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Cosmology in general massive gravity theories

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Abstract. We study the cosmological FRW flat solutions generated in general massive gravity theories. Such a model are obtained adding to the Einstein General Relativity action a peculiar non derivative potentials, function of the metric components, that induce the propagation of five gravitational degrees of freedom. This large class of theories includes both the case with a residual Lorentz invariance as well as the case with rotational invariance only. It turns out that the Lorentz-breaking case is selected as the only possibility. Moreover it turns out that that perturbations around strict Minkowski or dS space are strongly coupled. The upshot is that even though dark energy can be simply accounted by massive gravity modifications, its equation of state w_{eff} has to deviate from -1 . Indeed, there is an explicit relation between the strong coupling scale of perturbations and the deviation of w_{eff} from -1 . Taking into account current limits on w_{eff} and submillimeter tests of the Newton's law as a limit on the possible strong coupling scale, we find that it is still possible to have a weakly coupled theory in a quasi dS background. Future experimental improvements on short distance tests of the Newton's law may be used to tighten the deviation of w_{eff} form -1 in a weakly coupled massive gravity theory.

Keywords: modified gravity, cosmological perturbation theory, dark energy theory

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1 Introduction

Recently, a significant step forward in understanding massive gravity was made in a series of papers [1–3] by the nonperturbative construction of the most general theories with five propagating degrees of freedom (DoF).¹ Besides its theoretical interest, the main phenomenological goal is to investigate whether a modification of gravity at large distances and a massive graviton can be realized in a consistent theory which can also take care of the wealth of other observational tests of gravity, from the smallest (submillimeter) to largest (cosmological) scales.

In this work we study the cosmology of the massive gravity theories which propagate five DoF in a systematic and model independent way. The construction of [1–3] allows one to treat at once theories which possess a residual Lorentz invariance as those with a simpler rotational invariance. Lorentz invariant massive gravity [6, 7] is phenomenologically not very successful: in the ghost free version of massive gravity with graviton mass scale m the energy cutoff $\Lambda_3 = (m^2 M_{Pl})^{1/3}$ is too low [8], the theory is classically strongly coupled in the solar system and, as already predicted in [8], even the computation of the static potential in the vicinity of the earth is problematic due to possible large quantum corrections [9]. Cosmology is also definitely troublesome: spatially flat homogenous Friedmann-Robertson-Walker (FRW) solutions simply do not exist [10] and even allowing for open FRW solutions [11] strong coupling [12] and ghostlike instabilities [13] develop. In the bigravity formulation [14–18] FRW homogenous solutions do exist [19–21], however cosmological perturbations turn out to be strongly coupled [22]. On the other hand, things get better if one gives up Lorentz invariance and requires only rotational invariance [2, 23, 26]. Within the general class of theories which propagate five DOF found in [1–3], in the Lorentz breaking (LB) case most of the theories have much safer cutoff $\Lambda_2 = (m M_{Pl})^{1/2} \gg \Lambda_3$ and also avoid all of the phenomenological difficulties mentioned above. A recent comprehensive review of massive gravity can be found in [28].

We study, in full generality, the conditions for the existence of a homogeneous flat FRW cosmological background. The massive deformation of general relativity show up as an additional effective energy momentum tensor whose conservation has a crucial impact on the

¹See [4, 5] for an alternative analysis using Kuchar’s Hamiltonian formalism.

behaviour of perturbations. In particular, the present accelerated de Sitter (dS) phase of the Universe can be naturally accounted for by massive gravity, but it turns out that the effective equation of state of dark energy is directly connected to both the large distance scale of modification of gravity and the strong coupling scale of gravitational perturbations around quasi dS space.

The outline of the paper is the following. In section 2 we briefly recall the construction of the general massive gravity theories with five DoF. The existence of background FRW solutions are considered in section 3, where the conditions due to the Bianchi identity are studied and where we discuss the effective perfect fluid resulting from the massive deformation of gravity; some observational constraints are also discussed. Perturbations around FRW background and the relation with the effective gravitational dark fluid equation of state are discussed in section 4. Section 5 contains our conclusions.

2 Massive gravity with 5 DoF

Generic nonderivative deformations of GR are defined by adding to the Einstein-Hilbert action a potential V which depends on the metric $g_{\mu\nu}$,

$$S = M_{\text{Pl}}^2 \int d^3x \sqrt{g} \left[R - m^2 V(g) \right]; \quad (2.1)$$

the parameter m sets the scale for the graviton mass. The general features of such gravity modification were studied [1–3] at the nonperturbative level by using Hamiltonian analysis. The ADM decomposition [29] of the metric reads

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N^j \gamma_{ij} & \gamma_{ij} N^j \\ \gamma_{ij} N^j & \gamma_{ij} \end{pmatrix}, \quad (2.2)$$

and the potential V can be regarded as a function of lapse N , shifts N^i and spatial metric γ_{ij} .²

In [2, 3] the most general potential that propagates five DoFs at nonperturbative level was found, under the requirement that rotations are unbroken. It turns out that V is parametrized in terms of two arbitrary functions³ \mathcal{U} and \mathcal{E} of γ^{ij} and a set of new shift variables ξ^i which are related to N^i through the implicit relation

$$N^i - N \xi^i = \mathcal{U}^{ij} \mathcal{E}_{\xi^j} \equiv \mathcal{Q}^i(\gamma^{ij}, \xi^i). \quad (2.3)$$

Here, \mathcal{E} is a generic function $\mathcal{E}(\gamma^{ij}, \xi^i)$ of the spatial metric γ^{ij} and ξ^i , while \mathcal{U} is an arbitrary function of the special combination of variables $\gamma^{ij} - \xi^i \xi^j$, namely

$$\mathcal{U}(\mathcal{K}^{ij}), \quad \mathcal{K}^{ij} = \gamma^{ij} - \xi^i \xi^j. \quad (2.4)$$

²The action (2.1) is written in the so called unitary gauge, which we use throughout our work. By using a set of four additional (Stückelberg) scalar fields, V may be written as diffeomorphism invariant scalar function, see the detailed discussion in [3], and general frameworks in [8, 26, 27]. By definition in the unitary gauge the derivatives of the Stückelberg fields are trivial; as a result, V is function of the ADM variables only.

³Note, while in [2] the potential was parametrized in terms of *three* functions, in the following paper [3] it was shown that by solving the associated Monge-Ampere problem one of them is just a constant of integration, so that *two* functions only are sufficient. See also footnote 6 on page 10 in [3].

The notation \mathcal{E}_{ξ^i} denotes the partial derivative with respect to ξ^i , and \mathcal{U}^{ij} is the inverse of the Hessian matrix $\mathcal{U}_{ij} = \mathcal{U}_{\xi^i \xi^j}$. The bottom line of the canonical analysis is that all potentials which propagate five DoF are of the form

$$V(N, N^i, \gamma_{ij}) = \mathcal{U} + N^{-1} (\mathcal{E} + \mathcal{Q}^i \mathcal{U}_{\xi^i}), \quad (2.5)$$

which we will use in this work.

The general construction of V is rather powerful: a whole set of interesting physical implications can be worked out without even specifying the form of \mathcal{U} and \mathcal{E} . It is worth to stress that the Lorentz invariant ghost free massive gravity theory found in [6, 7] is of course of the form (2.5), see [3] for the details.⁴ As studied in [2, 3], besides the Lorentz-invariant case, a whole new class of interesting theories which are weakly coupled at phenomenologically interesting scales can be constructed. The existence of ghost-free weakly coupled massive gravity theories with only rotational invariance can also be important to avoid the issue of acausal propagation [30] which affects the already troubled Lorentz-invariant case, and appears to be due to the strong nonlinearities present there.

3 FRW background and the Bianchi identity

All observations [31–34] are consistent with the cosmological principle that we can slice our spacetime in spatial homogeneous and isotropic hypersurfaces associated with observers (cosmological observers) for whom the CMB is practically isotropic. Generically, massive modified gravity contains *a priori* nondynamical metric(s), much like the Nordstrom theory of gravity. The construction recalled above, allowing for Lorentz-breaking scenarios, relies mainly on a fiducial 3d metric δ_{ij} (see section 6 in [3]). As a result, there is also another special class of observers (preferred frame observers) for which some the dynamical metric has a preferred form. Whether the cosmological and fiducial frames coincide is a physical hypothesis. Working in a flat homogeneous space⁵ (with cartesian coordinates for simplicity) it is possible to bring, via a time redefinition, the 4-dimensional metric in the form

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 (d\vec{x})^2, \quad (3.1)$$

that matches with the ADM form of eq. (2.2). The lapse can be interpreted as the presence of a nontrivial temporal Stückelberg field, making explicit time reparametrizations. On this background, we have $N^i = \xi^i = \mathcal{Q}^i = 0$, while in general one can have $\partial_{N^j} \xi^i \neq 0$ and $\partial_{\xi^j} \mathcal{Q}^i \neq 0$. We also have $\mathcal{U}_{\gamma^{ij}} \propto \gamma_{ij}$ and we define the scalar quantity \mathcal{U}' by $\mathcal{U}_{\gamma^{ij}} \equiv \mathcal{U}' \gamma_{ij}$, and similarly for \mathcal{E}' by $\mathcal{E}_{\gamma^{ij}} = \mathcal{E}' \gamma_{ij}$. One also has $\mathcal{U}_{\xi^i \xi^j} = -2 \mathcal{U}' \gamma_{ij}$ and finally we use $H = \dot{a}/a$.

So our main hypothesis is that we preserve homogeneity and spatial isotropy both at the level of physical metric that for the non dynamical metric. Notice that also that homogeneity and isotropy forces the flat massive gravity preferred frame and our cosmological CMB frame to coincide.

⁴One can further take the extra metric dynamical by introducing an additional Einstein-Hilbert action, see for instance [14–18, 36–39], leading to a bimetric theory. We will not discuss this option here.

⁵We consider a flat spatial section, taking into account the overwhelming observational evidences in its favor. We leave the generalization to the non flat case for a future work. As well know there are no cosmological solutions for a flat FRW universe [10], while there are solutions for the open ($k = -1$) FRW cosmology [11]. Incidentally, we recall that for this Lorentz invariant cases, even resorting to nontrivial Stückelberg fields one has to face strong coupling and ghost instabilities [12, 13], for a review see [35].

The Einstein equations take the form

$$E_{\mu\nu} = \mathcal{T}_{\mu\nu} + 8\pi G T_{\mu\nu}^{(\text{matt})}, \quad (3.2)$$

where $T_{\mu\nu}^{(\text{matt})}$ is the matter energy momentum tensor (EMT) and $\mathcal{T}_{\mu\nu}$ is the contribution of the potential V . In particular, we have

$$\begin{aligned} \mathcal{T}_{00} &= m^2 \frac{N^2}{2} \mathcal{U} \equiv \rho_{\text{eff}} N^2 \\ \mathcal{T}_{ij} &= m^2 \gamma_{ij} \left[\mathcal{U}' - \frac{\mathcal{U}}{2} + \frac{1}{N} \left(\mathcal{E}' - \frac{\mathcal{E}}{2} \right) \right] \equiv p_{\text{eff}} \gamma_{ij}, \end{aligned} \quad (3.3)$$

where we used the fact that $\partial_t \mathcal{U} = \mathcal{U}_{,\gamma ij} \partial_t \gamma^{ij} = \mathcal{U}' \gamma_{ij} \partial_t \gamma^{ij} = -6 H \mathcal{U}'$.

We assume that matter is minimally coupled to gravity respecting general covariance, so that $T_{\mu\nu}^{(\text{matt})}$ is separately covariantly conserved. Therefore, also $\mathcal{T}_{\mu\nu}$ has to be separately conserved. The resulting Bianchi identity, $\nabla^\nu \mathcal{T}_{\mu\nu} = 0$, takes the simple form

$$\partial_t \rho_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0 \quad \rightarrow \quad H \left(\mathcal{E}' - \frac{\mathcal{E}}{2} \right) \Big|_{\xi^i=0} = 0. \quad (3.4)$$

Physically this condition can also be understood as the implementation of time reparametrization. While the \mathcal{U} part is automatically conserved (accordingly to time reparametrization \mathcal{U} appears linearly in N as $N\mathcal{U}$ in the action), the \mathcal{E} part is constrained by this equation. In fact, (3.4) has far reaching consequences. It implies that either $H = 0$ (i.e. a is constant in time) or that the second factor vanishes automatically, independently of a (otherwise it would imply an algebraic constraint on a , clearly incompatible with any sensible cosmology). Its automatic fulfillment poses a strong condition on the function \mathcal{E} , on the surface $\xi^i = 0$, which has to be satisfied by choosing some particular form of \mathcal{E} . For instance, one can take \mathcal{E} to be homogeneous of degree $-3/2$ in γ_{ij} ,

$$\mathcal{E} = \gamma^{-1/2} \mathcal{P} \left(\frac{\text{Tr}[\gamma^2]}{\text{Tr}[\gamma]^2}, \frac{\text{Tr}[\gamma^3]}{\text{Tr}[\gamma]^3}, \xi^i \right) \quad (3.5)$$

with any function \mathcal{P} , but other choices are possible.

Incidentally, we remark that the form of \mathcal{E} dictated by the dRGT Lorentz-invariant massive gravity, $\mathcal{E}_{(LI)} = (1 - \xi\gamma\xi)^{-1/2}$ (see [3]) does not satisfy (3.4) and thus no sensible cosmology is possible, as it was already realized in [10].⁶ We are thus led to the conclusion that Lorentz-breaking is necessary, in massive gravity with 5 DoF, to admit a nontrivial cosmology for a spatially flat Universe.

Before moving on, it is worth mentioning that our framework is the most general which preserves spatial homogeneity and isotropy, and has $\text{SO}(3)$ invariance in the theory. In fact, at most one might promote the 3D fiducial metric δ_{ij} used in the lagrangian to a time-dependent one, $\zeta(\tau)\delta_{ij}$, with ζ arbitrary. However, in this case the Bianchi identity becomes

$$\mathcal{H} \left(\mathcal{E}' - \frac{\mathcal{E}}{2} \right) + \frac{\dot{\zeta}}{\zeta} \mathcal{U}' = 0. \quad (3.6)$$

⁶Such a term corresponds to the potential $(\text{Tr}[X^{1/2}] - 3)$, one of the four possible pieces of the dRGT potential, see [6, 7] and [3].

So, instead of constraining the potential, the Bianchi identity results (for $\mathcal{U}' \neq 0$) in an algebraic equation that fixes the time expansion $a(\tau)$ of our universe in terms of the absolute field $\zeta(\tau)$. We think this scenario to be unphysical, in that we feel uneasy in relating time a dependent function introduced as a supposedly nondynamical fiducial metric to the physical scale factor that should depend on the matter content. For such a reason we do not consider this possibility further.

Generic background evolution. In the following we will assume that the Bianchi condition is automatically satisfied, with a nontrivial H , thanks to some particular choice of \mathcal{E} . In this case, \mathcal{E} effectively drops out of the background equations (3.3), and the contribution of V to the Einstein equations has the form of a “gravitational” perfect fluid, with effective density and pressure given by

$$\rho_{\text{eff}} \equiv m^2 \mathcal{U}, \quad p_{\text{eff}} \equiv m^2 (2\mathcal{U}' - \mathcal{U}). \quad (3.7)$$

Thus, the effective gravitational fluid mimics the following equation of state

$$w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \frac{2\mathcal{U}'}{\mathcal{U}}. \quad (3.8)$$

When $2\mathcal{U}'/\mathcal{U} < 1$, the gravitational fluid mimics dark energy; in particular if $2\mathcal{U}'/\mathcal{U} < 0$, one has $w < -1$, turning the gravitational fluid into a phantom one.

The fluid contributes to the expansion rate through the standard equation

$$3H^2 = N^2 \left(\frac{m^2 \mathcal{U}}{2} + 8\pi G \rho_m \right). \quad (3.9)$$

Now, given that the function \mathcal{U} is still generic, it can accommodate the most diverse cosmologies. In fact, it is sufficient to observe that given *any* cosmological history in terms of a positive $H(t)$ or equivalently in terms of $H(a)$, together with the matter content in terms of $\rho_m(a)$, the previous equation can always be solved by choosing a particular function $\mathcal{U}(\mathcal{K})$ so that on the background $\mathcal{K}^{ij} = a^{-2}\delta^{ij}$ it gives the correct a -dependence for (3.9). Clearly, any bizarre accelerating or decelerating or oscillating and bouncing cosmology can be present.

Constraints from the present epoch and early time. Having clarified that any cosmological evolution can result from the choice of the function \mathcal{U} , one can analyze the constraints on \mathcal{U} from present day ($a \sim 1$) and early time ($a \ll 1$) observational constraints, under some mild hypotheses for its form.

For instance, we can consider the early or late time expansions as $\mathcal{U} = \sum_{n=0}^{\infty} \tilde{\mathcal{U}}_n a^{-n} = \sum_{n=0}^{\infty} \bar{\mathcal{U}}_n (a-1)^n$, and assume there are no strong cancellations between the coefficients in the series. This allows us to put constraints on the coefficients $\tilde{\mathcal{U}}_n$ and $\bar{\mathcal{U}}_n$.

If the effective gravitational density is driving the present day observed acceleration, the background energy scale of the potential is fixed as (here $8\pi G = 1/2M_{\text{Pl}}^2$)

$$V = m^2 M_{\text{Pl}}^2 \bar{\mathcal{U}}_0 \sim \Omega_{\Lambda} \rho_c \quad \rightarrow \quad m^2 \bar{\mathcal{U}}_0 \sim H_0^2 \sim (10^{-34} \text{ eV})^2 \quad (3.10)$$

with H_0 the present Hubble parameter. The effective equation of state can also be expanded, as

$$\begin{aligned} w_{\text{eff}} &= -1 - \frac{1}{3} \frac{\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_0} - (1-a) \frac{\bar{\mathcal{U}}_1^2 - \bar{\mathcal{U}}_0 (2\bar{\mathcal{U}}_2 + \bar{\mathcal{U}}_1)}{3\bar{\mathcal{U}}_0^2} + \dots \\ &\equiv w_0 + (1-a)w_a + \dots, \end{aligned} \quad (3.11)$$

where w_0 is the present value of w_{eff} while w_a represents a possible time evolution of the equation of state. The combined data of Planck, WMAP low-multipole (WP) and baryonic acoustic oscillations (BAO) [40] give the following conservative observational constraints (at 95% C.L.)

$$\begin{aligned} w_0 &= -1.04_{-0.7}^{+0.7}, & w_a &< 1.32, \\ \text{i.e. } \frac{\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_0} &= 0.12 \pm 2.1, & \frac{\bar{\mathcal{U}}_2}{\bar{\mathcal{U}}_0} &< 2 \pm 3. \end{aligned} \quad (3.12)$$

On the other hand, the global analysis including supernovae data (figure 36 in [40]) has a mild preference for $w_a \simeq -1.6$, and 0 is disfavoured at 2σ . Any future hint like redshift dependent equation of state for dark energy can find in the massive gravity cosmology a simple explanation in terms of the nontrivial form of \mathcal{U} .

Focusing instead on the early Universe, each coefficient $\tilde{\mathcal{U}}_n$ mimics a different fluid with an equation of state $w_n = -1 + n/3$. For instance, $\tilde{\mathcal{U}}_1$ gives dark energy with $w = -2/3$, $\tilde{\mathcal{U}}_2$ behaves like spatial curvature, $\tilde{\mathcal{U}}_3$ like non relativistic matter and $\tilde{\mathcal{U}}_4$ like radiation. Defining the dimensionless constant $x \equiv m^2 M_{\text{Pl}}^2 / \rho_c = \Omega_\Lambda / \bar{\mathcal{U}}_0$, if we impose that the gravitational fluid does not alter the background evolution from big bang nucleosynthesis (BBN) on, a quick estimate assuming no cancelations between coefficients gives the following bounds

$$\begin{aligned} x \tilde{\mathcal{U}}_2 &\leq \Omega_K \sim 10^{-2}, \\ x \tilde{\mathcal{U}}_3 &\leq \Omega_{\text{mat}} \sim 10^{-2}, \\ x \tilde{\mathcal{U}}_4 &\leq \Omega_{\text{rad}} \sim 10^{-5}, \\ x \tilde{\mathcal{U}}_{4+n} &\ll z_{\text{bbn}}^{-n} \Omega_{\text{rad}} \sim 10^{-(5+8n)}, \quad (n \geq 0), \end{aligned} \quad (3.13)$$

where $z_{\text{bbn}} \sim 10^8$ is the redshift at BBN. Note that higher coefficients are much more constrained because their scale factor dependence would make their contribution more dominant at early times.

In the next section we discuss the minimal conditions under which the perturbations around the cosmological background and flat space are well behaved.

4 Impact on behaviour of perturbations

Exploiting the construction of the massive gravity potential we can study some very general aspects of perturbations around FRW and Minkowski backgrounds. Let us start by expanding the metric around the FRW background (3.1), choosing $\bar{N} = a$ (conformal time), so that the background metric is conformally flat:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} = a^2 (\eta_{\mu\nu} + h_{\mu\nu}). \quad (4.1)$$

The derivatives with respect to conformal time is denoted by $'$, in particular the conformal Hubble parameter will be denoted by $\mathcal{H} \equiv a'/a$ and the standard one is $H \equiv a'/a^2$. It is also convenient to decompose the perturbations according their transformation properties under rotations

$$h_{00} = \psi, \quad h_{ij} = \chi_{ij} + \partial_i s_j + \partial_j s_i + \delta_{ij} \tau + \partial_i \partial_j \sigma, \quad h_{0i} = u_i + \partial_i v, \quad (4.2)$$

with $\chi_{ij} \delta^{ij} = \partial_j \chi_{ij} = \partial_i s_i = \partial_i u_i = 0$.

Consider the case of a Universe dominated by dark energy induced by the massive gravity potential V . The quadratic action for the perturbation can be computed for a generic V and details are given in appendix A. Particularly useful is the quadratic expansion of the deforming potential parametrized in terms of the masses $m_0, \dots, m_4 \sim O(m)$ defined by [41]

$$M_{Pl}^2 \sqrt{g} (m^2 V - 6H^2) \equiv \frac{a^4 M_{Pl}^2}{4} \left(m_0^2 h_{00}^2 + 2 m_1^2 h_{0i}^2 - 2 m_4^2 h_{00} h_{ii} + m_3^2 h_{ii}^2 - m_2^2 h_{ij}^2 \right). \quad (4.3)$$

In particular, for a general massive gravity modification of GR, one has the following result

$$m_0^2 = 0, \quad m_1^2 = 2 m^2 \mathcal{U}' \mathcal{X}, \quad m_4^2 = m^2 \mathcal{U}', \quad \text{with } \mathcal{X} = \frac{2a\mathcal{U}'}{2a\mathcal{U}' + \mathcal{E}_v''}, \quad (4.4)$$

where we defined $\frac{\partial^2 \mathcal{E}}{\partial \xi^i \partial \xi^j |_{\bar{g}}} = \mathcal{E}_v'' \bar{\gamma}_{ij}$. The expressions for m_2 and m_3 are not needed here and are reported in appendix A. Using (4.3), the complete analysis of perturbations around a FRW background in massive gravity was given in [41]. Around a FRW background [41], as well as around flat space [18, 23, 26], when $m_0 = 0$ and $m_1 \neq 0$ the number of propagating degrees of freedom at linearized level is five; namely two tensors (χ_{ij}), two vectors (s_i) and a scalar (σ). Thus, the above result $m_0 = 0$ is in agreement with the nonperturbative canonical analysis, and the linearized approximation captures all the propagating DoF. This is a sign that the system can be analyzed by weak-field expansion (see also below).

The conditions for UV stability (no ghosts) are directly dictated by m_1^2 and m_4^2 [41], and turn out to be the following

$$\mathcal{E}_v'' > -2a\mathcal{U}', \quad \mathcal{E}_v''(\mathcal{E}_v'' - 2a\mathcal{U}' + 2a) > 0, \quad (4.5)$$

which can be always satisfied by choosing \mathcal{E} such that \mathcal{E}_v'' is sufficiently large and positive. Thus we conclude that the generic massive gravity theories are free of ghosts with mild assumptions on \mathcal{E} and \mathcal{U} .

Strong coupling. It is well known that the violation of diffeomorphisms by the potential implies that the new propagating modes can become strongly coupled at some energy or momentum scale. This can manifest already at the classical level for instance when perturbation series break down, and also at the quantum level, where in the spirit of the effective field theory, the possible effective operators suppressed by a cutoff scale become important. Let us first discuss the classical case.

In GR the classical perturbation expansion breaks down in the presence of a source of mass M at the Schwarzschild radius $r_s \sim M/M_{Pl}^2$. In massive gravity, since new fields mediate the gravitational interaction, one can expect deviations from the above behaviour. For instance, in Lorentz invariant (LI) massive gravity [6, 7], the perturbation expansion breaks down at a distance (Vainshtein scale) $\Lambda_* = (M_{Pl} m^2)^{1/3} (M/M_{Pl})^{1/3}$ from a source of mass M , which becomes untenable already for macroscopic objects like the Earth or Sun.

In the Lorentz breaking case the situation is completely different, and there is in fact no classical strong coupling apart from the Schwarzschild radius as in GR. To see this, we note that, for static configurations and also at large spatial momenta, the auxiliary fields vanish [41] and one can study interactions in terms of the sole physical propagating fields χ_{ij} ,

s_i, σ . Their quadratic Lagrangian in the UV limit is given by

$$L_\sigma = a^4 \Lambda_2^4 \nabla^2 \left(\frac{\mathcal{U}'}{2} \sigma'^2 - \frac{\lambda_{23}^2}{2} \nabla^2 \sigma^2 \right), \quad \lambda_{23}^2 \equiv \frac{m_2^2 - m_3^2}{m^2}, \quad (4.6)$$

$$L_{s_i} = a^4 \Lambda_2^4 \left(\frac{2\mathcal{U}'\mathcal{X}}{2} s_i'^2 - \frac{\lambda_2^2}{2} \nabla^2 s_i^2 \right), \quad \lambda_2^2 \equiv \frac{m_2^2}{m^2}. \quad (4.7)$$

where $\Lambda_2^2 = m M_{Pl}$ and we are using the spatial Fourier transform, defining $\nabla = \sqrt{-\Delta}$. Notice also that for $\mathcal{E} = Q_{\xi^j}^i = 0$, one has $\mathcal{X} = 1$.

Suppressing for a moment dimensionless factors, these fields are made canonical as follows

$$\sigma^c = \Lambda_2^2 \nabla \sigma, \quad s_i^c = \Lambda_2^2 s^i, \quad \chi_{ij}^c = M_{Pl} \chi_{ij}. \quad (4.8)$$

We can then study a generic Lagrangian term

$$\Lambda_2^4 (\nabla^2 \sigma)^n (\nabla s)^r \chi^p \sim \frac{(\nabla \sigma^c)^n (\nabla s^c)^r (\chi^c)^p}{\Lambda_2^{2(n+r)-4} M_{Pl}^p} \quad (4.9)$$

and we see that the leading operators are the ones with $p = 0$. Focusing on σ^c (the same holds for s_i^c), around a classical static source, the perturbative expansion works (no strong coupling) as long as

$$\nabla \sigma^c < \Lambda_2^2 = m M_{Pl}. \quad (4.10)$$

However, a crucial fact is that the canonical field σ^c induced by a static source $T_{\text{matter}} = M \delta^3(r)$ is given by, after integrating out the auxiliary fields,

$$\nabla^2 \sigma^c - \frac{m}{M_{Pl} \nabla} T_{\text{matter}} = 0 \quad \rightarrow \quad \nabla \sigma^c \sim \frac{m M}{M_{Pl} r}. \quad (4.11)$$

As a result, Λ_2 cancels out and one can use the weak-field expansion outside a static source at distances larger than a critical radius r_c which is simply given by

$$r_c \simeq r_s = \frac{M}{M_{Pl}^2} \quad (4.12)$$

i.e. the same as in GR. We conclude that the theory does not suffer from new classical nonlinearities near a source, and the perturbation series just break down at the Schwarzschild radius, as in GR. Notice that, at this radius all Lagrangian terms (4.9) become important. If one were to ignore quantum effects (see below), the perturbation series would be reliable even at distance scales smaller than $1/\Lambda_2$ (provided this were still larger than the Schwarzschild radius). This happens because the fields responsible for the strong coupling, σ and s_i , are not directly sourced by matter but only via a coupling with the standard GR fields, and the coupling itself is m -suppressed.

The absence of a Vainshtein strong coupling scale and the disappearance of Λ_2 from the classical perturbation series was also explicitly demonstrated with the first perturbative orders in [2]. This result is a remarkable fact, because it suggests that as a classical theory the present theory can be used perturbatively also at short distances.

Keeping track of the dimensionless coefficients in (4.6) and (4.7) does not alter the result, even in the limit of small \mathcal{U}' as required by cosmology. In fact, for static configurations the time-derivative kinetic term will clearly be less important. A possible worsening of the

nonlinearity scale, i.e. a larger Schwarzschild radius, might be present if λ_{23} is taken to be parametrically small.

A consequence of the requirement of small \mathcal{U}' is only present for time dependent classical solutions. In fact, the vanishing of the temporal kinetic terms for σ and s_i for $\mathcal{U}' \rightarrow 0$ (see (4.6) and (4.7)) tells us that, because the canonical analysis still requires five DoF, in the limit of $\mathcal{U}' \rightarrow 0$ these modes must propagate through interaction terms, i.e. become progressively strongly coupled. This will reflect on the classical response to highly oscillatory source, with frequency ω much larger than momentum k (and of course than H, m).

Let us now consider massive gravity as a quantum theory. In the spirit of effective field theory, a cutoff Λ is expected. Λ is the scale where we loose control of the theory and some UV completion becomes mandatory to make any physical prediction. In GR such scale is the the Planck mass, with no foreseeable phenomenological consequences. In the generic completions of the Fierz-Pauli massive gravity, the cutoff is as small as $\Lambda_5 = (M_{Pl}m^4)^{1/5}$ [8], or at best as in the dRGT theory $\Lambda_3 = (M_{Pl}m^2)^{1/3}$, at the price of an infinite number of fine tunings [8]. If $m \sim H_0$ then $\Lambda_3 \sim (1000\text{Km})^{-1}$, so that one loses control at macroscopic distances and even the static gravitational force in experiments around the earth is incalculable [9].

For the class of potentials (2.5) analyzed here with rotational invariance only, the situation is much more favorable with a reasonable cutoff $\Lambda_2 = (M_{Pl}m)^{1/2} \gtrsim (10^{-4}\text{mm})^{-1}$ in flat space [2]. This can be seen directly from the canonical fields or the Lagrangians (4.6) and (4.7). From those expression is also clear that taking a progressively small \mathcal{U}' will worsen the quantum cutoff. Indeed, without doing any actual loop computation, by rescaling time by a factor $\sqrt{\mathcal{U}'}$, the troublesome small dimensionless parameter is removed from the Lagrangian and the neat effect is to replace $\hbar \rightarrow \hbar/\sqrt{\mathcal{U}'}$ in the exponential of the action. Similar result can be understood by rescaling energy in the loops. As a result, the loop expansion will become less convergent.

Using standard arguments [43] one knows that given a (renormalized) classical nonlinearity scale f , the quantum corrections tend to become important at the scale $4\pi f/\sqrt{\hbar}$ (provided the generated operators contain even powers of a cutoff, as in (4.9)). In our case the scale of classical nonlinearity is Λ_2 , and together with the effect of \mathcal{U}' on \hbar , we expect a quantum cutoff of the order of $\Lambda \simeq 4\pi\Lambda_2(\mathcal{U}')^{1/4}$. To see quantitatively the effect, we can use the effective equation of state for the gravitational fluid (3.8) to solve for \mathcal{U}' and use the background equation $m^2\mathcal{U} = H^2$ to recast the cutoff Λ in terms of the deviation of w_{eff} from -1 :

$$\Lambda \simeq 4\pi\sqrt{M_{Pl}H}(w_{\text{eff}} + 1)^{1/4}. \quad (4.13)$$

Thus, the strong coupling scale is directly linked to $w_{\text{eff}} + 1$, and note that the graviton mass disappears in favour of the explicit appearance of w_{eff} . In any given massive gravity theory with five DoF, an accelerated expansion phase can exist, but as soon as the equation of state gets close to the DeSitter phase with $w_{\text{eff}} \approx -1$, gravitational perturbations tend to become progressively strongly coupled.⁷

How safe we are from large quantum effects depends on how far w_{eff} is from -1 and at the same time on the value of m . Note that $4\pi\sqrt{M_{Pl}H}$ at present time gives a cutoff at distances of the order of 10^{-2}mm at which no deviations from the Newton law have been found [42]. The present uncertainty on the deviation of w_{eff} from -1 is still of order one [40], and one can still be consistent both with experiments at small scales (test of the Newton's law) and at

⁷Clearly, for particular models the actual loop expansion could be still more convergent by virtue of the particular operator coefficients. The model-dependent analysis goes beyond the scope of this work.

large scales (cosmology). However, (4.13) gives an intriguing connection between small and large scales, and future progress on the determination of w_{eff} and on short distances test of gravity will turn (4.13) into a prediction, if one want to keep the theory calculable.

Finally, a brief comment comparing with previous analyses of massive gravity is in order. As we mentioned in the introduction, in all earlier approaches, homogeneous cosmology was always problematic: in the Lorentz invariant case it simply does not exist [10], while in the bigravity approach [19–21] it leads to strong coupling in perturbations [22]. In our case, a sensible stable and weakly coupled theory exists as long as $w_{\text{eff}} \neq -1$.

Impact on the static gravitational potential. Having the general expressions of the masses (4.4) in the Lorentz breaking (LB) case, we can study the consequences of the Bianchi identity and the existence of a nontrivial cosmology for the linearized gravitational potential. The gravitational potential is modified at large distances and amounts to a combination of two Yukawa terms [2, 3, 23–25]:

$$\Phi = \frac{GT_{00}}{2r} \left(A_1 e^{-M_1 r} + A_2 e^{-M_2 r} \right), \quad (4.14)$$

with $A_1 + A_2 = 1$, which implies that in the short distance limit $r \ll M_{1,2}$ the potential reduces to the Newtonian expression (absence of vDVZ discontinuity). The masses M_1, M_2 and the coefficients A_1, A_2 are given in terms of the LB masses, and their full expressions are not particularly illuminating.

What is interesting is that, after the discussion above, $\mathcal{U}' \sim w_{\text{eff}} + 1$ has to be sensibly smaller than 1 if cosmology requires w_{eff} to be near -1 . In this limit, we have (in the $m > H, H'$ hypothesis)

$$M_1^2 \simeq m^2 \frac{3\mathcal{U}'}{1+x} + O(\mathcal{U}'^2), \quad A_1 \simeq \frac{x}{\mathcal{U}'} + O(1), \quad (4.15)$$

$$M_2^2 \simeq m^2 \frac{3\mathcal{U}'}{1-x} + O(\mathcal{U}'^2), \quad A_2 \simeq -\frac{x}{\mathcal{U}'} + O(1), \quad (4.16)$$

where $x \equiv \sqrt{1 + 3\lambda_{23}/2\lambda_2} \sim O(1)$. The conclusion from the above $\mathcal{U}' \ll 1$ limit is that if one wishes to increase m to be larger than the Hubble scale, i.e. relevant for phenomenology, and at the same time keep $w_{\text{eff}} \simeq -1$, then the Yukawa distance scale is pushed again at the Horizon scale. Thus the requirement of approximately deSitter phase generically hinders a possible large distance modification of gravity.

The only corner in parameter space which may lead to nontrivial modifications of the gravitational potential at distances smaller than the Hubble scale can be reached with a tuning for $x \simeq 1$, in which case $M_2 \gg M_1$ and A_1, A_2 are large and of opposite sign. In this case, discussed in [2], the gravitational potential is Newtonian at short distances and Yukawa at very large distances, but there is a rise at intermediate radii which can mimic the presence of additional (dark) matter. The limit $x \simeq 1$ corresponds to $\lambda_{23} \simeq 0$, and notice that the Schwarzschild spatial cutoff discussed above can become larger, in this limit.

5 Conclusions

In this work we analyzed spatially flat FRW cosmology of massive gravity theories with five propagating degrees of freedom. The analysis is model independent and is based on the powerful nonperturbative results of [1–3], which enable one to express the deforming potential in terms of two functions \mathcal{U} and \mathcal{E} .

A first important result is that the existence of a nontrivial spatially flat FRW cosmological background requires, due to the Bianchi identities, a stringent condition on the potential, which selects a particular subclass of theories. The Lorentz-invariant DeRGT potential [6, 7] is not among those; as a result Lorentz-breaking in the gravitational sector is a consequence of requiring FRW cosmology to exist.

In this subclass which admits a nontrivial cosmology, the massive deformation of GR appears first of all as an effective “gravitational” perfect fluid with energy density and pressure determined solely by \mathcal{U} , with an effective equation of state $w_{\text{eff}} = -1 - \mathcal{U}'/2\mathcal{U}$. For instance, it can mimic dark energy when $2\mathcal{U}'/\mathcal{U} < 1$. Quite generally thus, massive gravity can easily account for the present acceleration of the Universe, possibly with a varying equation of state.

The study of static perturbations in these theories confirmed that from a phenomenological point of view the potentials with Lorentz breaking perform much better than the Lorentz invariant ones. The key point is that all the five required modes receive a kinetic term at linear level and are thus weakly coupled. In addition the classical strong coupling scale is screened in the weak-field expansion and one can use perturbation theory around a static source much like in GR: the scale at which the weak field expansion breaks down is the same as in GR, the Schwarzschild radius, and there is no Vainshtein strong coupling scale or phenomenon. If quantum effects are taken into account, an energy scale $\sim \Lambda_2 = (mM_{Pl})^{1/2}$ is the expected quantum cutoff and again the Lorentz-breaking theories are much better off with respect to the Lorentz invariant ones (where the cutoff is $\Lambda_3 = (m^2M_{Pl})^{1/3}$). In case the graviton mass is taken at cosmological scales, $\Lambda_2 \simeq (10^{-4} \text{ mm})^{-1}$ (while Λ_3 is of the order of 1000 Km). Thus in these theories quantum effects are automatically confined at submillimeter scales, possibly tested by future short-distance gravity probes.

The study of perturbations also allowed us to discover a general link between the cosmological background and their behaviour. We have shown that, if strict Minkowski space is a vacuum solution, then gravitational perturbations are strongly coupled, because the temporal kinetic terms of the vector and scalar perturbations vanish in this background. The same happens in strict de Sitter space. Therefore, some deviation from maximally symmetric backgrounds is required to have a healthy and calculable theory. It is remarkable that, in a quasi-dS universe dominated by the induced gravitational dark energy, one can find a simple relation between the cutoff scale of the theory and the deviation of w_{eff} from -1 , as $\Lambda \simeq 4\pi\sqrt{HM_{Pl}}(w_{\text{eff}} + 1)^{1/4}$. Thus, the requirement of the absence of strong coupling in the present quasi-dS phase can be used to predict the equation of state of dark energy.

Finally, and more broadly, it is a natural question whether, once flat-background Lorentz invariance is not imposed in the gravitational sector, there exist other massive deformations of gravity with a number of DoF different from five. The answer is positive and we will report on their general features in a separate publication [44].

A Quadratic action

The starting point is the following perturbed FRW background

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}, & h_{00} &= a^2 (N^i N^j \bar{\gamma}_{ij} - 2\bar{N} \delta N - \delta N^2), \\ h_{0i} &= a^2 (\bar{\gamma}_{ij} + \alpha_{ij}) N^j, & h_{ij} &= a^2 \alpha_{ij}. \end{aligned} \tag{A.1}$$

The of perturbations can be decomposed as

$$\begin{aligned} h_{00} &= \psi, & h_{ij} &= \chi_{ij} + \partial_i s_j + \partial_j s_i + \delta_{ij} \tau + \partial_i \partial_j \sigma, & h_{0i} &= u_i + \partial_i v; \\ \chi_{ij} \delta^{ij} &= \partial_j \chi_{ij} = \partial_i s_i = \partial_i u_i = 0. \end{aligned} \tag{A.2}$$

We have chosen $N = a$. The total action is given by

$$S = M_{\text{Pl}}^2 \int d^4x \sqrt{g} (R - m^2 V) \equiv M_{\text{Pl}}^2 \int d^4x \sqrt{g} [R - 6H^2 - (m^2 V - 6H^2)] . \quad (\text{A.3})$$

Using (A.1), the quadratic expansion of V , shifted by $6H^2$, can be written as

$$m^2 \sqrt{g} (V - 6H^2) = a^4 \left[t_V^{\mu\nu} h_{\mu\nu} + \frac{1}{4} (m_0^2 h_{00}^2 + 2m_1^2 h_{0i} h_{0i} - 2m_4^2 h_{00} h_{ii} + m_3^2 h_{ii}^2 - m_2^2 h_{ij} h_{ij}) \right] . \quad (\text{A.4})$$

The values of the various masses can be computed for any V of the form (2.5), the result is the following

$$\begin{aligned} m_0^2 &= 0, & m_1^2 &= m^2 2a \bar{U}' \mathcal{X}, & m_2^2 &= 2m^2 \left(\frac{\bar{\mathcal{E}}_t''}{a} + \bar{U}_t'' + 2\bar{U}' - \frac{\bar{\mathcal{E}}}{2a} \right), \\ m_3^2 &= 2m^2 \left(\bar{U}' + \frac{\bar{\mathcal{E}}}{4a} - \bar{U}_s'' - \frac{\bar{\mathcal{E}}_s''}{a} \right), & m_4^2 &= m^2 \bar{U}' ; \\ t_V^{00} &= \frac{m^2}{2} \bar{U}, & t_V^{ij} &= m^2 \left(\bar{U}' - \frac{\bar{U}}{2} \right) \delta^{ij} ; \end{aligned} \quad (\text{A.5})$$

where

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial \mathcal{K}^{ij}|_{\bar{g}}} &= \bar{U}' \bar{\gamma}_{ij}, & \frac{\partial^2 \mathcal{E}}{\partial \xi^i \partial \xi^j|_{\bar{g}}} &= \bar{\mathcal{E}}'' \bar{\gamma}_{ij}, & \frac{\partial \xi^m}{\partial N^i|_{\bar{g}}} \frac{\partial \xi^n}{\partial N^j|_{\bar{g}}} \bar{\gamma}_{mn} &= \bar{\gamma}_{ij} \kappa, \\ \frac{\partial^2 \mathcal{E}}{\partial \gamma_{ij} \partial \gamma_{mn}|_{\bar{g}}} &= \bar{\mathcal{E}}_s'' \gamma^{ij} \gamma^{mn} + \bar{\mathcal{E}}_t'' \frac{1}{2} (\gamma^{im} \gamma^{jn} + \gamma^{in} \gamma^{jm}) ; \\ \frac{\partial \mathcal{U}}{\partial \mathcal{K}^{ij} \partial \mathcal{K}^{mn}|_{\bar{g}}} &= \bar{U}_s'' \gamma_{ij} \gamma_{mn} + \bar{U}_t'' \frac{1}{2} (\gamma_{im} \gamma_{jn} + \gamma_{in} \gamma_{jm}) ; \\ \frac{1}{2} \left(\bar{\gamma}_{ik} \frac{\partial \xi^k}{\partial N^j|_{\bar{g}}} + \bar{\gamma}_{jk} \frac{\partial \xi^k}{\partial N^i|_{\bar{g}}} \right) &= \bar{\gamma}_{ij} \mathcal{X}. \end{aligned} \quad (\text{A.6})$$

Background quantities are denoted by a bar. In (A.5) the background Bianchi identity (3.4) has been used to eliminate $\bar{\mathcal{E}}'$. The value of \mathcal{X} can explicitly determined expanding ξ^i in powers of N^j , namely

$$\xi^i = A_j^i N^j + B_{jm}^i N^j N^m + C_j^i N^j \delta N \dots ; \quad (\text{A.7})$$

thus

$$\frac{\partial \xi^i}{\partial N^j|_{\bar{g}}} = A_j^i . \quad (\text{A.8})$$

From $Q^j \mathcal{U}_{ij} = -\mathcal{E}_i$ and using that $\mathcal{U}_{ij} = -2\bar{U}' \bar{\gamma}_{ij} + O(\xi)^2$ we have that

$$\frac{\partial \xi^i}{\partial N^j|_{\bar{g}}} = \delta_j^i \left(a + \frac{\bar{\mathcal{E}}_v''}{2\bar{U}'} \right)^{-1} ; \quad (\text{A.9})$$

thus

$$\mathcal{X} = \left(a + \frac{\bar{\mathcal{E}}_v''}{2\bar{U}'} \right)^{-1} . \quad (\text{A.10})$$

In particular

$$\mathcal{X}|_{Q=0} = a^{-1}. \quad (\text{A.11})$$

The quadratic expansion of total action for an Universe dominated by dark energy induced by massive gravity modification can be written as

$$\begin{aligned} S_{EH} &= \int d^4x \sqrt{g} M_{\text{Pl}}^2 R = \int d^4x M_{\text{Pl}}^2 [L_{(0)} + L_{(1)} + L_{(2)} + \dots]; \\ L_{(1)} &= L_{(1)}^{(s)} + L_{(1)}^{(v)} + L_{(1)}^{(t)}, \quad L_{(2)} = L_{(2)}^{(s)} + L_{(2)}^{(v)} + L_{(2)}^{(t)}. \end{aligned} \quad (\text{A.12})$$

The tensor part is given by

$$L_{(2)}^{(t)}{}_{\text{tot}} = \frac{a^2}{2} \chi'_{ij} \chi'_{ij} + \frac{a^2}{2} \chi_{ij} (\Delta + a^2 m_2^2) \chi_{ij}; \quad (\text{A.13})$$

For the vectors we get

$$L_{(2)}^{(v)}{}_{\text{tot}} = -\frac{a^2}{2} (u_i - s'_i) \Delta (u_i - s'_i) + 3 \frac{a^4}{2} m_1^2 u_i u_i - \frac{a^2}{2} s_i \Delta s_i m_2^2; \quad (\text{A.14})$$

Finally, for scalars we have

$$\begin{aligned} L_{(1)}^{(s)}{}_{\text{tot}} &= a^2 \left(\frac{m^2 a^2 \bar{\mathcal{U}}}{2} - 3\mathcal{H}^2 \right) \psi + a^2 \left[a^2 m^2 \left(\bar{\mathcal{U}}' - \frac{\bar{\mathcal{U}}}{2} \right) + \mathcal{H}^2 + 2\mathcal{H}' \right] 3\tau; \\ L_{(2)}^{(s)}{}_{\text{tot}} &= \frac{a^2}{4} \left\{ -6(\tau' + \mathcal{H}\psi)^2 + 2(2\psi - \tau)\Delta\tau + 4(\tau' + \mathcal{H}\psi)\Delta(2v - \sigma') \right. \\ &\quad \left. + a^2 \left[m_0^2 \psi^2 - 2m_1^2 v \Delta v - m_2^2 (\sigma \Delta^2 \sigma + 2\tau \Delta \sigma + 3\tau^2) \right. \right. \\ &\quad \left. \left. + m_3^2 (\Delta \sigma + 3\tau)^2 - 2m_4^2 \psi (\Delta \sigma + 3\tau) \right] \right\}. \end{aligned} \quad (\text{A.15})$$

We have set $8\pi G = 1/(2M_{\text{Pl}}^2)$. Notice that the linear term in the scalar action gives the background equations of motion. The tensor, vector and scalar parts of the action precisely coincides with ones studied in [41] and basically we can use all results in there using the values (A.5) for the masses (the only difference in comparison with [41] is that m_2^2 and m_3^2 are shifted in that work).

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