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Geometric Aspects of Quantum Mechanics and Quantum Entanglement

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Abstract. It is shown that the standard non-relativistic Quantum Mechanics gives rise to elegant and rich geometrical structures. The space of quantum states is endowed with nontrivial Fubini-Study metric which is responsible for the "peculiarities" of the quantum world. We show that there is also intricate connection between geometrical structures and quantum entanglement.

1. Introduction

It is well known that most theories of classical physics may be formulated in geometric language. The leading examples are Hamiltonian mechanics based on symplectic geometry, General Relativity based on Riemannian geometry and classical Yang-Mills theory which uses elegant theory of fibre bundles [1]. On the other hand Quantum Mechanics is intimately connected with linear operators in the Hilbert space and uses algebraic or functional analytic methods. We show that that the standard non-relativistic Quantum Mechanics may be formulated in the elegant geometric language. Moreover, it turns out that geometry enters Quantum Mechanics on the very fundamental level – the space of quantum states is endowed with rich and beautiful geometric structures.

Actually, geometry is present in quantum physics from the very beginning. Let us just mention celebrated Dirac magnetic pole. Dirac showed [2] that Quantum Mechanics leads to the celebrated charge quantization

$$e = \frac{n}{2}\,\frac{\hbar}{g}, \qquad n = 1, 2, 3, \ldots \tag{1}$$

where 'e' and 'g' stands for electric and magnetic charges, respectively. As wee shall see this condition is closely related to the geometrical properties of the qubit. Another well known example is provided by the Quantum Hall Effect [3]: the quantization of Hall conductance

$$\sigma_{\rm H} = n\,\frac{e^2}{h}\ , \qquad n = 1, 2, 3, \ldots \tag{2}$$

may be explained in topological terms. Geometry and topology are basic tools for Chern-Simons theory, Witten's topological quantum field theory and finally for string theory [4].

These examples shows that geometrical methods in quantum physics are not new. In what follows we shall concentrate on elementary Quantum Mechanics. In the next section we describe the geometrical structure of the space of quantum states – this space is endowed with so called *Fubini-Study* metric which enables one to measure a distance between quantum states. Actually, it turns out that the nontrivial geometry of the Fubini-Study metric is responsible for the "peculiarities" of the quantum world. Section 3 shows how the metric properties are connected with the quantum measurement and probabilistic interpretation of Quantum Mechanics. Section 4 relates the geometric properties of quantum states with elegant mathematical construction known as Hopf fibrations. Finally we show that there is also intricate connection between geometrical structures and quantum entanglement.

Geometric approach to Quantum Mechanics was originated in the work of Kibble [5] who showed how quantum theory could be formulated in the language of Hamiltonian phase- space dynamics. For recent investigation see [6, 7, 8] (see also [9]).

2. Space of quantum states

Any two vectors $\psi, \phi \in \mathcal{H}$ differing by a complex number $c \in \mathbb{C}$, i.e. $\psi = c\phi$, are physically equivalent $\psi \sim \phi$, that is, they define the same physical state. Therefore, the proper phase space of a quantum system is not the original Hilbert space $\mathcal{H}$ but rather the space of rays in $\mathcal{H}$:[1]

$$\mathcal{P}(\mathcal{H}) := \mathcal{H}/\sim \, , \tag{3}$$

called a *projective Hilbert space*. Points in $\mathcal{P}(\mathcal{H})$ are 1-dimensional rays in $\mathcal{H}$ or equivalently 1-dimensional projectors:

$$\psi \longrightarrow P_\psi := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \, . \tag{4}$$

It is evident that if $\psi \sim \phi$, then $P_\psi = P_\phi$.

In this paper we shall consider only finite dimensional case, i.e. we assume that $\mathcal{H} \cong \mathbb{C}^{n+1}$. Normalized vectors in $\mathbb{C}^{n+1}$ define a $2n+1$–dimensional sphere

$$S^{2n+1} = \left\{ \psi \in \mathbb{C}^{n+1} \;\middle|\; \langle\psi|\psi\rangle = 1 \right\} . \tag{5}$$

Now, two points ψ and ϕ in S^{2n+1} define the same quantum state iff $\phi = e^{i\alpha}\psi$. Hence, the corresponding projective Hilbert space reads

$$\mathcal{P}(\mathcal{H}) = S^{2n+1}/U(1) =: \mathbb{C}P^n \, , \tag{6}$$

and it is usually called *complex projective space*.

A well known example is $\mathbb{C}P^1$, i.e. space of quantum states of a 2-level system – a qubit – $\mathbb{C}P^1 = S^3/U(1) \cong S^2$. This 2D sphere, called usually a *Bloch sphere*, defines a *true* space of quantum states, cf. Fig. 1.

The shape of $\mathbb{C}P^n$ for $n > 1$ is not known (see however interesting analysis in [10]). It turns out that $\mathbb{C}P^n$ defines n-dimensional complex space (n is a complex dimension!). Moreover, it is equipped with rich geometrical structures: *Fubini-Study* metric $g_{\alpha\beta}$ and symplectic form $\omega_{\alpha\beta}$.[2] These structures are inherited from $\mathcal{H}$. Observe that the scalar product in $\mathcal{H}$ may be decomposed as follows:

$$\langle\psi|\phi\rangle = G(\psi,\phi) + i\,\Omega(\psi,\phi) \, , \tag{7}$$

[1] Throughout this paper we shall consider only pure states.
[2] Symplectic structure is equivalent to a Poisson bracket known from the Hamiltonian mechanics [11, 12].

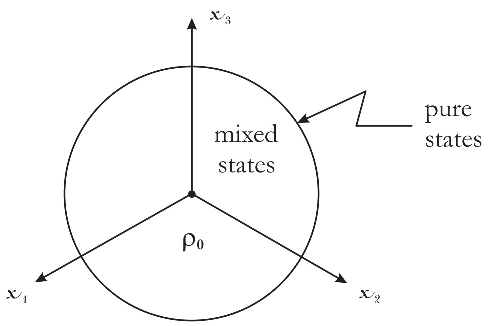

Figure 1. Bloch ball. Pure qubit states lie on the boundary $\mathbb{C}P^1 \cong S^2$. The maximally mixed state $\rho_0 = \frac{1}{2}\mathbb{1}$ lies in the center.

where $G(\psi, \phi) = \mathrm{Re}\,\langle\psi|\phi\rangle$, and $\Omega(\psi, \phi) = \mathrm{Im}\,\langle\psi|\phi\rangle$. These objects enjoy the following properties:

$$G(\psi, \phi) = G(\phi, \psi) \ , \qquad \Omega(\psi, \phi) = -\Omega(\phi, \psi) \ , \qquad \Omega(\psi, \phi) = G(\psi, i\,\phi) \ . \tag{8}$$

It shows that both $\mathcal{H}$ and $\mathcal{P}(\mathcal{H})$ define so called *Kähler spaces.*

Now, if e_α defines an orthonormal base in $\mathbb{C}^{n+1}$, i.e. any ψ may be decomposed as follows $\psi = \sum_\alpha \psi_\alpha e_\alpha$, then the Fubini-Study metric $g_{\alpha\beta}$ is given by the following formula

$$g_{\alpha\beta} = \frac{\langle\psi|\psi\rangle\delta_{\alpha\beta} - \psi_{(\alpha}\overline{\psi}_{\beta)}}{\langle\psi|\psi\rangle^2} \ , \tag{9}$$

that is, the infinitesimal distance is defined by $ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, d\psi_\alpha\, d\overline{\psi}_\beta$. For $n = 1$ (qubit) the Fubini-Study metric reproduces the standard *round* metric on S^2. It turns out that the nontrivial geometry of the Fubini-Study metric is responsible for the "peculiarities" of the quantum world.

3. Quantum measurement

In the standard formulation of Quantum Mechanics an observable is represented by a self-adjoint operator $\hat{F}$. Measuring $\hat{F}$ one obtains one of its eigenvalues:

$$\hat{F}\psi_k = \lambda_k \psi_k \ . \tag{10}$$

If the original state is represented by ψ then one measures λ_k with probability $p_k = |\langle\psi|\psi_k\rangle|^2$ and just after the measurement the state of the system jumps to ψ_k.[3]

Now, we show how to describe this process on the projective space $\mathcal{P}(\mathcal{H})$. Observe that $\hat{F}$ gives rise to the function f on $\mathcal{P}(\mathcal{H})$ defined by:

$$f(P) = \mathrm{Tr}(P\hat{F}) \ , \tag{11}$$

where P is a quantum state in $\mathcal{P}(\mathcal{H})$ represented by a 1-dimensional projector (density matrix). Clearly, not all functions on $\mathcal{P}(\mathcal{H})$ have the above form. If the function 'f' may be represented by (11) we call it a *quantum observable.* This is one of the peculiarity of the quantum world: in classical mechanics all function on the corresponding phase-space are *classical observables.* In Quantum Mechanics it is no longer true. The natural question is therefore how to check whether a function 'f' is a quantum observable. It turns out that the appropriate test is provided by

[3] For simplicity we assume that λ_k are not degenerate.

the Fubini-Study metric. Let Δ be a Laplace operator on the projective space.[4] Now, 'f' is a quantum observable iff

$$\Delta f = 0 \ , \qquad \text{or} \qquad \Delta f = -2nf \ , \tag{12}$$

that is, 'f' is either a zero mode of Δ or is a eigenvector of Δ corresponding to the first non-vanishing eigenvalue '$-2n$'.

To see how this test works let us consider the simplest case $n = 1$. The corresponding projective space is given by the Bloch sphere S^2 and the eigenvalue problem $\Delta f = \lambda f$ is well known from the theory of angular momentum. One has

$$\Delta Y_{lm} = -l(l+1)Y_{lm} \ , \tag{13}$$

where Y_{lm} are spherical harmonics and the integer m runs from $-l$ to l. Note, that (12) implies that $l = 0$ or $l = 1$. In the first case Y_{00} defines a constant function on S^2 whereas in the second case we have three independent dipole functions

$$Y_{11} = x \ , \quad Y_{1-1} = y \ , \quad Y_{10} = z \ . \tag{14}$$

A constant function corresponds via (11) to the identity operator $\mathbb{1}$. The reader easily checks that dipole functions correspond to Pauli matrices: σ_x, σ_y and σ_z. This reproduces the well known fact that for 2-level system one has 4 independent observables.

Having described the structure of quantum observables let us see how to obtain a quantum spectrum $\{\lambda_1, \lambda_2, \dots\}$ from (10). It turns out that the stationary states P_k are critical points of 'f', i.e. $f'(P_k) = 0$. Moreover, P_k correspond to projectors onto the rays generated by ψ_k, i.e. $P_k = |\psi_k\rangle\langle\psi_k|$. Now, the 'spectrum' of 'f' immediately follows:

$$\lambda_k = f(P_k) \ , \tag{15}$$

i.e. possible outcomes in measuring 'f' are values of 'f' in its critical points P_k.

Finally, let us see how the probability enters the game. Suppose that just before the measurement of 'f' the system was in the state P. Denote by $\gamma(P, P_k)$ a distance between P and P_k defined in terms of the Fubini-Study metric, i.e. it is the length of a geodesic connecting P and P_k. The corresponding probability p_k is given by the following formula:

$$p_k = \cos^2\{\gamma(P, P_k)\} \ , \tag{16}$$

i.e. a transition probability is determined by the distance between corresponding states. The system is more likely to collapse to a nearby state than to a distant one. The miracle of the Fubini-Study metric implies that these probabilities sum to 1!

4. Hopf fibrations

The map defining projection from S^{2n+1} to $\mathbb{C}P^n$ defines so called *Hopf fibration* or equivalently a *Hopf bundle*. The simplest and the most popular corresponds to $n = 1$. In this case $\mathbb{C}P^1 \cong S^2$ and hence it gives rise to a map $S^3 \longrightarrow S^2$. An example of such a map is easy to construct: let ψ be a normalized state vector of a qubit

$$\psi = \alpha|0\rangle + \beta|1\rangle \ ,$$

[4] Δ depends upon the metric. If x^k are local coordinates, then the formula for Δ reads as follows:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g}\, g^{kl} \frac{\partial f}{\partial x^l} \right)$$

where $g = |\det g_{kl}|$.

with $|\alpha|^2 + |\beta|^2 = 1$. Let us define

$$x_1 = \langle\psi|\sigma_1|\psi\rangle \ , \quad x_2 = \langle\psi|\sigma_2|\psi\rangle \ , \quad x_3 = \langle\psi|\sigma_3|\psi\rangle \ . \tag{17}$$

It is easy to see that $x_1^2 + x_2^2 + x_3^2 = 1$, and hence the above assignment establishes a map from S^3 to S^2. To visualize the Hopf fibration $S^3 \longrightarrow S^2$ one may perform a stereographic projection from S^3 to $\mathbb{R}^3$. Each circular fibre of S^3 is mapped onto a circle in $\mathbb{R}^3$, cf. Figure 2.

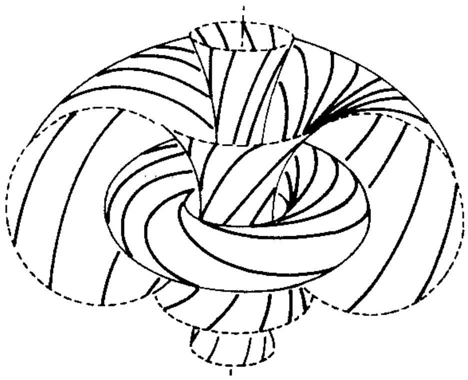

Figure 2. $S^3 \longrightarrow S^2$ Hopf fibration after a stereographic projection to $\mathbb{R}^3$. Circular fibres are grouped into a continuous family of nested tori.

Maps $S^3 \longrightarrow S^2$ were studied by Hopf [13] who found that any such map belongs to a class uniquely characterized by an integer number m – so called *Hopf number* of the map:

$$f \ : \ S^3 \ \longrightarrow \ S^2 \quad \Longrightarrow \quad \text{Hopf}(f) = m \ ,$$

with $m \in \mathbb{Z}$. Any two maps f_1 and f_2 which can be continuously deformed one into another have the same Hopf numbers, i.e. $\text{Hopf}(f_1) = \text{Hopf}(f_2)$. One calls such maps homotopically equivalent. It turns out that the map defined by the qubit state $\psi \longrightarrow \langle\psi|\overrightarrow{\sigma}|\psi\rangle$ satisfies $\text{Hopf}(f) = 1$, i.e. it belongs to the simplest nontrivial class of maps.

Actually, the Hopf map $S^3 \longrightarrow S^2$ is closely related to the mechanism of charge quantization developed by Dirac [2], see e.g. [4].

Now, let us recall that only three spheres S^n can be equipped with the group structures:

(i) S^1 which corresponds to normalized complex numbers,
(ii) S^3 which corresponds to normalized quaternions; $S^3 \cong SU(2)$,
(iii) S^7 which corresponds to normalized octonions.

Now, the Hopf map $S^3 \longrightarrow S^2$ corresponds to the coset space S^3/S^1. Another coset space S^7/S^3 gives rise to the following family of Hopf fibrations

$$f \ : \ S^7 \ \longrightarrow \ S^4 \ .$$

It turns out that this family finds interesting applications in Yang-Mills theory (it describes so called instanton configuration of the classical Yang-Mills field) and in quantum entanglement.

5. Quantum entanglement

Consider two quantum systems A and B and let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the corresponding Hilbert spaces.[5] Suppose now that we are interested in the composite system AB made up of A and B. The composite system Hilbert space $\mathcal{H}_{AB}$ is a tensor product of $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \ , \tag{18}$$

[5] In quantum information theory one usually speaks about Alice and Bob systems, see [14].

which means that if $|A\rangle \in \mathcal{H}_A$ and $|B\rangle \in \mathcal{H}_B$ denote state vectors of A and B, respectively, then the joint system is in the state

$$|AB\rangle \equiv |A\rangle \otimes |B\rangle \in \mathcal{H}_{AB} \ . \tag{19}$$

Note however, that a general vector from $\mathcal{H}_{AB}$ can not be written this way. We call an element $\psi \in \mathcal{H}_{AB}$ a *separable state* if there are $\psi_A \in \mathcal{H}_A$ and $\psi_B \in \mathcal{H}_B$ such that $\psi = \psi_A \otimes \psi_B$. If this is not the case we call ψ an *entangled state* or simply nonseparable state.

Recall, that if $(e_1, \ldots, e_N)$ denote the basis in $\mathcal{H}_A$ and $(f_1, \ldots, f_M)$ the basis in $\mathcal{H}_B$, then

$$e_\mu \otimes f_\nu \ , \qquad \mu = 1, \ldots, N, \ \nu = 1, \ldots, M \ , \tag{20}$$

define a basis in $N \cdot M$ dimensional space $\mathcal{H}_{AB}$. It means that an arbitrary vector $\psi \in \mathcal{H}_{AB}$ may be represented as follows:

$$\psi = \sum_{\mu=1}^{N} \sum_{\nu=1}^{M} \psi_{\mu\nu}\, e_\mu \otimes f_\nu \ , \tag{21}$$

with $\psi_{\mu\nu} \in \mathbb{C}$. Now, given a state $\psi \in \mathcal{H}_{AB}$ is it separable or entangled? To answer this question let us observe that for every ψ in $\mathcal{H}_{AB}$ there exist orthonormal basis $\{\tilde{e}_\mu\}_{\mu=1}^N$ and $\{\tilde{f}_\mu\}_{\nu=1}^M$ such that

$$\psi = \sum_{\alpha=1}^{K} a_\alpha\, \tilde{e}_\alpha \otimes \tilde{f}_\alpha \ , \tag{22}$$

where $a_\alpha > 0$ with $\sum_{\alpha=1}^K a_\alpha^2 = 1$, and $K \leq \min\{N, M\}$. Formula (22) defines *Schmidt decomposition* of ψ.

The state $\psi \in \mathcal{H}_{AB}$ is separable if and only if its Schmidt decomposition contains only one terms, i.e. $K = 1$. As an example consider two qubits, that is $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$. Denoting by $|0\rangle$ and $|1\rangle$ the standard orthonormal basis in $\mathbb{C}^2$ one introduces so called *Bell states*

$$|\psi^\pm\rangle := \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \ , \tag{23}$$

and

$$|\phi^\pm\rangle := \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \ , \tag{24}$$

where $|01\rangle := |0\rangle \otimes |1\rangle$, etc. These four vectors define an orthonormal basis in $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. Moreover, they are all entangled states.[6]

6. Entanglement and geometry

Consider now a composite 2-qubit system. Any normalized 2-qubit state may be written as follows

$$\psi = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle \ , \tag{25}$$

[6] The notion of separability and entanglement may be easily generalized to mixed states of the composed system [15]. A mixed state represented by a density matrix ρ in $\mathcal{H}_{AB}$ is called separable if and only if it can be represented as

$$\rho = \sum_{\alpha=1}^{K} p_\alpha\, \rho_\alpha \otimes \sigma_\alpha \ ,$$

where $p_\alpha > 0$ with $\sum_{\alpha=1}^K p_\alpha = 1$, and ρ_α and σ_α are mixed states of A and B, respectively. Surprisingly, contrary to the case of pure states the criterion of separability is still unknown for composite mixed states, see [16, 17] for more details.

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ together with

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \ . \tag{26}$$

The above formula defines a unit 7D sphere S^7. A 2-qubit state ψ is separable if it is a tensor product of two 1-qubit states $\psi = \psi_1 \otimes \psi_2$. Representing ψ_k in $(|0\rangle, |1\rangle)$ basis

$$\psi_1 = a\,|0\rangle + b\,|1\rangle \ , \qquad \psi_2 = c\,|0\rangle + d\,|1\rangle \ , \tag{27}$$

one finds for separable state

$$\psi = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle \ . \tag{28}$$

Hence, a 2-qubit state represented by (25) is separable if and only if

$$\alpha\delta = \beta\gamma \ . \tag{29}$$

Let us introduce $\mathbf{E} : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$ by

$$\mathbf{E}\,\psi = (\sigma_y \otimes \sigma_y)\,\psi^* \ , \tag{30}$$

and following [18] define

$$x_0 = \langle\psi|\,\sigma_z \otimes \mathbb{1}\,|\psi\rangle \ , \quad x_1 = \langle\psi|\,\sigma_x \otimes \mathbb{1}\,|\psi\rangle \ , \quad x_2 = \langle\psi|\,\sigma_y \otimes \mathbb{1}\,|\psi\rangle \ ,$$

and

$$x_3 = \mathrm{Re}\,\langle\psi|\,\mathbf{E}\,|\psi\rangle \ , \quad x_4 = \mathrm{Im}\,\langle\psi|\,\mathbf{E}\,|\psi\rangle \ .$$

Using (25) one shows that $x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and hence the above map $\psi \longrightarrow (x_0, x_1, x_2, x_3, x_4)$ defines Hopf fibration $S^7 \longrightarrow S^4$. Actually,

$$C(\psi) = |\langle\psi|\,\mathbf{E}\,|\psi\rangle| \ , \tag{31}$$

defines a Wootters concurence [19] which defines an entanglement measure for 2 qubit case.

Note that

$$x_3 = 2\mathrm{Re}\,(\alpha\delta - \beta\gamma) \ , \quad x_4 = 2\mathrm{Im}\,(\alpha\delta - \beta\gamma) \ ,$$

which shows that for separable states $x_3 = x_4 = 0$. Therefore, the Hopf map $S^7 \longrightarrow S^4$ is entanglement sensitive and may be used to detect quantum entanglement in a 2-qubit system: a state is separable iff its image under the Hopf map belongs to the equatorial 2D sphere defined by the intersection of S^4 with the 3D plane $x_3 = x_4 = 0$.

For other studies on geometric approach to quantum entanglement see e.g. [7, 10, 20].

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References

[1] Nash N and Sen S 1983 *Topology and Geometry for Physicists* (London: Academic)
Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Adam Hilger)
Eguchi T Gilkey P G and Hanson J 1980 *Phys. Rep.* **66** 213

[2] Dirac P A M 1931 *Proc. Roy. Soc. London* **A 133** 60
Dirac P A M 1948 *Phys. Rev.* **74** 817

[3] Prange R E and Girvin S M 187 *The Quantum Hall Effect* (New York: Springer)
Morandi G 1988 *Quantum Hall Effect* (Naples: Bibliopolis)

[4] Morandi G 1992 *The Role of Topology in Classical and Quantum Physics* Lecture Notes in Physics m7 (Berlin: Springer)

[5] Kibble T W B 1978 *Commun. Math. Phys.* **64** 73
Kibble T W B 1979 *Commun. Math. Phys.* **65** 189

[6] Ashtekar A and Schilling T A 1998 *On Einsteins Path* (Berlin: Springer) ed Harvey A (*Preprint* gr-qc/9706069)

[7] Brody D C and Hughston L P 2001 *J. Geom. Phys.* **38** (Berlin: Springer)

[8] Cirelli R Mania A and Pizzocchero L 1990 *J. Math. Phys.* **31** 2891

[9] Chruściński D and Jamiołkowski A 2004 *Geometric Phases in Classical and Quantum Mechanics* (Boston: Birkhäuser)

[10] Bengtsson I Brännlund J and Życzkowski K 2001 *CP^n, or, Entanglement Illustrated* Preprint quant-ph/0108064

[11] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (New York: Addison-Wesley)

[12] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[13] Hopf H 1931 *Mat. Annalen.* **104** 637

[14] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[15] Werner R F 1989 *Phys. Rev.* **A 40** 4277

[16] Peres A 1995 *Quantum Theory: Concepts and Methods*, (Dordrecht: Cluwer Academic Publisher)

[17] Horodecki M Horodecki P and Horodecki R 2001 *Mixed-state entanglement and quantum communication* in *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments* eds Alber G Beth T Horodecki M Horodecki P Horodecki R Rotteler M Weinfurter H Werner R and Zeilinger A (Berlin: Springer)

[18] Mosseri R and Dandaloff R 2001 *J. Math. Phys. A: Math. Gen.* **34** 10243

[19] Wootters K W 1998 *Phys. Rev. Lett.* **80** 2245

[20] Kuś M and Życzkowski K 2001 *Phys. Rev.* **A 63** 032307