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Octonions, E_6 , and particle physics

Corinne A Manogue¹ and Tevian Dray²

¹ Department of Physics, Oregon State University, Corvallis, OR 97331, USA

² Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA

E-mail: corinne@physics.oregonstate.edu, tevian@math.oregonstate.edu

Abstract. In 1934, Jordan et al. gave a necessary algebraic condition, the Jordan identity, for a sensible theory of quantum mechanics. All but one of the algebras that satisfy this condition can be described by Hermitian matrices over the complexes or quaternions. The remaining, exceptional Jordan algebra can be described by 3×3 Hermitian matrices over the octonions.

We first review properties of the octonions and the exceptional Jordan algebra, including our previous work on the octonionic Jordan eigenvalue problem. We then examine a particular real, noncompact form of the Lie group E_6 , which preserves determinants in the exceptional Jordan algebra.

Finally, we describe a possible symmetry-breaking scenario within E_6 : first choose one of the octonionic directions to be special, then choose one of the 2×2 submatrices inside the 3×3 matrices to be special. Making only these two choices, we are able to describe many properties of leptons in a natural way. We further speculate on the ways in which quarks might be similarly encoded.

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1. Introduction

A personal note from Corinne: During the academic year 1986/87, Tevian and I were living in York, newly married and young postdocs. Tevian was working in the mathematics department there, doing research in general relativity, and I was working in Durham, with David Fairlie, just beginning my research into the octonionic structures reported here. Imagine my pleasure, when I found out that York had its own resident expert on the octonions! I returned to York the following summer, to work with Tony on an attempt to describe the superstring using octonions [1]. I will be forever grateful to him, not only for the generous way in which he shared his vast knowledge and experience in this field, but also for the friendship, respect, collegiality, and mentorship, which he also generously shared.

2. Exceptional Quantum Mechanics

In the Dirac formulation of quantum mechanics, a quantum mechanical state is represented by a *complex* vector v , often written as $|v\rangle$, which is usually normalized such that $v^\dagger v = 1$. In the Jordan formulation [2, 3, 4, 5], the same state is instead represented by the Hermitian matrix vv^\dagger , also written as $|v\rangle\langle v|$, which squares to itself and has trace 1. The matrix vv^\dagger is thus the projection operator for the state v , which can also be viewed as a pure state in the density matrix formulation of quantum mechanics. Note that the usual phase freedom in v is no longer present in vv^\dagger , which is uniquely determined by the state (and the normalization condition).

A fundamental object in the Dirac formalism is the probability amplitude $v^\dagger w$, or $\langle v|w\rangle$, which is not however measurable; it is the squared norm $|\langle v|w\rangle|^2 = \langle v|w\rangle\langle w|v\rangle$ of the probability amplitude which yields measurable probabilities. One of the basic observations which leads to the Jordan formalism is that these probabilities can be expressed entirely in terms of the Jordan product of projection operators, since

$$\langle v|w\rangle\langle w|v\rangle = (v^\dagger w)(w^\dagger v) \equiv \text{tr}(vv^\dagger \circ ww^\dagger) \quad (1)$$

where \circ denotes the *Jordan product* [2, 3]

$$\mathcal{A} \circ \mathcal{B} = \frac{1}{2}(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) \quad (2)$$

which is commutative but not associative.

Remarkably, the Jordan formulation of quantum mechanics does not require (v and) \mathcal{A} to be complex, but only that the *Jordan identity*

$$(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{A}^2 = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{A}^2) \quad (3)$$

hold for two Hermitian matrices \mathcal{A} and \mathcal{B} . As shown in [3], the Jordan identity (3) is equivalent to power associativity, which ensures that arbitrary powers of Jordan matrices — and hence of quantum mechanical observables — are well-defined.

The Jordan identity (3) is the defining property of a *Jordan algebra* [2], and is clearly satisfied if the operator algebra is associative, which will be the case if the elements of the Hermitian matrices \mathcal{A} , \mathcal{B} themselves lie in an associative algebra. Remarkably, one further possibility exists, for which the elements of the Hermitian matrices do *not* lie in an associative algebra. This example is the Albert algebra (also called the exceptional Jordan algebra) $H_3(\mathbb{O})$ of 3×3 octonionic Hermitian matrices [3, 6].¹ In what follows we will restrict our attention to this exceptional case.²

3. Quaternions and Octonions

The Hurwitz Theorem states that the real numbers \mathbb{R} , complexes \mathbb{C} , quaternions \mathbb{H} , and octonions \mathbb{O} are the *only* (normed) division algebras (over the real numbers).³ The quaternions and octonions are extensions of the familiar real and complex numbers. A quaternion is an arbitrary real linear combination of the real identity element 1 and three different square roots of minus one, which are conventionally called $\{i, j, k\}$ and satisfy the multiplication table given in Figure 1. Similarly, the octonions are formed from seven square roots of minus one which we will call $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$, whose multiplication table is summarized in Figure 2. In these multiplication tables, each point corresponds to an imaginary unit. Each line or circle corresponds to a quaternionic triple with the arrow giving the orientation. For example,

$$k \ell = k\ell \quad (4)$$

$$\ell k\ell = k \quad (5)$$

$$k\ell k = \ell \quad (6)$$

and each of these products anticommutes, that is, reversing the order contributes a minus sign.

¹ The 2×2 octonionic Hermitian matrices $H_2(\mathbb{O})$ also form a Jordan algebra, but, even though the octonions are not associative, it is possible to find an associative algebra that leads to the same Jordan algebra [3, 7].

² In this case, the equivalence in (1) fails; it is the right-hand side which provides the correct generalization.

³ A *division algebra* is a *vector space* over a *field* (in this case \mathbb{R}) which is also a *ring with identity* under multiplication, and in which $ax = b$ can be uniquely solved for x (unless $a = 0$). A *normed* division algebra satisfies (12) in addition, and is therefore also an *integral domain*, that is, a ring in which $ab = 0$ implies $a = 0$ or $b = 0$.

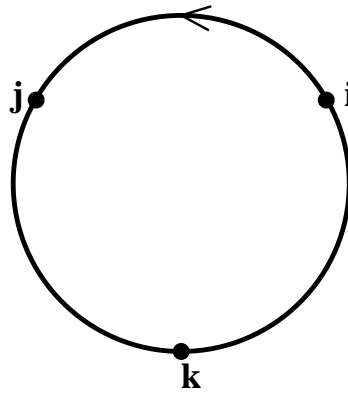


Figure 1. The quaternionic multiplication table.

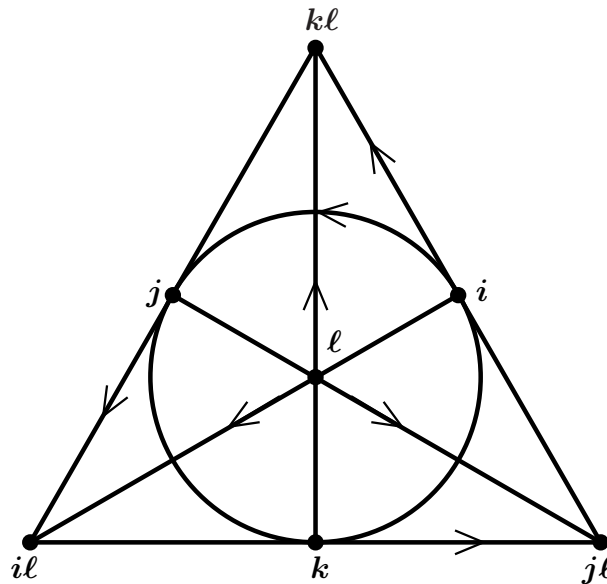


Figure 2. The octonionic multiplication table. Each line segment should be thought of as circle, identical to the quaternionic multiplication table in Figure 1.

We define the *conjugate* \bar{a} of a quaternion or octonion a as the (real) linear map which reverses the sign of each imaginary unit. Thus, if

$$a = a_1 + a_2 i + a_3 j + a_4 k + a_5 kl + a_6 jl + a_7 il + a_8 l \quad (7)$$

then

$$\bar{a} = a_1 - a_2 i - a_3 j - a_4 k - a_5 kl - a_6 jl - a_7 il - a_8 l \quad (8)$$

Direct computation shows that

$$\overline{ab} = \bar{b}\bar{a} \quad (9)$$

The *norm* $|a|$ of an octonion a is defined by

$$|a|^2 = a\bar{a} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 \quad (10)$$

The only octonion with norm 0 is 0, and every nonzero octonion has a unique inverse, namely

$$a^{-1} = \frac{\bar{a}}{|a|^2} \quad (11)$$

For all the normed division algebras, the norm satisfies the identity

$$|ab| = |a||b| \quad (12)$$

A remarkable property of the octonions is that they are *not* associative! For example, compare

$$(i j)(\ell) = +(k)(\ell) = +k\ell \quad (13)$$

$$(i)(j \ell) = (i)(j\ell) = -k\ell \quad (14)$$

However, the octonions are *alternative*, that is, products involving no more than 2 independent octonions do associate. The *commutator* of two octonions a, b is given as usual by

$$[a, b] = ab - ba \quad (15)$$

and we define the *associator* of three octonions a, b, c by

$$[a, b, c] = (ab)c - a(bc) \quad (16)$$

which quantifies the lack of associativity. More generally, both the commutator and associator are *antisymmetric*, that is, interchanging any two arguments changes the result by a minus sign; replacing any argument by its conjugate has the same effect, because the real parts don't contribute to the associator.

The units $i, j, k, k\ell, j\ell, i\ell$, and ℓ are by no means the only square roots of -1 . Rather, *any* pure imaginary quaternion or octonion squares to a negative number, so it is only necessary to choose its norm to be 1 in order to get a square root of -1 . The imaginary quaternions of norm 1 form a 2-sphere in the 3-dimensional space of imaginary quaternions. The imaginary octonions of norm 1 form a 6-sphere in the 7-dimensional space of imaginary octonions.

Any such unit imaginary quaternion or octonion \hat{s} can be used to construct a complex subalgebra of \mathbb{H} or \mathbb{O} , which we will also denote by \mathbb{C} , and which takes the form

$$\mathbb{C} = \{a_R + a_s \hat{s}\} \quad (17)$$

with $a_R, a_s \in \mathbb{R}$. Regarding \hat{s} as the complex unit, we have the familiar Euler identity

$$e^{\hat{s}\theta} = \cos \theta + \hat{s} \sin \theta \quad (18)$$

so that *any* quaternion or octonion can be written in the form

$$a = r e^{\hat{s}\theta} \quad (19)$$

where

$$r = |a| \quad (20)$$

Any *two* unit imaginary octonions \hat{s} and \hat{t} that point in independent directions determine a *quaternionic* subalgebra of \mathbb{O} .

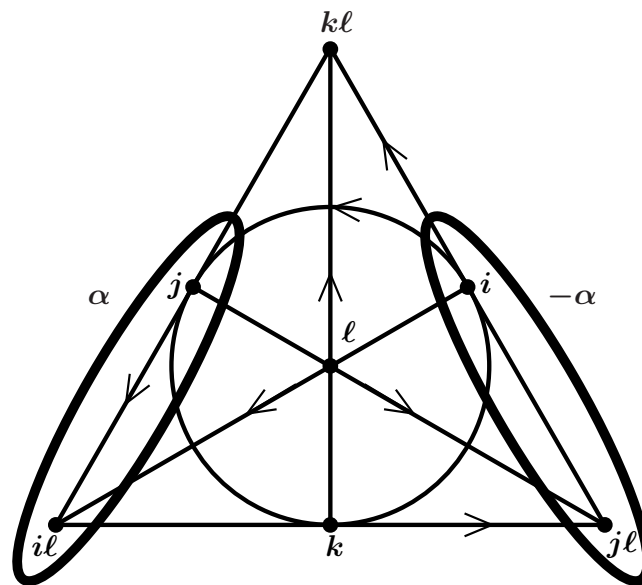


Figure 3. One class of elements of G_2 . These transformations are also contained in the preferred $SU(3)$ that fixes l .

4. The Structure of G_2 and $SU(3)$

The freedom to choose an entire 2-sphere or 6-sphere of square roots of minus one within the 3-dimensional space of the pure imaginary quaternions or the 7-dimensional space of pure imaginary octonions leads one to investigate the transformations that preserve the corresponding multiplication table. These transformations form the automorphism group of the corresponding division algebra.

In the case of the quaternions, one can imagine rotating i to any pure imaginary point on the 2-sphere (2 degrees of freedom). Then j can be chosen to be any direction perpendicular to the direction of i , i.e. on the equator of the resulting 2-sphere (1 degree of freedom). The direction of k is determined by the multiplication table. The 3-dimensional automorphism group of the quaternions is therefore seen to be $SO(3)$.

For the octonions, one can again imagine rotating i to any pure imaginary point on the 6-sphere (6 degrees of freedom). Then j must again be perpendicular to i (5 degrees of freedom) and the direction of k is fixed by the multiplication table. But l is now free to be any direction perpendicular to all of the i , j , and k directions (3 degrees of freedom) and the directions of the remaining units are determined by the multiplication table. This 14-dimensional Lie group turns out to be the exceptional group G_2 .

Another way of envisioning the transformations in the group G_2 was first shown to us by Sudbery [8]. Consider the octonionic unit kl at the top of the multiplication table shown in Figure 3. There are three pairs of octonionic units that form quaternionic subalgebras with kl , i.e. $\{j, il\}$, $\{jl, i\}$, and $\{k, l\}$. We call these the pairs that “point to” kl . If the elements of two of these pairs are rotated into one another oppositely, for instance, if the $\{j, il\}$ -plane is rotated by an angle α , and the $\{jl, i\}$ -plane is rotated by the angle $-\alpha$, then it turns out that the multiplication table is preserved. We have thus constructed a 1-parameter family of automorphisms. There are three ways of pairing up the three pairs of units in this way, but only two are independent. Since there are 7 different units that can be pointed to, the dimension of this group is again 14.

In what follows, we will need not only G_2 , but also $SU(3)$, the subgroup of G_2 that fixes one

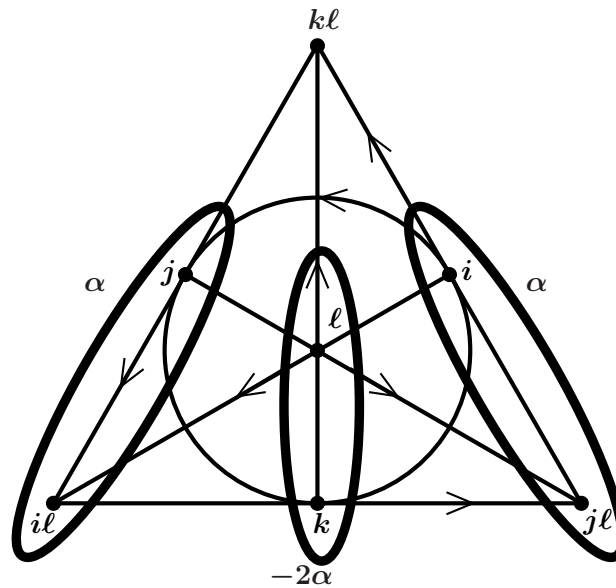


Figure 4. A second class of elements of G_2 . These transformations are not contained in the preferred $SU(3)$ that fixes ℓ .

of the octonionic units. Since ℓ is in the middle of our multiplication table, we will, without loss of generality, choose it to be the unit that is fixed. We see that the G_2 transformation in Figure 3 fixes ℓ and is therefore in $SU(3)$, but a G_2 transformation involving either of the other two pairs that point to $k\ell$ will not fix ℓ . To be symmetric, we choose the linear combination of transformations shown in Figure 4 ($\{j, i\ell\}$ and $\{j\ell, i\}$ both rotate by α and $\{k, \ell\}$ rotates by -2α) to be the G_2 transformation that points to $k\ell$ that is *not* in $SU(3)$. If we choose to point in turn to each of the six units that are *not* ℓ , we have six G_2 transformations that are in $SU(3)$ and six that are not. What about the remaining two G_2 transformations? These are transformations that point to ℓ . One such transformation, shown in Figure 5, rotates $\{i\ell, i\}$ by α and $\{j\ell, j\}$ by $-\alpha$. There are three transformations of this type, all of which fix ℓ and are therefore elements of $SU(3)$, but only two are linearly independent. Any two of these transformations complete the 8-dimensional Lie group $SU(3)$.

Yet another way to describe G_2 is in terms of *inner automorphisms*, that is, transformations of the form

$$x \mapsto axa^{-1} \tag{21}$$

Inner automorphisms always preserve an associative multiplication rule, since

$$(axa^{-1})(aya^{-1}) = a(xy)a^{-1} \tag{22}$$

However, this condition is nontrivial over the octonions, since the parentheses cannot be moved. As shown in [9], (22) holds for all $x, y \in \mathbb{O}$ if and only if a is a sixth root of unity. That is, the inner automorphisms of the octonions are given by (21) where

$$a = e^{n\pi\hat{s}/3} \tag{23}$$

where $n \in \mathbb{Z}$ and \hat{s} is any pure imaginary unit octonion. As further discussed in [9], any G_2 transformation can in fact be generated by a finite sequence of nested transformations of the form (21), with a given by (23).

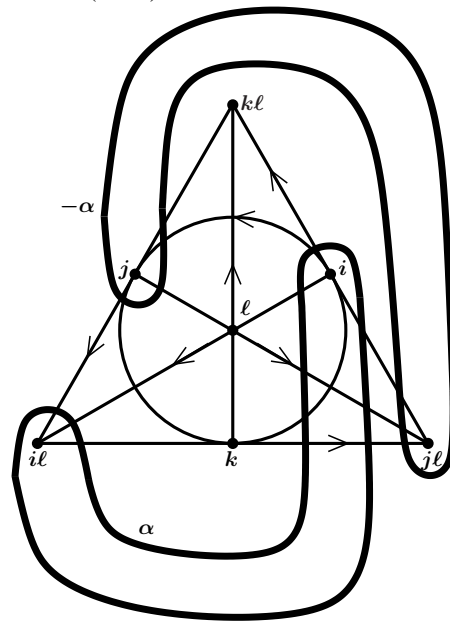


Figure 5. A third class of elements of G_2 . These transformations are contained in the preferred $SU(3)$ that fixes ℓ .

5. The Jordan Eigenvalue Problem

In previous work [5], we solved the *Jordan eigenvalue problem*, namely the eigenmatrix problem

$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V} \tag{24}$$

where \mathcal{A} and \mathcal{V} are both 3×3 octonionic Hermitian matrices. Unlike the right eigenvalue problem $\mathcal{A}v = v\lambda$ considered in [10], the Jordan eigenvalue problem (24) admits only real eigenvalues, which do solve the characteristic equation for \mathcal{A} , namely [11]

$$-\det(\mathcal{A} - \lambda \mathcal{I}) = \lambda^3 - (\text{tr } \mathcal{A}) \lambda^2 + \sigma(\mathcal{A}) \lambda - (\det \mathcal{A}) \mathcal{I} = 0 \tag{25}$$

where \mathcal{I} denotes the identity matrix, $\sigma(\mathcal{A})$ is defined by

$$\sigma(\mathcal{A}) = \frac{1}{2} ((\text{tr } \mathcal{A})^2 - \text{tr}(\mathcal{A}^2)) = \text{tr}(\mathcal{A} * \mathcal{A}) \tag{26}$$

the operation $*$ denotes the Freudenthal product

$$\mathcal{A} * \mathcal{B} = \mathcal{A} \circ \mathcal{B} - \frac{1}{2} (\mathcal{A} \text{tr}(\mathcal{B}) + \mathcal{B} \text{tr}(\mathcal{A})) + \frac{1}{2} (\text{tr}(\mathcal{A}) \text{tr}(\mathcal{B}) - \text{tr}(\mathcal{A} \circ \mathcal{B})) \mathcal{I} \tag{27}$$

and the determinant is defined unambiguously by

$$\det(\mathcal{A}) = \frac{1}{3} \text{tr} ((\mathcal{A} * \mathcal{A}) \circ \mathcal{A}) \tag{28}$$

The Jordan and Freudenthal products are generalizations of the standard dot and cross products.

Just as in the more familiar complex case, normalized eigenmatrices for each nondegenerate eigenvalue are primitive idempotents, and the degenerate case can be handled using Gram-Schmidt orthogonalization. Furthermore, each primitive idempotent is in fact an element of the *Cayley-Moufang plane* $\mathbb{O}\mathbb{P}^2$, which can be characterized as

$$\begin{aligned} \mathbb{O}\mathbb{P}^2 &:= \{ \mathcal{V} \in H_3(\mathbb{O}) : \mathcal{V} \circ \mathcal{V} = \mathcal{V} \text{ and } \text{tr } \mathcal{V} = 1 \} \\ &\equiv \{ \mathcal{V} \in H_3(\mathbb{O}) : \mathcal{V} * \mathcal{V} = 0 \text{ and } \text{tr } \mathcal{V} = 1 \} \end{aligned} \tag{29}$$

It is straightforward to show from the first condition in (29) that the components of any element $\mathcal{V} \in \mathbb{O}\mathbb{P}^2$ must lie in some quaternionic subalgebra of \mathbb{O} , which of course depends on \mathcal{V} . Put differently, the associator of the (independent, off-diagonal) components of \mathcal{V} , denoted $[\mathcal{V}]$, must vanish. But quaternionic primitive idempotents have (nonunique) “square roots”, $\mathcal{V} = \Psi\Psi^\dagger$, so that we can also write

$$\mathbb{O}\mathbb{P}^2 = \{\Psi\Psi^\dagger : \Psi \in \mathbb{O}^3, [\Psi] = 0, \Psi^\dagger\Psi = 1\} \quad (30)$$

where $[\Psi]$ denotes the associator of the components of Ψ . We refer to such 3-component octonionic column vectors Ψ as *Cayley spinors*.

Putting this all together, any 3×3 octonionic Hermitian matrix \mathcal{A} can be expressed as the sum of the squares of quaternionic columns, which are orthogonal under the Jordan product, that is

$$\mathcal{A} = \sum_{i=1}^3 \lambda_i \mathcal{V}_i \quad (31)$$

in terms of primitive idempotents $\mathcal{V}_i = \Psi_i\Psi_i^\dagger \in \mathbb{O}\mathbb{P}^2$ and their corresponding eigenvalues λ_i .

6. The Structure of E_6

The automorphism group of the Jordan product (2) (and consequently also of the Freudenthal product (27)) is the exceptional group F_4 , and the group which leaves the determinant (28) invariant is a particular real form of the exceptional group E_6 . These groups can be interpreted as $F_4 = SU(3, \mathbb{O})$ and $E_6 = SL(3, \mathbb{O})$, as we now show; for further details see [12].

In previous work [9], Manogue and Schray showed how to write the (double cover of the) Lorentz group $SO(9, 1)$ as $SL(2, \mathbb{O})$, with the action given by

$$\mathbf{X} \mapsto \mathbf{M}\mathbf{X}\mathbf{M}^\dagger \quad (32)$$

where $\mathbf{X} \in H_2(\mathbb{O})$, the 2×2 Hermitian matrices with octonionic components. The key to that work was to give an explicit set of basis transformations — the rotations and boosts in coordinate planes — which were *compatible* with the spinor representation in the sense that if $\theta \in \mathbb{O}^2$ transforms like

$$\theta \mapsto \mathbf{M}\theta \quad (33)$$

then there are no associativity problems in the vector transformation

$$\mathbf{M}(\theta\theta^\dagger)\mathbf{M}^\dagger = (\mathbf{M}\theta)(\mathbf{M}\theta)^\dagger \quad (34)$$

Any such basis transformation $\mathbf{M} \in SL(2, \mathbb{O})$ can be immediately reinterpreted as a 3×3 transformation \mathcal{M} via

$$\mathcal{M} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & 1 \end{pmatrix} \quad (35)$$

and it is straightforward to verify that any such \mathcal{M} preserves the determinant of $\mathcal{X} \in H_3(\mathbb{O})$ and is therefore an element of E_6 .

How many such transformations are there? We first give the basis transformations for the simpler case of $SL(2, \mathbb{C})$, adapted from [9] and rewritten as elements of E_6 . When interpreting these transformations, it is helpful to recall that, in this case,

$$\mathbf{X} = \begin{pmatrix} t + z & x - \ell y \\ x + \ell y & t - z \end{pmatrix} \quad (36)$$

We have the three rotations

$$\mathcal{M}_{xy} = \begin{pmatrix} e^{-\ell\theta/2} & 0 & 0 \\ 0 & e^{\ell\theta/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (37)$$

$$\mathcal{M}_{yz} = \begin{pmatrix} \cos \frac{\theta}{2} & -\ell \sin \frac{\theta}{2} & 0 \\ -\ell \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{M}_{zx} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & 0 \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (38)$$

and the three boosts

$$\mathcal{M}_{tz} = \begin{pmatrix} e^{\beta/2} & 0 & 0 \\ 0 & e^{-\beta/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (39)$$

$$\mathcal{M}_{tx} = \begin{pmatrix} \cosh \frac{\beta}{2} & \sinh \frac{\beta}{2} & 0 \\ \sinh \frac{\beta}{2} & \cosh \frac{\beta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{M}_{ty} = \begin{pmatrix} \cosh \frac{\beta}{2} & -\ell \sinh \frac{\beta}{2} & 0 \\ \ell \sinh \frac{\beta}{2} & \cosh \frac{\beta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (40)$$

written in terms of the single imaginary unit ℓ . To generate a 3×3 representation of $SL(2, \mathbb{O})$, start by replacing ℓ in turn by each of the other imaginary units, yielding 18 new transformations, for a total of 24. The remaining 21 transformations in $SL(2, \mathbb{O})$ are precisely the rotations of the imaginary units with each other, which make up an $SO(7)$ (really $Spin(7)$; we are being casual about double covers). As shown in [9], these rotations are obtained by *nesting*, that is by transformations of the form

$$\mathcal{X} \mapsto \mathcal{M}_2 \left(\mathcal{M}_1 \mathcal{X} \mathcal{M}_1^\dagger \right) \mathcal{M}_2^\dagger \quad (41)$$

where each corresponding \mathbf{M} represents a “flip”, that is, a pure imaginary multiple of the (2×2) identity matrix. Thus, a typical \mathcal{M} takes the form

$$\mathcal{M} = \begin{pmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (42)$$

where it is important to note that \mathcal{M} is *not* a multiple of the (3×3) identity matrix.

We are now ready to count the basis transformations of E_6 . At first sight, it appears we have three copies of $SL(2, \mathbb{O})$ — simply repeat the embedding (35) with the two other obvious block structures. We call these three copies type I, II, and III. However, this yields $3 \times 45 = 135$ transformations, and, while these transformations do indeed generate all of E_6 , it is clear that they can not be a basis, since the dimension of E_6 is only 78.

Let’s try again. Each of these three copies of $SL(2, \mathbb{O}) = SO(9, 1)$ contains a copy of $SO(8)$. A famous property of $SO(8)$ called *triality* asserts in this context that these three copies of $SO(8)$ in fact consist of the same E_6 transformations (but labeled differently), so we should count these copies only once. But $SO(8)$ has 28 elements, to which we must add 3 copies of the 8 rotations needed to get to $SO(9)$, then 3 copies of the 9 boosts needed to get to $SO(9, 1)$, resulting in $28 + 3 \times 8 + 3 \times 9 = 79$ transformations. A bit of thought reveals that the 3 copies of the tz -boost (39) are not independent; removing one of them correctly yields an explicit set of 78 basis transformations for E_6 , also justifying the interpretation $E_6 = SL(3, \mathbb{O})$.

It is worth pointing out that, due to triality, only the 14 G_2 transformations need to be written in the nested form (41). Remarkably, the remaining 14 $SO(8)$ transformations can all be expressed using the type I transformation (37) and its types II and III variants. The former are just the usual 7 rotations needed to get from $SO(7)$ to $SO(8)$, but the latter yield an unnested description of the 7 non- G_2 transformations in $SO(7)$, which take the form

$$\mathcal{M}_\ell = \begin{pmatrix} e^{\ell\theta/2} & 0 & 0 \\ 0 & e^{\ell\theta/2} & 0 \\ 0 & 0 & e^{-\ell\theta} \end{pmatrix} \quad (43)$$

Each of these transformations rotates 3 octonionic planes by the same amount, and can therefore be thought of as a “phase” transformation.

What about F_4 ? Note that we have described $27 - 1 = 26$ boosts, and $78 - 26 = 52$ rotations. So our E_6 is the real representation with 26 boosts, commonly written as $E_{6(-26)}$, with the number in parentheses denoting the number of boosts minus the number of rotations. It is straightforward to show that F_4 preserves the trace of elements of $H_3(\mathbb{O})$, corresponding to the timelike direction; this is the compact representation of F_4 , consisting precisely of the rotation subgroup of this real form of E_6 . These considerations justify the interpretation $F_4 = SU(3, \mathbb{O})$.

Returning to the characteristic equation (25), not only does E_6 preserve the determinant, it also preserves the condition $\sigma = 0$. But these two coefficients control the number of nonzero eigenvalues — 3 if $\det \mathcal{A} \neq 0$, 2 if $\det \mathcal{A} = 0 \neq \sigma(\mathcal{A})$, and 1 if $\det \mathcal{A} = 0 = \sigma(\mathcal{A})$. Thus, E_6 preserves the number of nonzero eigenvalues of \mathcal{A} , and hence the number of terms in the decomposition (31) with nonzero eigenvalue.

7. Symmetry Breaking and Particle Physics

In order to apply this formalism to elementary particle physics, we break the full symmetry and explore how a Jordan matrix transforms under various subgroups of E_6 . One such symmetry breaking occurs when we choose a preferred $SL(2, \mathbb{O})$ subgroup of $SL(3, \mathbb{O})$, as in (35). This leads us to impose a block structure on $H_3(\mathbb{O})$, so that

$$\mathcal{P} = \begin{pmatrix} \mathbf{P} & \psi \\ \psi^\dagger & n \end{pmatrix} \quad (44)$$

where $\mathbf{P} \in H_2(\mathbb{O})$ transforms like a 10-dimensional momentum vector, $\psi \in \mathbb{O}^2$ transforms like a (Majorana-Weyl) spinor, and $n \in \mathbb{R}$ is a scalar. Direct computation shows that

$$\mathcal{P} * \mathcal{P} = \begin{pmatrix} \widetilde{\psi\psi^\dagger} - n\tilde{\mathbf{P}} & \tilde{\mathbf{P}}\psi \\ (\tilde{\mathbf{P}}\psi)^\dagger & \det \mathbf{P} \end{pmatrix} \quad (45)$$

where tilde denotes trace reversal, that is, $\tilde{\mathbf{P}} = \mathbf{P} - \text{tr}(\mathbf{P})\mathbf{I}$. As shown in [14], the massless, momentum-space Dirac equation in 10 dimensions can be written

$$\tilde{\mathbf{P}}\psi = 0 \quad (46)$$

which implies the nonlinear constraint

$$\det \mathbf{P} = 0 \quad (47)$$

The general solution of (46) and (47) is of the form

$$\psi = \theta\xi \quad (48)$$

$$\mathbf{P} = \theta\theta^\dagger \quad (49)$$

where the components of $\theta \in \mathbb{O}^2$ lie in the complex subalgebra of \mathbb{O} determined by \mathbf{P} and $\xi \in \mathbb{O}$ is arbitrary. Using (45), these equations are seen to be precisely the same as

$$\mathcal{P} * \mathcal{P} = 0 \quad (50)$$

where

$$n = |\xi|^2 \quad (51)$$

Thus, (normalized) solutions of the Dirac equation are precisely elements of $\mathbb{O}\mathbb{P}^2$, and therefore the squares of Cayley spinors.

In previous work [14], we discussed solutions of the Dirac equation in the form (46). Remembering that solutions of (50) are quaternionic, and reducing 10 spacetime dimensions to 4 by the simple expedient of choosing a preferred complex subalgebra of \mathbb{O} , we used spin eigenstates and particle/antiparticle projection operators as usual to identify particle states within our division algebra formalism. This procedure identified a spin- $\frac{1}{2}$ massive particle with two spin states, namely

$$e_{\uparrow} = \begin{pmatrix} 1 \\ k \end{pmatrix} \quad e_{\uparrow}e_{\uparrow}^{\dagger} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix} \quad (52)$$

$$e_{\downarrow} = \begin{pmatrix} -k \\ 1 \end{pmatrix} \quad e_{\downarrow}e_{\downarrow}^{\dagger} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix} \quad (53)$$

where the direction of the arrow indicates the z -component of the spin, and where the second equality in each case gives the momentum vector. Comparison with (36) shows that the x , y , and z components of the momentum vanish; these states are given at rest. Similarly, there is an analogous antiparticle with two spin states. The procedure also identified a left-handed massless particle, which when moving in the z -direction takes the form

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \quad \nu_z\nu_z^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (54)$$

together with a massless particle of the opposite helicity, which when moving in the z -direction takes the form

$$\emptyset_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \emptyset_z\emptyset_z^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (55)$$

Any other 2-component quaternionic column can be identified as an appropriately rotated and/or boosted superposition of these particles in the usual way. Noting that the first two particles carry an octonionic label (k), it is straightforward to generalize these particles to 3 generations of leptons, labeled by i, j, k ; the remaining particle does not have an octonionic label. Thus, an octonionic description of the Dirac equation in 10 dimensions yields a particle spectrum containing precisely 3 generations of leptons, each with a single-helicity, massless neutrino, together with a single “sterile” neutrino of the opposite helicity but with no generation structure.

8. Discussion

We have shown how to break the symmetry group E_6 so that a Lorentzian 3 + 1 dimensional momentum space emerges, together with internal symmetries that describe the correct spin/helicity transformations on Cayley spinors to describe leptons. Furthermore, precisely three generations of such leptons exist which respect the octonionic structure of the transformations. Contained naturally within this description of leptons and their symmetries are three massless left-handed neutrinos and a single, sterile, right-handed neutrino.

We have learned several important lessons along the way. First, in spite of the non-commutativity and non-associativity of the octonions, everything that one might want to do can be made to work if one defines everything carefully. Second, when working with the octonions, it is important to make the Lie group structure primary, rather than the Lie algebra structure. Some of the nested group transformations described in (41) cannot be described in terms of the exponentials of any Lie algebra transformations. Finally, in order to keep a complex structure on the Lie algebra from interfering with the octonionic units in the matrices, it is important in the symmetry-breaking process to look at *real* Lie subalgebras rather than complexified ones. We

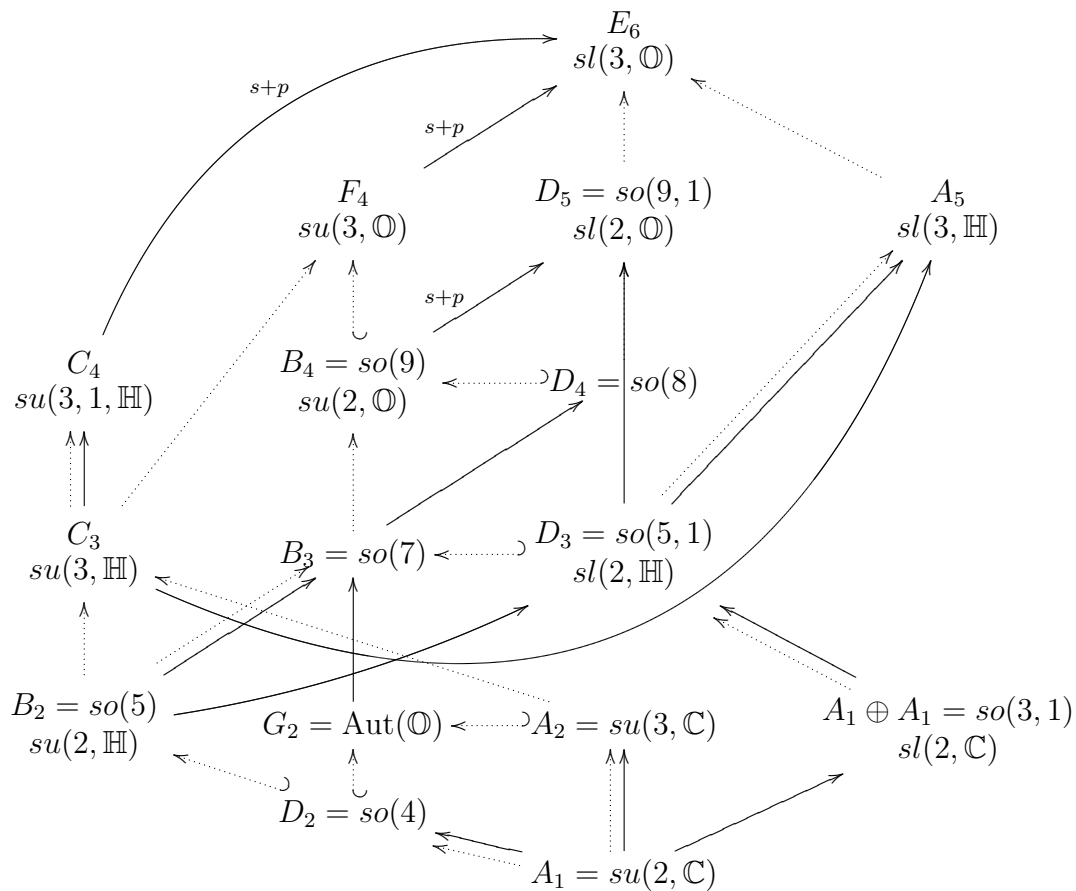


Figure 6. A map of E_6 , taken from [13].

call the reader’s attention to the recent work of Aaron Wangberg [13], in which the important real forms of the subgroups of E_6 were identified, using a division-algebra perspective. A map of these subgroups, taken from [13], appears in Figure 6.

We conclude with several speculative questions:

- (i) In this paper, we have described leptons in terms of “1-squares,” i.e. the squares of Cayley spinors. But the most general octonionic hermitian matrix can be a linear combination of up to three such squares. Is it possible to describe the meson and baryon sectors of standard particle physics as “2-squares” and “3-squares,” respectively?
- (ii) Does the group E_6 , broken in the way we have discussed, describe both the standard model interactions and Lorentz transformations in $3 + 1$ -dimensions?
- (iii) After the symmetry breaking described in Section 7, some of the Lie group transformations of the form (21), (23) are no longer connected to the identity. Do these discrete transformations correspond to discrete conserved quantities such as charge?

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