compactified Kaluza-Klein theory have demonstrated essential gauge properties and fully geometrical interactions. Here, a geometrical basis for Dirac theory is added. A recent discussion [1] gives a general overview and some preliminaries to the inclusion of spin.

Certain concepts are essential. 1. A single wave function is used as the defining characteristic of the geometry, rather than an entire Hilbert space. The quantum mechanics and the geometry meet through the quantum field equation in differential form. 2. An individual metric is assigned to each particle. Under these conditions, the universal field assumption of general relativity fails as different gravitational forces must act between overlapping parts of individual wave functions. 3. Interactions are described by curvilinear motion without the use of a point vertex approximation. 4. An absolute formulation of equivalence is assumed. Internal geometrical transformations convert between different types of force. These may include coordinate transformations and often contribute to the gauge structure as the particle is projected onto space-time. 5. A single universal coordinate system is assumed and a first quantized structure is generated without active quantization [2]. 6. The wave functions, and also the interactions, are associated with a conformal factor. Linear wave equations for the conformal factor come from the invariant curvature scalar, thus assuring compatibility with relativity. 7. Mass appears during the reduction of the five dimensional representation to space-time. The resulting mechanical inertia is suitable and gravitational equivalence is satisfied. 8. Fundamental time-symmetry is assumed throughout. Distant absorbing particles are assigned to account for apparent retarded behavior. 9. The resulting structure is a set of mathematically determined, non-linear, interacting wave fields. 10. Second quantization is treated as a formalism to account for multiple particles.

11. The fields evolve without any assumed fundamental statistics. 12. Photons and gravitons acquire their discrete properties entirely from the countability of the emitting particles.

Because the eight dimensional description connects through the five dimensional theory [3], a short review is included here. The arguments proceed almost entirely from general covariance. The required representations follow from the initial development of spin by Dirac [4, 5], combined with later studies [6]. The physical content of a geometry of eight dimension should be much greater than for four or five. The gauge complexity should increase as well. New interactions may appear without any explicit introduction.

2. Five-Dimensional Theory

Because some of the properties of five dimensional theory are required, a short summary is given here. The five coordinates, x^m , $m = 0, \dots, 4$, can, in a local Riemannian system, be taken equal to (t, x, y, z, τ) . The usual four-metric is

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

and becomes ¹

$$0 = g_{\mu\nu} dx^\mu dx^\nu - (-\mathcal{A}_\mu dx^\mu - d\tau)^2 \equiv \gamma_{mn} dx^m dx^n \quad (2)$$

which gives

$$\gamma_{mn} \equiv \begin{pmatrix} g_{\mu\nu} - \mathcal{A}_\mu \mathcal{A}_\nu & -\mathcal{A}_\mu \\ -\mathcal{A}_\nu & -1 \end{pmatrix} \quad (3)$$

where, in this gauge,

$$\mathcal{A}_\nu = \frac{1}{m} \left[\frac{\partial}{\partial x^\nu} \Im(\ln \psi) - eA_\nu \right] \quad (4)$$

All curvature is taken to be conformally generated, so that the Ricci tensor of $\omega\gamma^{mn}$ for some function ω is identically zero. This implies that source terms exist and that they can be calculated from ω . A particular choice gives the gravitational and electrodynamic field equations. In four-dimensional form these are

$$R^{\alpha\beta} = 8\pi\kappa \left(F^\alpha{}_\mu F^{\mu\beta} + m|\psi|^2 \mathcal{A}^\alpha \mathcal{A}^\beta + m|\psi|^2 \frac{1 - \mathcal{A}^2}{2 - \mathcal{A}^2} g^{\alpha\beta} \right) \quad (5)$$

and

$$F^{\beta\mu}{}_{|\mu} = 4\pi e|\psi|^2 \mathcal{A}^\beta. \quad (6)$$

The quantum field equation can be found by working from the invariant curvature scalar. Setting it to zero gives

$$\frac{1}{\sqrt{-\dot{g}}} \left(i\hbar \frac{\partial}{\partial x^\mu} - eA_\mu \right) \sqrt{-\dot{g}} g^{\mu\nu} \left(i\hbar \frac{\partial}{\partial x^\nu} - eA_\nu \right) \psi = \left[m^2 + \frac{3}{16} \left(\dot{R} - \frac{e^2}{4m^2} F_{\alpha\beta} F^{\alpha\beta} \right) \right] \psi. \quad (7)$$

Here, the fifth dimension has been reduced by setting derivatives of the conformal factor with respect to the proper time equal to the mass. This reduction insures that identical particles are represented by identical rest masses.

¹ The sign convention is difficult for historical reasons. \mathcal{A}_μ is to be interpreted as a quantum-corrected kinetic four velocity. It must have the opposite sign as the vector potential because the classical five metric is defined using the vector potential A_μ .

3. Eight-Dimensional Derivation

The structure of the eight dimensional theory is entirely analogous to that above for five dimensions [7]. Motion is conformally generated. The scalar curvature is expected to supply a linear quantum wave equation and the Ricci tensor should define an interaction structure.

First, the Dirac equation must be written in five-dimensional form. The fifth anti-commuting spin matrix is assigned to the fifth coordinate. The commutation properties are extended to the five metric:

$$\gamma^{mn}\delta_A^B = \frac{1}{2}\{\gamma^m, \gamma^n\} \equiv \frac{1}{2}(\gamma^m{}_A{}^C\gamma^n{}_C{}^B + \gamma^n{}_A{}^C\gamma^m{}_C{}^B) \\ \text{for } m, n = 0, \dots, 4, \text{ and } A, B, C = 1, \dots, 4. \quad (8)$$

As described in reference [1], by using a five dimensional wave function and applying a similarity transformation, the usual four-dimensional dirac equation can be rewritten,

$$\gamma^m{}_A{}^B \frac{\partial}{\partial x^m} \Psi_B = 0 \quad \text{or} \quad \gamma^m \frac{\partial}{\partial x^m} \Psi = 0. \quad (9)$$

Here, it is intended that the spin vector be derived naturally from the eight dimensional coordinate geometry. To relate this to properties of conformal transformations in eight space, we choose eight real coordinates ξ_r^A, ξ_i^A for $A = 1 \dots 4$, and consider the associated conformally flat space. Let the conformal factor be denoted by ω . The condition that the scalar curvature is zero constitutes a second-order differential equation for ω and provides a Riemannian characterization of the conformal waves. The resulting equation can be written in modern spinor notation. The eight coordinates must be arranged in complex pairs.

$$\xi^A = \xi_r^A + i\xi_i^A, \xi^{\bar{A}} = \xi_r^A - i\xi_i^A, \quad A = 1 \dots 4 \quad (10)$$

where the bar of conjugation is applied to the index for consistency with existing conventions [8]. The fundamental form is $d\xi^A d\xi^{\bar{B}} \epsilon_{A\bar{B}}$ with spinor metric

$$\epsilon_{A\bar{B}} = \epsilon^{A\bar{B}} = \text{diag}(1, 1, -1, -1). \quad (11)$$

The wave equation, as it corresponds to zero scalar curvature, is

$$\epsilon^{\bar{A}B} \frac{\partial}{\partial \xi^{\bar{A}}} \frac{\partial}{\partial \xi^B} \Psi = 0 \quad (12)$$

where $\omega = \Psi^p$ with $p = 4/(n-2) = 2/3$.

Locally, the coordinates in the neighborhood of a point in five-space are related to a corresponding point in eight-space by the differential relation

$$dx^m = \zeta^A \gamma^m{}_A{}^B d\xi^{\bar{C}} \epsilon_{\bar{C}B} + d\xi^A \bar{\gamma}^m{}_A{}^B \zeta^{\bar{C}} \epsilon_{\bar{C}B} \quad (13)$$

Because the eight-dimensional displacements are summed with conjugates, the five space displacements are always real. Furthermore, Lorentz rotations, extended to five dimensions, are mapped in the usual way from the spinor space of displacements $d\xi^A, d\xi^{\bar{A}}$ to dx^m . It is the simplest coordinate transformation law which satisfies these conditions. The parameter ζ specifies the relative orientation of the spin space. It is equivalent to the use of a four- or five-dimensional spin frame.

The quantity

$$\Psi_B = \frac{\partial \Psi}{\partial \xi^B} \quad (14)$$

is taken as the Dirac spinor wave function which, following equation (12), satisfies the first order equation

$$\epsilon^{\bar{A}B} \frac{\partial \Psi_B}{\partial \xi^{\bar{A}}} = 0 \quad (15)$$

Using the chain rule

$$\frac{\partial}{\partial \xi^{\bar{A}}} = \frac{\partial x^m}{\partial \xi^{\bar{A}}} \frac{\partial}{\partial x^m}$$

from the coordinate transformation (13), this becomes

$$\epsilon^{\bar{A}B} \gamma^m{}_D{}^C \zeta^D \epsilon_{\bar{A}C} \frac{\partial \Psi_B}{\partial x^m} \equiv \zeta^D \left(\gamma^m{}_D{}^B \frac{\partial \Psi_B}{\partial x^m} \right) = 0 \quad (16)$$

The complex conjugates $\xi^{\bar{A}}$ are treated as independent of the ξ^A 's during differentiation. The quantity in parenthesis is identified with equation (9), and is interpreted as characterizing the eight-dimensional space in an orientation independent way. Equation (16) should be satisfied for any value of ζ .

Using equation (14), a local plane wave solution of (7) can be converted to a solution of (9) by differentiation.

$$\Psi = e^{i(\omega t - \vec{k} \cdot \vec{x} - m\tau)} \equiv e^{ik_m x^m}, \quad k_m = (\omega, \vec{k}, m) \quad (17)$$

$$\Psi_A \equiv \frac{\partial \Psi}{\partial \xi^A} = \Psi i k_m \frac{\partial x^m}{\partial \xi^A} = i \Psi (k_m \gamma^{\dagger m}) \zeta^\dagger \quad (18)$$

This is the form for a free electron or positron with arbitrary spin when written in the basis implied by the matrices (9). As a calculation done in a local frame, it shows that during interaction, the particle remains a spinning electron.

This derivation involves a minimum of physical assumptions and has no explicitly performed first quantization. If the five- and eight-dimensional spaces are conformally flat, integrability issues are minimized. Equation (13) can be made to hold globally. Gravitational and electromagnetic interactions are included when the $\gamma^m{}_A{}^B$ matrices are defined. Moreover, the conformal variations that generate external source terms in the five-dimensional theory imply, through (13), that they are equivalent to analogous conformal transformations in the eight-dimensional space. Forces other than electromagnetism or gravitation must be present because the matrix of second order coordinate derivatives has become larger. Weak interactions are expected because they are known to be described with the Dirac theory, and equivalence requires that such additional forces be gauged to the others.

In local Cartesian coordinates, and using standard notation for the γ 's, the six quantities

$$\begin{aligned} A^\mu &= \psi^\dagger \gamma^\mu \psi, \text{ for } \mu = 0, 1, 2, 3, \\ \text{with } A^4 &= i\psi^\dagger \gamma^5 \psi, \text{ and } A^5 = i\psi^\dagger \psi \end{aligned} \quad (19)$$

following [9, 10] combine into a quadratic invariant which, in modern notation, is

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = (A^4)^2 + (A^5)^2. \quad (20)$$

Term by term, in the classical limit, equation (20) corresponds to the relation $E^2 - p^2 = m^2$. The last two of the six quantities must make up the mass. If these two quantities cancel, a zero-mass particle results. The condition is

$$\begin{aligned} 0 &= (A^4)^2 + (A^5)^2 \equiv (A^5 - iA^4)(A^5 + iA^4) \\ &\equiv [\psi^\dagger (1 + \gamma^5) \psi] \cdot [\psi^\dagger (1 - \gamma^5) \psi] \end{aligned} \quad (21)$$

One of the factors must be zero, giving either a neutrino or an anti-neutrino. Equation (15) has a zero-mass solution, and it must transform into the electron solution under a hyper-rotation. This inter-transformation is identified with weak isospin. Because (7) and (9) can only apply to a system of particles of fixed charge-to-mass ratio, neutrinos and anti-neutrinos must have, respectively, the same charge-to-mass ratio as the electron and positron. Particle transmutation by isospin hyper-rotation is qualitatively distinct from changes of bound composite systems. A more detailed representation of the weak force as a conformal effect is under study.

4. Discussion

Because this derivation does not proceed from a putative classical particle, a more sophisticated internal mass structure can be used. Additive terms to the square of the electron mass, such as are present in equation (7), will affect particle motion when energy densities are sufficiently high. The propagational mass is distinct from the rest mass. Based on the coefficients of the terms in the scalar theory, the effect is unobservably small for electrons, but might be measurable for neutrinos. It may even be possible to use gradients in the square of the electromagnetic field strength to deflect or focus them. In any case however, a more careful study of equation (12) is required to verify the presence of analogous terms that are expected to appear as corrections to the Dirac equation. Strong interactions as well may contribute to mass corrections, even for leptons. Moreover, the theoretical implications for renormalization and regularization, have not been investigated.

There are three hierarchies in the solutions of equation (12). The massless particles correspond to neutrinos. The fixed mass solutions are assigned to electrons. Additional solutions may not have well defined, separable, mass terms. If these are allowed, they are a much more complicated type of spin one-half particle. All of these are fermionic because the spinor wave functions are odd in the spin coordinates. Some of the interaction properties can be derived from the geometry. The gravitational equivalence requires that all forces come from the Ricci tensor. The coordinate transformation (13) implies a relationship of gravitational or electromagnetic effects as they might be represented in correspondence between the eight and five dimensional formalism. The interconversion of electrons and neutrinos, that characterizes weak isospin may be complicated by the presence of other forces. Can these be strong forces? A possible representation of quarks and gluons is an ongoing investigation.

The proposed continuing methodology is to write strong interactions in eight-dimensional form and to assume a universal geometry. The usual Yang-Mills derivative is written

$$\partial_\mu \rightarrow \partial_\mu - i\frac{g}{2}\lambda_a G_\mu^a(x) \quad a = 1, \dots, 8 \quad (22)$$

where $\lambda_a = [SU(3)_c]$ is a three dimensional matrix representation of color $SU(3)$. Usually, it is assumed that a triplet of basis states for the gauge group is formed from three distinct Dirac spinors.

$$\begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}' = (SU(3)_c) \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix} \quad (23)$$

The question is how this matrix could be placed in the geometry. Certain advantages are obtained if it can be placed interior to the spinor. If each one of the eight $SU(3)_c$ matrices is associated with one of the spinor coordinate directions, the gluon sum can be written $\lambda_a G_\mu^a \rightarrow \lambda_{\hat{A}} G_{\hat{B}}^{\hat{A}}$ where the eight sum is denoted by a hat over the index. However, if in addition,

the $SU(3)_c$ basis is internal to the spinor, so that

$$\begin{pmatrix} \dots \\ \psi^k \\ \psi^{k+1} \\ \psi^{k+2} \\ \dots \end{pmatrix}' = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \lceil & \dots & \rceil & \dots \\ \dots & & SU(3)_c & & \dots \\ \dots & \lfloor & \dots & \rfloor & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \dots \\ \psi^k \\ \psi^{k+1} \\ \psi^{k+2} \\ \dots \end{pmatrix} \quad (24)$$

then the covariant derivative can be written in the Riemannian form

$$D_{\hat{A}}\psi^{\hat{B}} = \partial_{\hat{A}}\psi^{\hat{B}} - \Gamma_{\hat{A}\hat{C}}^{\hat{B}}\psi^{\hat{C}} \quad (25)$$

Elements of the eight dimensional geometry may appear as an internal structure in space-time. Consider a local linear map between five space and eight space. The signatures are $(++++-- --)$ and $(+----)$. The three extra positive elements of the eight-metric cannot map to any corresponding terms in the five geometry. An internal group, such as $SU(3)$, may be present. Because the absolute sign of the spinor space is not identified, the signature could also be taken as $(----++++)$, more complicated gauge structures may be involved. Such a construction may allow the gauge group of the strong interactions to be part of a single unifying geometry.

5. Conclusion

Fundamental gauge structures and the extension of quantum-geometrical ideas to spin has brought in the possibility that all gauge theories are of geometrical origin. The reformulated Dirac theory, as a restriction on an eight dimensional conformal space, allows for additional particles beyond the electron. The accepted electron theory results if the eight coordinates are combined into four complex pairs and identified with the known spinor structure. The anti-commuting Dirac matrices are used to map local complex displacements onto five-dimensional space. The wave function is represented by the eight-gradient of a conformal parameter. A quantum field equation is specified by setting the scalar curvature to zero. The Dirac equation is a result of this geometrical condition. Neutrino solutions satisfy the same equation and transform into electrons under hyper-rotation. The isospin-changing weak interactions are given an explicit mechanical model. The eight dimensional formalism may be suitable for strong interactions. The theoretical appeal of this approach is that it satisfies mass-energy gravitational equivalence by design. The fundamental assumption of geometry may lead to theories that naturally contain all gauge transformations.

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