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# Extrapolation of Tikhonov and Lavrentiev regularization methods

**Uno Hämarik, Reimo Palm and Toomas Raus**

Faculty of Mathematics and Informatics, University of Tartu, Liivi 2, 50409 Tartu, Estonia

E-mail: uno.hamarik@ut.ee, reimo.palm@ut.ee, toomas.raus@ut.ee

**Abstract.** We consider solution of linear ill-posed problem $Au = f$ by Tikhonov method and by Lavrentiev method. For increasing the qualification and accuracy of these methods we use extrapolation, taking for the approximate solution linear combination of $n \geq 2$ approximations of Tikhonov or Lavrentiev methods with different parameters and with proper coefficients. If the solution $u_*$ belongs to $\mathcal{R}((A^*A)^n)$ and instead of $f$ noisy data $f_\delta$ with $\|f_\delta - f\| \leq \delta$ are available, maximal guaranteed accuracy of Tikhonov and Lavrentiev approximations is $O(\delta^{2/3})$ and $O(\delta^{1/2})$, respectively, versus accuracy $O(\delta^{2n/(2n+1)})$ and $O(\delta^{n/(n+1)})$ of corresponding extrapolated approximations. We propose several new rules for a posteriori choice of the regularization parameter, including modifications of the monotone error rule. Extensive numerical experiments show that in case $u_* \in \mathcal{R}(A^*)$ the extrapolated Tikhonov approximation with a posteriori parameter choice (not using any smoothness information) is typically more accurate than Tikhonov approximation with optimal parameter.

## 1. Introduction

We consider an operator equation

$$Au = f, \quad f \in \mathcal{R}(A),$$

where $A \in L(H, F)$ is a linear continuous operator between Hilbert spaces $H$ and $F$. We suppose that instead of exact $f \in F$ noisy data $f_\delta \in F$ with $\|f_\delta - f\| \leq \delta$ are available. For approximation to the solution $u_* \in H$ we use in case $F = H$, $A = A^* \geq 0$ (the selfadjoint case) the Lavrentiev method $u_\alpha = (\alpha I + A)^{-1} f_\delta$ and in general case (the non-selfadjoint case) the Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1} A^* f_\delta$. Here $\alpha > 0$ and $I$ is the identity operator. These approximations have low accuracy: if

$$u_* \in \mathcal{R}((A^*A)^{p/2}), \tag{1}$$

then for $u_{\mathrm{appr}} = u_\alpha$ the order optimal error estimate

$$\|u_{\mathrm{appr}} - u_*\| \leq \mathrm{const}\ \delta^{p/(p+1)} \tag{2}$$

can be reached only for small $p$: in Lavrentiev method for $p \leq 1$, in Tikhonov method for $p \leq 2$. We propose to use for $u_{\mathrm{appr}}$ a proper linear combination of $n \geq 2$ terms $u_{\alpha_i}$

which has in case (1) the error estimate (2) for $p \leq n$ in selfadjoint case and for $p \leq 2n$ in non-selfadjoint case. Note that in a posteriori choice of the regularization parameter $\alpha$ often several approximations with different parameters are computed and then computation of their linear combination is an easy task.

Extrapolation is widely used in well-posed problems: in discretization methods, numerical integration, interpolation etc [1,2]. Extrapolation for increasing the accuracy of regularization methods is much less studied. In case of exact data the extrapolated Tikhonov method was studied in [2–6] for systems of linear algebraic equations and in [7] for operator equations, the extrapolated Lavrentiev method was studied for linear systems [2, 6] and in case $n = 2$ for Fredholm integral equations of the first kind [8]. In case of noisy data extrapolation of Lavrentiev and Tikhonov methods and iterated versions of these methods for operator equations was briefly discussed in [9, 10], and a more detailed treatment of this subject was given in [11]. In this paper we give a survey of the results of [11], propose several rules for choosing a proper extrapolated approximation and present numerical examples.

Note also that extrapolation algorithms do not use any a priori information about solution as some other algorithms do (e.g [12] uses information (1)).

## 2. Parameter choice in the Tikhonov method

For a posteriori choice of the regularization parameter $\alpha$ in Tikhonov method several rules are proposed. In discrepancy principle [13, 14] for Tikhonov approximation such $\alpha$ is chosen, for which $\|Au_\alpha - f_\delta\| = b\delta$, $b \geq 1$. In the modified discrepancy principle [15, 16] $\alpha_{\mathrm{MD}}$ and in the monotone error rule [17] (ME-rule) $\alpha_{\mathrm{ME}}$ are chosen from equations

$$(Au_\alpha - f_\delta, Au_{2,\alpha} - f_\delta) = b\delta,\ b \geq 1, \quad u_{2,\alpha} = (\alpha I + A^*A)^{-1}(\alpha u_\alpha - A^* f_\delta), \tag{3}$$

$$\frac{(Au_\alpha - f_\delta, Au_{2,\alpha} - f_\delta)}{\|Au_{2,\alpha} - f_\delta\|} = b\delta,\ b \geq 1, \tag{4}$$

respectively. Here $u_{2,\alpha}$ is the approximation of the iterated Tikhonov method. Discrepancy principle, the modified discrepancy principle and the ME-rule guarantee in case (1) the error estimate (2) for $p \leq 1$, $p \leq 2$ and $p \leq 2$, respectively. The name of the ME-rule is justified by the property

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\|u_\alpha - u_*\| \geq 0 \qquad \text{for all } \alpha \in (\alpha_{\mathrm{ME}}, \infty).$$

It means that the optimal parameter $\alpha_{\mathrm{opt}} = \operatorname{argmin}\{\|u_\alpha - u_*\|, \alpha > 0\} \leq \alpha_{\mathrm{ME}}$. Usually $\alpha_{\mathrm{ME}} < 1$, hence there exists $c \geq 1$ with $\alpha_{\mathrm{opt}} = \alpha_{\mathrm{ME}}^c$. In numerical experiments of Section 8 we get good results with estimated parameter $\alpha_{\mathrm{MEE}} = \alpha_{\mathrm{ME}}^{1.09}$.

Recently many papers [18–29] advocate the balancing principle (called also Lepskii principle). Here the approximations $u_{\alpha_i}$ are computed for values $\alpha_1 = \delta^2$ and $\alpha_k = \alpha_1 q^{k-1}$, $k = 2, 3, \ldots, M$, where $q > 1$ and $M$ is such that $\alpha_{M-1} < 1 \leq \alpha_M$. For the regularization parameter $\alpha_m$ is taken, where $m$ is the first index, for which a certain condition is fulfilled. For Tikhonov method this condition is in [26, 27]

$$\|u_{\alpha_{m+1}} - u_{\alpha_m}\| > \frac{c\delta}{\sqrt{\alpha_m}} \tag{5}$$

with $c = 2$ and in [22, 29]

$$\exists j \in \{1, \ldots, m\} : \ \|u_{\alpha_{m+1}} - u_{\alpha_j}\| > \frac{c\delta}{\sqrt{\alpha_j}} \tag{6}$$

with $c = 2$. In [24] for nonlinear problem condition (6) with $c \geq 8$ is used. However, evidently a proper $c$ must depend on $q$, in such way that $c \to 0$ as $q \to 1$. Otherwise, after finding $\alpha_m \approx \alpha_{\text{opt}}$ on coarse mesh and then refining the mesh, left hand side of (5) tends to zero as $q \to 1$, hence $\alpha_m$ chosen by (5) increases. More exactly, one can show that if in conditions (5), (6) constant $c$ satisfies $c > q - 1$ and $c > q - q^{j-m}$, respectively, then the error of Tikhonov approximation is a monotonically increasing function of $c$. As proven in [30], the balancing principle with condition (5) is quasioptimal if $c \geq 3\sqrt{3}(q-1)/(16\sqrt{q})$. We recommend to use the last constant in condition (5) and $c = (q - q^{j-m})/(4\sqrt{q^{j-m+1}})$ in condition (6). However, in our numerical experiments with the last $c$ in (6) always such $\alpha_m$ was chosen for which (6) was satisfied with $j = m - 1$ or $j = m$. Condition (6) needs huge computation time, and resulted in large error in numerical experiments, therefore we used in Section 8 instead of (6) the condition

$$\exists j \in \{m-1, m\} : \ \frac{4\|u_{\alpha_{m+1}} - u_{\alpha_j}\| \sqrt{q^{j-m+1}}}{q - q^{j-m}} > \frac{\delta}{\sqrt{\alpha_j}}. \tag{7}$$

In Lavrentiev method $\alpha_{\text{MD}}$ from the modified discrepancy principle [31] is found as the solution of the equation

$$\|Au_{2,\alpha} - f_\delta\| = b\delta, \ b \geq 1, \quad u_{2,\alpha} = (\alpha I + A)^{-1}(\alpha u_\alpha - f_\delta). \tag{8}$$

## 3. Extrapolation of methods of Tikhonov and Lavrentiev

Let $u_{\alpha_i}$, $i = 1, \ldots, n$ be approximations of Tikhonov or Lavrentiev methods with different parameters $\alpha_i = q_i\alpha$ ($q_j \neq q_j$, if $i \neq j$). The corresponding extrapolated approximation $v_{n,\alpha}$ has the form

$$v_{n,\alpha} = \sum_{i=1}^{n} d_i u_{\alpha_i}, \quad d_i = \prod_{j=1, j\neq i}^{n} (1 - \alpha_i/\alpha_j)^{-1}. \tag{9}$$

As shown in [11], extrapolated Lavrentiev approximation (9) and extrapolated Tikhonov approximation (9) coincide with corresponding approximations

$$u_n = (\alpha_n I + A)^{-1}(\alpha_n u_{n-1} + f_\delta), \quad (n = 1, 2, \ldots, \ u_0 = 0)$$
$$u_n = (\alpha_n I + A^*A)^{-1}(\alpha_n u_{n-1} + A^* f_\delta) \quad (n = 1, 2, \ldots, \ u_0 = 0)$$

of nonstationary implicit iterative methods. In $v_{n,\alpha}$ both indexes can be viewed as regularization parameters.

## 4. Choice of parameters in extrapolated approximation

In the following we consider separately the cases, when one of parameters $n$ and $\alpha$ is fixed and other parameter is regularization parameter.

1) Let the sequence $\alpha_1, \alpha_2, \ldots$ be given ($\alpha$ is fixed) and consider choice of $n$ in extrapolated Tikhonov approximation $v_{n,\alpha}$. We give condition for checking, whether $v_{n,\alpha}$ is more accurate solution than $v_{n-1,\alpha}$. Denote $r_n \equiv Av_{n,\alpha} - f_\delta$. Let $C = \text{const} > 1$.

**Theorem 1.** *[11]. The functions $d_D(n) = \|r_n\|$, $d_{ME}(n) = (r_n + r_{n+1}, r_{n+1})/(2\|r_{n+1}\|)$ are monotonically decreasing and $d_D(n+1) < d_{ME}(n) < d_D(n)$ for all $n$. Let $n_D$, $n_{ME}$ be the first numbers with $d_D(n) \leq C\delta$, $d_{ME}(n) \leq C\delta$ respectively. Then $n_D - 1 \leq n_{ME} \leq n_D$ and*

$$\|v_{n,\alpha} - u_*\| < \|v_{n-1,\alpha} - u_*\| \ \textit{for}\ n = 1, 2, \ldots, n_{ME}.$$

*If the monotonically decreasing infinite sequence* $\alpha_1, \alpha_2, \ldots$ *satisfies conditions*

$$\sum_{i=1}^{\infty} \alpha_i^{-1} = \infty, \quad \alpha_n^{-1} \leq const \sum_{i=1}^{n-1} \alpha_i^{-1},$$

*then existence of finite* $n_D$ *and* $n_{ME}$ *is guaranteed and for* $n \in \{n_D, n_{ME}\}$, $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ *and in case* (1) *the error estimate* (2) *holds for all* $p > 0$.

In extrapolated Lavrentiev method we recommend to choose $n$ by the discrepancy principle: $n = n(\delta)$ is the first $n$ with $\|Av_{n,\alpha} - f_\delta\| \leq C\delta$. It guarantees the convergence $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ and under assumption (1) the error estimate (2) for all $p > 0$.

2) Let $n \geq 2$ and $q_1, \ldots, q_{n+1}$ be fixed. Consider choice of $\alpha$ in extrapolated approximation $v_{n,\alpha}$.

**Theorem 2.** *[11] The functions* $d_D(\alpha) = \|Av_{n,\alpha} - f_\delta\|$, $d_{MD}(\alpha) = (Av_{n,\alpha} - f_\delta, Av_{n+1,\alpha} - f_\delta)$ *are monotonically decreasing. If* $\alpha$ *is chosen from the discrepancy principle* $d_D(n) = C\delta$, *then* $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ *and in case* (1) *for* $u_{appr} = v_{n,\alpha}$ *the error estimate* (2) *holds in extrapolated Tikhonov method with* $p \leq 2n - 1$ *and in extrapolated Lavrentiev with* $p \leq n - 1$. *If* $\alpha$ *in extrapolated Tikhonov method is chosen from the modified discrepancy principle* $d_{MD}(\alpha) = C\delta$, *then* $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ *and in case* (1) *for* $u_{appr} = v_{n,\alpha}$ *the error estimate* (2) *holds with* $p \leq 2n$. *If* $\alpha$ *in extrapolated Lavrentiev method is chosen from the modified discrepancy principle* $\|Av_{n+1,\alpha} - f_\delta\| = C\delta$, *then* $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ *and in case* (1) *for* $u_{appr} = v_{n,\alpha}$ *the error estimate* (2) *holds with* $p \leq n$.

## 5. Extrapolation of iterated methods of Lavrentiev and Tikhonov

Consider now extrapolation of $m$ times iterated methods of Lavrentiev and Tikhonov

$$u_\alpha = u_{m,\alpha}, \qquad u_{k,\alpha} = (\alpha I + A)^{-1}(\alpha u_{k-1,\alpha} + f_\delta) \quad (k = 1, \ldots, m),\ u_{0,\alpha} = 0,$$
$$u_\alpha = u_{m,\alpha}, \qquad u_{k,\alpha} = (\alpha I + A^*A)^{-1}(\alpha u_{k-1,\alpha} + A^* f_\delta) \quad (k = 1, \ldots, m),\ u_{0,\alpha} = 0.$$

For different $\alpha_i = q_i \alpha$ $(i = 1, \ldots, n)$ different number of iterations $m_1, \ldots, m_n$ may be used. We take for approximate solution (see [11])

$$v_{n,\alpha} = \sum_{i=1}^{n} \sum_{k=1}^{m_i} d_{i,k} u_{k,\alpha_i},$$

where the coefficients $d_{i,k}$ can be uniquely determined from relation

$$\sum_{i=1}^{n} \sum_{k=1}^{m_i} d_{i,k} (1 + \lambda/q_i)^{-k} = \prod_{i=1}^{n} (1 + \lambda/q_i)^{-m_i} \quad (\forall \lambda \in \mathbb{R}).$$

**Theorem 3.** *[11] If* $n$ *and* $q_1, \ldots, q_n$ *are fixed and* $\alpha$ *is chosen from the discrepancy principle* $d_D(n) = C\delta$, *then* $\|v_{n,\alpha} - u_*\| \to 0$ $(\delta \to 0)$ *and in case* (1) *for* $u_{appr} = v_{n,\alpha}$ *the error estimate* (2) *holds in non-selfadjoint case with* $p \leq 2(m_1 + m_2 + \cdots + m_n) - 1$ *and in selfadjoint case with* $p \leq m_1 + m_2 + \cdots + m_n - 1$.

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## 6. Use of extrapolation for parameter choice in Tikhonov method

Consider a posteriori choice of the regularization parameter $\alpha$ in methods of Lavrentiev and Tikhonov. In rules (3), (4), (8) of Section 3 iterated approximation $u_{2,\alpha}$ is used, hence one additional equation must be solved. Following theorems show that order optimal error estimates for source-like solutions remain true, if in these rules $u_{2,\alpha}$ is replaced by a proper linear combination of two approximations.

**Theorem 4.** *[11] Let $u_\alpha$ and $u_{q\alpha}$ with $q < 1$ be approximate solutions of $Au = f$, found by Tikhonov method and let $v_{2,\alpha}$ be their linear combination*

$$v_{2,\alpha} = \left(1 - q^{-1}\right)^{-1} u_\alpha + (1 - q)^{-1} u_{q\alpha}. \tag{10}$$

*Let us choose $\alpha = \alpha(\delta)$ in $u_\alpha$ according to the rule*

$$b_1\delta \le (Au_\alpha - f_\delta, Av_{2,\alpha} - f_\delta)^{1/2} \le b_2\delta, \quad b_2 \ge b_1 > 1 \tag{11}$$

*or to the rule*

$$b_1\delta \le (Au_\alpha - f_\delta, Av_{2,\alpha} - f_\delta) \,/\, \|Av_{2,\alpha} - f_\delta\| \le b_2\delta, \quad b_2 \ge b_1 > 1/q. \tag{12}$$

*Then $\|u_\alpha - u_*\| \to 0$ as $\delta \to 0$. In case (1) the error estimate (2) holds with $p \le 2$.*

**Theorem 5.** *[11] Let $F = H$, $A = A^* \ge 0$. Let $u_\alpha$ and $u_{q\alpha}$ be approximate solutions of $Au = f$, found by Lavrentiev method. Let us choose the regularization parameter $\alpha = \alpha(\delta)$ in $u_\alpha$ according to the rule*

$$b_1\delta \le \|Av_{2,\alpha} - f_\delta\| \le b_2\delta, \quad b_2 \ge b_1 > 1. \tag{13}$$

*Then $\|u_\alpha(\delta) - u_*\| \to 0$ as $\delta \to 0$. In case (1) the error estimate (2) holds with $p \le 1$.*

## 7. The monotone error rule for choosing an approximation from sequence

In balancing principle a sequence of approximate solutions $\{u_{\alpha_i}\}$ is computed and a rule for choice of one approximation $u_{\alpha_i}$ is given. It motivates us to give another rule, the monotone error rule, for choice of proper approximation from sequence.

**Theorem 6.** *Let $u_i = A^* w_i$, $i = 1, 2, \ldots$ be a sequence of approximations to solution $u_*$ of the equation $Au = f$. Let $i_{ME}$ be the first index $i$ satisfying*

$$d_{ME}(i) = \frac{(Au_i - f_\delta + Au_{i+1} - f_\delta, w_{i+1} - w_i)}{2\|w_{i+1} - w_i\|} \le \delta.$$

*Then*

$$\|u_i - u_*\| \le \|u_{i-1} - u_*\| \qquad \textit{for all } i = 2, \ldots, i_{ME}.$$

*Proof.* We have

$$\begin{aligned}
\|u_i - u_*\|^2 - \|u_{i-1} - u_*\|^2 &= (u_{i-1} + u_i - 2u_*, u_i - u_{i-1}) \\
&= (Au_{i-1} + Au_i - 2f, w_i - w_{i-1}) \\
&= (Au_{i-1} - f_\delta + Au_i - f_\delta + 2(f_\delta - f), w_i - w_{i-1}) \\
&\le 2\|w_i - w_{i-1}\|[\delta - d_{\mathrm{ME}}(i-1)].
\end{aligned}$$

Therefore, if $d_{\mathrm{ME}}(i-1) > \delta$, then $\|u_i - u_*\| < \|u_{i-1} - u_*\|$. □

For using the functional $d_{\mathrm{ME}}(i)$ elemets $w_i$ are needed. They may be get, computing at first $w_i$ and on final step $u_i = A^* w_i$. Last theorem may be applied for many kind of approximations: for approximations $u_i = u_{\alpha_i}$ with decreasing parameters $\alpha_1 > \alpha_2 > \ldots$ in Tikhonov method or in iterated Tikhonov method. In extrapolated Tikhonov method $i$ in $u_i$ may refer to number of terms $n$ in linear combination (9) or to $\alpha_i$ in (9) or to some other element in arbitrary sequence of extrapolated approximations.

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## 8. Numerical experiments

We solved 12 test problems, 10 of which were from [32] and the other two were slight modifications of these. We used discretization parameter 100 and if the problem had more parameters, these were taken 1 (except the problem deriv2). Besides solutions $u_*$ of [32] we used smoothened solutions $(A^*A)^{p/2}u_*$ with $p = 0.25$, 0.5, 1, 1.5, 2, 4, 8 and the right-hand side was computed as $f = A(A^*A)^{p/2}u_*$. All problems were normalized in such way that the norms of operator and right-hand side were 1.

Instead of exact data $f$ randomly perturbed data $f_\delta$ were used with error $\|f - f_\delta\| = \delta$, where values of $\delta$ were 0.5 and $10^{-i}$, $i = 1, \ldots, 7$. The problems were regularized by the Tikhonov method. As in balancing principle, Tikhonov approximations were computed on the set of alpha-values $\Omega = \{\alpha_i\}$ with $\alpha_N = 1$, $\alpha_{i-1} = \alpha_i/q$, $N = 1000$. For $q$ we took $\sqrt[16]{2} \approx 1.04$, $\sqrt[4]{2} \approx 1.19$, and 2. In case $q = 2$ and $\delta \in \{0.5, 0.1\}$ we used also the values $\alpha_i$ with $i \leq N + 5$.

For choice of $\alpha_i \in \Omega$, the following rules were used.

1) Discrepancy principle: $\alpha_{\mathrm{D}}$ is the first $\alpha_i$ in the sequence $\alpha_N, \alpha_{N-1}, \ldots$, for which $\|Au_{\alpha_i} - f_\delta\| \leq \delta$; $\alpha_{\mathrm{DE}}$ is the nearest alpha-value to $\alpha_{\mathrm{D}}^{0.86}$ in $\Omega$.

2) Monotone error rule: $\alpha_{\mathrm{ME}}$ is the first $\alpha_i$ in the sequence $\alpha_N, \alpha_{N-1}, \ldots$, for which $(Au_{\alpha_i} - f_\delta, h_i)/\|h_i\| \leq \delta$, $h_i = A\left(u_{\alpha_{i-1}}/(1-q^{-1}) + u_{\alpha_i}/(1-q)\right) - f_\delta$ (see (10)); $\alpha_{\mathrm{MEE}}$ is the nearest alpha-value to $\alpha_{\mathrm{ME}}^{1.09}$ in $\Omega$.

3) Balancing principle: $\alpha_{\mathrm{L1a}}$ and $\alpha_{\mathrm{L1b}}$ were chosen as the first $\alpha_m$ in the sequence $\alpha_1, \alpha_2, \ldots$, for which (5) holds with $c = 3\sqrt{3}(q-1)/(16\sqrt{q})$ and $c = 2$, respectively; $\alpha_{\mathrm{L2}}$ was chosen as the first $\alpha_m$ in the sequence $\alpha_1, \alpha_2, \ldots$, for which (7) holds.

We computed the extrapolated approximations

$$v_n^{(k)} = \sum_{i=k-n+1}^{k} d_i u_{\alpha_i}, \quad d_i = \prod_{j=1, j\neq i}^{n} \frac{\alpha_j}{\alpha_j - \alpha_i} = \prod_{j=1, j\neq i}^{n} \frac{1}{1 - q^{i-j}}. \tag{14}$$

For $n = 2, \ldots, 5$, various rules for choosing $k$ lead to the following extrapolated approximations.

1) $v_{n\mathrm{ME}} = v_n^{(n\mathrm{ME})}$, where $N - n\mathrm{ME} + 1$ is index $i_{\mathrm{ME}}$ from Theorem 6, applied to the sequence $u_i = v_n^{(N-i+1)}$.

2) $v_{n\mathrm{MEE}} = v_n^{(n\mathrm{MEE})}$, where $n$MEE is the nearest index for alpha-value $\alpha_{n\mathrm{ME}}^{1.083}$.

3) $v_{n\mathrm{D}} = v_n^{(n\mathrm{D})}$, where $n$D is the first $k$ in sequence $N, N-1, \ldots$, for which $\|Av_n^{(k)} - f_\delta\| \leq \delta$.

4) $v_{n\mathrm{DE}} = v_n^{(n\mathrm{DE})}$, where $n$DE is the nearest index for alpha-value $\alpha_{n\mathrm{D}}^{c_{n,1}}\alpha_{\mathrm{D}}^{c_{n,2}}$, where $(c_{21}, c_{22}) = (1.22, -0.12)$, $(c_{31}, c_{32}) = (1.16, -0.04)$, $(c_{41}, c_{42}) = (1.11, -0.01)$, $(c_{51}, c_{52}) = (1.1, 0)$. These constants and constants for $\alpha_{\mathrm{DE}}$, $\alpha_{\mathrm{MEE}}$, and $v_{\mathrm{maxDE}}$ below were found by optimization on a large data set. The exponent less than 1 for $\alpha_{\mathrm{D}}$ was good only for $p \geq 1$.

We computed also $v_{\mathrm{maxD}} = v_n^{(N)}$, choosing $n$ as $n_{\mathrm{D}}$ in Theorem 1, and $v_{\mathrm{maxDE}} = v_n^{(N)}$ with $n$ as the nearest integer to $1.1(N - n_{\mathrm{maxD}})$.

In model equations the exact solutions are known. We found $\alpha_{\mathrm{opt}}$ as $\alpha_i \in \Omega$ with the smallest error: $\|u_{\alpha_{\mathrm{opt}}} - u_*\| = \min\{\|u_{\alpha_i} - u_*\|, \alpha_i \in \Omega\}$. We solved these problems 10 times. Tables 1 and 2 show the averages (over all problems, all $q$, all $\delta$ and 10 runs) of error ratios $e_{\mathrm{D}} = \|u_{\alpha_{\mathrm{D}}} - u_*\|/\|u_{\alpha_{\mathrm{opt}}} - u_*\|$, $e_{\mathrm{ME}} = \|u_{\alpha_{\mathrm{ME}}} - u_*\|/\|u_{\alpha_{\mathrm{opt}}} - u_*\|$, $\ldots$, $e_{\mathrm{vmax}} = \|u_{\alpha_{\mathrm{vmax}}} - u_*\|/\|u_{\alpha_{\mathrm{opt}}} - u_*\|$. Table 2 does not contain results for $v_{4\mathrm{MEE}}$ and $v_{5\mathrm{MEE}}$, which were by about 0.05 larger than the ratios for $v_{4\mathrm{DE}}$ and $v_{5\mathrm{DE}}$.

In Table 3 error ratios of $v_{3\mathrm{DE}}$ for every problem are given. In most problems the ratios decreased for increasing $p$.

As tables 2–4 show, in case $u_* \in \mathcal{R}(A)$ the error of extrapolated approximation was in most cases smaller than the error of the best single Tikhonov approximation.

**Table 1.** Means of error ratios.

| $p$ | $e_{\mathrm{D}}$ | $e_{\mathrm{ME}}$ | $e_{\mathrm{L1a}}$ | $e_{\mathrm{L1b}}$ | $e_{\mathrm{L2}}$ | $e_{\mathrm{2D}}$ | $e_{\mathrm{3D}}$ | $e_{\mathrm{4D}}$ | $e_{\mathrm{5D}}$ | $e_{\mathrm{maxD}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.302 | 1.483 | 2.460 | 12.013 | 2.193 | 1.319 | 1.339 | 1.344 | 1.346 | 1.349 |
| 0.25 | 1.719 | 1.863 | 3.344 | 17.688 | 2.983 | 1.590 | 1.615 | 1.623 | 1.623 | 1.630 |
| 0.5 | 1.957 | 2.003 | 3.937 | 22.623 | 3.395 | 1.607 | 1.628 | 1.638 | 1.643 | 1.646 |
| 1 | 2.451 | 1.612 | 3.486 | 28.701 | 2.991 | 1.052 | 1.075 | 1.084 | 1.090 | 1.092 |
| 1.5 | 3.180 | 1.423 | 3.066 | 30.211 | 2.598 | 0.721 | 0.720 | 0.734 | 0.727 | 0.724 |
| 2 | 3.575 | 1.362 | 2.850 | 29.784 | 2.403 | 0.571 | 0.541 | 0.537 | 0.542 | 0.536 |
| 4 | 3.864 | 1.340 | 2.824 | 29.496 | 2.385 | 0.450 | 0.333 | 0.312 | 0.314 | 0.300 |
| 8 | 3.889 | 1.338 | 2.823 | 29.515 | 2.384 | 0.443 | 0.308 | 0.291 | 0.267 | 0.248 |
| mean | 2.742 | 1.553 | 3.099 | 25.004 | 2.666 | 0.969 | 0.945 | 0.945 | 0.944 | 0.941 |

**Table 2.** Means of error ratios (cont).

| $p$ | $e_{\mathrm{DE}}$ | $e_{\mathrm{MEE}}$ | $e_{\mathrm{2DE}}$ | $e_{\mathrm{2MEE}}$ | $e_{\mathrm{3DE}}$ | $e_{\mathrm{3MEE}}$ | $e_{\mathrm{4DE}}$ | $e_{\mathrm{5DE}}$ | $e_{\mathrm{maxDE}}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.714 | 1.234 | 1.206 | 1.216 | 1.219 | 1.222 | 1.219 | 1.222 | 1.231 |
| 0.25 | 2.339 | 1.484 | 1.435 | 1.445 | 1.452 | 1.460 | 1.450 | 1.454 | 1.471 |
| 0.5 | 2.634 | 1.480 | 1.371 | 1.391 | 1.384 | 1.396 | 1.393 | 1.398 | 1.413 |
| 1 | 1.539 | 1.222 | 0.954 | 0.969 | 0.945 | 0.962 | 0.943 | 0.941 | 0.948 |
| 1.5 | 1.351 | 1.137 | 0.694 | 0.717 | 0.664 | 0.684 | 0.665 | 0.655 | 0.651 |
| 2 | 1.430 | 1.114 | 0.562 | 0.588 | 0.517 | 0.543 | 0.514 | 0.500 | 0.492 |
| 4 | 1.514 | 1.110 | 0.448 | 0.445 | 0.345 | 0.369 | 0.334 | 0.316 | 0.299 |
| 8 | 1.522 | 1.110 | 0.440 | 0.430 | 0.320 | 0.337 | 0.300 | 0.274 | 0.253 |
| mean | 1.755 | 1.236 | 0.889 | 0.900 | 0.856 | 0.872 | 0.852 | 0.845 | 0.845 |

**Table 3.** Means of error ratios for $v_{\mathrm{3DE}}$ by problems.

| $p$ | baart | baart2 | deriv2 | foxgood | gravity | heat | heat2 | ilaplace | phillips | shaw | spikes | wing |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.42 | 2.19 | 0.86 | 1.39 | 1.11 | 1.05 | 0.89 | 1.17 | 1.05 | 1.37 | 1.00 | 1.12 |
| 0.25 | 2.33 | 1.38 | 0.81 | 1.51 | 1.08 | 1.01 | 0.84 | 1.16 | 0.99 | 1.47 | 1.69 | 3.17 |
| 0.5 | 3.23 | 0.95 | 0.73 | 1.44 | 1.00 | 0.97 | 0.76 | 1.11 | 0.91 | 1.42 | 1.82 | 2.26 |
| 1 | 1.45 | 0.70 | 0.47 | 0.89 | 0.78 | 0.85 | 0.69 | 0.93 | 0.72 | 1.12 | 1.35 | 1.38 |
| 1.5 | 0.97 | 0.58 | 0.33 | 0.54 | 0.55 | 0.75 | 0.64 | 0.73 | 0.51 | 0.78 | 0.98 | 0.61 |
| 2 | 0.56 | 0.49 | 0.30 | 0.37 | 0.44 | 0.63 | 0.60 | 0.54 | 0.41 | 0.58 | 0.81 | 0.49 |
| 4 | 0.31 | 0.35 | 0.29 | 0.28 | 0.35 | 0.37 | 0.38 | 0.36 | 0.34 | 0.35 | 0.48 | 0.29 |
| 8 | 0.31 | 0.34 | 0.29 | 0.28 | 0.34 | 0.33 | 0.33 | 0.34 | 0.33 | 0.35 | 0.35 | 0.28 |
| mean | 1.32 | 0.87 | 0.51 | 0.84 | 0.71 | 0.74 | 0.64 | 0.79 | 0.66 | 0.93 | 1.06 | 1.20 |

**Table 4.** Means (over all $p$) of error ratios and errors for problem phillips in case $q = \sqrt[16]{2}$.

| $\delta$ | $e_{\mathrm{D}}$ | $e_{\mathrm{ME}}$ | $e_{\mathrm{MEE}}$ | $e_{\mathrm{2MEE}}$ | $e_{\mathrm{2DE}}$ | $e_{\mathrm{3DE}}$ | $e_{\mathrm{4DE}}$ | $e_{\mathrm{5DE}}$ | $e_{\mathrm{maxDE}}$ | $\Vert u_{\alpha_{\mathrm{opt}}-u_*}\Vert$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1.029 | 1.196 | 1.122 | 0.950 | 0.942 | 0.905 | 0.854 | 0.831 | 0.840 | $1.99 \cdot 10^{-1}$ |
| $10^{-1}$ | 1.059 | 1.295 | 1.137 | 0.885 | 0.865 | 0.824 | 0.815 | 0.812 | 0.781 | $6.81 \cdot 10^{-2}$ |
| $10^{-2}$ | 1.367 | 1.305 | 1.073 | 0.726 | 0.715 | 0.664 | 0.658 | 0.651 | 0.658 | $1.71 \cdot 10^{-2}$ |
| $10^{-3}$ | 1.981 | 1.238 | 1.022 | 0.629 | 0.631 | 0.581 | 0.577 | 0.573 | 0.589 | $5.36 \cdot 10^{-3}$ |
| $10^{-4}$ | 2.503 | 1.200 | 1.026 | 0.560 | 0.576 | 0.521 | 0.501 | 0.494 | 0.485 | $1.67 \cdot 10^{-3}$ |
| $10^{-5}$ | 3.274 | 1.210 | 1.048 | 0.556 | 0.576 | 0.527 | 0.505 | 0.500 | 0.495 | $5.44 \cdot 10^{-4}$ |
| $10^{-6}$ | 4.660 | 1.223 | 1.070 | 0.551 | 0.564 | 0.535 | 0.505 | 0.501 | 0.489 | $1.80 \cdot 10^{-4}$ |
| $10^{-7}$ | 6.954 | 1.224 | 1.119 | 0.554 | 0.556 | 0.560 | 0.513 | 0.508 | 0.472 | $6.24 \cdot 10^{-5}$ |
| mean | 2.853 | 1.236 | 1.077 | 0.676 | 0.678 | 0.640 | 0.616 | 0.609 | 0.601 | $3.65 \cdot 10^{-2}$ |

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