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Quasi-lognormal distribution tails in many-body stochastic processes with weighted multiplicative interactions

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Abstract. Many-body stochastic processes with weighted multiplicative interactions are investigated analytically in a particular case that systems grow and a weight parameter of a interaction kernel is negative. In consequence, it is found that a probability distribution function of the processes obeys a log-normal type distribution. The variance of the log-normal type distribution is derived explicitly. These results would give new understanding of log-normal distributions especially observed in aggregation processes in nature.

1. Introduction

Log-normal distributions have been reported in various fields of sciences, such as the size distributions of small grains, aerosol, clouds, foams, galaxies, the abundance of species, the number of employees in manufacturing plants, the prices of insurance claims, and the size of firms, and so on[1, 2, 3]. Nevertheless, theoretical understanding of the log-normal distributions are still quite poor. The only known theory is that in systems of a single degree of freedom, a multiplicative stochastic process generates a log-normal distribution[4]. In most of actual processes, however, the effect of interactions between components is inevitable. Therefore, systems must be treated with multiple degrees of freedom so as to understand those well. In this sense, theoretical understanding of lognormal distributions in many-body systems is needed. In this paper, we investigate many-body stochastic processes with weighted multiplicative interactions analytically. Consequently, we find that in a particular case, a probability distribution function(PDF) of the processes obeys a log-normal type distributions.

2. Definition of the processes

We consider a system of N particles with positive quantities $x_i(>0)$ $(i = 1, \ldots, N)$. At each time step, the system evolves with a binary interaction between particles labeled by i and j $(i \neq j)$, and two quantities x_i, x_j are converted into x'_i, x'_j by the rule $x'_i = \alpha x_i + \beta x_j, x'_j = \beta x_i + \alpha x_j$, where α, β (> 0) are positive interaction parameters. In the limit $N \to \infty$, the processes are governed by the master equation

$$\begin{aligned}\frac{\partial f(z,t)}{\partial t} &= \int_0^\infty dx \int_0^\infty dy\ f(x,t)\ f(y,t)\ K(x,y) \\ &\times \frac{1}{2}\left[\,\delta(z-(\alpha x+\beta y))+\delta(z-(\beta x+\alpha y))-\delta(z-x)-\delta(z-y)\,\right], \qquad (1)\end{aligned}$$

where $f(x,t)$ is a PDF and $K(x,y)$ is a kernel representing interaction rates expressed explicitly as

$$K(x,y)=\frac{x^w\ y^w}{(m_w)^2}, \qquad m_w(t)=\int_0^\infty x^w f(x,t)dx, \qquad (2)$$

where w is a weight parameter and $m_w(t)$ is a $w-$th order moment. Hereafter, we deal with a particular case that the system grows($\alpha > 1$ or $\beta > 1$) and the weight parameter is negative($w < 0$).

3. Derivation of a log-normal distribution in many-body systems

We attempt to find a scaling solution of Eq. (1) and assume scaling relations as $z = \xi \exp(\gamma t)$, $f(z,t) = \Psi(\xi)\exp(-\gamma t)$. A growth rate of systems are defined by $\gamma \equiv (\alpha+\beta-1)\mu_{1+w}/\mu_1\mu_w$, where μ_p is a scaled p-th moment. By estimating moment relations obtained from a scaled master equation, it is found that when $\alpha > \beta$, the following moment function is satisfied asymptotically(For more details, please see [5]).

$$\mu_p = \exp(ap^2 - bp\ln p + cp), \qquad (3)$$

$$a=\frac{\ln\alpha}{2|w|}, \quad b=\frac{1}{|w|}, \quad c=\frac{1}{|w|}\ln\frac{(1+\delta_{\alpha\beta})\mu_1}{(\alpha+\beta-1)\mu_{w+1}}+\frac{\ln\alpha}{2}+\frac{1}{|w|}, \qquad (4)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Inverse Mellin transform of the most leading term in Eq. (3) leads to a log-normal distribution.

$$\Psi(\xi) \simeq \frac{1}{\sqrt{4a\pi}\xi}\exp\left(-\frac{(\ln\xi)^2}{4a}\right), \qquad (\xi \gg 1). \qquad (5)$$

The variance of this log-normal distribution is given by the parameter $a = \ln(\alpha)/2|w|$, which is independent of β.

4. Conclusion and Discussion

To our knowledge, this is the first analytical derivation of log-normal distributions in many-body systems. It can be concluded that the following two conditions are essential for the emergence of the log-normal distributions in many-body systems: (i) systems grow with multiplicative interactions, (ii) interactions of particles having large quantities are inhibited. In the many-body processes, the correspondences between microscopic parameters α, β, w and macroscopic statistics a, γ are clarified. These results would give new understanding of log-normal distributions especially observed in aggregation processes in nature.

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