

## Decoherence from ensembles of two-level fluctuators

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2006 New J. Phys. 8 1

(<http://iopscience.iop.org/1367-2630/8/1/001>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 38.107.179.213

The article was downloaded on 20/02/2012 at 10:28

Please note that [terms and conditions apply](#).

## Decoherence from ensembles of two-level fluctuators

Josef Schrieffer<sup>1</sup>, Yuriy Makhlin<sup>2</sup>, Alexander Shnirman<sup>1</sup>  
and Gerd Schön<sup>1</sup>

<sup>1</sup> Institut für Theoretische Festkörperphysik, Universität Karlsruhe,  
D-76128 Karlsruhe, Germany

<sup>2</sup> Landau Institute for Theoretical Physics, Kosygin st. 2, 119 334 Moscow,  
Russia

E-mail: [shnirman@tfp.uni-karlsruhe.de](mailto:shnirman@tfp.uni-karlsruhe.de)

*New Journal of Physics* **8** (2006) 1

Received 21 October 2005

Published 20 January 2006

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/8/1/001

**Abstract.**  $1/f$  noise, the major source of dephasing in Josephson qubits, may be produced by an ensemble of two-level systems. Depending on the statistical properties of their distribution, the noise distribution can be Gaussian or non-Gaussian. The latter situation is realized, for instance, when the distribution of coupling strengths has a slowly decaying power-law tail. In this regime, questions of self-averaging and sample-to-sample fluctuations become crucial. We study the dephasing process for a class of distribution functions and analyse the self-averaging properties of the results.

**Contents**

<b>1. Introduction</b>	<b>2</b>
<b>2. <math>1/f</math> noise from two-level fluctuators (TLFs)</b>	<b>3</b>
<b>3. Distribution of coupling strengths, self-averaging</b>	<b>3</b>
<b>4. Longitudinal and transverse noise coupling</b>	<b>5</b>
<b>5. Bloch–Redfield theory</b>	<b>5</b>
<b>6. Pure dephasing for Gaussian noise, <math>\mu &gt; 2</math></b>	<b>6</b>
6.1. $1/f$ spectrum . . . . .	7
<b>7. Individual fluctuators</b>	<b>8</b>
<b>8. Non-Gaussian effects, <math>\mu &gt; 2</math></b>	<b>9</b>
8.1. Free induction decay . . . . .	9
8.2. Echo signal decay . . . . .	10
<b>9. Quadratic coupling</b>	<b>10</b>
9.1. $1/f$ noise . . . . .	11
9.2. Quasi-static case. . . . .	11
<b>10. Conclusions</b>	<b>13</b>
<b>Acknowledgments</b>	<b>13</b>
<b>References</b>	<b>13</b>

**1. Introduction**

In Josephson qubits, dephasing is dominated by low-frequency noise, often with a  $1/f$  power spectrum, due to fluctuations of background charges, magnetic fluxes or critical currents [1]–[3]. While irrelevant for the relaxation process with a timescale  $T_1$ , low-frequency noise dominates the dephasing time  $T_2^*$ . Standard NMR echo techniques allow one to reduce dephasing by rendering the low-frequency spectrum ineffective [1]. Operation at optimal bias points, chosen such that the linear longitudinal coupling of the qubit to the  $1/f$  noise source vanishes, proved to be very successful in increasing the dephasing time [2]. Further progress in this direction may require an improved understanding of the mechanisms causing  $1/f$  noise and of its statistical properties. It was realized recently that qubits themselves can be used to study the noise properties of their environment [4, 5], and an interesting relation between the low-frequency  $1/f$  and the high-frequency charge noise was observed [6]. An extensive study of dephasing due to both charge and flux noise was undertaken in [7].

Still, many questions remain open. If the number of fluctuators contributing to the  $1/f$  noise is large, one could expect Gaussian statistics [1, 8]. In [9] and following work [10], the role of individual, strongly coupled fluctuators was emphasized, and it was suggested that even ensembles of many fluctuators may produce strong non-Gaussian effects, emerging as a result of rare configurations in which dephasing is dominated by a small number of very strongly coupled fluctuators. As far as we can judge, the decay laws observed in [7] cannot be fully explained by either of these theories.

As the experiments are performed on individual systems with a particular configuration of the fluctuators, it is important to understand whether the predicted decay laws are self-averaging or have strong sample-to-sample fluctuations. Here we will analyse a class of distribution functions

for the coupling strengths of the fluctuators. We determine the ensemble-averaged decay laws (extending the results of [10]) and analyse which of them are self-averaging. We study both dephasing due to linear longitudinal coupling and dephasing at the optimal point where the coupling is quadratic.

## 2. $1/f$ noise from two-level fluctuators (TLFs)

$1/f$  noise is often attributed to a collection of bistable systems, switching randomly between two states [11]. On one hand, such a model provides a natural explanation of  $1/f$  noise. On the other hand, in many samples distinct TLFs were detected. In metals, this switching causes conductance fluctuations [12, 13] and, consequently,  $1/f$  noise of transport current. In Josephson junctions, it makes the critical current to fluctuate [14, 15]. More generally, spin bath environments were analysed in [24]. In charge qubits, the TLFs contribute to the fluctuations of the gate charge controlling the qubit. The TLFs are characterized by their coupling strengths to the qubit,  $v_n$ , which may vary depending on the location of the respective TLF. The fluctuating quantity that couples to the qubit,  $X(t)$ , contains contributions from all TLFs:

$$X(t) = \sum_n v_n \sigma_{n,z}(t). \quad (1)$$

Each fluctuator switches randomly between two positions, denoted by  $\sigma_{n,z} = \pm 1$ , with a rate  $\gamma_n$  (for simplicity, we assume equal rates in both directions for the relevant TLFs) and thus contributes to the noise power  $S_X(\omega) = \sum_n S_n(\omega)$  with

$$S_n(\omega) = \frac{2\gamma_n v_n^2}{\omega^2 + \gamma_n^2}. \quad (2)$$

A set of TLFs produces  $1/f$  noise when the switching rate  $\gamma$  depends exponentially on a physical quantity,  $l$ , with a smooth distribution. For instance,  $\gamma \propto e^{-l/l_0}$ , with  $l$  distributed uniformly over a range much wider than  $l_0$ , translates in a log-uniform distribution of the switching rates, with probability density  $P(\gamma) \propto 1/\gamma$  in the corresponding exponentially wide range  $\gamma_{\min} \ll \gamma \ll \gamma_{\max}$ . In this range the total noise power thus scales as

$$S_X(\omega) \propto \int \frac{d\gamma}{\gamma} \frac{2v^2\gamma}{\gamma^2 + \omega^2} \propto \frac{v^2}{|\omega|}. \quad (3)$$

An example is a particle trapped in a double-well potential, whose tunnelling rate through the potential barrier depends exponentially on both the height and the width of the barrier, leading to  $1/f$  noise. Another example is thermally activated tunnelling with rate  $\gamma_0 e^{-E/k_B T}$ , where  $E$  denotes an activation energy. In this way, the  $1/f$  power spectrum observed in metals can be attributed to a broad (much wider than  $k_B T$ ) distribution of activation energies [16].

## 3. Distribution of coupling strengths, self-averaging

The analysis of decoherence in the presence of many fluctuators requires the study of probability distributions of coupling strengths and switching rates. In each particular sample, one deals with

specific fluctuators, i.e., with a realization of the set of parameters  $v$  and  $\gamma$ , drawn from this distribution. One should distinguish between quantities averaged over a statistical ensemble of samples and the results for a specific sample. This difference is essential, if the quantity under consideration is not self-averaging, i.e., if it has considerable sample-to-sample fluctuations. Such a situation arises if a quantity is dominated by contributions from a small number of TLFs.

In [10], a continuous distribution of the parameters  $v_n$  and  $\gamma_n$  was considered, with a long tail of the distribution of coupling strengths  $v_n$ , such that rare configurations with very large  $v_n$  dominate certain ensemble properties. It arises, e.g., from a uniform spatial distribution of fluctuators on a  $d$ -dimensional surface and a power-law TLF–qubit coupling [10],  $v(r) \propto 1/r^b$ . This results in a distribution of coupling strengths  $P(v) \propto 1/v^{1+d/b}$ . The joint distribution  $P(v, \gamma)$ , defined in the domain  $[v_{\min}, \infty) \times [\gamma_{\min}, \gamma_{\max}]$  and normalized to describe  $N$  fluctuators, is thus

$$P(v, \gamma) = \frac{c}{\gamma} \frac{\mu \eta^\mu}{v^{1+\mu}}. \quad (4)$$

Here  $\mu = d/b > 0$ ,  $c = 1/\ln(\gamma_{\max}/\gamma_{\min})$  and  $\eta = v_{\min} N^{1/\mu}$ . One can also allow for fluctuations of  $N$ , but this does not change the results significantly.

We consider a  $d$ -dimensional volume of typical size  $r_{\max}$  around the qubit containing a uniform distribution of TLFs. The typical distance between the strongest (closest) fluctuator and the qubit thus scales as  $r_{\min} \sim (V/N)^{1/d} \sim r_{\max}/N^{1/d}$ . On the other hand, since the coupling strength was assumed to decay as  $v(r) \propto 1/r^b$ , the relation between the strongest and weakest coupling strength is given by  $v_{\max}/v_{\min} = (r_{\max}/r_{\min})^b$ . Combining both results, we find that the typical maximal coupling strength scales as  $v_{\max}^{\text{typ}} \sim v_{\min} N^{1/\mu}$ . This does not exclude the existence of fluctuators with  $v \gg v_{\max}^{\text{typ}}$  in certain realizations, as the long tail of the distribution function suggests.

As examples of averaging over the distribution of coupling strengths and switching rates, we calculate the noise produced by the ensemble of fluctuators,

$$S_X(\omega) = \int dv d\gamma P(v, \gamma) \frac{2v^2 \gamma}{\gamma^2 + \omega^2} \quad (5)$$

distinguishing two cases: in one case, for  $\mu < 2$  the integral over  $v$  diverges at the upper limit. Hence the noise is dominated by the strongest fluctuator(s). Thus the result is sensitive to the properties of one or a few fluctuators and is therefore not self-averaging. Estimates below are based on cutting the integral at  $v = \eta = v_{\max}^{\text{typ}}$  but one has to remember that for a comparison with experiment averaging (including the averaging over  $\gamma$ ) makes little sense, since only a few TLFs contribute.

In contrast, for  $\mu > 2$  the integral is dominated by fluctuators with  $v < \eta$ . The weak fluctuators are most important, and due to their large number the noise is given by a sum of many comparable independent contributions. Consequently, the result is self-averaging, i.e., in different samples or runs of the experiment with  $\mu > 2$  one should observe the same noise amplitude.

We now summarize the typical/average results for the noise, retaining only the leading contributions

$$S_X(\omega) = \frac{2\pi A}{|\omega|} \quad \text{with } A = \begin{cases} c \frac{\mu}{2-\mu} \eta^2: & \mu < 2 \text{ (typically)}, \\ c N \langle v^2 \rangle: & \mu > 2. \end{cases} \quad (6)$$

For  $\mu > 2$  we defined the average coupling strength of the TLFs,  $\langle v^2 \rangle = \frac{1}{N} \int dv P(v)v^2$ . Note that (6) is only valid for frequencies  $\gamma_{\min} \ll |\omega| \ll \gamma_{\max}$ . At lower frequencies,  $|\omega| < \gamma_{\min}$ ,  $S_X(\omega)$  tends to a constant, whereas at higher frequencies,  $|\omega| > \gamma_{\max}$ ,  $S_X(\omega)$  crosses over to a faster power-law decay  $\propto 1/\omega^2$ .

#### 4. Longitudinal and transverse noise coupling

We consider a qubit controlled (for simplicity) by a single parameter  $\lambda$  and Hamiltonian

$$H_{qb} = -\frac{1}{2} \vec{H}_0(\lambda) \vec{\sigma}. \quad (7)$$

After an initial preparation in a coherent superposition of the qubit's eigenstates, the effective spin precesses under the influence of the static field  $\vec{H}_0$ , set by the control parameter  $\lambda_0$ . Coupling to the environment disturbs this evolution, leading to decoherence. In many cases the effect of the environment can be modelled by classical and quantum fluctuations of  $\lambda(t) = \lambda_0 + X(t)$ , where  $X(t)$  fluctuates. For instance, in a charge qubit, electromagnetic fluctuations of the control circuit as well as the background charge noise influence the gate voltage which controls the qubit.

To proceed we expand the Hamiltonian  $H_{qb}$  to second order in the perturbation  $X$ ,

$$H_{qb} = -\frac{1}{2} \left[ \vec{H}_0(\lambda_0) + \frac{\partial \vec{H}_0}{\partial \lambda} X + \frac{\partial^2 \vec{H}_0}{\partial \lambda^2} \frac{X^2}{2} + \dots \right] \vec{\sigma}. \quad (8)$$

Introducing the notations  $\vec{D}_\lambda \equiv (1/\hbar) \partial \vec{H}_0 / \partial \lambda$  and  $\vec{D}_{\lambda^2} \equiv (1/\hbar) \partial^2 \vec{H}_0 / \partial \lambda^2$ , we find in the eigenbasis of  $\vec{H}_0(\lambda_0) \vec{\sigma}$ :

$$H_{qb} = -\frac{1}{2} \hbar (\omega_{01} \sigma_z + \delta\omega_z \sigma_z + \delta\omega_\perp \sigma_\perp), \quad (9)$$

where  $\hbar\omega_{01} \equiv |\vec{H}_0(\lambda_0)|$ ,  $\delta\omega_z \equiv D_{\lambda,z} X + D_{\lambda^2,z} X^2/2 + \dots$  and  $\delta\omega_\perp \equiv D_{\lambda,\perp} X + \dots$ . Here  $\sigma_\perp$  denotes the transverse spin components (i.e.,  $\sigma_x$  or  $\sigma_y$ ). The coefficients  $D$  are related to the derivatives of  $\omega_{01}(\lambda)$ :

$$\frac{\partial \omega_{01}}{\partial \lambda} = D_{\lambda,z} \quad (10)$$

and

$$\frac{\partial^2 \omega_{01}}{\partial \lambda^2} = D_{\lambda^2,z} + \frac{\vec{D}_{\lambda,\perp}^2}{\hbar\omega_{01}}. \quad (11)$$

Thus, in general, the coupling of noise to the qubit contains both transverse ( $\delta\omega_z$ ) and longitudinal ( $\delta\omega_\perp$ ) parts, and both may have linear as well as higher order (e.g., quadratic) contributions.

#### 5. Bloch–Redfield theory

For weak, short-correlated noise the dynamics of the two-level systems (spins, qubits) can be summarized by Bloch's equations [17, 18] in terms of two rates: the longitudinal relaxation

(depolarization) rate  $\Gamma_1 = T_1^{-1}$ , and the transverse relaxation (dephasing) rate  $\Gamma_2 = T_2^{-1}$ . Evaluated perturbatively, using the golden rule, the rates are given by

$$\Gamma_1 = \frac{1}{2} S_{\delta\omega_{\perp}}(\omega = \omega_{01}) = \frac{1}{2} D_{\lambda,\perp}^2 S_X(\omega = \omega_{01}), \quad (12)$$

and

$$\Gamma_2 = \frac{1}{2} \Gamma_1 + \Gamma_{\varphi}, \quad (13)$$

where

$$\Gamma_{\varphi} = \frac{1}{2} S_{\delta\omega_z}(\omega = 0) = \frac{1}{2} D_{\lambda,z}^2 S_X(\omega = 0). \quad (14)$$

The dephasing process (13) is a combination of depolarization effects ( $\Gamma_1$ ) and of the so-called ‘pure’ dephasing, characterized by the rate  $\Gamma_{\varphi} = T_2^{*-1}$ . The pure dephasing is usually associated with the inhomogeneous level broadening in ensembles of spins, but occurs also for a single spin due to the longitudinal low-frequency noise.

## 6. Pure dephasing for Gaussian noise, $\mu > 2$

If there are sufficiently many fluctuators in the environment, the central limit theorem (CLT) applies, and the noise is Gaussian. More specifically, since the CLT applies to a large collection of equally distributed random quantities, one needs to have a large number of TLFs of each (relevant) ‘kind’ (i.e., for each pair  $\nu, \gamma$ ). This implies a regular distribution of coupling strengths, so that the relevant physical quantities are not dominated by a few TLFs at a boundary of the distribution.

In particular, the distribution (4) gives rise to Gaussian noise if  $\mu > 2$ . We will discuss now the pure dephasing derived from such Gaussian noise. The random phase accumulated at time  $t$ ,

$$\Delta\phi = D_{\lambda,z} \int_0^t dt' X(t'),$$

is then also Gaussian distributed. Hence the decay law, due to longitudinal noise (coupling to  $\sigma_z$ ) in a free induction decay (Ramsey signal) is given by  $f_R(t) = \langle \exp(i\Delta\phi) \rangle = \exp(-(1/2)\langle \Delta\phi^2 \rangle)$ . Averaging here is over the different trajectories of  $X(t)$  in repeated runs of the dephasing experiment. We obtain

$$f_R(t) = \exp \left[ -\frac{t^2}{2} D_{\lambda,z}^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_X(\omega) \text{sinc}^2 \frac{\omega t}{2} \right], \quad (15)$$

where  $\text{sinc } x \equiv \sin x/x$ . If most of the noise power is concentrated at frequencies  $\omega \ll 1/t$  (static noise), then one can approximate  $\text{sinc} \frac{\omega t}{2} \approx 1$  and obtain

$$f_R^{\text{stat}}(t) = \exp \left( -\frac{t^2}{2} D_{\lambda,z}^2 \sigma_X^2 \right), \quad (16)$$

where  $\sigma_X^2 = \int_{-\infty}^{+\infty} (d\omega/2\pi) S_X(\omega)$  is the dispersion of  $X$ . In general, for static noise with (not necessarily Gaussian) distribution function  $P(X)$ , the Ramsey decay is given by

$$f_R^{\text{stat}}(t) = \int d(X) P(X) e^{iD_{\lambda,z} X t}, \quad (17)$$

i.e., by the Fourier transform of  $P(X)$ . Static noise corresponds to a situation when  $X$  is constant during each run of the experiment but fluctuates between different runs.

In an echo experiment, the phase acquired is the difference between the two free evolution periods:

$$\Delta\phi_E = -\Delta\phi_1 + \Delta\phi_2 = -D_{\lambda,z} \int_0^{t/2} dt' X(t') + D_{\lambda,z} \int_{t/2}^t dt' X(t'), \quad (18)$$

which after averaging over the trajectories of  $X(t)$  gives

$$f_E(t) = \exp \left[ -\frac{t^2}{2} D_{\lambda,z}^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_X(\omega) \sin^2 \frac{\omega t}{4} \text{sinc}^2 \frac{\omega t}{4} \right]. \quad (19)$$

### 6.1. $1/f$ spectrum

Here and below we assume that the  $1/f$  law extends over a wide range of frequencies, limited by infrared and ultraviolet cut-offs,

$$S_X(\omega) = \frac{2\pi A}{|\omega|} = \frac{A}{|\nu|}, \quad \text{for } \omega_{\text{ir}} \ll |\omega| \ll \omega_c. \quad (20)$$

The infrared cut-off  $\omega_{\text{ir}}$  is usually determined by the measurement protocol, as discussed further below. The decay rates typically depend only logarithmically on  $\omega_{\text{ir}}$ , and details of the noise power below  $\omega_{\text{ir}}$  are irrelevant to logarithmic accuracy. For most of our analysis, the same applies to the ultraviolet cut-off  $\omega_c$ . However, for some specific questions considered below, frequency integrals may be dominated by  $\omega \approx \omega_c$ , and thus the detailed behaviour near and above  $\omega_c$  (i.e. the ‘shape’ of the cut-off) is relevant. We will refer to an abrupt suppression above  $\omega_c$  ( $S(\omega) \propto \theta(\omega_c - |\omega|)$ ) as a ‘sharp cut-off’, and to a crossover at  $\omega \sim \omega_c$  to a faster decay  $1/\omega \rightarrow 1/\omega^2$  (motivated by modelling of the noise via a set of bistable fluctuators, see below), as a ‘soft cut-off’.

For  $1/f$  noise, at times  $t \ll 1/\omega_{\text{ir}}$ , the free induction (Ramsey) decay is dominated by the frequencies  $\omega < 1/t$ , i.e., by the quasi-static contribution [19], and (15) reduces to

$$f_R(t) = \exp \left[ -t^2 D_{\lambda,z}^2 A \left( \ln \frac{1}{\omega_{\text{ir}} t} + O(1) \right) \right]. \quad (21)$$

Here the logarithmically large part of the exponent originates from a static contribution of frequencies  $\omega < 1/t$ . Indeed, it can be obtained from equation (16) with  $\sigma_X^2 = 2 \int_{\omega_{\text{ir}}}^{1/t} (d\omega/2\pi) S_X(\omega) = A \ln(1/\omega_{\text{ir}} t)$ . This contribution dominates the decay of  $f_R(t)$ .

For the echo decay, we obtain

$$f_E(t) = \exp(-t^2 D_{\lambda,z}^2 A \cdot \ln 2). \quad (22)$$

The echo method thus increases the decay time only by a logarithmic factor. This low efficiency of the echo has its origin in the high-frequency tail of the  $1/f$  noise, which, as we note, influences the results strongly. For  $1/f$  noise with a low cut-off  $\omega_c$ , the integral in equation (19) over the interval  $\omega \lesssim \omega_c$  is dominated by the upper limit. For instance, in the case of a sharp cut-off, i.e.,  $S = (A/|\omega|)\theta(\omega_c - \omega)$ , we obtain

$$f_E(t) = \exp\left(-\frac{1}{32}D_{\lambda,z}^2 A \omega_c^2 t^4\right). \quad (23)$$

On the other hand, for a soft cut-off, which we expect when the noise is produced by a collection of bistable fluctuators with Lorentzian spectrum, the integral in equation (19) is dominated by frequencies  $\omega_c < \omega < 1/t$ , and we find  $\ln f_E(t) \propto D_{\lambda,z}^2 A \omega_c t^3$ . In either case, one finds that the decay is slower by a factor  $\sim (\omega_c t)^2$  or  $\omega_c t$ , respectively, than for  $1/f$  noise with a high cut-off,  $\omega_c > D_{\lambda,z} A^{1/2}$ .

## 7. Individual fluctuators

We consider a single fluctuator coupled longitudinally to the qubit, whose contribution to the level splitting,  $v_n(t) \equiv v_n \sigma_{n,z}(t)$ , switches between  $\pm v_n$ . For this case, the free induction (Ramsey) and echo decays have been evaluated in [9, 10]. In the limit of high effective temperature, i.e., when the transition rates in both directions are equal, the decay functions, obtained by averaging over the switching history of  $\sigma_{n,z}(t)$ , are given by

$$f_{R,n}(t) = e^{-\gamma_n t} \left( \cos \mu_n t + \frac{\gamma_n}{\mu_n} \sin \mu_n t \right), \quad (24)$$

and

$$f_{E,n}(t) = e^{-\gamma_n t} \left[ 1 + \frac{\gamma_n}{\mu_n} \sin \mu_n t + \frac{\gamma_n^2}{\mu_n^2} (1 - \cos \mu_n t) \right], \quad (25)$$

where  $\mu_n \equiv \sqrt{(Dv_n)^2 - \gamma_n^2}$  and  $D \equiv D_{\lambda,z}$ . In order to derive these expressions, we introduce the averaged phase factors  $\chi_{\pm}(t) = \langle \exp(i \int^t dt' D v_n(t')) \rangle$ , averaged over the switching histories ending at  $v_n(t) = +v_n$  or  $-v_n$ , respectively. Their dynamics are governed by the rate equations

$$\begin{aligned} \dot{\chi}_+ &= iDv_n \chi_+ - \gamma_n \chi_+ + \gamma_n \chi_-, \\ \dot{\chi}_- &= -iDv_n \chi_- - \gamma_n \chi_- + \gamma_n \chi_+. \end{aligned} \quad (26)$$

The solution for  $f_{R,n}(t) = \chi_+(t) + \chi_-(t)$  is obtained by solving the coupled equations for the initial conditions  $\chi_{\pm} = 1/2$ , which gives equation (24). Similarly, for more general protocols, we have to analyse phase factors  $\langle \exp(i \int^t dt' D g(t') v_n(t')) \rangle$  with appropriate time dependence of  $g(t)$ . In this case the first terms on the right-hand side of equations (26) are modified accordingly. For the echo experiment we obtain in this way, equation (25).

The decay produced by a number of fluctuators is the product of the individual contributions, i.e.,  $f_R(t) = \prod_n f_{R,n}(t)$  and  $f_E(t) = \prod_n f_{E,n}(t)$ . If the noise is dominated by a few fluctuators (this includes the case of many fluctuators in total, but a few of them with similar rates  $\gamma$ ), the fluctuations of  $X(t)$  may be strongly non-Gaussian.

## 8. Non-Gaussian effects, $\mu > 2$

Since we consider uncorrelated TLFs the total decay of coherence is the product of all single-TLF contributions,  $f(t) = \prod_n f_n(t)$ , where  $f_n(t)$  is given by (24) and (25) for the free induction and echo experiment, respectively. In [10] an ensemble-averaged value of  $\ln f(t)$ , denoted as  $\langle \ln f(t) \rangle_F$ , was calculated for  $\mu = 1$ . Here  $\langle \dots \rangle_F$  denotes the average over the distribution of coupling strengths and switching rates (4). Both free induction decay and the ‘phase memory decay’ (a protocol similar but not equivalent to the spin echo decay) were analysed in the regimes  $t < \gamma_{\max}^{-1}$  and  $t > \gamma_{\max}^{-1}$ . Below we will generalize these results to the range  $0 < \mu < 2$ .

As discussed above, the quantity  $\langle \ln f(t) \rangle_F$  is relevant for experiments with specific samples only if the sample-to-sample fluctuations of  $\ln f(t)$  are weak, i.e., if  $\ln f(t)$  is self-averaging. Then, experimentally observable decay law  $f(t)$  would be well approximated by  $\exp(\langle \ln f(t) \rangle_F)$ . In [10] the self-averaging was numerically confirmed for the phase memory decay in the regime  $t < \gamma_{\max}^{-1}$ . Here we analyze the self-averaging in four regimes: for the free induction and the echo cases, both in the limits  $t < \gamma_{\max}^{-1}$  and  $t > \gamma_{\max}^{-1}$ . Specifically, we evaluate the ensemble average  $\langle \ln f(t) \rangle_F$ , given by an integral over the  $(v, \gamma)$ -space. In some cases this integral is dominated by a range in the ‘bulk’ of the distribution, which contains many fluctuators on average; this indicates that sample-to-sample fluctuations are weak. In other cases, the integral is dominated by the boundary of the distribution, indicating that the studied quantity is not self-averaging. Our analysis confirms the conclusion given in [10], obtained in one regime: for the echo decay at short times  $t < \gamma_{\max}^{-1}$ . We show further that in all other three regimes investigated, the dephasing law is not self-averaging. In the calculations we assume that  $v_{\min}$  and  $\gamma_{\min}$  are very low-frequency scales, and  $1/t$  always exceeds them.

### 8.1. Free induction decay

For short times,  $t < \gamma_{\max}^{-1}$ , we are effectively in the static regime, and the ensemble-averaged free induction decay is described by

$$\langle \ln |f_R(t)| \rangle_F \propto -(D_{\lambda,z} \eta t)^\mu. \quad (27)$$

This result is dominated by the fluctuators with strength of order  $v \sim v_{\max}^{\text{typ}} \sim \eta$  and thus is not self-averaging. For an experiment with a specific sample, the results should be fitted by a contribution of one (24) or a few fluctuators, rather than by the ensemble-averaged behaviour (27). We can also estimate the typical decay law for short times  $t < \eta^{-1}$ . In every realization, there will be a few strongest fluctuators, typically with strength  $v_{\max}$ . For  $t \ll v_{\max}^{-1}$ , we obtain  $\ln |f_R(t)| \approx -D_{\lambda,z}^2 t^2 \sum_n v_n^2$ . For distributions with  $\mu < 2$ , the sum  $\sum_n v_n^2$  is dominated by the largest  $v_n$ ’s, and thus,  $\ln |f_R(t)| \propto -D_{\lambda,z}^2 t^2 v_{\max}^2$ . Finally, we can calculate the distribution function for the strength of the strongest fluctuator  $v_{\max}$  and obtain

$$P(v_{\max}) = \frac{\mu \eta^\mu}{v_{\max}^{1+\mu}} e^{-(\eta/v_{\max})^\mu}. \quad (28)$$

Most of the weight of this distribution is around  $v_{\max} \sim \eta$ . Thus, in a typical sample for  $t < \eta^{-1}$ , the decay is given by  $\ln |f_R(t)| \propto -(D_{\lambda,z} \eta t)^2$  rather than by (27). To understand the difference, we note that the average decay law (27) can also be obtained by averaging the realization-dependent

$-D_{\lambda,z}^2 v_{\max}^2 t^2$  (valid for  $t < v_{\max}^{-1}$ ) over the distribution (28). This average is dominated by rare samples with a fluctuator of strength  $v_{\max} \sim 1/t$  rather than by typical samples.

For longer times,  $t > \gamma_{\max}^{-1}$ , the integration gives

- for  $1 \leq \mu < 2$

$$\langle \ln |f_R(t)| \rangle_F \propto - \left| \frac{\ln \gamma_{\min} t}{\ln(\gamma_{\max}/\gamma_{\min})} \right| (D_{\lambda,z} \eta t)^\mu, \quad (29)$$

- for  $\mu < 1$

$$\langle \ln |f_R(t)| \rangle_F \propto -c(D_{\lambda,z} \eta / \gamma_{\max})^\mu \gamma_{\max} t. \quad (30)$$

Both results are not self-averaging.

## 8.2. Echo signal decay

For short times,  $t < \gamma_{\max}^{-1}$ , we find

$$\langle \ln |f_E(t)| \rangle_F \propto -c(D_{\lambda,z} \eta)^\mu \gamma_{\max} t^{1+\mu}, \quad (31)$$

For  $c^{1/\mu} D_{\lambda,z} \eta > \gamma_{\max}$ , the echo decay is dominated by this quasi-static contribution; the decay takes place on the timescale shorter than the flip time of the fastest fluctuators,  $1/\gamma_{\max}$ . In this regime ( $c^{1/\mu} D_{\lambda,z} \eta > \gamma_{\max}$ ), the result is self-averaging since it is dominated by fluctuators with  $D_{\lambda,z} v \sim (c D_{\lambda,z}^\mu \eta^\mu \gamma_{\max})^{1/(1+\mu)} < c^{1/\mu} D_{\lambda,z} \eta < D_{\lambda,z} v_{\max}^{\text{typ}}$ .

For longer times,  $t > \gamma_{\max}^{-1}$ , the dephasing is due to multiple flips of the fluctuators. These times are relevant if  $c^{1/\mu} D_{\lambda,z} \eta < \gamma_{\max}$ . The decay law is given by

- for  $1 < \mu < 2$

$$\langle \ln |f_E(t)| \rangle_F \propto -c(D_{\lambda,z} \eta t)^\mu, \quad (32)$$

- for  $\mu = 1$

$$\langle \ln |f_E(t)| \rangle_F \propto -c(D_{\lambda,z} \eta t) \ln(\gamma_{\max} t), \quad (33)$$

- for  $\mu < 1$

$$\langle \ln |f_E(t)| \rangle_F \propto -c(D_{\lambda,z} \eta / \gamma_{\max})^\mu \gamma_{\max} t. \quad (34)$$

All these results are not self-averaging.

## 9. Quadratic coupling

At the optimal working point, the first-order longitudinal coupling  $D_{\lambda,z}$  vanishes. Thus, to first order, the decay of the coherent oscillations is determined by the relaxation processes and for regular power spectra at low frequencies one expects from equation (13) that  $\Gamma_2 = \Gamma_1/2$ . On the other hand, for power spectra which are singular at low frequencies, the second-order

contribution of the longitudinal noise can be comparable or even dominate over  $\Gamma_1/2$ . To evaluate this contribution, one has to calculate

$$f_2(t) = \left\langle \exp \left( i \frac{1}{2} \frac{\partial^2 \omega_{01}}{\partial \lambda^2} \int_0^t g(t') X^2(t') dt' \right) \right\rangle, \quad (35)$$

where for the analysis of the free induction decay (Ramsey signal) one sets  $g(t') = 1$ , while for decay of the echo signal  $g(t' < t/2) = -1$  and  $g(t' > t/2) = 1$ .

### 9.1. $1/f$ noise

The free induction decay for the  $1/f$  noise with a high cut-off  $\omega_c$  (the highest energy scale in the problem) has been analysed in [20]. Depending on the time  $t$ , the decay is dominated by low- or high-frequency noise, and the decay law can be approximated by a product of low-frequency ( $\omega < 1/t$ , quasi-static) and high-frequency ( $\omega > 1/t$ ) contributions,  $f_{2,R}(t) = f_{2,R}^{\text{lf}}(t) \cdot f_{2,R}^{\text{hf}}(t)$ . The contribution of low frequencies is given by [20]–[22]

$$f_{2,R}^{\text{lf}}(t) = \frac{1}{\sqrt{1 - i(\partial^2 \omega_{01}/\partial \lambda^2) \sigma_X^2 t}}. \quad (36)$$

For  $1/f$  noise the variance of the low-frequency fluctuations is  $\sigma_X^2 = 2A \ln(1/\omega_{\text{ir}} t)$ . This contribution dominates at short times  $t < [(\partial^2 \omega_{01}/\partial \lambda^2) A/2]^{-1}$ . At longer times, the high-frequency contribution

$$\ln f_{2,R}^{\text{hf}}(t) \approx -t \int_{\sim 1/t}^{\infty} \frac{d\omega}{2\pi} \ln \left[ 1 - 2\pi i \frac{\partial^2 \omega_{01}}{\partial \lambda^2} S_X(\omega) \right], \quad (37)$$

takes over. When  $t \gg [(\partial^2 \omega_{01}/\partial \lambda^2) A/2]^{-1}$  (provided  $\omega_c \gg \pi (\partial^2 \omega_{01}/\partial \lambda^2) A$ ), we obtain asymptotically  $\ln f_{2,R}^{\text{hf}}(t) \approx -(\pi/2)(\partial^2 \omega_{01}/\partial \lambda^2) A t$ . Otherwise, the quasi-static result (36) is valid at all relevant times. One can also evaluate the pre-exponential factor in the long-time decay. This pre-exponent decays very slowly (algebraically) but differs from 1 and thus further reduces  $f_{2,R}(t)$  [23].

### 9.2. Quasi-static case

In this case, i.e., when the cut-off  $\omega_c$  is lower than  $1/t$  for all relevant times, the Ramsey decay is simply given by the static contribution (36). At all relevant times, the decay is algebraic and the crossover to the exponential law is not observed. More generally, in the static approximation with a distribution  $P(\delta\lambda)$ , the dephasing law is given by the Fresnel-type integral transform,

$$f_{2,R}^{\text{st}}(t) = \int d(X) P(X) \exp \left( i \frac{1}{2} \frac{\partial^2 \omega_{01}}{\partial \lambda^2} X^2 t \right), \quad (38)$$

which reduces to equation (36) for a Gaussian  $P(X) \propto \exp(-X^2/2\sigma_X^2)$ . In general, any distribution  $P(X)$ , finite at  $X = 0$ , yields a long-time decay of  $f_{2,R}^{\text{st}}$  proportional to  $t^{-1/2}$ .

For  $\mu < 2$  the analysis is technically more complicated. In that case the distribution of initial conditions  $P(X_0)$  and equivalently the sum  $X_0 = \sum_n v_n \sigma_{n,z}(t=0)$  are no longer Gaussian distributed and, in particular, they cannot be characterized by a typical width  $\sigma$ , due to the divergence of the second moment  $\langle v^2 \rangle$  of (4). The generalized CLT tells us that  $x = X_0/\eta$  is then distributed according to a Lévy distribution  $L_{\mu,0}(x)$  and consequently, according to (38), the free induction decay in the quasi-static regime is given by

$$f_{2,R}^{\text{st}}(t) = \int dx L_{\mu,0}(x) \exp\left(\frac{i}{2} \frac{\partial^2 \omega_{01}}{\partial \lambda^2} \eta^2 t \cdot x^2\right). \quad (39)$$

For some values of  $\mu$  explicit expressions are known. An example is the Cauchy distribution,  $L_{1,0}(x) = 1/[\pi(1+x^2)]$ . Using (39), the free induction decay in the static regime,  $t < \gamma_{\text{max}}^{-1}$ , is then given by

$$f_{2,R}^{\text{st}}(t) = e^{-i\alpha t} \left[ 1 - \Phi\left(\sqrt{\alpha t/i}\right) \right]. \quad (40)$$

Here we introduced the rate

$$\alpha = \frac{1}{2} \frac{\partial^2 \omega_{01}}{\partial \lambda^2} \eta^2, \quad (41)$$

and  $\Phi(z) = 2\pi^{-1/2} \int_0^z dx e^{-x^2}$  denotes the error function. One can expand  $\Phi(z)$  in (40) to find the asymptotic behaviour of  $f_{2,R}^{\text{st}}(t)$  for  $\mu = 1$ :

$$|f_{2,R}^{\text{st}}(t)| = \begin{cases} 1 - \left(\frac{2}{\pi}\alpha t\right)^{1/2} & \text{for } t \ll \alpha^{-1}, \\ (\pi\alpha t)^{-1/2} & \text{for } t \gg \alpha^{-1}. \end{cases} \quad (42)$$

The initial decay for  $t \ll \alpha^{-1}$  is thus very fast, but at times  $t \approx \alpha^{-1}$  the decay crosses over to a much slower power law  $\propto 1/\sqrt{t}$ . The dephasing time scales as  $\alpha^{-1}$ , but with a relatively large prefactor due to the slow algebraic decay. For other values of  $\mu < 2$ , the asymptotic behaviour of  $f_{2,R}^{\text{st}}$  has been obtained in [23]:

$$|f_{2,R}^{\text{st}}(t)| = \begin{cases} 1 - C(\mu) (\alpha t)^{\mu/2} & \text{for } t \ll \alpha^{-1}, \\ D(\mu) (\alpha t)^{-1/2} & \text{for } t \gg \alpha^{-1}, \end{cases} \quad (43)$$

where  $C(\mu)$  and  $D(\mu)$  are factors of order 1.

Let us now discuss the shape of the decay of  $f_R^{\text{st}}$  qualitatively and comment on their validity. The initial decay of  $f_R^{\text{st}}$  for  $\mu < 2$  is singular and thus very fast compared to the Gaussian case  $\mu > 2$ . This initial decay is dominated by strongly coupled fluctuators, i.e., by the tail of the distribution (4). It is, thus, not self-averaging.

On the other hand, for longer times,  $t \gg \alpha^{-1}$ , the decay goes over to a much slower power law. The exponent  $-1/2$  is independent of  $\mu$  and coincides even with the prediction of the Gaussian model ( $\mu > 2$ ). Hence, the  $1/\sqrt{t}$  decay law appears to be universal in the presence of quasi-static noise, independent of the considered statistics. For low enough  $\gamma_{\text{max}}$  such that  $\alpha \gg \gamma_{\text{max}}$ , the free induction signal decays already in the quasi-static regime,  $t < \gamma_{\text{max}}^{-1}$ , and is

thus given by (43). Otherwise, further analysis characterizing the contribution of fast fluctuators,  $\gamma > t^{-1}$ , is needed to describe decoherence.

## 10. Conclusions

We have shown that non-Gaussian  $1/f$  noise of ensembles of TLFs frequently leads to non-self-averaging dephasing laws. Non-Gaussian noise arises, for instance, when the distribution of coupling strengths between the TLFs and a qubit has a long algebraic tail. In this case, since experiments are performed on specific samples, one should study the typical rather than ensemble-averaged behaviour. Interestingly, in certain regimes, e.g., for short-time echo decay, the decay law is self-averaging.

## Acknowledgments

The work is part of the CFN of the DFG and of the EU IST Project SQUBIT. YM acknowledges support from the Dynasty Foundation and the grant MD-2177.2005.2.

## References

- [1] Nakamura Y, Pashkin Yu A, Yamamoto T and Tsai J S 2002 Charge echo in a Cooper-pair box *Phys. Rev. Lett.* **88** 047901
- [2] Vion D, Aassime A, Cottet A, Joyez P, Pothier H, Urbina C, Esteve D and Devoret M H 2002 Manipulating the quantum state of an electrical circuit *Science* **296** 886
- [3] VanHarlingen D J, Robertson T L, Plourde B L T, Reichardt P A, Crane T A and Clarke J 2004 Decoherence in Josephson-junction qubits due to critical current fluctuations *Phys. Rev. B* **70** 064517
- [4] Aguado R and Kouwenhoven L P 2000 Double quantum dots as detectors of high-frequency quantum noise in mesoscopic conductors *Phys. Rev. Lett.* **84** 1986
- [5] Schoelkopf R J, Clerk A A, Girvin S M, Lehnert K W and Devoret M H 2003 Qubits as spectrometers of quantum noise *Quantum Noise in Mesoscopic Physics* ed Y V Nazarov (Dordrecht: Kluwer) pp 175–203 (Preprint [cond-mat/0210247](http://arxiv.org/abs/cond-mat/0210247))
- [6] Astafiev O, Pashkin Yu A, Nakamura Y, Yamamoto T and Tsai J S 2004 Quantum noise in the Josephson charge qubit *Phys. Rev. Lett.* **93** 267007
- [7] Ithier G *et al* 2005 Decoherence in a superconducting quantum bit circuit *Phys. Rev. B* **72** 134519
- [8] Shnirman A, Makhlin Yu and Schön G 2002 Noise and decoherence in quantum two-level systems *Phys. Scr. T* **102** 147
- [9] Paladino E, Faoro L, Falci G and Fazio R 2002 Decoherence and  $1/f$  noise in Josephson qubits *Phys. Rev. Lett.* **88** 228304
- [10] Galperin Y M, Altshuler B L and Shantsev D V 2004 Low-frequency noise as a source of dephasing of a qubit *Fundamental Problems of Mesoscopic Physics* ed I V Lerner, B L Altshuler and Yu Gefen (Dordrecht: Kluwer) (Preprint [cond-mat/0312490](http://arxiv.org/abs/cond-mat/0312490))
- [11] Bernamont J 1937 *Ann. Phys. (Leipzig)* **7** 71
- [12] Ludviksson A, Kree R and Schmid A 1984 Low-frequency  $1/f$  fluctuations of resistivity in disordered metals *Phys. Rev. Lett.* **52** 950
- [13] Kogan Sh M and Nagaev K E 1984 On the low-frequency current noise in metals *Solid State Commun.* **49** 387
- [14] Kozub V I 1984 Properties of superconducting S-I-N, S-I-S, and S-C-S structures with amorphous weak coupling *Sov. Phys.—JETP* **60** 810

- [15] Galperin Y M, Gurevich V L and Kozub V I 1989 Disorder-induced low-frequency noise in small systems—point and tunnel contacts in the normal and superconducting state *Europhys. Lett.* **10** 753
- [16] Dutta P and Horn P M 1981 Low-frequency fluctuations in solids:  $1/f$  noise *Rev. Mod. Phys.* **53** 497
- [17] Bloch F 1957 Generalized theory of relaxation *Phys. Rev.* **105** 1206
- [18] Redfield A G 1957 On the theory of relaxation processes *IBM J. Res. Dev.* **1** 19
- [19] Cottet A 2002 Implementation of a quantum bit in a superconducting circuit *PhD Thesis* Université Paris VI
- [20] Makhlin Yu and Shnirman A 2004 Dephasing of solid-state qubits at optimal points *Phys. Rev. Lett.* **92** 107001
- [21] Rabenstein K, Sverdlov V A and Averin D V 2004 Qubit decoherence by Gaussian low-frequency noise *JETP Lett.* **79** 783
- [22] Falci G, D'Arrigo A, Mastellone A and Paladino E 2005 Initial decoherence in solid state qubits *Phys. Rev. Lett.* **94** 167002
- [23] Schrieffl J 2005 Dekohärenz in Josephson quantenbits *PhD Thesis* University Karlsruhe
- [24] Prokof'ev N V and Stamp P C E 2000 Theory of the spin bath *Rep. Prog. Phys.* **63** 669