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Generalized clocks in timeless canonical formalism

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Generalized clocks in timeless canonical formalism

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Abstract. Hamiltonian dynamics is recast in a timeless formalism in which parameter time σ is derived from the generalized coordinates, the Hamiltonian invariance on trajectories, and the Maupertuis principle. In order to define a time variable T in macroscopic systems, the cyclicity in the phase space replaces the self consistent assumption of time periodicity generally adopted for real clocks. Generalized clocks are defined in physical systems of sufficient complexity. Under suitable assumptions, physical systems can be separated in a subsystem to be dynamically described, and another cyclic subsystem which has the role of generalized clock. The latter provides a discrete approximation of the parameter time, called metric time. The stability prescription of generalized clocks guarantees that dynamics is expressed by the same equations of motion parametrized by the parameter time, in terms of metric time at the desired approximation. The timeless Hamiltonian framework, together with the definition of generalized clock, provide a ground to account the fundamental timelessness of nature, and the experimental evidence of time evolution in macroscopic systems experienced by the observers.

1. Introduction

After the pioneering work of De Witt [1], timeless approach to fundamental physics [2] has recently obtained a renewed interest[3, 4, 5]. The odd nature of time arises from a large number of reasons, first of all the discrepancy between the satisfactory description of dynamics in terms of time evolution, and the fundamental timelessness of general relativity and canonical quantum gravity [1, 4, 5, 6, 7, 8, 9]. Furthermore, several contradictions emerge when different theories are compared, like classical and quantum mechanics, for which time is an external parameter to generate a strongly continuous unitary group of transformations $U(1)$ [10], special and general relativity, quantum field theory, where time is the negative metric signature coordinate of a 4-dimensional differential manifold [11, 12] and canonical quantum gravity. On the other hand, in metrology clock time is a metric time operatively defined at rest. Time is also sometimes associated to the concept of irreversibility and entropy [13, 14, 15]. Such variety of properties and domains of applicability reveals the lack of a satisfactory and universal well posed definition of time.

When considering the problem of the nature of time, there are implicitly two distinct problems to address. The first is about whether or not time must be included in the list of fundamental quantities of Nature, well defined at all the possible energy and length scales. The second is the description of the emergence of time metrology based on operatively defined clocks, and the explanation of time evolution experienced by an observer. The present work strongly supports the view that time is not a fundamental quantity of Nature. The starting point is to observe that time is used both to indicate the parameter used in dynamic equations to describe change in the phase space, and the discretized quantity measured by some macroscopic instruments treated as reference clocks. In most of the scientific literature, the two are implicitly considered the same quantity. Hamiltonian mechanics, which governs the dynamics of generalized coordinates or quantum fields, can be rigorously well defined without the concept of time. As a consequence, it becomes apparently even more difficult to solve the second problem, because time disappears from the list of observable quantities, even if there is a field of metrology entirely devoted to time and frequency measurements conceptually based on the common experience of time.

A good theoretical model capable to be predictive and satisfactory without time answers only partly to the problem of the nature of time: it says what time is not [7]. In the present work the answer to the first problem is addressed, with a particular attention in the definition of the time parameter in a Hamiltonian system in terms of other quantities. Next, also the second part of the problem is addressed, in order to account the experimentally measured and experienced clock time. The connection between the experimental clock time and the theoretical parameter time is the main goal of the present contribute.

As already pointed out by Newton itself, the fact that time is not a measurable quantity [1, 16, 17, 18] can be clarified as follows. One observes that a clock measures with some uncertainty an hypothetical 'true' external time t as a classical quantity $T_i(t)$ where the index i spans the clocks. The other observable quantities $O_j(t)$ are detected as $O_j(T_1)$, $O_j(T_2)$,.... where j spans the observables. However, the clock used to label the dynamical quantities of the system is in turn object of a measurement which establishes its value, accuracy and stability, by means of another clock. Consequently, being the second clock subject to the same check by the first, a two-clock time measurement is required to determine the fractional frequency stability from the Allan variance of both. The reference standard is expressed by $T_2(T_1)$ and $T_1(T_2)$,[7, 19] without any explicit use of t. In other words, a clock is not capable to measure such hypothetical external parameter time, but only self consistent quantities assisted by the recursive definition of *period*.

In the following we addresses the problem of explaining the macroscopic correspondence of clock time with parameter time of dynamics and we provide a universal definition of time for a Hamiltonian system in terms of generalized coordinates change in the phase space.

I will concentrate on a variational approach which enables the introduction of time in a physical theory in two steps. The method provides a parameter, called parameter time, which does not correspond to a specific observable quantity. However it can be put in correspondence with measurable quantities via cyclic phenomena. This is achieved by dividing a system in opportune subsystems. The present approach partially recalls the distinction presented in Ref.[4] between parametric (proper) time and discrete physical time. Differently from there, here no compactified extra-dimensions are required to introduce a detector operator, neither a lapse function or other parameters to appear in the Lagrange function.

In Section II the definition of parameter time in the framework of timeless Hamiltonian theories is presented. Time emerges as the natural parameter after one imposes a variational principle on a timeless action. The approach is applied in the subsection 2.1 to classical mechanics, and extended to quantum field theory in the subsection 2.2. Section III is devoted to connect the parametric time with clock metric time measured by means of realistic devices. In Section IV the conclusions are briefly discussed.

2. Parameter time in a Hamiltonian timeless scenario

The Maupertuis [20, 21, 22] action principle generates the dynamics without explicitly using time in the Hamilton. The interest is restricted to closed systems, so parameter independent Hamiltonians are considered. The variational principle, the Hamiltonian and the generalized coordinates are consequently expressed in a timeless framework. In the following I will show that the imposition of both the variational principle and the stationarity of the Hamiltonian individuate a special parametrization among all the possible parametrizations, which is the one commonly used to describe dynamics. In the following the corresponding parameter is indicated by σ and corresponds to the parameter τ of Ref. [20], and to parameter time t of ix_0 in ordinary Hamiltonian theory. The main difference from the latter is given by its derivation in a timeless framework. The capability of defining Hamiltonian mechanics without the concept of time will require consequently that some extra hypothesis are assumed in order to provide a definition of clock time. Its correspondence with the parameter σ is defined and discussed in the next section.

2.1. Parameter time in timeless classical mechanics

The Hamilton equations are expressed in timeless formalism from a variational principle on asynchronous varied trajectories. The time independent Hamiltonian $H(\mathbf{p},\mathbf{q})$ is a function of the generalized three dimensional coordinates $\mathbf{p}$ and $\mathbf{q}$. The independence of H from time reduces the degrees of freedom to $2n-1$. It is necessary to assume that it exists a set of trajectories in the coordinates space μ for which H is constant.

In order to determine the parametrization imposed by the stationarity of the action, a generic parametrization of the points of the trajectories is first assumed. Such arbitrary parametrization λ gives $q_i = q_i(\lambda)$ and $p_i = p_i(\lambda)$ where all the functions belong to C^2 on the interval $[\lambda_A, \lambda_B] \in \mathcal{R}$. The Hamiltonian $H(\mathbf{p},\mathbf{q})$ does not depend explicitly on λ. In order to impose a variational principle on the trajectory it is now considered a variation that is normally used to impose asynchronous varied trajectories in canonical formalism to derive Hamilton equation from the Maupertuis principle. A new parametrization σ of the generalized coordinates and of λ is now defined, under the condition that $\frac{d\lambda}{d\sigma} \neq 0$ on $[\sigma_A, \sigma_B]$.

Such distinction between λ and σ represents a subtle principle and technical difference from the approach of Ref.[23, 24]. The stationarity of the action is imposed:

$$A = \int p_i dq_i \tag{1}$$

where the Einstein summation on the repeated indexes is adopted and $i = 1,2,3$. The Maupertuis variational principle reads

$$\delta A = \delta \int p_i dq_i = 0 \tag{2}$$

The imposition of the stationarity of the action is given by the variation of the trajectories. Neglecting as usual second order perturbations and integrating by parts where necessary, one has:

$$d\sigma = \left(\frac{\partial H}{\partial p_i}\right)^{-1} dq_i = -\left(\frac{\partial H}{\partial q_i}\right)^{-1} dp_i \tag{3}$$

under the hypothesis that $\left(\frac{\partial H}{\partial p_i}\right) \neq 0$ and $\left(\frac{\partial H}{\partial q_i}\right) \neq 0$. They differ from the Hamilton equations since σ does not represent the macroscopic metric time. On the contrary, it only represents the natural parameterization of the system imposed by the energy conservation. *σ has nothing to do with the quantity measured by clocks.*

2.2. Parameter time in quantum field theory

The most convenient formalism to extend the action principle to general relativity and to quantum mechanics is the extended presymplectic approach [7]. There, dynamics is expressed on the unparameterized curve γ in the relativistic configuration space $C = \mathcal{R} \times C_0$, where C_0 is the m-dimensional space of coordinates q^i, which extremizes the integral

$$A[\gamma] = \int_\gamma \theta \tag{4}$$

where

$$\theta = p_i dq^i + p_t dt \tag{5}$$

is the natural one-form defined on the cotangent space T^*C and the constraint

$$H(q^i, t, p_i, p_t) = 0 \tag{6}$$

where H is the relativistic Hamiltonian. In the extended presymplectic formalism, the variational principle reads:

$$\delta A[\gamma] = \delta \int_\gamma \theta = 0 \tag{7}$$

Such principle allows a quantum extension, which goes beyond the scopes of the present section. Both the lagrangian and the extended presymplectic formalism consider time as a part of the manifold where physics is defined. Time t or x_0 assumes a role comparable to that of space, even when starting with an unparameterized curve as happens in presymplectic approach. Technically, since the action admits invariance under reparameterization of time (spacetime in relativistic domain), it does not represent a problem. Here, in order to avoid the use of the concept of time, the configuration space is only C_0 instead of $C = \mathcal{R} \times C_0$ and the extended configuration space will only include fields and their conjugate momenta (generalized fields).

A Hamiltonian operator $H = \int d^3x\mathcal{H}$ is given, where $\mathcal{H}$ is the Hamiltonian density. The Hamiltonian operator H acts as a constraint for quantum field dynamics. The action, in terms of a quantum fields $\psi_i(x)$ and the conjugate coordinates $\pi_i(x)$, can be re-expressed as:

$$A = \int d^3x \int d\psi_i \pi_i \tag{8}$$

where the Einstein summation on the repeated indexes is adopted. The roman index spans on the space dimensions 1, 2, and 3. To define time as the natural parameterization of change in the generalized coordinate space μ_Q, the points of the trajectories $f(q_i, p_i) = 0$ are replaced in QFT by space configurations of the generalized field $Q = (\psi_i(\mathbf{x}), \pi_i(\mathbf{x}))$ in μ_Q . In the classical case neighboring position and momentum states are associated to the parameter σ, while in QFT σ labels the generalized field with support in $\mathcal{R}^3$. Two arrays of fields variate the quantum fields and their conjugate fields respectively. As in the previous case, the extremality of the action is obtained under the condition that:

$$d\sigma = \left(\frac{\delta\mathcal{H}}{\delta\pi_i}\right)^{-1}_{\psi_i} d\psi_i(\mathbf{x}) = -\left(\frac{\delta\mathcal{H}}{\delta\psi_i}\right)^{-1}_{\pi_i} d\pi_i(\mathbf{x}) \tag{9}$$

The parameter σ belongs to $\mathcal{R}$ by construction. The parameterization of the field distribution is locally achieved by tagging neighboring configurations with the parameter σ.

3. Definition of generalized clock time

σ has the property of providing a special parameterization suitable for describing dynamics, but it is not an observable quantity. In order to explain the macroscopic experience of time in complex systems, an observable quantity T is built. T realizes an experimentally measurable discrete approximation of σ. Since (metric) time is operatively defined by clock standards based on the period of an oscillator, it is only defined in such systems complex enough to contain a subsystem acting as such a clock. Unfortunately the definition of periodicity implicitly assumes that an external time is available in order to compare a period with the next one, which is meaningless in a timeless framework. Consequently, the concept of *period* is relaxed to the concept of *cycle* in the phase space μ or in μ_Q. Defining the *clock time* T, measured for example by atomic clocks, corresponds to label simultaneous occurences in the phase space of two or more subsystems where one is identified as the clock. The clock corresponds to the cyclic subsystem, as defined below. The dynamics of the i-th observable O_i will consequently be expressed by the simple law involving σ:

$$O_i(T) \cong O_i(\sigma) \tag{10}$$

Let's consider a Hamiltonian system S separable in two independent subsystems S_1 and S_2, so that all the states are represented by factorized (eigen)states of their respective Hamiltonian $\psi_1 \otimes \psi_2 \in H_1 \otimes H_2$ where H_1 and H_2 are the Hilbert spaces of the subsystems 1 and 2 respectively. From the previous analysis, the system S owns a unique natural parameter time σ which is well defined also separately for the two subsystems by construction. We now define the properties required by the system S_1 to act as a clock in S in order to describe dynamics in S_2. For a given $\bar{\sigma}$, a state $\psi \in H_1 \otimes H_2$ consists of the tensor product of the state $\psi_1(\bar{\sigma}) \in H_1$ and the state $\psi_2(\bar{\sigma}) \in H_2$. We say that $\bar{\psi}_1$ has multiplicity κ_{AB} on the interval (σ_A, σ_B) if there are κ_{AB} values of $\tilde{\sigma}_i \in (\sigma_A, \sigma_B)$ such that $\psi_1(\tilde{\sigma}_i) = \bar{\psi}_1$ where $i \in (0, \kappa_{AB})$. We say that the subsystem S_1 is cyclic in the phase space if

(i) its path in the phase space is closed,

(ii) its velocity $|dQ/d\sigma| \neq 0$ and it is smooth,

(iii) the multiplicity κ_{AB} of a state vector in the System 1 monotonically grows with the interval (σ_A, σ_B) and it tends to infinity when $\sigma_A \to -\infty \wedge \sigma_B \to +\infty$.

The second requirement grants that the realizations of two contiguous states occur along the σ axis by respecting the order of the parameter σ. The third requirement that the clock never stops and its velocity in the phase space is enough to grant that the number of cycles is not finite.

Given the interval (σ_A, σ_B), it is now defined the set $\Omega(\sigma_A, \sigma_B) \subset H_2$:

$$\Omega(\sigma_A, \sigma_B) = \{\psi_2(\sigma) \in H_2 | \sigma \in (\sigma_A, \sigma_B)\} \tag{11}$$

An arbitrary origin σ_0 is fixed for the parameter time. We associate to such origin the arbitrary initial states $\bar{\psi}_1 = \psi_1(\sigma_0)$ and $\bar{\psi}_2 = \psi_2(\sigma_0)$. Macroscopic time duration $T^{(S_1)}$ of the interval (σ_A, σ_B) measured by the cyclic subsystem S_1 is given by the number k_{AB} of states $\psi_2(\sigma) \in \Omega$ so that $\psi_1(\sigma) = \bar{\psi}_1$. More explicitly, one has

$$T^{(S_1)}_{AB} \equiv k_{AB} \tag{12}$$

A good clock has the property of being *stable* (small standard deviation) and *accurate* (high Q factor of the resonance associated to the clock) [19, 25]. Since the accuracy refers to the arbitrary resonance frequency of the time standard (for example the Cesium resonance frequency), the present analysis considers only the requirement of stability. Given a target standard deviation Σ required in an experiment performed on the subsystem S_2 in the interval (σ_A, σ_B), for an integration time τ, the *generalized clock* has to fulfill the following prescription:

$$\epsilon \equiv E^2 \left[T^{(S_1)}_{i,i+1}\right] < \Sigma \tag{13}$$

where E^2 is the standard deviation and

$$\sigma_{i+1} = \sigma_i + \tau \tag{14}$$

where $i = 0...N_{AB}$ with $N_{AB} = (\sigma_B - \sigma_A)/\tau$. Consequently, the definition of clock metric time loses of validity for time intervals $T^{(S_1)}$ comparable with the clock period, and for shorter time intervals. Under such hyphotesis, dynamics of observables in the interval (σ_A, σ_B) is approximated by the discrete valued equations:

$$x_\rho(T_i) \cong x_\rho(\sigma_i \pm \epsilon) = x_\rho(\sigma_i) \pm O_{x\rho}[\sigma_i, \epsilon] \tag{15}$$

$$p_\rho(T_i) \cong p_\rho(\sigma_i \pm \epsilon) = p_\rho(\sigma_i) \pm O_{p\rho}[\sigma_i, \epsilon] \tag{16}$$

where $\rho = 1, 2, 3$, and $O_{x\rho}[\sigma, \epsilon]$ and $O_{p\rho}[\sigma, \epsilon]$ are higher order quantities in ϵ. Such equations provide the bridge between parameter time of Hamiltonian timeless formalism, and the experimentally defined clock time experienced by observers.

4. Conclusion

I've presented an approach to provide the correspondence between timeless physics in the microscopic domain, and macroscopic time metrology. Consistently with the discussion, some considerations on the use of the concept of time in theoretical physics is implied. Since clock time is by definition fundamentally discrete and it depends on the specific fabrication of the clock, a (macroscopic) measurement of time below one cycle (period) of the time standard is meaningless. At the present time the most advanced available clock technology is given by single ion atomic clocks based on Al^+/Hg^+ with a fractional uncertainty of about $1-2\times10^{17}$ [26]. The presented approch implies for example that Planck time scale is an extrapolation, an extension of the concept of clock time beyond its field of definition. To conclude an explicit Hamiltonian framework, entirely developed without the concept of time, has been defined. The time and frequency metrology has been mapped in the equations of motion expressed as a function of the time parameter obtained in the timeless framework, by defining cyclic subsystems capable to account the (discrete) definition of clock time. The present work provides a framework capable to account the timelessness of nature at a fundamental level, and to explain how clock time can be defined in metrology and experiments, consistently with the dynamics of relations between variables and parameter time evolution itself.

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