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Replica symmetry breaking in mean-field spin glasses through the Hamilton–Jacobi technique

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Abstract. During the last few years, through the combined effort of the insight coming from physical intuition and computer simulation, and the exploitation of rigorous mathematical methods, the main features of the mean-field Sherrington–Kirkpatrick spin glass model have been firmly established. In particular, it has been possible to prove the existence and uniqueness of the infinite-volume limit for the free energy, and its Parisi expression, in terms of a variational principle involving a functional order parameter. Even the expected property of ultrametricity, for the infinite-volume states, seems to be near to a complete proof.

The main structural feature of this model, and related models, is the deep phenomenon of spontaneous replica symmetry breaking (RSB), discovered by Parisi many years ago. By expanding on our previous work, the aim of this paper is to investigate a general framework, where replica symmetry breaking is embedded in a kind of mechanical scheme of the Hamilton–Jacobi type. Here, the analog of the 'time' variable is a parameter characterizing the strength of the interaction, while the 'space' variables rule out quantitatively the broken replica symmetry pattern. Starting from the simple cases, where annealing is assumed, or replica symmetry, we build up a progression of dynamical systems, with an increasing number of space variables, which allow us to weaken the effect of the potential in the Hamilton–Jacobi equation as the level of symmetry breaking is increased.

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This new machinery allows us to work out mechanically the general K-step RSB solutions, in a different interpretation with respect to the replica trick, and easily reveals their properties such as existence or uniqueness.

Keywords: rigorous results in statistical mechanics, cavity and replica method, disordered systems (theory), spin glasses (theory)

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Contents

1. Introduction

In the past 20 years the statistical mechanics of disordered systems has earned an always increasing weight as a powerful framework by which to analyze the world of complexity [5, 6, 11, 14, 28, 29, 31].

The basic model of this field of research is the Sherrington–Kirkpatrick (SK) model [26] for a spin glass, for which several methods of investigation have been tested over the years [3, 4, 9, 13, 17, 23, 24, 36, 37]. The first method developed has been the *replica trick* [27, 32] which, in a nutshell, consists in expressing the quenched average of the logarithm of the partition function $Z(\beta)$ in the form $\mathbb{E} \ln Z(\beta) = \lim_{n\to 0} \mathbb{E}(Z(\beta)^n - 1)/n$. Since the averages are easily calculated for integer values of n, the problem is to find the right analytical continuation allowing us, in some way, to evaluate the $n \to 0$ limit, at least for the case of large systems. Such an analytical continuation is extremely complex, and many efforts have been necessary to examine this problem in the light of theoretical physics tools, such as symmetries and their breaking [33, 34]. In this scenario a solution has been proposed by Parisi, with the well-known replica symmetry breaking scheme (RSB), both solving the SK model by showing a peculiar 'picture' of the organization of the underlying microstructure of this complex system [28], as well as conferring a key role on the replica-trick method itself [39].

J. Stat. Mech. (2010) P09006

The physical relevance, and deep beauty, of the results obtained in the framework of the replica trick has prompted a wealth of further research, in particular towards the objective of developing rigorous mathematical tool for the study of these problems. Let us recall, very schematically, some of the results obtained along these lines. Ergodic behavior has been confirmed in [16, 23], the lack of self-average for the order parameter has been shown in [35], the existence of the thermodynamic limit in [22], the universality with respect to the coupling's distribution in [15], the correctness of the Parisi expression for the free energy in [21, 38], the critical behavior in [1], the constraints to the free overlap fluctuations in [2, 25], and so many other contributions developed to give rise even to textbooks (see, for example, [12, 18, 39]).

Very recently, new investigations on ultrametricity started [7, 8] and allowed even strong statements dealing with the latter [30], highlighting as a consequence the inquiry for techniques to prove the uniqueness of the Parisi solution, step by step.

In this paper we match two other techniques, the broken replica symmetry bound [21] and the Hamilton–Jacobi method [20, 10, 19], so as to obtain a unified and stronger mathematical tool to work out free energies at various levels of RSB, whose properties are easily available as consequences of simple analogies with purely mechanical systems [19]. We stress that, within this framework, the improvement of the free energy by increasing the replica symmetry breaking steps is transparent.

In this first paper we show the method in complete detail and pedagogically apply it for recovering the annealed and the replica symmetric solutions. Then we work out the first level of RSB and show how to obtain the 1-RSB Parisi solution with its properties.

This paper is organized as follows. In section 2 the SK model is introduced together with its related statistical mechanics definitions. In section 3 the broken replica mechanical analogy is outlined in complete detail (minor calculations are reported in the appendix), while sections 4–6 are respectively dedicated to the annealed, the replica symmetric and the 1-RSB solutions of the SK model using our approach. Section 7 deals with the properties of the solutions and section 8 is left for the outlook and conclusions.

2. The Sherrington–Kirkpatrick mean-field spin glass

The generic configuration of the Sherrington–Kirkpatrick model [26, 27] is determined by the N Ising variables $\sigma_i = \pm 1$, $i = 1, 2, \ldots, N$. The Hamiltonian of the model, in some external magnetic field h, is

$$H_N(\sigma, h; J) = -\frac{1}{\sqrt{N}} \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - h \sum_{1\le i\le N} \sigma_i. \tag{1}$$

The first term in (1) is a long range random two-body interaction, while the second represents the interaction of the spins with the magnetic field h. The external quenched disorder is given by the $N(N-1)/2$ independent and identically distributed random variables J_{ij}, defined for each pair of sites. For the sake of simplicity, denoting the average over this disorder by $\mathbb{E}$, we assume each J_{ij} to be a centered unit Gaussian with averages

$$\mathbb{E}(J_{ij}) = 0, \qquad \mathbb{E}(J_{ij}^2) = 1.$$

For a given inverse temperature[4] β, we introduce the disorder-dependent partition function $Z_N(\beta, h; J)$, the quenched average of the free energy per site $f_N(\beta, h)$,

[4] Here and in the following, we set the Boltzmann constant k_B equal to one, so that $\beta = 1/(k_B T) = 1/T$.

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J. Stat. Mech. (2010) P09006

the associated averaged normalized log-partition function $\alpha_N(\beta,h)$ and the disorder-dependent Boltzmann–Gibbs state ω, according to the definitions

$$Z_N(\beta,h;J)=\sum_{\sigma}\exp(-\beta H_N(\sigma,h;J)), \tag{2}$$

$$-\beta f_N(\beta,h)=\frac{1}{N}\mathbb{E}\ln Z_N(\beta,h)=\alpha_N(\beta,h), \tag{3}$$

$$\omega(A)=Z_N(\beta,h;J)^{-1}\sum_{\sigma}A(\sigma)\exp(-\beta H_N(\sigma,h;J)), \tag{4}$$

where A is a generic function of σ.

Let us now introduce the important concept of replicas. Consider a generic number n of independent copies of the system, characterized by the spin configurations $\sigma^{(1)},\ldots,\sigma^{(n)}$, distributed according to the product state

$$\Omega=\omega^{(1)}\times\omega^{(2)}\times\cdots\times\omega^{(n)},$$

where each $\omega^{(\alpha)}$ acts on the corresponding $\sigma_i^{(\alpha)}$ variables, and all are subject to the *same* sample J of the external disorder. These copies of the system are usually called *real replicas*, to distinguish them from those appearing in the *replica trick*, [28], which requires a limit towards zero number of replicas ($n\to 0$) at some stage.

The overlap between two replicas a,b is defined according to

$$q_{ab}(\sigma^{(a)},\sigma^{(b)})=\frac{1}{N}\sum_{1\le i\le N}\sigma_i^{(a)}\sigma_i^{(b)}, \tag{5}$$

and satisfies the obvious bounds

$$-1\le q_{ab}\le 1.$$

For a generic smooth function A of the spin configurations on the n replicas, we define the averages $\langle A\rangle$ as

$$\langle A\rangle=\mathbb{E}\Omega A(\sigma^{(1)},\sigma^{(2)},\ldots,\sigma^{(n)}), \tag{6}$$

where the Boltzmann–Gibbs average Ω acts on the replicated σ variables and $\mathbb{E}$ denotes, as usual, the average with respect to the quenched disorder J.

3. Thermodynamics through a broken replica mechanical analogy

Once we have introduced the model, let us briefly discuss the plan we are going to follow.

In the broken replica symmetry bound (BRSB) [21] it has been shown that the Parisi solution is a bound for the true free energy (the opposite bound has been achieved in [38]). This has been done by introducing a suitable recursive interpolating scheme that we are going to recall hereafter.

In the Hamilton–Jacobi technique instead [20], it has been shown, by introducing a simple two-parameter interpolating function, how to recover the replica symmetric solution through a mechanical analogy, offering as a sideline a simple prescription, once the bridge to mechanics was achieved, to prove the uniqueness of the replica symmetric solution.

The main result of this paper is that the two approaches can be merged such that even the recursive interpolating structure of the BRSB obeys a particular Hamilton–Jacobi description. This result has both theoretical and practical advantages: the former is a clear bridge to improving the approximation of the free energy solution and increasing the levels of RSB, while the latter is a completely autonomous mechanical tool by which to obtain solutions at various RSB steps in further models.

The task is, however, not trivial: the motion is no longer on a $1+1$ Euclidean spacetime as in [20] but lives in $K+1$ dimensions such that momenta and a mass matrix need to be introduced.

To start showing the whole procedure, let us introduce the following *Boltzmann factor*:

$$B(\{\sigma\};\mathbf{x},t) = \exp\left(\sqrt{\frac{t}{N}}\sum_{(ij)} J_{ij}\sigma_i\sigma_j + \sum_{a=1}^{K}\sqrt{x_a}\sum_i J_i^a\sigma_i\right) \tag{7}$$

where both J_{ij}s and J_i^as are standard Gaussian random variables $\mathcal{N}[0,1]$ i.i.d. The t parameter and each of the x_a may be tuned in $\mathbb{R}^+$. We will use both the symbol $\mathbf{x}$ as well as $(x_1,\ldots,x_K)$ to label the K interpolating real parameters coupling the one-body interactions. K represents the dimensions, corresponding to the RSB steps in the replica trick. Let us denote via E_a each of the averages with respect to each of the J_as and E_0 is the one with respect to the whole J_{ij} random couplings. Through equation (7) we are allowed to define the following partition function $\tilde{Z}_N(t;x_1,\ldots,x_K)$ and, iteratively, all the other BRSB approximating functions for $a=0,\ldots,K$:

$$Z_K \equiv \tilde{Z}_N = \sum_\sigma B(\{\sigma\};\mathbf{x},t), \tag{8}$$

$$\ldots$$

$$Z_{a-1}^{m_a} \equiv E_a\left(Z_a^{m_a}\right), \tag{9}$$

$$\ldots$$

$$Z_0^{m_1} \equiv E_1(Z_1^{m_1}). \tag{10}$$

We need further to introduce the following interpolating function:

$$\tilde{\alpha}_N(t;x_1,\ldots,x_K) \equiv \frac{1}{N}E_0\log Z_0, \tag{11}$$

and define, for $a=1,\ldots,K$, the random variables

$$f_a \equiv \frac{Z_a^{m_a}}{E_a(Z_a^{m_a})}, \tag{12}$$

and the generalized states

$$\tilde{\omega}_a(.) \equiv E_{a+1}\cdots E_K(f_{a+1}\cdots f_K\omega(.)), \tag{13}$$

the whole in complete analogy with the 'broken prescriptions' [21].

Of course, the corresponding replicated states Ω_a are immediately generalized with respect to each of the ω_a states introduced above.

Overall, for $a=0,\ldots,K$, we further need the averages

$$\langle . \rangle_a \equiv E\left(f_1\cdots f_a\tilde{\Omega}_a(.)\right). \tag{14}$$

J. Stat. Mech. (2010) P09006

While it is clear that, when evaluated at $t=\beta^2$ and $\mathbf{x}=0$, our interpolating function $\tilde{\alpha}(t,\mathbf{x})$ reproduces the definition of the quenched free energy, when evaluated at $t=0$ (which is a proper choice for the $\mathbf{x}$ parameters that we are going to show), it reproduces the Parisi trial solution $f(q=0,y=h)$ at the given K level of RSB:

$$\tilde{\alpha}_N(t=0;x_1,\ldots,x_K)=\frac{1}{N}E_0 \times \log\left[E_1\cdots\left[E_K\left(\sum_\sigma \exp\left(\sum_{a=1}^K\sqrt{x_a}\sum_i J_i^a\sigma_i\right)\right)^{m_K}\right]^{1/m_K}\cdots\right]^{1/m_1}. \quad (15)$$

Even though this is far from being trivial, this is an essential feature of mean-field behavior even in the disordered framework; in fact, in the thermodynamic limit the connected correlation inside pure states should go to zero, bridging the two-body problem to a (collection of) one-body model, or better ‘high temperature model’, whose partition function factorizes:

$$\sum_\sigma \exp\left(\sum_{a=1}^K\sqrt{x_a}\sum_i J_i^a\sigma_i\right)=2^N\prod_i\cosh\left(\sum_{a=1}^K\sqrt{x_a}J_i^a\right), \quad (16)$$

such that, averaging over J_i^K, we get

$$E_K\left(\sum_\sigma \exp\left(\sum_{a=1}^K\sqrt{x_a}\sum_i J_i^a\sigma_i\right)\right)^{m_K}=2^{Nm_K}\prod_i\int \mathrm{d}\,\mu(z_K)\cosh^{m_K} \times\left(\sum_{a=1}^{K-1}\sqrt{x_a}J_i^a+z_K\sqrt{x_K}\right), \quad (17)$$

and so on. Even taking the external field h, which is again encoded in a single-body interaction and is simply added into the hyperbolic cosine, we get

$$\tilde{\alpha}_N(t=0;x_1,\ldots,x_K)=\log 2 + \log\left[\int \mathrm{d}\mu(z_1)\cdots\left[\int \mathrm{d}\mu(z_K)\cosh^{m_K}\left(\sum_{a=1}^K\sqrt{x_a}z_a+\beta h\right)\right]^{1/m_K}\cdots\right]^{1/m_1}.$$

In the case where $x_a=\beta^2(q_a-q_{a-1})$ the second term does coincide sharply with the solution of the Parisi equation [28].

Let us now define $S(t,\mathbf{x})$ as the principal Hamilton function (PHF) for our problem:

$$S(t;x_1,\ldots,x_K)=2\left(\alpha(t;x_1,\ldots,x_K)-\tfrac{1}{2}\sum_{a=1}^K x_a-\tfrac{1}{4}t\right). \quad (18)$$

As proved in the appendix, the (x,t)-streaming of $S(t;x_1,\ldots,x_K)$ is then

$$\partial_t S(t;x_1,\ldots,x_K)=-\tfrac{1}{2}\sum_{a=0}^K(m_{a+1}-m_a)\langle q_{12}^2\rangle_a, \quad (19)$$

$$\partial_a S(t;x_1,\ldots,x_K)=-\tfrac{1}{2}\sum_{b=a}^K(m_{b+1}-m_b)\langle q_{12}\rangle_b. \quad (20)$$

It is then possible to introduce a Hamilton–Jacobi structure for $S(x,t)$, which implicitly defines a potential $V(t;x_1,\ldots,x_K)$, so we can write

$$\partial_t S(t,\mathbf{x}) + \tfrac{1}{2}\sum_{a,b=1}^{K}\partial_a S\,(M^{-1})_{ab}\,\partial_b S + V(t,\mathbf{x}) = 0. \tag{21}$$

The kinetic term is then

$$\begin{aligned} T &\equiv \tfrac{1}{2}\sum_{a,b=1}^{K}\partial_a S(t;x_1,\ldots,x_K)\,(M^{-1})_{ab}\,\partial_b S(t;x_1,\ldots,x_K) \\ &= \tfrac{1}{2}\sum_{a,b=1}^{K}(M^{-1})_{ab}\sum_{c\ge a}^{K}\sum_{d\ge b}^{K}(m_{c+1}-m_c)\langle q_{12}\rangle_c(m_{d+1}-m_d)\langle q_{12}\rangle_d \\ &= \tfrac{1}{2}\sum_{c,d=1}^{K}D_{cd}\,(m_{c+1}-m_c)\langle q_{12}\rangle_c(m_{d+1}-m_d)\langle q_{12}\rangle_d, \end{aligned} \tag{22}$$

where we defined

$$D_{cd} \equiv \sum_{a=1}^{c}\sum_{b=1}^{d}(M^{-1})_{ab}. \tag{23}$$

By the inversion of the mass matrix

$$D_{cd}(m_{c+1}-m_c) = \delta_{cd} \tag{24}$$

we obtain the expression

$$T = \tfrac{1}{2}\sum_{c=1}^{K}(m_{c+1}-m_c)\langle q_{12}\rangle_c^2 \tag{25}$$

$$= \tfrac{1}{2}\sum_{c=0}^{K}(m_{c+1}-m_c)\langle q_{12}\rangle_c^2 - \tfrac{1}{2}(m_1-m_0)\langle q_{12}\rangle_0^2. \tag{26}$$

Condition (24) determines the elements of the inverse of the mass matrix M^{-1}.

In particular, we stress that it is symmetric and the non-zero values are only on the diagonal and all of them respecting $(M^{-1})_{a,a+1} = (M^{-1})_{a+1,a}$:

$$(M^{-1})_{11} = \frac{1}{m_2-m_1}, \tag{27}$$

$$(M^{-1})_{a,a} = \frac{1}{m_{a+1}-m_a} + \frac{1}{m_a-m_{a-1}}, \tag{28}$$

$$(M^{-1})_{a,a+1} = -\frac{1}{m_{a+1}-m_a}, \tag{29}$$

all the others being zero.

J. Stat. Mech. (2010) P09006

The elements of the mass matrix M are determined by the equation

$$\sum_b M_{ab}(M^{-1})_{bc} = \delta_{ac}, \tag{30}$$

and it is immediate to verify that the following representation holds:

$$M_{ab} = 1 - m_{(a\wedge b)}. \tag{31}$$

With this expression for the matrix elements, by substituting equations (19) and (26) into (21) we obtain the expression for the potential such that overall

$$\begin{aligned} &\partial_t S(t; x_1, \ldots, x_K) + \tfrac{1}{2}\sum_{a,b=1}^{K} \partial_a S\,(M^{-1})_{ab}\,\partial_b S + V(t; x_1, \ldots, x_K) = 0, \\ &V(t; x_1, \ldots, x_K) = \tfrac{1}{2}\sum_{a=0}^{K}(m_{a+1} - m_a)(\langle q_{12}^2\rangle_a - \langle q_{12}\rangle_a^2) + \tfrac{1}{2}(m_1 - m_0)\langle q_{12}\rangle_0^2. \end{aligned} \tag{32}$$

Once the mechanical analogy is built, it is, however, prohibitive solving the problem as it is (i.e. integrating the equations of motion); instead we propose an iterative scheme that mirrors the replica symmetry breaking one: at first, by choosing $K = 1$, we solve the free-field solution (we impose $V(t, \mathbf{x}) = 0$) and we recover the annealed expression for the free energy. This is consistent with neglecting the potential as it turns out to be the squared overlap. Then, we avoid the perturbation scheme to deal with the source but we enlarge our Euclidean space by considering $K = 2$. Again we work out the free-field solution to obtain the replica symmetric expression for the free energy, consistent with neglecting the potential; in fact, the source we avoid this time is the variance of the overlap: a much better approximation with respect to $K = 1$. We go further explicitly by considering the $K = 3$ case and we get the 1-RSB solution in the same way (and so on). Interestingly we discover that there is a one-to-one connection among the steps of replica symmetry breaking in the replica trick and the Euclidean dimension in the broken replica mechanical analogy. The latter, however, incorporates, in a single scheme, even the annealed and the replica symmetry solutions.

4. $K = 1$, annealed free energy

Let us now recover some properties of disordered thermodynamics by studying the $K = 1$ case so to show how the solution of the free problem coincides with the annealed expression.

We assume $x(q) = m_1 = 1$ in the whole interval $[0, 1]$.

We show now that, within our approach, this implies a reduction in the degrees of freedom where the Hamilton–Jacobi action lives, such that the PHF depends on t only.

The dynamics involves a $1 + 1$ Euclidean space–time such that

$$Z_1 \equiv Z_K \equiv \tilde{Z}_N \equiv \sum_\sigma \exp(\sqrt{t/N}H_N(\sigma; J) + \sqrt{x}\sum_i J_i\sigma_i). \tag{33}$$

Z_0 is consequently given by

$$Z_0 \equiv E_1 Z_1 = \exp\left(\frac{N}{2}x\right)\sum_\sigma \exp(\sqrt{t/N}H_N(\sigma; J)). \tag{34}$$

This implies, in the interpolating function, a linear and separate dependence by the x:

$$\tilde{\alpha}(t,x) = \frac{x}{2} + \frac{1}{N}E_0 \log \sum_{\sigma} \exp(\sqrt{t/N} H_N(\sigma;J)). \tag{35}$$

The x derivative of $\tilde{\alpha}(t,\mathbf{x})$ is immediate, while for the t one we can use the general expression previously obtained (cf equations (19) and (20)):

$$\partial_t \tilde{\alpha} = \tfrac{1}{4}[1 - \langle q_{12}^2 \rangle_0], \tag{36}$$

$$\partial_x \tilde{\alpha} = \tfrac{1}{2}. \tag{37}$$

As a straightforward but interesting consequence, PHF does not depend on x and we get

$$\begin{aligned} S(t,x) &= 2\tilde{\alpha}(t,x) - x - \frac{t}{2} \\ &= \frac{2}{N}E_0 \log \sum_{\sigma} \exp(\sqrt{t/N} H_N(\sigma;J)) - \frac{t}{2}, \end{aligned} \tag{38}$$

$$\partial_t S = -\tfrac{1}{2}\langle q_{12}^2 \rangle_0, \tag{39}$$

$$\partial_x S \equiv v(t,x) = 0, \tag{40}$$

where $v(t)$ defines the velocity field, which is identically zero such that $x(t) \equiv x_0$.

In this simplest case, the potential is trivially the t derivative of $S(t,\mathbf{x})$ with a change in the sign (the averaged squared overlap):

$$V(t,\mathbf{x}) = \tfrac{1}{2}\langle q_{12}^2 \rangle_0. \tag{41}$$

Now we want to deal with the solution of the statistical mechanics problem. As we neglect the source (we are imposing $\langle q_{12}^2 \rangle_0 = 0$), we can take the initial value for $S(x,t)$ as it must be constant over all spacetime:

$$\bar{S} = S(0) = 2\log 2, \tag{42}$$

and, consequently, we can write the solution of the problem as

$$\bar{\alpha}(t,x) = \log 2 + \frac{x}{2} + \frac{t}{4}. \tag{43}$$

At this point it is straightforward to obtain the statistical mechanics by posing $t = \beta^2$ and $x = 0$:

$$\alpha_N(\beta) = \log 2 + \frac{\beta^2}{4}, \tag{44}$$

which is exactly the annealed free energy.

J. Stat. Mech. (2010) P09006

5. $K = 2$, replica symmetric free energy

In this section, by adding another degree of freedom to our mechanical analogy, we want to reproduce the replica symmetric solution of the statistical mechanics problem.

We deal with $K = 2$. The order parameter is now taken as

$$x(q) = x_{\bar{q}}(q) = \begin{cases} 0 & \text{if } \ q \in [0, \bar{q}), \\ 1 & \text{if } \ q \in [\bar{q}, 1]. \end{cases} \tag{45}$$

So

$$q_1 = \bar{q}, \qquad q_2 = q_K \equiv 1 \tag{46}$$

$$m_0 = m_1 = 0, \qquad m_2 = m_K = 1, \qquad m_3 = m_{K+1} \equiv 1. \tag{47}$$

The auxiliary partition function depends on t and on the two spatial coordinates x_1 and x_2:

$$\tilde{Z}_N(t; x_1, x_2) \equiv \sum_\sigma \exp\left(\sqrt{t/N} H_N(\sigma; J) + \sqrt{x_1} \sum_i J_i^1 \sigma_i + \sqrt{x_2} \sum_i J_i^2 \sigma_i\right), \tag{48}$$

and with the latter, recursively, we obtain Z_0:

$$Z_K \equiv Z_2 \equiv \tilde{Z}_N, \tag{49}$$

$$Z_1 \equiv (E_2 Z_2^{m_2})^{1/m_2} = E_2 Z_2, \tag{50}$$

$$Z_0 = (E_1 Z_1^{m_1})^{1/m_1}. \tag{51}$$

The function Z_1 can be immediately evaluated by standard Gaussian integration as

$$Z_1 = \exp\left(N \frac{x_2}{2}\right) \sum_\sigma \exp\left(\sqrt{t/N} H_N(\sigma; J) + \sqrt{x_1} \sum_i J_i^1 \sigma_i\right). \tag{52}$$

Concerning the function Z_0 we can write

$$\begin{aligned} (E_1 Z_1^{m_1})^{1/m_1} &= \exp\left[\frac{1}{m_1} \log E_1[\exp(m_1 \log Z_1)]\right] \\ &= \exp\left[\frac{1}{m_1} \log E_1[1 + m_1 \log Z_1 + o(m_1^2)]\right] \\ &= \exp\left[\frac{1}{m_1}[m_1 E_1 \log Z_1 + o(m_1^2)]\right] \\ &= \exp E_1 \log Z_1 + o(m_1), \end{aligned}$$

and consequently

$$Z_0 = \exp E_1 \log Z_1. \tag{53}$$

In this case, our interpolating function is

$$\tilde{\alpha}(t, x_1, x_2) = \frac{x_2}{2} + \frac{1}{N} E_0 E_1 \log\left[\sum_\sigma \exp\left(\sqrt{t/N} H_N(\sigma; J) + \sqrt{x_1} \sum_i J_i^1 \sigma_i\right)\right]. \tag{54}$$

Again by using the general formulae sketched in the first section (cf equations (19) and (20)) we get for the derivatives

$$\partial_t \tilde{\alpha} = \tfrac{1}{4}[1 - \langle q_{12}^2 \rangle_1], \tag{55}$$

$$\partial_{x_1} \tilde{\alpha} = \tfrac{1}{2}[1 - \langle q_{12} \rangle_1], \tag{56}$$

$$\partial_{x_2} \tilde{\alpha} = \tfrac{1}{2}. \tag{57}$$

Evaluating our function at $t = 0, x_1 = x_1^0, x_2 = x_2^0$ we easily find

$$\tilde{\alpha}(0; x_1^0, x_2^0) = \frac{x_2^0}{2} + \log 2 + \int \mathrm{d}\mu(z) \log\cosh\left(\sqrt{x_1^0}\, z\right). \tag{58}$$

Let us introduce now the $K = 2$ PHF:

$$S(t; x_1, x_2) = 2\left(\tilde{\alpha} - \frac{x_1}{2} - \frac{x_2}{2} - \frac{t}{4}\right), \tag{59}$$

together with its derivatives

$$\partial_t S = -\tfrac{1}{2}\langle q_{12}^2 \rangle_1, \tag{60}$$

$$\partial_{x_1} S = v_1(t, x_1) = -\langle q_{12} \rangle_1, \tag{61}$$

$$\partial_{x_2} S = 0. \tag{62}$$

We observe that, even in this case, there is no true dependence by one of the spatial variables (x_2): this is due to the constant value of the last interval $m_K = m_2$ where the order parameter equals one and can be Gaussian-integrated out immediately into the corresponding Z_2, getting the pre-factor $\exp(\frac{1}{2}Nx_2^0)$.

As a consequence, we can forget the mass matrix as there is no true multidimensional space.

Let us write down the Hamilton–Jacobi equation:

$$\partial_t S(t, x_1) + \tfrac{1}{2}(\partial_{x_1} S(t, x_1))^2 + V(t, x_1) = 0. \tag{63}$$

The potential is given by the function

$$V(t, x_1) = \tfrac{1}{2}(\langle q_{12}^2 \rangle_1 - \langle q_{12} \rangle_1^2), \tag{64}$$

where

$$\langle q_{12}^2 \rangle_1 = E_0 E_1 f_1 \Omega_1(q_{12}^2) = E_0 E_1 f_1 \frac{1}{N^2} \sum_{ij} (E_2 f_2 \omega(\sigma_i \sigma_j))^2. \tag{65}$$

When taking $x_1 = 0$ and $t = \beta^2$ the variance of the overlap becomes the source of the streaming:

$$V(\beta^2, 0) = \tfrac{1}{2}(\langle q_{12}^2 \rangle - \langle q_{12} \rangle^2). \tag{66}$$

As usual in our framework, we kill the source (i.e. $V(t, \mathbf{x}) = 0$) and obtain for the velocity

$$\bar{q}(x_1^0) \equiv -v_1(0, x_1^0) = \int \mathrm{d}\mu(z) \tanh^2(z\sqrt{x_1^0}). \tag{67}$$

This is the well-known self-consistency relation of Sherrington and Kirkpatrick, namely

$$\bar{q}(\beta) = \int \mathrm{d}\mu(z)\ \tanh^2(\beta\sqrt{\bar{q}}z). \tag{68}$$

The free-field solution of the Hamilton–Jacobi equation is then the solution in a particular point (and of course the choice is $\bar{S}(0, x_1^0)$ which requires only a one-body evaluation) plus the integral of the Lagrangian over time (which is trivially built by the kinetic term alone when considering free propagation). Overall the solution is

$$\bar{S}(t, x_1) = \bar{S}(0, x_1^0) + \tfrac{1}{2}\bar{q}^2(x_1^0)t, \tag{69}$$

by which statistical mechanics is recovered as usual, obtaining for the pressure

$$\bar{\alpha}(t; x_1, x_2) = \log 2 + \int \mathrm{d}\mu(z)\ \log\cosh\left(\sqrt{x_1^0}z\right) + \frac{t}{4}(1-\bar{q})^2 + \frac{x_2}{2}, \tag{70}$$

which corresponds exactly to the replica symmetric solution once evaluated at $x_1 = x_2 = 0$ and $t = \beta^2$ and noticing that $0 = x(t) = x_1^0 - \bar{q}t$.

Within our description it is not surprising that the replica symmetric solution is a better description with respect to the annealing. In fact, while annealing is obtained by neglecting the whole squared overlap $\langle q_{12}^2\rangle$ as a source term, the replica symmetric solution is obtained when neglecting only its variance.

Of course, neither the former nor the latter may correspond to the true solution. However, we understand that increasing the Euclidean dimensions (the RSB steps in the replica framework) corresponds to lessening the potential in the Hamilton–Jacobi framework and consequently reducing the error of the free-field approximation towards the true solution.

6. $K = 3$, 1-RSB free energy

The simplest expression of $x(q)$ which breaks replica symmetry is obtainable when considering $K = 3$:

$$0 = q_0 < q_1 < q_2 < q_3 = 1, \tag{71}$$

$$0 = m_1 < m_2 \equiv m < m_3 = 1. \tag{72}$$

With this choice for the parametrization of $x(q)$ the solution of the Parisi equation

$$\partial_q f + \tfrac{1}{2}\partial_y^2 f + \tfrac{1}{2}x(\partial_y f)^2 = 0 \tag{73}$$

is given by

$$f(0, h; x, \beta) = \frac{1}{m}\int \mathrm{d}\mu(z_1)\log\int \mathrm{d}\mu(z_2)\cosh^m[\beta(\sqrt{q_1}z_1 + \sqrt{q_2 - q_1}z_2 + h)] + \tfrac{1}{2}\beta^2(1 - q_2), \tag{74}$$

J. Stat. Mech. (2010) P09006

and, using a label P to emphasize that we are considering the Parisi prescription, the pressure becomes

$$\begin{aligned}\alpha_P(\beta,h;x) &= \log 2 + f(0,h;x,\beta) - \tfrac{1}{2}\beta^2\int_0^1 q\,x(q)\,\mathrm{d}q \\ &= \log 2 - \tfrac{1}{4}\beta^2[(m-1)q_2^2 - 1 - mq_1^2 + 2q_2] \\ &\quad + \frac{1}{m}\int \mathrm{d}\mu(z_1)\log\int \mathrm{d}\mu(z_2)\cosh^m[\beta(\sqrt{q_1}z_1+\sqrt{q_2-q_1}z_2+h)].\end{aligned} \tag{75}$$

Now we want to see how it is possible to obtain this solution by analyzing the geodetics of our free mechanical propagation in $3+1$ dimensions.

Let us define

$$\tilde{Z}_N(t;x_1,x_2,x_3) \equiv \sum_\sigma \exp\left[\sqrt{t/N}H_N(\sigma;J) + \sum_{a=1}^{3}\sqrt{x_a}\sum_i J_i^a\sigma_i\right], \tag{76}$$

by which

$$Z_3 \equiv Z_K \equiv \tilde{Z}_N, \tag{77}$$

$$Z_2 = E_3 Z_3 = \exp\left(\frac{Nx_3}{2}\right)\sum_\sigma \exp\left[\sqrt{t/N}H_N(\sigma;J) + \sum_{a=1}^{2}\sqrt{x_a}\sum_i J_i^a\sigma_i\right], \tag{78}$$

$$Z_1 = (E_2 Z_2^m)^{1/m}, \tag{79}$$

$$Z_0 = (E_1 Z_1^{m_1})^{1/m_1} = \exp(E_1\log Z_1) = \exp\left[\frac{1}{m}E_1\log E_2 Z_2^m\right]. \tag{80}$$

For the interpolating function we get in this way

$$\begin{aligned}\tilde{\alpha}_N(t;x_1,x_2,x_3) &\equiv \frac{1}{N}E_0\log Z_0 = \frac{x_3}{2} + \frac{1}{Nm}E_0E_1 \\ &\quad\times\ \log\left\{E_2\left[\sum_\sigma \exp\left(\sqrt{t/N}H_N(\sigma;J) + \sum_{a=1}^{2}\sqrt{x_a}\sum_i J_i^a\sigma_i\right)\right]^m\right\},\end{aligned} \tag{81}$$

while for the derivatives we can use the general formulae so as to obtain

$$\partial_t\tilde{\alpha} = \tfrac{1}{4}[1 - m\langle q_{12}^2\rangle_1 - (1-m)\langle q_{12}^2\rangle_2], \tag{82}$$

$$\partial_1\tilde{\alpha} = \tfrac{1}{2}[1 - m\langle q_{12}\rangle_1 - (1-m)\langle q_{12}\rangle_2], \tag{83}$$

$$\partial_2\tilde{\alpha} = \tfrac{1}{2}[1 - (1-m)\langle q_{12}\rangle_2], \tag{84}$$

$$\partial_3\tilde{\alpha} = \tfrac{1}{2}. \tag{85}$$

Then we need to evaluate the interpolating function at the starting time:

$$\begin{aligned}\tilde{\alpha}_N(0;x_1^0,x_2^0,x_3^0) &= \frac{x_3}{2} + \log 2 \\ &\quad + \frac{1}{m}\int \mathrm{d}\mu(z_1)\log\left[\int \mathrm{d}\mu(z_2)\cosh^m\left(\sqrt{x_1^0}z_1 + \sqrt{x_2^0}z_2\right)\right].\end{aligned} \tag{86}$$

J. Stat. Mech. (2010) P09006

The $K = 3$ PHF, as usual and previously explained for the $K = 1, 2$ cases, does not depend on the last coordinate (i.e. x_3), such that we can ignore it when studying the properties of the solution:

$$S(t; x_1, x_2) = \frac{2}{Nm} E_0 E_1 \log E_2 \left[\sum_\sigma \exp\left(\sqrt{tN/2} K(\sigma) + \sum_{a=1}^{2} \sqrt{x_a} \sum_i J_i^a \sigma_i \right) \right]^m - x_1 - x_2 - t/2. \quad (87)$$

and the derivatives, implicitly defining the momenta (labeled by p_1, p_2), are given by

$$\partial_t S = -\frac{m}{2} \langle q_{12}^2 \rangle_1 - \frac{1-m}{2} \langle q_{12}^2 \rangle_2, \quad (88)$$

$$\partial_1 S \equiv p_1(t; x_1, x_2) = -m \langle q_{12} \rangle_1 - (1-m) \langle q_{12} \rangle_2, \quad (89)$$

$$\partial_2 S \equiv p_2(t; x_1, x_2) = -(1-m) \langle q_{12} \rangle_2. \quad (90)$$

The kinetic energy consequently turns out to be

$$T = \frac{m}{2} \langle q_{12} \rangle_1^2 + \frac{1-m}{2} \langle q_{12} \rangle_2^2, \quad (91)$$

and the potential, which we are going to neglect as usual, is given by

$$V(t; x_1, x_2) = \tfrac{1}{2} [m(\langle q_{12}^2 \rangle_1 - \langle q_{12} \rangle_1^2) + (1-m)(\langle q_{12}^2 \rangle_2 - \langle q_{12} \rangle_2^2)]. \quad (92)$$

By having two spatial degrees of freedom, the mass matrix has a 2×2 structure now:

$$M^{-1} = \begin{pmatrix} 1/m & -1/m \\ -1/m & 1/[m(1-m)] \end{pmatrix}, \quad (93)$$

$$M = \begin{pmatrix} 1 & 1-m \\ 1-m & 1-m \end{pmatrix}. \quad (94)$$

Note that the eigenvalues of the mass matrix are always positive defined for $m \in [0, 1]$.

We can determine now the velocity field

$$v_1(t; x_1, x_2) = \sum_{b=1}^{2} (M^{-1})_{1b} \, p_b = -\langle q_{12} \rangle_1, \quad (95)$$

$$v_2(t; x_1, x_2) = \sum_{b=1}^{2} (M^{-1})_{2b} \, p_b = \langle q_{12} \rangle_1 - \langle q_{12} \rangle_2. \quad (96)$$

So we get all the ingredients for studying the free-field solution (the one we get neglecting the source). In this case the equations of motion are

$$x_1(t) = x_1^0 - \langle q_{12} \rangle_1 (0; x_1^0, x_2^0) \, t \equiv x_1^0 - \bar{q}_1 t \quad (97)$$

$$x_2(t) = x_2^0 + \left(\bar{q}_1 - \langle q_{12} \rangle_1 (0; x_1^0, x_2^0) \right) \, t \equiv x_2^0 + (\bar{q}_1 - \bar{q}_2) t \quad (98)$$

J. Stat. Mech. (2010) P09006

and we can see that $\bar{q}_1$ and $\bar{q}_2$ satisfy the self-consistency relations in agreement with the replica trick predictions:

$$q_1 = \int \mathrm{d}\mu(z) \left[D^{-1}(z) \int \mathrm{d}\mu(y) \cosh^m \theta(z,y) \tanh \theta(z,y) \right]^2, \tag{99}$$

$$q_2 = \int \mathrm{d}\mu(z) \left[D^{-1}(z) \int \mathrm{d}\mu(y) \cosh^m \theta(z,y) \tanh^2 \theta(z,y) \right], \tag{100}$$

$$\theta(z,y) = \beta(\sqrt{q_1} z + \sqrt{q_2 - q_1} y), \tag{101}$$

$$D(z) = \int \mathrm{d}\mu(y) \cosh^m \theta(z,y). \tag{102}$$

The PHF is obtained in coherence with the previous cases and obeys

$$\bar{S}(t; x_1, x_2) = \bar{S}(0; x_1^0, x_2^0) + T(0; x_1^0, x_2^0)t, \tag{103}$$

by which

$$\bar{\alpha}(t; x_1, x_2, x_3) - \frac{x_1}{2} - \frac{x_2}{2} - \frac{t}{4} = \bar{\alpha}(0; x_1^0, x_2^0, x_3^0) - \frac{x_1^0}{2} - \frac{x_2^0}{2} + T(0; x_1^0, x_2^0)\frac{t}{2}, \tag{104}$$

and, remembering that

$$x_1 - x_1^0 = -\bar{q}_1 t, \tag{105}$$

$$x_2 - x_2^0 = (\bar{q}_1 - \bar{q}_2)t, \tag{106}$$

we get the thermodynamic pressure in the space–time coordinates:

$$\begin{aligned} \bar{\alpha}(t; x_1, x_2, x_3) &= \frac{x_3}{2} + \log 2 - \frac{t}{4}[-1 + 2\bar{q}_2 - m\bar{q}_1^2 - (1-m)\bar{q}_2^2] \\ &+ \frac{1}{m} \int \mathrm{d}\mu(z_1) \log \left[\int \mathrm{d}\mu(z_2) \cosh^m \left[\sqrt{x_1^0} z_1 + \sqrt{x_2^0} z_2 \right] \right]. \end{aligned} \tag{107}$$

In order to get the statistical mechanics result, as usual, we need to evaluate the latter in $t = \beta^2$, $x_1 = x_2 = x_3 = 0$, from which $x_1^0 = \bar{q}_1 t$ and $x_2^0 = (\bar{q}_1 - \bar{q}_2)t$, gaining once again (75).

7. Properties of the $K = 1, 2, 3$ free energies

In the previous sections, we obtained solutions for the Hamilton–Jacobi equation in the $K = 1, 2, 3$ cases without saying anything about uniqueness. For $K = 1$, the annealed case, there is no true motion so it is clear that there is just a single straight trajectory, identified by the initial point $x_0 = x$, intersecting the generic point (x, t), with $x, t > 0$.

In the $K = 2$ problem, well studied in [20], one can show uniqueness by observing that the function $t(x_0)$, representing the point at which the trajectory intersects the x axis, is a monotone increasing one of the initial point x_0, so that given $x, t > 0$, there is a unique point x_0 (and velocity $\bar{q}(x_0)$, of course) from which the trajectory starts.

For $K = 3$, the problem becomes much complicated, because we now have to consider motion in a three-dimensional Euclidean space, proving that given the generic point

J. Stat. Mech. (2010) P09006

(x_1, x_2, t), with $x_1 > 0$, $x_2 > 0$, $t > 0$, there exists a unique line passing in (x_1, x_2) at time t. So let us consider the functions

$$F(x_1, t; x_1^0, x_2^0) \equiv x_1 - x_1^0 + \bar{q}_1(x_1^0, x_2^0)t, \tag{108}$$

$$G(x_2, t; x_1^0, x_2^0) \equiv x_2 - x_2^0 + \bar{q}_2(x_1^0, x_2^0)t - \bar{q}_1(x_1^0, x_2^0)t. \tag{109}$$

These functions vanish in the points corresponding to the solutions of the equations of motion, and in particular for all the $A_t \equiv (x_1 = 0, x_2 = 0, t > 0; x_1^0 = 0, x_2^0 = 0)$. Labeling with ∂_1 and ∂_2 the partial derivatives with respect to x_1^0 and x_2^0, the Dini prescription tells us that, if the determinant of the Hessian matrix

$$\frac{\partial(F, G)}{\partial(x_1^0, x_2^0)} = \begin{vmatrix} \partial_1 F & \partial_2 F \\ \partial_1 G & \partial_2 G \end{vmatrix} \tag{110}$$

is different from zero in a neighborhood of A_t, then we can explicate x_1^0 and x_2^0 as functions of x_1, x_2 and t in such a neighborhood. This means that the initial point and the velocities, which depend on it, are univocally determined by x_1, x_2 and t via the equations of motion.

Calculating the determinant we find

$$\begin{aligned} \frac{\partial(F, G)}{\partial(x_1^0, x_2^0)} &= (-1 + \partial_1\bar{q}_1 t)(-1 + \partial_2\bar{q}_2 t - \partial_2\bar{q}_1 t) - (\partial_2\bar{q}_1 t)(+\partial_1\bar{q}_2 t - \partial_1\bar{q}_1 t) \\ &= 1 + (\partial_2\bar{q}_1 - \partial_1\bar{q}_1 - \partial_2\bar{q}_2)t + (\partial_1\bar{q}_1\partial_2\bar{q}_2 - \partial_2\bar{q}_1\partial_1\bar{q}_2)t^2. \end{aligned} \tag{111}$$

so we should ask, for all $x_1^0 > 0$ and $x_2^0 > 0$:

$$\Delta \equiv (\partial_2\bar{q}_1 - \partial_1\bar{q}_1 - \partial_2\bar{q}_2)^2 - 4(\partial_1\bar{q}_1\partial_2\bar{q}_2 - \partial_2\bar{q}_1\partial_1\bar{q}_2) \tag{112}$$

to be negative, or in the case $\Delta \geq 0$, the zeros of

$$t_\pm = \frac{-\partial_2\bar{q}_1 + \partial_1\bar{q}_1 + \partial_2\bar{q}_2 \pm \sqrt{\Delta}}{2(\partial_1\bar{q}_1\partial_2\bar{q}_2 - \partial_2\bar{q}_1\partial_1\bar{q}_2)} \tag{113}$$

correspond to non-invertibility points.

The expression we obtain for the determinant is quite intractable. However, we can show uniqueness in a neighborhood of the initial point $x_1^0 = 0$, $x_2^0 = 0$. The motion starting from this point has zero velocity and we saw that it gives the high temperature solution for the mean-field spin glass model. Remembering that the transition to low temperature is continuous, we can expand the Hessian for small values of x_1^0 and x_2^0 and observe that, for $x_1 = 0$, $x_2 = 0$, $t = \beta^2$, the equations of motions become

$$x_1^0 = \beta^2\bar{q}_1 \tag{114}$$

$$x_2^0 = \beta^2(\bar{q}_2 - \bar{q}_1). \tag{115}$$

When $x_1^0 \to 0$ and $x_2^0 \to 0$ we have also $\bar{q}_1 \to 0$ and $\bar{q}_2 \to 0$, so we have an expansion close to the critical point (which is the only region where the control of the unstable 1-RSB solution makes sense for the SK, the latter being ∞-RSB).

For $\bar{q}_1$ and $\bar{q}_2$ we have, retaining terms until the second order:

$$\bar{q}_1(x_1^0, x_2^0) \approx x_1^0 - 2(1 - m)x_1^0x_2^0 - 2(x_1^0)^2 \tag{116}$$

$$\bar{q}_2(x_1^0, x_2^0) \approx x_1^0 + x_2^0 + mx_2^0(x_2^0 + 2x_1^0) \tag{117}$$

J. Stat. Mech. (2010) P09006

and consequently

$$\partial_1 \bar{q}_1(x_1^0, x_2^0) \approx 1 - 2(1-m)x_2^0 - 4x_1^0 \tag{118}$$

$$\partial_2 \bar{q}_1(x_1^0, x_2^0) \approx -2(1-m)x_1^0 \tag{119}$$

$$\partial_1 \bar{q}_2(x_1^0, x_2^0) \approx 1 + 2mx_2^0 \tag{120}$$

$$\partial_2 \bar{q}_2(x_1^0, x_2^0) \approx 1 + 2mx_1^0 + 2mx_2^0. \tag{121}$$

Substituting in (111) we find

$$\begin{aligned}\frac{\partial(F,G)}{\partial(x_1^0,x_2^0)} \approx {} & 1 - 2[1 - x_1^0 - (1-2m)x_2^0]t \\ & + [1 - (2-m)x_1^0 - 2(1-2m)x_2^0 + 2m(m-4)x_1^0x_2^0 + \\ & - 8m(x_1^0)^2 - 4m(1-m)(x_2^0)^2]t^2\end{aligned} \tag{122}$$

and, for $x_1^0, x_2^0 = 0$ (which corresponds to expand the velocities up to first order in x_1^0 and x_2^0), we simply obtain

$$\frac{\partial(F,G)}{\partial(x_1^0,x_2^0)} \approx (1-t)^2. \tag{123}$$

This means that in a neighborhood of $x_1 = x_2 = 0$ we have uniqueness, provided that we are not exactly at the critical point $t = \beta^2 = 1$.

8. Outlook and conclusions

In this paper we pioneered one step forward with respect to the previously investigated Hamilton–Jacobi structure for free energy in the thermodynamics of complex systems (tested on the paradigmatic SK model). This has been achieved by merging this approach with the broken replica symmetry bound technique.

At the mathematical level the main achievement is the development of a new method which is autonomously able to give the various steps of replica symmetry breaking (of the replica trick counterpart). At the actual level, our method can be thought of as a 'trick' for deriving the various steps of RSB, as an alternative to the replica trick or the cavity fields. For instance, when working with the replica trick, annealing is a regime far from being incorporated 'naturally' in the RSB mechanism; instead, in our approach, it is simply the 'zero level' of approximation of a single scheme which merges the various approximations in a unifying framework. Then, what can be proven within this formalism—at this stage—are the various upper bounds of the corresponding K levels of RSB with respect to the true free energy (full RSB)—this is essentially because this version of the Hamilton–Jacobi streaming implicitly incorporates the broken replica symmetry bounds. Surely the limit of K diverging is expected to offer the true solution (the full Parisi scheme). However, rigorous control is still lacking nowadays. Surely future research by our staff will be developed in that direction. However, the method itself deserves attention because there are several models apart from the paradigmatic SK which are finite-RSB (i.e. from the—still pedagogical—Gardner P-spin to the whole plethora of applications in quantitative biology or computer science which actually are strongly based on cavity and belief propagation), to which the method can be directly applied.

At a physical level this method highlights an alternative perspective by which we understand that increasing the steps of RSB improves the achieved approximating thermodynamics; this is achieved by mirroring these increments in diminishing the approximation of a free-field propagation in a Euclidean spacetime for an extended free energy, which recovers the proper one of statistical mechanics as a particular, well-defined, limit.

However, when increasing the steps of RSB (and so making smaller the potential that we neglect, and so, the smaller the error) there is a price to pay: each step of replica symmetry breaking enlarges by one dimension the space for the motion of the mechanical action. As a consequence the full RSB theory should live on some infinite-dimensional space which deserves more analysis.

We need to investigate both the $K \to \infty$ limit to complete the theory as well as its immediate applications, primarily diluted systems.

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Appendix. Streaming of the interpolating function $\tilde{\alpha}(t, \mathbf{x})$

In this appendix we show in full detail how to get the streaming of the interpolating function (11).

The t-streaming of the interpolating function $\tilde{\alpha}(t, \mathbf{x})$ is given by the following formula:

$$\partial_t \tilde{\alpha}_N(\mathbf{x}, t) = \tfrac{1}{4}\left(1 - \sum_{a=0}^{K}(m_{a+1} - m_a)\langle q_{12}^2(\mathbf{x}, t)\rangle_a\right). \tag{A.1}$$

To get this result, let us start by

$$\partial_t \tilde{\alpha}_N(\mathbf{x}, t) = \frac{1}{N} E_0 Z_0^{-1}(\mathbf{x}, t)\partial_t Z_0(\mathbf{x}, t), \tag{A.2}$$

and, as it is straightforward to show that

$$Z_a^{-1}(\mathbf{x}, t)\partial_t Z_a(\mathbf{x}, t) = E_{a+1}(f_{a+1} Z_{a+1}^{-1}(\mathbf{x}, t)\partial_t Z_{a+1}(\mathbf{x}, t)), \tag{A.3}$$

by iteration, we get

$$Z_0^{-1}(\mathbf{x}, t)\partial_t Z_0(\mathbf{x}, t) = E_1 \cdots E_K(f_1 \cdots f_K Z_K^{-1}(\mathbf{x}, t)\partial_t Z_K(\mathbf{x}, t)). \tag{A.4}$$

The t-derivative of Z_K is then given by

$$Z_K^{-1}(\mathbf{x},t)\partial_t Z_K(\mathbf{x},t) = \frac{1}{4\sqrt{tN}}\sum_{ij} J_{ij}\omega(\sigma_i\sigma_j), \tag{A.5}$$

from which

$$Z_0^{-1}(\mathbf{x},t)\partial_t Z_0(\mathbf{x},t) = \frac{1}{4\sqrt{tN}}\sum_{ij} E(f_1\cdots f_K J_{ij}\omega(\sigma_i\sigma_j)), \tag{A.6}$$

where we labeled with E the global average over all the random variables as there is no danger of confusion. All the terms in the sum can be worked out by integrating by parts:

$$\begin{aligned}E(f_1\cdots f_K J_{ij}\omega(\sigma_i\sigma_j)) = &\sum_{a=1}^{K} E(f_1\cdots \partial_{J_{ij}} f_a \cdots f_K\omega(\sigma_i\sigma_j))\\ &+ E(f_1\cdots f_K\partial_{J_{ij}}\omega(\sigma_i\sigma_j)).\end{aligned} \tag{A.7}$$

So we need to calculate the explicit expression of the derivatives with respect to J_{ij} of both f_a as well as $\omega(\sigma_i\sigma_j)$. For the latter, it is easy to check that

$$\partial_{J_{ij}}\omega(\sigma_i\sigma_j) = \sqrt{\frac{t}{N}}(1-\omega^2(\sigma_i\sigma_j)), \tag{A.8}$$

while for the f_as we have

$$\partial_{J_{ij}} f_a = m_a f_a\left(Z_a^{-1}(\mathbf{x},t)\partial_{J_{ij}} Z_a(\mathbf{x},t)\right) - m_a f_a E_a f_a\left(Z_a^{-1}(\mathbf{x},t)\partial_{J_{ij}} Z_a(\mathbf{x},t)\right). \tag{A.9}$$

By using the analogy of (A.3) we get

$$\begin{aligned}Z_a^{-1}(\mathbf{x},t)\partial_{J_{ij}} Z_a(\mathbf{x},t) &= E_{a+1}\dots E_K\left(f_{a+1}\dots f_K Z_K^{-1}\partial_{J_{ij}} Z_K\right)\\ &= \sqrt{\frac{t}{N}}\tilde{\omega}_a(\sigma_i\sigma_j),\end{aligned} \tag{A.10}$$

such that

$$\partial_{J_{ij}} f_a = m_a f_a\sqrt{\frac{t}{N}}(\tilde{\omega}_a(\sigma_i\sigma_j) - \tilde{\omega}_{a-1}(\sigma_i\sigma_j)). \tag{A.11}$$

Substituting (A.8) and (A.11) into (A.7) we obtain

$$\begin{aligned}E(f_1\dots f_K J_{ij}\omega(\sigma_i\sigma_j)) = &\sqrt{\frac{t}{N}}\sum_{a=1}^{K} m_a[E(f_1\cdots f_a\tilde{\omega}_a(\sigma_i\sigma_j)\dots f_K\omega(\sigma_i\sigma_j))\\ &- E(f_1\cdots f_{a-1}\tilde{\omega}_{a-1}(\sigma_i\sigma_j)\cdots f_K\omega(\sigma_i\sigma_j))]\\ &+ \sqrt{\frac{t}{N}}E(f_1\cdots f_K(1-\omega^2(\sigma_i\sigma_j))).\end{aligned} \tag{A.12}$$

Overall, an explicit expression for the equation (A.2) is given by

$$\begin{aligned}\partial_t\tilde{\alpha} = &\frac{1}{4N^2}\sum_{a=1}^{K}\sum_{ij} m_a[E_0\dots E_a f_1\cdots f_a\tilde{\omega}_a(\sigma_i\sigma_j)E_{a+1}\cdots E_K f_{a+1}\dots f_K\omega(\sigma_i\sigma_j)\\ &- E_0\cdots E_{a-1}f_1\cdots f_{a-1}\tilde{\omega}_{a-1}(\sigma_i\sigma_j)E_a\cdots E_K f_a\cdots f_K\omega(\sigma_i\sigma_j)]\\ &+ \frac{1}{4N^2}Ef_1\cdots f_K\sum_{ij}(1-\omega^2(\sigma_i\sigma_j)).\end{aligned} \tag{A.13}$$

Once the overlap is introduced, we can write the result:

$$\begin{aligned}\partial_t\tilde{\alpha} &= \tfrac{1}{4}\sum_{a=1}^{K} m_a(\langle q_{12}^2\rangle_a - \langle q_{12}^2\rangle_{a-1}) + \tfrac{1}{4}(1-\langle q_{12}^2\rangle_K)\\ &= \tfrac{1}{4}\left(\sum_{a=1}^{K} m_a\langle q_{12}^2\rangle_a - \sum_{a=0}^{K} m_{a+1}\langle q_{12}^2\rangle_a + m_{K+1}\langle q_{12}^2\rangle_K + 1 - \langle q_{12}^2\rangle_K\right)\\ &= \tfrac{1}{4}\left(1-\sum_{a=0}^{K}(m_{a+1}-m_a)\langle q_{12}^2\rangle_a\right). \end{aligned} \tag{A.14}$$

Now let us focus on the x-streaming of the interpolating function $\tilde{\alpha}(t,\mathbf{x})$ and show that it is given by the following formula:

$$\partial_a\tilde{\alpha}_N(\mathbf{x},t) = \tfrac{1}{2}\left(1-\sum_{b=a}^{K}(m_{b+1}-m_b)\langle q_{12}(\mathbf{x},t)\rangle_b\right). \tag{A.15}$$

In analogy with the t-streaming we have

$$\partial_a\tilde{\alpha}_N(\mathbf{x},t) = \frac{1}{N}E_0 Z_0^{-1}(\mathbf{x},t)\partial_a Z_0(\mathbf{x},t), \tag{A.16}$$

$$Z_b^{-1}(\mathbf{x},t)\partial_a Z_b(\mathbf{x},t) = E_{b+1}(f_{b+1}Z_{b+1}^{-1}(\mathbf{x},t)\partial_a Z_{b+1}(\mathbf{x},t)), \tag{A.17}$$

$$\Rightarrow Z_0^{-1}(\mathbf{x},t)\partial_a Z_0(\mathbf{x},t) = E_1\ldots E_K(f_1\ldots f_K Z_K^{-1}(\mathbf{x},t)\partial_a Z_K(\mathbf{x},t)), \tag{A.18}$$

$$Z_K^{-1}(\mathbf{x},t)\partial_a Z_K(\mathbf{x},t) = \frac{1}{2\sqrt{x_a}}\sum_i J_i^a\tilde{\omega}(\sigma_i), \tag{A.19}$$

by which

$$\partial_a\tilde{\alpha} = \frac{1}{N}\frac{1}{2\sqrt{x_a}}\sum_i E(f_1\cdots f_K J_i^a\tilde{\omega}(\sigma_i)). \tag{A.20}$$

Again by integrating by parts we have

$$\partial_a\tilde{\alpha} = \frac{1}{N}\frac{1}{2\sqrt{x_a}}\sum_{i=1}^{N}[\sum_{b=1}^{K} E(f_1\cdots\partial_{J_i^a}f_b\cdots f_K\tilde{\omega}(\sigma_i)) + E(f_1\cdots f_K\partial_{J_i^a}\tilde{\omega}(\sigma_i))]. \tag{A.21}$$

Let us work out J_i^a by remembering that Z_bs, and consequently f_bs, do not depend on $J_i^{b+1},\ldots,J_i^K$:

$$\partial_{J_i^a}f_b = \begin{cases} 0 & \text{if } a>b\\ m_a f_a(Z_a^{-1}(\mathbf{x},t)\partial_{J_i^a}Z_a(\mathbf{x},t)) & \text{if } a=b\\ m_b f_b(Z_b^{-1}(\mathbf{x},t)\partial_{J_i^a}Z_b(\mathbf{x},t)) - m_b f_b E_b f_b(Z_b^{-1}(\mathbf{x},t)\partial_{J_i^a}Z_b(\mathbf{x},t)) & \text{if } a<b. \end{cases} \tag{A.22}$$

The same recursion relationship holds in this case as well:

$$Z_b^{-1}(\mathbf{x},t)\partial_{J_i^a}Z_b(\mathbf{x},t) = E_{b+1}\cdots E_K(f_{b+1}\cdots f_K Z_K^{-1}(\mathbf{x},t)\partial_{J_i^a}Z_K(\mathbf{x},t)). \tag{A.23}$$

J. Stat. Mech. (2010) P09006

Furthermore

$$Z_K^{-1}(\mathbf{x},t)\partial_{J_i^a} Z_K(\mathbf{x},t) = \sqrt{x_a}\tilde{\omega}(\sigma_i), \tag{A.24}$$

from which we get

$$Z_b^{-1}(\mathbf{x},t)\partial_{J_i^a} Z_b(\mathbf{x},t) = \sqrt{x_a}\tilde{\omega}_b(\sigma_i). \tag{A.25}$$

Consequently, equations (A.22) can be written as

$$\partial_{J_i^a} f_b = \begin{cases} 0 & \text{if } \ a > b \\ m_a f_a \sqrt{x_a}\tilde{\omega}_a(\sigma_i) & \text{if } \ a = b \\ \sqrt{x_a} m_b f_b (\tilde{\omega}_b(\sigma_i) - \tilde{\omega}_{b-1}(\sigma_i)) & \text{if } \ a < b. \end{cases} \tag{A.26}$$

The last thing missing is evaluating the derivative of the state

$$\partial_{J_i^a}\omega(\sigma_i) = \sqrt{x_a}(1-\omega^2(\sigma_i)), \tag{A.27}$$

so as to write, via the overlap, the analogous terms for the generalized states. Substituting equations (A.27) and (A.26), once expressed via overlaps, into (A.21) we obtain equation (A.15).

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J. Stat. Mech. (2010) P09006