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Quantum master equation descriptions of a nanomechanical resonator coupled to a single-electron transistor

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Abstract. We analyse the quantum dynamics of a nanomechanical resonator coupled to a normal-state single-electron transistor (SET). Starting from a microscopic description of the system, we derive a master equation for the SET island charge and resonator which is valid in the limit of weak electromechanical coupling. Using this master equation we show that, apart from brief transients, the resonator always behaves like a damped harmonic oscillator with a shifted frequency and relaxes into a thermal-like steady state. Although the behaviour remains qualitatively the same, we find that the magnitude of the resonator damping rate and frequency shift depend very sensitively on the relative magnitudes of the resonator period and the electron tunnelling time. Maximum damping occurs when the electrical and mechanical timescales are the same, but the frequency shift is greatest when the resonator moves much more slowly than the island charge. We then derive reduced master equations which describe just the resonator dynamics. By making slightly different approximations, we obtain two different reduced master equations for the resonator. Apart from minor differences, the two reduced master equations give rise to a consistent picture of the resonator dynamics which matches that obtained from the master equation including the SET island charge.

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1. Introduction

A very interesting class of nanoelectromechanical systems (NEMS) is composed of a mesoscopic conductor, such as a quantum dot, point contact, or single-electron transistor (SET), coupled electrostatically to a nanomechanical resonator [1]–[17]. Such devices have potentially important applications as ultra-sensitive measuring devices [1, 4, 8], as well as being interesting dynamical systems in their own right. A question of particular interest for these systems is under what circumstances the mechanical degrees of freedom require quantum mechanics for their proper description.

When a resonator is coupled electrostatically to a mesoscopic conductor as a mechanically compliant voltage-gate, the position of the resonator affects the current flowing through the conductor and hence the latter can be used to monitor the motion of the resonator. However, the movement of electrons through the conductor necessarily acts back on the resonator affecting its dynamics in important ways. For weak electromechanical coupling, the back-action of the conductor on the resonator is typically analogous to an equilibrium thermal bath [3, 6, 7, 9, 11, 13], [15]–[17]. The stochastic motion of electrons through the conductor gives rise to a force on the resonator which leads to fluctuations in its state that can be described by an effective temperature. Furthermore, the conductor can also damp the oscillations of the resonator, so that it reaches a thermal-like steady state.¹

The damping and heating effects of a conductor on a nearby nanomechanical resonator were first reported by Mozyrsky and Martin [3] who examined the quantum dynamics of a resonator coupled to a quantum point contact (QPC). In this device [3, 11, 16], the tunnelling matrix elements of a tunnel junction depend linearly on the resonator position. The back-action of the QPC on the resonator was obtained by tracing over all the electronic degrees of freedom to derive a reduced density matrix for the resonator. The resulting resonator master equation was found to be of the Caldeira–Leggett form and hence an effective temperature and damping constant arising from the back-action of the electrons could be ascribed to the resonator.

¹ Recently, it has been shown that more complex conductors like superconducting SETs can also give rise to *negative* damping of the resonator under certain circumstances [17].

The dynamics of a resonator coupled to an SET has also been studied theoretically [6, 7, 9, 17]. In the simplest case, electrons tunnel sequentially through a metallic island or quantum dot gated by a mechanical resonator [7, 9]. A classical master equation description was proposed for this system in [7] and it was found that the resonator dynamics could be described by a Fokker–Planck equation [15] with the SET electrons again acting like a thermal bath. However, apart from weak electromechanical coupling, it was also assumed that the resonator moved very slowly on the timescale of the electron tunnelling time and that the energy associated with the SET bias voltage was much greater than the resonator quanta. A subsequent quantum mechanical study of the closely related system of a resonator coupled to a quantum dot [9] also showed that the back-action of the electrons could give rise to resonator damping, even without assuming a slow resonator or very high-bias voltage, but with a very different rate to that found in the classical study.

In this paper, we begin from a microscopic description of the SET–resonator system and proceed to derive reduced master equations which describe the dynamics of the resonator alone. In contrast to the QPC–resonator system, this derivation can proceed in two stages. In the first stage, we trace over the microscopic energy levels in the leads and the island having made an assumption of weak tunnelling through the SET junctions, obtaining a master equation that describes the behaviour of both the resonator and the SET charge. In the second stage, we then trace over the macroscopic charge degree of freedom of the SET, for which an assumption of weak electromechanical coupling between the SET and the resonator is essential. As well as emphasizing the different approximations made, deriving the resonator-only master equation in two stages allows us to investigate its accuracy by comparing its dynamics to that of the SET–resonator equation.

By first tracing over the microscopic electronic levels, we obtain a master equation which describes both the resonator and the excess charge on the SET island. We show that although this master equation is essentially equivalent to that proposed in the classical description [7], it can nevertheless be used to investigate the interesting question of what happens when the resonator period matches the electronic tunnelling time. By solving equations of motion for the resonator moments, we find that the resonator is damped and reaches a thermal-like steady state even when the resonator period is of order or less than the electron tunnelling time, with maximum damping occurring when the electrical and mechanical timescales are equal.

Having obtained a master equation describing the SET island charge and resonator, we then go on to derive, via different approximations, two reduced master equations for the resonator alone. Although the two reduced master equations have differences in form, we find that (within their domains of validity) they both lead to a description of the resonator dynamics which closely matches that obtained from the master equation for the resonator and SET island charge. In this framework, the two different expressions for the resonator damping obtained before in quantum [9] and classical treatments [7] emerge naturally as limiting values for the cases where the average tunnelling rate of the electrons is much less than the resonator frequency, and the opposite case where the resonator frequency is much smaller than the tunnel rate, respectively.

The outline of this paper is as follows. In section 2, we outline the details of the nanomechanical–SET system we consider and derive a master equation which describes the dynamics of the resonator and the excess charge on the SET island for linear electromechanical coupling. In section 3, we obtain equations of motion for the charge and resonator moments and hence explore how the resonator dynamics depend on important parameters such as the ratio of the relevant electrical and mechanical timescales. Then, in section 4, we trace over the

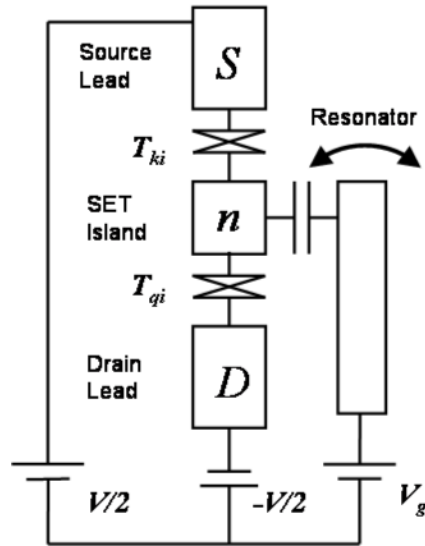


Figure 1. Schematic circuit diagram for the SET resonator system.

SET island charge to derive a reduced master equation for the resonator alone, and compare the resulting resonator dynamics with that obtained in the previous section. Next, in section 5, we use a rotating wave approximation (RWA) to obtain an alternative reduced master equation for the resonator, which is valid in the limit where the resonator frequency is much higher than the electron tunnelling rate. Again we compare the resonator dynamics predicted by this equation with the results of previous sections. Finally, in section 6, we present our conclusions and a brief discussion of our results. The appendices contain more details on certain aspects of the calculations.

2. Master equation for resonator and SET island charge

The nanomechanical resonator–SET system we consider is shown schematically in figure 1. The SET consists of a metallic island, with total capacitance C_{Σ} , connected via tunnel junctions to two leads, with a bias voltage V applied across it. The nanomechanical resonator is adjacent to the SET island and is coated with a metal layer so that it forms a mechanically compliant voltage gate to which a voltage V_g is applied. Motion of the resonator modulates the gate capacitance and hence the charging energy of the SET island, while changes in the charge on the SET island modulate the equilibrium position of the resonator. The resonator has a mass m and is treated as a single-mode harmonic oscillator with frequency ω_0 . Here, we are primarily interested in the effect of the SET on the resonator and hence do not include any other influences on the resonator’s dynamics in our description.

Because the SET is metallic, the island as well as the source and drain leads contain many (microscopic) electron energy levels. The source (drain) lead has energy levels $\epsilon_{k(q)}$ and a chemical potential $\mu_{S(D)} = -eV/2(+eV/2)$. The island has energy levels ϵ_i and we set its chemical potential to zero for simplicity. The resistances of the tunnel junctions, $R_{S(D)}$, are taken to be much larger than the quantum resistance, $R_Q = h/e^2$, so we can neglect higher order processes such as co-tunnelling [18, 19].

The dynamics of the resonator is affected by the overall charge on the SET island, rather than the details of which of the microscopic electron levels are occupied, hence we introduce the macroscopic charge operator, N , for the total number of excess electrons on the SET island. Using this macroscopic charge variable, we can integrate over the microscopic electronic states, whilst still keeping track of the overall island charge [18]. The operator ϕ , conjugate to N , can be used to form the operators $e^{i\phi}$ and $e^{-i\phi}$ which increase and decrease the overall charge on the island respectively.

The overall electrostatic energy of the SET island is determined not just by the island charge, but also by the gate voltage (which we write in dimensionless units as n_g) and the position of the resonator. We assume that the electromechanical coupling is weak, as has been the case in recent experiments [4, 8], and hence we consider only a linear coupling between the SET island charge and the resonator position [13]. Bearing these details in mind, we write the Hamiltonian for this system as [9,18, 19]:

$$H = \sum_k \epsilon_k c_k^\dagger c_k + \sum_q \epsilon_q c_q^\dagger c_q + \sum_i \epsilon_i c_i^\dagger c_i + E_C (N - n_g)^2 - \chi N (a^\dagger + a) + \hbar \omega_0 a^\dagger a + \sum_k T_{ki} (c_k^\dagger c_i e^{-i\phi} + c_k c_i^\dagger e^{+i\phi}) + \sum_q T_{qi} (c_q^\dagger c_i e^{-i\phi} + c_q c_i^\dagger e^{+i\phi}), \quad (1)$$

where $E_C = e^2/2C_\Sigma$ is the charging energy of the SET island, a is the resonator annihilation operator, c_k, c_q and c_i are the annihilation operators acting on the electron levels in the two leads and the island of the SET respectively, and $T_{k(q)i}$ are the tunnelling matrix elements between the microscopic states in the leads and the SET island. The coupling between the SET island charge and the resonator is given by $\chi = m\omega_0^2 x_0 x_q$, where x_0 is the shift in the equilibrium position of the resonator when an electron is added to the island (which in turn depends on details such as the gate capacitance and voltage as well as the resonator–island separation [7]), and $x_q = \sqrt{\hbar/2m\omega_0}$ is the zero-point position uncertainty of the resonator [10].

Assuming that the charging energy of the SET island, E_C , is much greater than the thermal energy of the electrons, and that the drain–source voltage is not too large (i.e. $eV/2 \lesssim E_C$), then only the two consecutive charge states $|N_0\rangle, |N_1\rangle$ closest to n_g will be involved in the dynamics of the system. In what follows, we will restrict the charge state basis to just these two states, and introduce the convenient operators $n = |N_1\rangle\langle N_1|$, $\sigma^+ = |N_1\rangle\langle N_0|$ and $\sigma^- = |N_0\rangle\langle N_1|$. Choosing the origin of the resonator’s position coordinate so that $\langle a^\dagger + a \rangle = 0$ corresponds to the equilibrium position when there are N_0 electrons on the SET island, then the effective electrostatic energy of the SET can be written (discarding constants) as $E_C(1 - 2n'_g)$, where $n'_g = n_g - N_0[1 - \lambda^2 \hbar \omega_0 / E_C]$. The parameter n'_g ranges from 0 to 1 and determines the difference in electrostatic energy of the two charge states.

The form of the Hamiltonian (equation (1)) is, apart from the leads, essentially that of the independent boson model [9, 20] and hence we make the canonical transformation usually applied to such systems to eliminate the term in the Hamiltonian describing the SET–resonator coupling. The operators in this canonically transformed picture are given by $\bar{A} = e^{-\lambda n (a^\dagger - a)} A e^{\lambda n (a^\dagger - a)}$, where we have defined a dimensionless electromechanical coupling

parameter, $\lambda = \chi/\hbar\omega_0 = x_0/(2x_q)$. The canonically transformed Hamiltonian is then

$$\begin{aligned} \bar{H} = \bar{H}_S + H_B + \sum_k T_{ki} [c_k^\dagger c_i \sigma^- e^{\lambda(a^\dagger - a)} + c_k c_i^\dagger \sigma^+ e^{-\lambda(a^\dagger - a)}] \\ + \sum_q T_{qi} [c_q^\dagger c_i \sigma^- e^{\lambda(a^\dagger - a)} + c_q c_i^\dagger \sigma^+ e^{-\lambda(a^\dagger - a)}], \end{aligned} \quad (2)$$

where the transformed system Hamiltonian is

$$\bar{H}_S = E_{ch}n + \hbar\omega_0 a^\dagger a, \quad (3)$$

with

$$E_{ch} = E_C(1 - 2n'_g) - \hbar\omega_0\lambda^2 \quad (4)$$

and the bath Hamiltonian describing the microscopic energy levels on the metallic leads and the island is given by

$$H_B = \sum_k \epsilon_k c_k^\dagger c_k + \sum_q \epsilon_q c_q^\dagger c_q + \sum_i \epsilon_i c_i^\dagger c_i. \quad (5)$$

Converting to the interaction picture, $\bar{A}_I(t) = e^{i(\bar{H}_S + H_B)t/\hbar} \bar{A} e^{-i(\bar{H}_S + H_B)t/\hbar}$, we obtain

$$\bar{H}_I = \sum_l T_{li} [c_l^\dagger(t) c_i(t) \sigma^-(t) e^{\lambda(a^\dagger(t) - a(t))} + c_l(t) c_i^\dagger(t) \sigma^+(t) e^{-\lambda(a^\dagger(t) - a(t))}], \quad (6)$$

where the sum over l runs over k and q .

We wish to find an equation of motion for the density matrix of our system, $\rho(t)$, which we define to be the resonator plus the macroscopic charge degree of freedom. For high-resistance junctions, the tunnel couplings between the leads and the SET island (T_{li}) are weak; thus we assume that the microscopic energy levels in the leads and island remain in equilibrium throughout, with occupancies set by the Fermi–Dirac distribution function (with the relevant chemical potential) so that $\Xi(t) = \Xi(0)$ for all time, t , where $\Xi(t)$ is the density matrix of the microscopic levels in the leads and the island (Born approximation [9, 21, 22]). (The full density matrix is assumed to factorize at $t = 0$, when it takes the form $\rho(0) \Xi(0)$.) Assuming that the electrons in the leads relax on a timescale much faster than that over which $\dot{\rho}_I$ evolves, we also make a Markov approximation [22] and hence arrive at a master equation of the form,

$$\dot{\rho}_I(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}[\bar{H}_I(t), [\bar{H}_I(t'), \bar{\rho}_I(t) \Xi]], \quad (7)$$

where the trace is over the microscopic electronic levels in the leads and island.

Defining the bath operators $B_{il} = c_i c_l^\dagger$ and the system operator $S = \sigma^- e^{\lambda(a^\dagger + a)}$, we can write the equation of motion for the system density matrix as:

$$\begin{aligned} \dot{\rho}_I(t) = & -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr} \sum_{l,i} |T_{li}|^2 [B_{il}(t) S(t), [B_{il}^\dagger(t') S^\dagger(t'), \bar{\rho}_I(t) \Xi]] \\ & -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr} \sum_{l,i} |T_{li}|^2 [B_{il}^\dagger(t) S^\dagger(t), [B_{il}(t') S(t'), \bar{\rho}_I(t) \Xi]]. \end{aligned} \quad (8)$$

Notice that we need only sum over the microscopic levels once, as only matched pairs of creation and annihilation operators contribute when the electronic levels are in equilibrium. The time dependence of the system operator $S(t) = \sigma^-(t)e^{\lambda(a^\dagger(t)-a(t))}$ is somewhat complicated, and so we write this as a sum of eigen operators (defined by $[\bar{H}_S, S_m] = -\hbar\omega_m S_m$), each with a time dependence of the form $e^{-i\omega_m t}$, where ω_m is an integer multiple of ω_0 . Anticipating the approximation we will make later that the electromechanical coupling is weak, we write out $S(t)$ as a series in λ :

$$\begin{aligned} S(t) &= \sum_m \sigma^-(t) S_m e^{-i\omega_m t} \\ &= \sigma^- e^{-iE_{ch}t} \left(1 + \lambda a^\dagger e^{i\omega_0 t} - \lambda a e^{-i\omega_0 t} + \frac{\lambda^2}{2} a^\dagger a^\dagger e^{2i\omega_0 t} + \frac{\lambda^2}{2} a a e^{-2i\omega_0 t} - \frac{\lambda^2}{2} a^\dagger a - \frac{\lambda^2}{2} a a^\dagger + \dots \right). \end{aligned} \quad (9)$$

The operators S_m are products of different numbers of the operators a and a^\dagger , with ω_m determined by the number of each. Putting the time dependence explicitly into the equation of motion we obtain,

$$\begin{aligned} \dot{\bar{\rho}}_I(t) &= -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr} \sum_{l,i} |T_{li}|^2 \sum_{m,n} e^{-i(t-t')[\epsilon_i - \epsilon_l + E_{ch}]/\hbar} e^{-i(\omega_m t - \omega_n t')} [B_{il} \sigma^- S_m, [B_{il}^\dagger \sigma^+ S_n^\dagger, \bar{\rho}_I(t) \Xi]] \\ &\quad - \frac{1}{\hbar^2} \int_0^t dt' \text{Tr} \sum_{l,i} |T_{li}|^2 \sum_{m,n} e^{i(t-t')[\epsilon_i - \epsilon_l + E_{ch}]/\hbar} e^{i(\omega_m t - \omega_n t')} [B_{il}^\dagger \sigma^+ S_m^\dagger, [B_{il} \sigma^- S_n, \bar{\rho}_I(t) \Xi]]. \end{aligned} \quad (10)$$

We now assume that we can replace the integral over t' with an integral over $\tau = (t - t')$ and that the resulting integrand is sufficiently peaked about $\tau = 0$ that we can extend the upper limit to infinity (assumptions consistent with the Markov approximation [9, 21]). The time integrals can be evaluated with the help of the expression: $\int_0^\infty d\tau e^{i\omega\tau} = \pi\delta(\omega) + iPV(1/\omega_0)$. The principal value terms lead to coherent corrections to the evolution of the density matrix, but since we are working in the limit where the junction resistances are relatively high ($R_{S(D)} \gg R_Q$), these corrections are small and can be neglected [9, 18, 23].

Cyclicly permuting the bath operators under the trace, the master equation can be rewritten in the form:

$$\begin{aligned} \dot{\bar{\rho}}_I(t) &= \frac{2\pi}{\hbar} \text{Tr} \sum_{l,i} |T_{li}|^2 \sum_{m,n} \delta(\epsilon_i - \epsilon_l + E_{ch} + \hbar\omega_n) (B_{il}^\dagger B_{il} \mathcal{F}[\sigma^- S_n(t), \sigma^+ S_m^\dagger(t)] \\ &\quad + B_{il} B_{il}^\dagger \mathcal{F}[\sigma^+ S_n^\dagger(t), \sigma^- S_m(t)]) \bar{\rho}_I(t) \Xi, \end{aligned} \quad (11)$$

where we have defined the super-operator,

$$\mathcal{F}[X, Y]\rho = \frac{1}{2}(X\rho Y + Y^\dagger \rho X^\dagger) - \frac{1}{2}(YX\rho + \rho X^\dagger Y^\dagger). \quad (12)$$

We now perform the trace over the bath, using $\text{Tr}[c_k^\dagger c_k \Xi] = f(\epsilon_k - \mu_S)$, etc, and convert the sums over the electron levels (i.e. over k, q, i) to integrals. After integrating over ϵ_k and ϵ_q ,

the master equation becomes

$$\begin{aligned}
\dot{\bar{\rho}}_I(t) = & g_D \sum_{m,n} \int d\epsilon_i f(\epsilon_i) [1 - f(\epsilon_i + E_{ch} + \hbar\omega_n - \mu_D)] \mathcal{F}[\sigma^- S_n(t), \sigma^+ S_m^\dagger(t)] \bar{\rho}_I(t) \\
& + g_D \sum_{m,n} \int d\epsilon_i [1 - f(\epsilon_i)] f(\epsilon_i + E_{ch} + \hbar\omega_n - \mu_D) \mathcal{F}[\sigma^+ S_n^\dagger(t), \sigma^- S_m(t)] \bar{\rho}_I(t) \\
& + g_S \sum_{m,n} \int d\epsilon_i f(\epsilon_i) [1 - f(\epsilon_i + E_{ch} + \hbar\omega_n - \mu_S)] \mathcal{F}[\sigma^- S_n(t), \sigma^+ S_m^\dagger(t)] \bar{\rho}_I(t) \\
& + g_S \sum_{m,n} \int d\epsilon_i [1 - f(\epsilon_i)] f(\epsilon_i + E_{ch} + \hbar\omega_n - \mu_S) \mathcal{F}[\sigma^+ S_n^\dagger(t), \sigma^- S_m(t)] \bar{\rho}_I(t), \quad (13)
\end{aligned}$$

where the quantities $g_{S(D)}$ are defined by the relation

$$g_{S(D)} = \frac{1}{R_{S(D)} e^2} = \frac{2\pi}{\hbar} |T_{k(q)i}|^2 D_{k(q)} D_i \quad (14)$$

and it has been assumed that the tunnelling matrix elements $T_{k(q)i}$ and the densities of states of the leads, $D_{k(q)}$, and island, D_i are independent of energy over the relevant range of $k(q)$, i .

So long as the electron temperature in the leads is much less than the other relevant energy scales, the Fermi functions can be replaced by simple step functions. Furthermore, for sufficiently large bias voltages (such that $E_D = eV/2 - E_{ch} > \hbar\omega_n$ and $E_S = eV/2 + E_{ch} > -\hbar\omega_n$) and sufficiently weak electromechanical coupling (we require $\lambda \ll 1$, so that the series can be truncated after a finite number of terms), tunnelling in the opposite direction to that set by the bias voltage cannot occur and hence the master equation can be written in the simplified form

$$\begin{aligned}
\dot{\bar{\rho}}_I(t) = & g_S \sum_{m,n} (E_S + \hbar\omega_n) \mathcal{F}[\sigma^- S_n(t), \sigma^+ S_m^\dagger(t)] \bar{\rho}_I(t) \\
& + g_D \sum_{m,n} (E_D - \hbar\omega_n) \mathcal{F}[\sigma^+ S_n^\dagger(t), \sigma^- S_m(t)] \bar{\rho}_I(t). \quad (15)
\end{aligned}$$

Converting the master equation out of the interaction and canonically transformed pictures, using $\sigma_{CT}^- = \sigma^- e^{\lambda(a^\dagger - a)}$ and $a_{CT} = a + \lambda n$, and working to order λ^2 , we find

$$\begin{aligned}
\dot{\rho}(t) = & -\frac{i}{\hbar} [H_S, \rho(t)] + g_S \mathcal{F}[\sigma^- (E_S - \lambda\hbar\omega_0(a^\dagger + a) + \lambda^2\hbar\omega_0), \sigma^+] \rho(t) \\
& + g_D \mathcal{F}[\sigma^+ (E_D + \lambda\hbar\omega_0(a^\dagger + a) - \lambda^2\hbar\omega_0), \sigma^-] \rho(t), \quad (16)
\end{aligned}$$

where

$$H_S = E_C(1 - 2n'_g)n + \hbar\omega_0[a^\dagger a - \lambda n(a^\dagger + a)]. \quad (17)$$

The assumption that the electromechanical coupling is weak compared to the resonator quantum, $\lambda \ll 1$ (i.e. $\chi \ll \hbar\omega_0$) is the principal approximation made about the resonator in deriving this master equation. The derivation we have given here is convenient for later calculation, but it is also possible to derive equation (16) under the less restrictive weak-coupling approximation, $\chi \ll eV$, as we discuss in appendix A. By casting the master equation in its Wigner function form (see appendix A), we see that it is essentially equivalent to the classical one proposed in [7], even for intermediate voltages ($eV \gtrsim \hbar\omega_0$) and fast oscillator motion ($\Gamma \ll 1$).

3. Resonator and charge dynamics

The quantum master equation we have derived for the SET island charge and resonator, equation (16), is readily integrated numerically to give a complete description of the combined dynamics of the resonator and the macroscopic SET charge. However, for the normal state SET–resonator system we consider (unlike its superconducting counterpart [17]), it is also possible to obtain a closed set of equations of motion for the moments of the electrical and mechanical degrees of freedom.² In this section, we use a combination of analytical and numerical methods to investigate the dynamical and steady state properties of the resonator and SET moments. In later sections, we make further approximations in order to derive more compact analytical descriptions of the resonator dynamics and comparison with the results obtained in this section play an important role in judging their fidelity.

We can calculate the equations of motion for the moments by multiplying the master equation by the appropriate operator and tracing over the resonator and charge states. Assuming for simplicity that $g_S = g_D = g$, a closed set of equations for the first moments is obtained,

$$\frac{d}{dt}\langle n \rangle = g[eV/2 - E_C(1 - 2n'_g) + \lambda\hbar\omega_0\langle a^\dagger + a \rangle] - geV\langle n \rangle, \quad (18)$$

$$\frac{d}{dt}\langle a^\dagger + a \rangle = i\omega_0\langle a^\dagger - a \rangle, \quad (19)$$

$$\frac{d}{dt}\langle a^\dagger - a \rangle = i\omega_0[\langle a^\dagger + a \rangle - 2\lambda\langle n \rangle]. \quad (20)$$

These equations of motion are naturally equivalent to those for a classical system [7]. Here, we investigate their behaviour over a wide range of parameter values and find in each case that the resonator motion is closely analogous to that of a damped harmonic oscillator with a shifted frequency. Of particular interest is the effect of varying the ratio of electrical and mechanical timescales, $\Gamma = geV/\omega_0$, as only the regime $\Gamma \gg 1$ was described for classical treatments of the system [7, 15].

Figure 2 shows the time evolution of the moments $\langle a + a^\dagger \rangle$ and $\langle n \rangle$, obtained numerically, for a particular choice of parameters. For the relatively fast oscillator chosen ($\Gamma = 0.6$), the average resonator position shows a clear initial transient before decaying exponentially as if it were a damped harmonic oscillator. The average SET island charge, $\langle n \rangle$, relaxes on two timescales: initially, the charge relaxes rapidly towards its steady state value over a time $(geV)^{-1}$, it then undergoes weak oscillations (following those of the resonator) which are damped out slowly.

Although the dynamics of the resonator is qualitatively the same for the whole range of parameters we have studied, there are important quantitative differences. In figures 3 and 4, we plot the damping rate and frequency shift (extracted from fits to the average resonator position as a function of time) for different choices of eV , λ and for a wide range of Γ values. Strikingly, we find that the data all fall very close to universal curves. When time is measured in units of the resonator period, the form of the universal curves for damping and frequency shift are determined by Γ and $\kappa = 2\lambda^2\hbar\omega_0/eV = m\omega_0^2x_0^2/eV$, which gives a measure of the SET–resonator coupling strength compared to the bias voltage. As we shall see later on, the universal curves emerge naturally from a reduced master equation for the resonator.

² We did check by numerical integration that for Gaussian initial states of the resonator, the master equation preserves positivity (even though it is not of the Lindblad form) and relaxes to a Gaussian steady state.

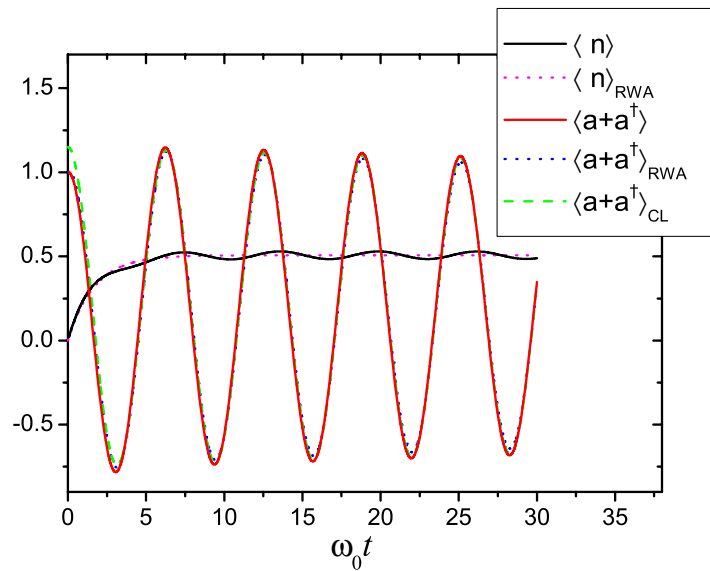


Figure 2. Evolution of the average resonator position, $\langle a + a^\dagger \rangle$, and SET island charge, $\langle n \rangle$. Results from a numerical integration of the full master equation are compared with results calculated using the Caldeira–Leggett (CL) and RWA master equations (described in sections 4 and 5, respectively). Notice that the Caldeira–Leggett curves used ‘slipped’ initial conditions [24] chosen to fit the long-time behaviour of the numerical results. The parameters used are $eV/\hbar\omega_0 = 6$, $g = 0.1/\hbar$ (i.e., $\Gamma = 0.6$) and $\lambda = 0.2$.

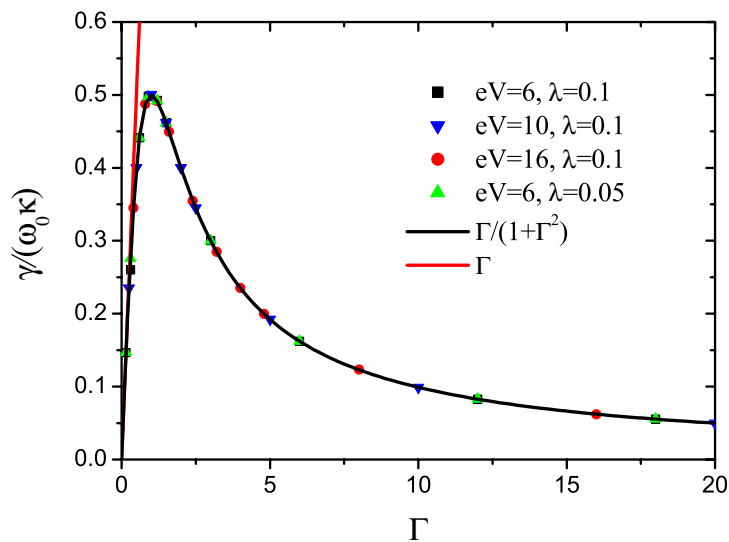


Figure 3. Plot of the resonator damping rate as a function of the average tunnel rate, Γ . For simplicity, the junction resistances are assumed to be equal, $g_D = g_S = g$, and we choose $n'_g = 1/2$ (the degeneracy point). The voltages given for the various data sets are measured in units of $\hbar\omega_0$. Also shown are analytic curves which match expressions derived in sections 4 and 5.

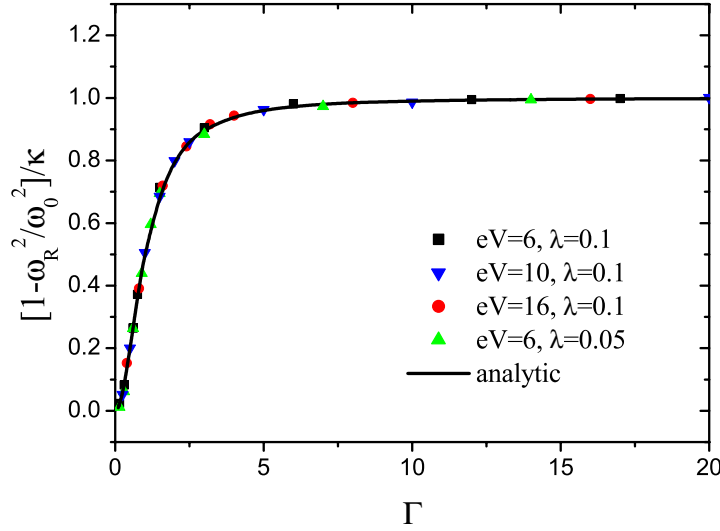


Figure 4. Plot of the relative frequency shift (squared) as a function of Γ . The junctions are chosen to have equal resistance $g_D = g_S = g$, and we choose $n'_g = 1/2$ (the degeneracy point). The voltages given for the various data sets are measured in units of $\hbar\omega_0$. The analytic curve comes from equation (24) discussed in section 4.

From figure 3, we can see that the damping rate reaches a maximum when the resonator period matches the tunnelling rate, and drops to zero when the tunnelling rate is either much faster or much slower than the oscillator. We also note that the damping rate is proportional to Γ for $\Gamma \ll 1$ and inversely proportional for $\Gamma \gg 1$, with the latter matching the results obtained in previous classical treatments of the system [7, 15] while the former is consistent with results found using a quantum treatment in [9]. The frequency shift follows a different pattern, increasing monotonically with Γ until saturating at a constant value for $\Gamma \gg 1$.

Physically, the frequency shift and damping of the resonator arise because the charge on the SET is a function of both the resonator position and (indirectly) velocity and the resonator in turn experiences a force proportional to the charge on the resonator. Thus the resonator-SET coupling results in forces on the resonator proportional to its position (frequency shift) and to its velocity (damping) [14]. The velocity-dependent term is due to the fact that the SET takes a finite amount of time to respond to the resonator. If the SET responds instantly (which is equivalent to taking the limit $\Gamma \rightarrow \infty$), there is no damping. Conversely, if the SET takes an essentially infinite amount of time to respond to the resonator position (i.e. in the limit $\Gamma \rightarrow 0$), then the damping must also disappear.

The second moments of the resonator can also be calculated from equation (16), and solved to find the steady state variances of the resonator. We find,

$$\frac{m \langle \delta u^2 \rangle_{ST}}{1 - \kappa} = m \omega_0^2 \langle \delta x^2 \rangle_{ST} = eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}], \quad (21)$$

where

$$\langle n \rangle_{ST} = \frac{eV - 2E_C(1 - 2n'_g)}{2eV(1 - \kappa)} \quad (22)$$

is the steady state value of the island occupation (given by equations (18)–(20)). In the weak-coupling limit, $\kappa \rightarrow 0$, equipartition is recovered and if we also take the high-voltage limit, $eV \gg \hbar\omega_0$, then³ we can identify an effective temperature for the resonator which matches previous classical results [7]: $k_B T_{SET} = eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}]$. Note that these expressions for the steady state variances are not valid for arbitrarily small voltages (and hence do not violate the uncertainty relations) as they were derived assuming a bias voltage large enough to ensure that no back-tunnelling occurs. Specifically, equation (21) is valid for $eV > 2(E_{ch} + 2\hbar\omega_0)$ and $eV > 2(-E_{ch} + 2\hbar\omega_0)$, and for weak electromechanical coupling.

4. Caldeira–Leggett equation for the resonator

In section 2, we traced over the microscopic electronic degrees of freedom to obtain a master equation for a system consisting of the resonator and the macroscopic charge of the SET island. A natural further step is to trace over the remaining electronic degree of freedom and hence obtain a reduced master equation for the resonator alone [15]. The advantage of this reduced master equation is that it gives a compact description of the resonator dynamics and makes clear the analogy between the effect of the SET electrons and a thermal bath. In order to trace over the macroscopic charge variable, we treat the SET as the bath and make many of the same kinds of assumptions as were used in deriving equation (16) (details of the calculation are given in appendix B). In particular, we assume that the tunnelling rate, geV , is fast compared to the rate of change of the resonator’s reduced density matrix in the interaction picture (Markov approximation). Whilst we cannot be completely sure when such an approximation is valid before carrying out the calculation, we can judge its validity by comparing the predictions of the resulting master equation with those of equation (16) described in the previous section. It is also necessary to assume that the coupling is weak compared to the bias voltage (Born approximation), $\chi \ll eV$, but this is not restrictive given the weak-coupling assumption used in the derivation of equation (16).⁴ We also assume that the SET and resonator density matrices factorize.

The reduced master equation for the resonator we arrive at by means of the Born–Markov approximations takes a very similar form to that of the Caldeira–Leggett equation which describes quantum Brownian motion [26]–[28],

$$\dot{\rho}(t) = -\frac{i}{\hbar} [\tilde{H}_r, \rho_r(t)] - \frac{i}{\hbar} \frac{\gamma_{CL}}{2} [x, \{p, \rho_r(t)\}] - D[x, [x, \rho_r(t)]] - \frac{f}{\hbar} [x, [p, \rho_r(t)]], \quad (23)$$

where \tilde{H}_r is the Hamiltonian of an harmonic oscillator with the renormalized frequency,

$$\omega_R = \omega_0 \left(1 - \kappa \frac{\Gamma^2}{1 + \Gamma^2} \right)^{\frac{1}{2}} \quad (24)$$

³ Away from the high-voltage limit, the value of the SET effective temperature will depend on the resonator frequency, as pointed out in [16].

⁴ It should be noted that in tracing over the microscopic electronic degrees of freedom in section 2, we assumed that the tunnel couplings were small in making a Born approximation.

and the damping constant is⁵

$$\gamma_{CL} = \omega_0 \frac{\kappa\Gamma}{1 + \Gamma^2}. \quad (25)$$

The normal and anomalous diffusion constants [15, 27] are

$$D = \frac{m\gamma_{CL}eV}{\hbar^2} \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}] \quad (26)$$

and

$$\hbar f = -\frac{\kappa}{(1 + \Gamma^2)} eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}], \quad (27)$$

respectively.

The steady state resonator variances are readily obtained from the reduced master equation and hence we obtain

$$m\omega_R^2 \langle \delta x^2 \rangle_{ST} = m \langle \delta u^2 \rangle_{ST} - \hbar f, \quad (28)$$

$$m \langle \delta u^2 \rangle_{ST} = eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}], \quad (29)$$

$$m\omega_0^2 \langle \delta x^2 \rangle_{ST} = eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}] \left(1 + \frac{\kappa}{1 + \Gamma^2}\right) \left(1 - \frac{\kappa\Gamma^2}{1 + \Gamma^2}\right)^{-1}. \quad (30)$$

These values are somewhat different to those obtained before the macroscopic charge variable was traced out. In particular, the presence of the anomalous diffusion term means that the variances in the position and velocity are no longer directly proportional to one another.⁶ Such differences must be interpreted as artefacts of our Born–Markov approximations. However, in the limit $\kappa \rightarrow 0$ these values converge to the same limits as before.

Having compared the steady state properties of the resonator given by the reduced master equation with the ‘true’ values given by the master equation for the resonator and SET island charge, we now compare the dynamics. In figures 3 and 4, we plot the analytic Caldeira–Leggett values of frequency shift and damping (equations (24) and (25)) together with values extracted from numerical integrations of the full master equation. The analytic results match up very well with the numerics in the weak electromechanical coupling limit. Furthermore, this good agreement extends over the whole range of Γ showing that our Markov approximation remains valid even for $\Gamma \ll 1$.

Although the reduced master equation successfully describes the long-time dynamics of the resonator, it fails to capture certain transient features which are present in equations (18)–(20) over timescales of order of the electron tunnelling time, $(geV)^{-1}$. In figure 2, we compare the time evolution of the moments calculated from equation (23) to those obtained from the full

⁵ For comparison, the damping and renormalized frequency found in previous classical calculations [7, 15, 14] were $\gamma = \kappa\omega_0/\Gamma$ and $\omega_R = \omega_0(1 - \kappa)^{1/2}$, respectively.

⁶ The presence of an anomalous diffusion term in the reduced master equation of a resonator coupled to a point contact was reported in [16]. However, in that case there is no analogue of the resonator–SET charge master equations which can be used to determine the ‘true’ dynamics of the resonator.

master equation. In order to match the long-time behaviour of equations (18)–(20), we have had to ‘slip’ the initial conditions for the reduced master equation [24]. The failure of a reduced master equation to capture transient features in the motion of the system is often a consequence of Born–Markov approximations used in their derivation [24, 25], and it seems very likely that the same has happened here.

An additional factor that may introduce further complications is the presence of an external bath, which has been neglected in this treatment. While we might expect that a thermal environment could simply be incorporated into the reduced master equation of the resonator (equation (23)) by the addition of further Caldeira–Leggett type terms [16], the correlation time of the environment provides an additional timescale which might affect the validity of the assumptions used in deriving the reduced master equation in the first place [29]. A more detailed treatment is therefore required, for instance including the environment at a more microscopic level as a bath of harmonic oscillators in the original Hamiltonian of the system.

5. RWA

An alternative way of deriving a reduced master equation for the resonator alone is to make use of a RWA [21, 30]. In RWA, terms with an $e^{i\omega_0 t}$ dependence are assumed to oscillate very rapidly compared to the rate of change of $\bar{\rho}_I$ and hence effectively average to zero. As with the validity of the Markov approximation used in deriving the Caldeira–Leggett type master equation, we cannot be sure in advance whether this condition will be satisfied, but we can verify that it is *a posteriori* from the rate of change of $\bar{\rho}_I$ in the resulting master equation.

Returning to equation (15), implementing the RWA means dropping any term where $(\omega_m - \omega_n) \neq 0$. Expanding up to order λ^2 , the resulting master equation is

$$\begin{aligned} \dot{\bar{\rho}}_I = & g_S E_S \mathcal{F}[\sigma^-, \sigma^+ \{1 - \lambda^2 (a^\dagger a + a a^\dagger)\}] \bar{\rho}_I \\ & + g_S \{ (E_S - \hbar\omega_0) \lambda^2 \mathcal{F}[a^\dagger \sigma^-, a \sigma^+] + (E_S + \hbar\omega_0) \lambda^2 \mathcal{F}[a \sigma^-, a^\dagger \sigma^+] \} \bar{\rho}_I \\ & + g_D E_D \mathcal{F}[\sigma^+, \sigma^- \{1 - \lambda^2 (a^\dagger a + a a^\dagger)\}] \bar{\rho}_I \\ & + g_D \{ (E_D - \hbar\omega_0) \lambda^2 \mathcal{F}[a^\dagger \sigma^+, a \sigma^-] + (E_D + \hbar\omega_0) \lambda^2 \mathcal{F}[a \sigma^+, a^\dagger \sigma^-] \} \bar{\rho}_I. \end{aligned} \quad (31)$$

This master equation is closely related to that found in [9] where a master equation for a quantum dot gated by a nanomechanical resonator was derived.

When we make the simplifying assumption $g_S = g_D = g$, the dynamics of the resonator and SET island charge decouple. The equation of motion for the average charge then takes the simple form,

$$\frac{d}{dt} \langle n \rangle = -g [E_S - \hbar\omega_0 \lambda^2] \langle n \rangle + g [E_D - \hbar\omega_0 \lambda^2] (1 - \langle n \rangle). \quad (32)$$

The behaviour of $\langle n \rangle$ according to this equation is compared with the ‘‘true’’ behaviour given by equations (18)–(20) in figure 2. It is clear that although the fixed point value of $\langle n \rangle$ remains the same, the dynamics is different as it no longer displays oscillations correlated to the resonator motion.

Assuming that the density matrix factorizes into charge and resonator parts, the evolution of the resonator alone is given by a reduced master equation which has the Lindblad form

$$\begin{aligned} \dot{\bar{\rho}}_{r,I} = & g \lambda^2 \{ \langle n \rangle (E_S - E_D) + (E_D - \hbar\omega_0) \} \mathcal{F}[a^\dagger, a] \bar{\rho}_{r,I} \\ & + g \lambda^2 \{ \langle n \rangle (E_S - E_D) + (E_D + \hbar\omega_0) \} \mathcal{F}[a, a^\dagger] \bar{\rho}_{r,I}, \end{aligned} \quad (33)$$

where

$$\bar{\rho}_{r,I} = \langle N_0 | \bar{\rho}_I | N_0 \rangle + \langle N_1 | \bar{\rho}_I | N_1 \rangle. \quad (34)$$

The average charge appears in equation (33) as a parameter though it is not necessary to assume that $\langle n \rangle$ has reached its fixed point value. Instead, we can use the solution of equation (32) to obtain a reduced master equation for the resonator capable of capturing transient behaviour arising from different initial conditions, albeit one with time-dependent coefficients.

The decoupling between the occupancy of the SET island and the resonator motion occurs because we are working in the limit where the resonator oscillations are much faster than the timescale for electron tunnelling events and hence the SET only experiences the average of the resonator position. It should be noted, however, that this decoupling only occurs in the strict RWA limit, i.e. $\Gamma \rightarrow 0$: for any non-zero values of Γ there is a residual correlation between the resonator motion and the average charge. The oscillations in $\langle n \rangle$ (obtained from equations (18)–(20)) shown in figure 2, for which we set $\Gamma = 0.6$, clearly follow those of the average resonator position, but this behaviour disappears completely in the RWA limit.

Equations of motion for the moments of the resonator are readily obtained from equation (33) or from equation (31),

$$\frac{d}{dt} \langle a^\dagger + a \rangle = -i\omega_0 \langle a - a^\dagger \rangle - g\lambda^2 \hbar \omega_0 \langle a^\dagger + a \rangle + 2g\lambda^3 \hbar \omega_0 \langle n \rangle + 2\lambda \frac{d}{dt} \langle n \rangle, \quad (35)$$

$$i \frac{d}{dt} \langle a^\dagger - a \rangle = -\omega_0 \langle a^\dagger + a \rangle + 2\lambda \omega_0 \langle n \rangle - ig\lambda^2 \hbar \omega_0 \langle a^\dagger - a \rangle. \quad (36)$$

In order to capture the long-time dynamics of the resonator as simply as possible, it is sufficient to assume that the average island charge has reached its steady state value. We can then obtain the following equation of motion for the resonator position about its fixed point value,

$$\langle \ddot{x} \rangle = -\omega_0^2 \langle x \rangle - 2g\lambda^2 \hbar \omega_0 \langle \dot{x} \rangle, \quad (37)$$

where we have discarded terms $\mathcal{O}(\lambda^3)$ and higher. From this equations it is clear that the resonator is damped at a uniform rate $\gamma_{RW} = 2g\lambda^2 \hbar \omega_0 = \kappa \Gamma \omega_0$, but there is no shift in its frequency. The steady state values of the resonator variances take the form,

$$m\omega_0^2 \langle \delta x^2 \rangle_{ST} = eV \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}] + \lambda^2 m\omega_0^2 x_q^2, \quad (38)$$

$$m \langle \delta u^2 \rangle_{ST} = (eV - 4\lambda^2 m\omega_0^2 x_q^2) \langle n \rangle_{ST} [1 - \langle n \rangle_{ST}] + \lambda^2 m\omega_0^2 x_q^2. \quad (39)$$

As with the variances obtained from the Caldeira–Leggett master equation, these differ slightly from the ‘true’ values (equation (21)), but they do approach the same weak-coupling limit ($\lambda \rightarrow 0$).

In figure 2, we compare the RWA evolution of the system moments with the full case. We see that the agreement is very close, including the short-time transient, for the value $\Gamma = 0.6$ used and in fact it becomes exact in the limit $\Gamma \rightarrow 0$. The RWA values for the damping and temperature (and the absence of a frequency shift) are consistent with those calculated from equations (15) and (16) in the limit $\Gamma \ll 1$, as can be clearly seen from figures 3 and 4. This implies that the correct criteria for the validity of RWA in this system is $\Gamma \ll 1$. This is in contrast to the case of just a harmonic oscillator coupled to a bath [21], where the criteria for the validity of the RWA is simply that the resonator frequency should be much larger than the damping rate

due to the bath. It is clear that the presence of the SET in our system introduces an additional timescale over which $\bar{\rho}_I$ evolves and hence the period of the resonator oscillations must be much shorter than both the damping time and the electron tunnelling time for the RWA approximation to be valid.

It is worth noting that equation (31) seems to suggest that the damping is caused by terms like $\mathcal{F}[a^\dagger\sigma^+, a\sigma^-]\bar{\rho}_I$, which add or remove an energy quantum from the resonator during a tunnelling event (for discussions of energy exchange in similar master equations see [9, 30]). Although this mechanism seems intrinsically quantum mechanical, involving the exchange of energy in units of $\hbar\omega_0$, the RWA master equation was derived from an equation describing essentially classical behaviour. However, we must remember that the equation we have derived is an *unconditional* master equation. Even though the terms in equation 31 describe the transfer of individual quanta, the fact we are considering an ensemble of systems means that the density matrix changes continuously. A full description of damping that proceeds by transfer of discrete units of $\hbar\omega_0$ (as equation (31) seems to imply) would require quantum trajectory methods [22].

6. Conclusions and discussion

Starting from a microscopic formulation, we have derived a quantum master equation for the SET–resonator system which describes the coupled dynamics of the SET island charge and the resonator in the limit of weak electromechanical coupling and intermediate voltages (i.e. $eV \gtrsim \hbar\omega_0$). By investigating the dynamics of the charge and resonator moments, we find that the resonator is damped, undergoes a shift in frequency and eventually reaches a thermal-like steady state. These results agree with those found in a classical description of the same problem [7, 15], but are more general. In particular, we find that the resonator is damped by the SET even when the electron tunnelling time is of order of the resonator period or longer. Furthermore, maximum damping of the resonator by the SET electrons occurs when the electron tunnelling time matches the resonator period.

Starting with the resonator–island charge master equation, we have derived reduced master equations for the resonator alone in two different ways. The first reduced master equation is very similar to the Caldeira–Leggett master equation and is obtained by tracing over the island charge, treating it like an external bath (i.e. making Born–Markov approximations). The second type of reduced master equation is found by making the RWA so that the charge dynamics decouples from the resonator motion. Apart from an initial transient, the Caldeira–Leggett type master equation describes the long-time dynamics of the resonator faithfully over the whole range of relative electrical and mechanical time scales. In contrast, the master equation obtained via the RWA only captures the resonator dynamics when its period is much shorter than the electron tunnelling time, though unlike the Caldeira–Leggett master equation it captures the transient motion and is of the Lindblad form so is guaranteed to preserve the positivity of the density matrix.

The derivation of a reduced master equation for the resonator in the SET–resonator system makes an interesting comparison with better known dissipative systems as it proceeds via a two-stage process: tracing over the microscopic electronic levels gives rise to a master equation which still contains the SET island charge, and further approximations are needed to obtain a description of the resonator alone. This is in contrast to other NEMS such as the resonator–point contact system [3, 11, 16], where tracing over the microscopic electronic states leads directly

to a Caldeira–Leggett type master equation for the resonator. However, the intermediate stage for the SET–resonator system (consisting of a master equation for the resonator and SET island charge) leads rather fortuitously to closed sets of equations of motion for the resonator and charge moments. Thus it is possible to characterize the dynamics of the resonator using simple numerical and analytical techniques before making the additional approximations required to obtain the reduced master equations of the resonator. Thus we are able to show that the anomalous diffusion term which arises in the Caldeira–Leggett type reduced master equation is not a feature of the ‘true’ resonator dynamics, but rather an artefact of the approximations we make.

The quantum master equations we derive here for the SET–resonator system will provide useful tools for further investigations into the quantum dynamics of this system. In particular, we can use them as a starting point for investigations of how an individual SET and resonator (rather than an ensemble of such systems) evolves. However, it will also be interesting to try to extend the ensemble-averaged master equations derived here to obtain descriptions of the low-bias and strong-coupling regimes where the close connection between quantum and classical treatments found here is unlikely to persist.

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Appendix A. Alternative derivation of the resonator and island charge master equation

In the body of the paper, we rewrote equation (15) to get a master equation in a simple form (equation (16)). Although the derivation of equation (16) required an expansion in λ , this simple master equation is in fact more general. Here, we carry out a slightly different derivation that does not require $\lambda \ll 1$, and is therefore valid in the fully classical limit where both the coupling χ and the bias voltage V are much larger than $\hbar\omega_0$. Instead, we make a different weak-coupling approximation and assume that the coupling is small compared to the bias voltage, $\chi \ll eV$. In addition, we assume that the resonator moves slowly on the relaxation time of the electrons in the leads.

We start from the Born–Markov master equation,

$$\dot{\bar{\rho}}_I(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}[\bar{H}_I(t), [\bar{H}_I(t'), \bar{\rho}_I(t) \Xi]],$$

where the interaction picture Hamiltonian (equation (6)), is

$$\bar{H}_I = \sum_l T_{li} [c_l^\dagger(t) c_i(t) \sigma^-(t) e^{\lambda(a^\dagger e^{i\omega_0 t} - a e^{-i\omega_0 t})} + c_l(t) c_i^\dagger(t) \sigma^+(t) e^{-\lambda(a^\dagger e^{i\omega_0 t} - a e^{-i\omega_0 t})}]. \quad (\text{A.1})$$

Using the Baker–Campbell–Hausdorff theorem [21] to rearrange the terms under the integral so that the time dependence in the exponentials is expressed in terms of $\tau = t - t'$, we then assume that the resonator evolves slowly on the timescale of the bath correlation functions (i.e., the relaxation time of the electrons in the leads) and hence make the approximation $e^{i\omega_0 \tau} \simeq 1 + i\omega_0 \tau$.

Then we perform the integration over τ to obtain

$$\begin{aligned} \dot{\bar{\rho}}_I(t) &= \frac{1}{\hbar^2} \text{Tr} \sum_{l,i} |T_{il}|^2 B_{il} B_{il}^\dagger \mathcal{F}[S'^{\dagger}(t) e^{-\lambda(a^\dagger - a)}, S'(t)] \\ &\quad \times \delta(\epsilon_i - \epsilon_l + E_{ch} - \lambda \hbar \omega_0 (a^\dagger + a) - \lambda^2 \hbar \omega_0) e^{\lambda(a^\dagger - a)} \bar{\rho}_I(t) \Xi \\ &\quad + \frac{1}{\hbar^2} \text{Tr} \sum_{l,i} |T_{il}|^2 B_{il}^\dagger B_{il} \mathcal{F}[S'(t) e^{\lambda(a^\dagger - a)}, S'^{\dagger}(t) e^{-\lambda(a^\dagger - a)}] \\ &\quad \times \delta(\epsilon_i - \epsilon_l + E_{ch} - \lambda \hbar \omega_0 (a^\dagger + a) - \lambda^2 \hbar \omega_0) \bar{\rho}_I(t) \Xi, \end{aligned} \quad (\text{A.2})$$

where $S'(t) = \sigma^- e^{-i(E_{ch}/\hbar - \lambda \omega_0 (a^\dagger + a) - \lambda^2 \omega_0)t}$. Although this appears unwieldy, converting back from the interaction and canonically transformed picture using the definition, $\rho(t) = e^{\lambda(a^\dagger - a)} e^{-iH_S t} \rho_I(t) e^{+iH_S t} e^{-\lambda(a^\dagger - a)}$, the above equation reduces to a simple form. After tracing over the bath, performing the integrals over the Fermi functions (following the same approach as in section 2), and assuming that the electromechanical coupling is sufficiently weak to ensure that no back-tunnelling occurs, we regain equation (16):

$$\begin{aligned} \dot{\rho}(t) &= -\frac{i}{\hbar} [H_S, \rho(t)] + g_S \mathcal{F}[\sigma^-(E_S - \lambda \hbar \omega_0 (a^\dagger + a) + \lambda^2 \hbar \omega_0), \sigma^+] \rho(t) \\ &\quad + g_D \mathcal{F}[\sigma^+(E_D + \lambda \hbar \omega_0 (a^\dagger + a) - \lambda^2 \hbar \omega_0), \sigma^-] \rho(t). \end{aligned}$$

Note that in this case the condition on the coupling required to ensure that no back-tunnelling occurs is $\lambda \ll eV/\hbar\omega_0$ (i.e. $\chi \ll eV$).

Writing the charge-diagonal matrix elements $\langle N_0 | \rho | N_0 \rangle$ and $\langle N_1 | \rho | N_1 \rangle$ of equation (16) in Wigner function form [21] results in a pair of coupled equations equivalent to the classical master equations proposed in [7]

$$\begin{aligned} \dot{W}_{00}(x, u) &= \left(\omega_0^2 x \frac{\partial}{\partial u} - u \frac{\partial}{\partial x} \right) W_{00}(x, u) + g_S [E_S - m\omega^2 x_0 x + \lambda^2 \hbar \omega_0] W_{11}(x, u) \\ &\quad - g_D [E_D + m\omega^2 x_0 x - \lambda^2 \hbar \omega_0] W_{00}(x, u), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \dot{W}_{11}(x, u) &= \left(\omega_0^2 (x - x_0) \frac{\partial}{\partial u} - u \frac{\partial}{\partial x} \right) W_{11}(x, u) - g_S [E_S - m\omega^2 x_0 x + \lambda^2 \hbar \omega_0] W_{11}(x, u) \\ &\quad + g_D [E_D + m\omega^2 x_0 x - \lambda^2 \hbar \omega_0] W_{00}(x, u). \end{aligned} \quad (\text{A.4})$$

Appendix B. Caldeira–Leggett equation for the resonator

It has been shown [15] that when the resonator coupling is small compared to the gate voltage ($\chi \ll eV$), and the separation of timescales is large ($\Gamma \gg 1$), a further Born–Markov approximation can be performed on the classical master equation [7], to get a Fokker–Planck equation for the resonator alone that nevertheless takes into account the effect of the SET on the resonator (to second order in χ). We can perform a closely analogous procedure, starting with equation (16), to get a master equation for the resonator alone, but we do not make the assumption $\Gamma \gg 1$.

Firstly, we note that the diagonal elements of the SET $\rho_{00} = \langle N_0 | \rho | N_0 \rangle$ and $\rho_{11} = \langle N_1 | \rho | N_1 \rangle$ decouple from the off-diagonal terms. Writing the density matrix as a vector $\rho = (\rho_{00}, \rho_{11})$, we can rewrite equation (16) as,

$$\begin{aligned} \dot{\rho} = & -\frac{i}{\hbar}[H_r, \rho] - i\omega_0 \left[\frac{\langle a^\dagger + a_{ST} \rangle}{2} - \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] [(a^\dagger + a), \rho] \\ & + g \begin{pmatrix} -E'_D & E'_S \\ E'_D & -E'_S \end{pmatrix} \rho + \frac{g\lambda\hbar\omega_0}{2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \{(a^\dagger + a), \rho\}, \end{aligned} \quad (\text{B.1})$$

where $H_r = \hbar\omega_0 a^\dagger a$, the curly braces, $\{., .\}$, indicate an anti-commutator and we have shifted the origin of the oscillator position by the steady state value $\langle a^\dagger + a \rangle_{ST}$ so that the new fixed point will be zero. The electrostatic energy differences associated with tunnelling are given by

$$E'_D = E_D - (\lambda^2\hbar\omega_0 + \lambda\hbar\omega_0\langle a^\dagger + a_{ST} \rangle) = \frac{eV}{2} - E_C(1 - n'_g) + \lambda\hbar\omega_0\langle a^\dagger + a \rangle_{ST}, \quad (\text{B.2})$$

$$E'_S = E_S + (\lambda^2\hbar\omega_0 + \lambda\hbar\omega_0\langle a^\dagger + a_{ST} \rangle) = \frac{eV}{2} + E_C(1 - n'_g) - \lambda\hbar\omega_0\langle a^\dagger + a \rangle_{ST}. \quad (\text{B.3})$$

To obtain equation (16), we traced over the electron levels to get an effective master equation for the resonator and the charge on the SET island. Now, we wish to trace over the remaining charge degree of freedom to leave a master equation for just the resonator. We switch to an interaction picture representation defined by $\tilde{\rho}(t) = e^{gH_{SET}t} e^{-iH_r t/\hbar} \rho(t) e^{iH_r t/\hbar}$, where

$$H_{SET} = \begin{pmatrix} -E'_D & E'_S \\ E'_D & -E'_S \end{pmatrix},$$

is the non-Hermitian ‘‘Hamiltonian’’ that describes the non-unitary evolution of the SET in the absence of interaction with the resonator. We define the SET-only operators as

$$\dot{B}_1 = \frac{g\lambda\hbar\omega_0}{2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (\text{B.4})$$

$$B_2 = \hbar\omega_0 \left[\frac{\langle a^\dagger + a \rangle_{ST}}{2} - \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]. \quad (\text{B.5})$$

In this interaction picture, the equation of motion for the density matrix is

$$\dot{\rho}_I(t) = -\frac{i}{\hbar} B_2(t) [(a^\dagger + a)(t), \rho_I(t)] + B_1(t) \{(a^\dagger + a)(t), \rho_I(t)\}. \quad (\text{B.6})$$

Following a very similar procedure to that used to obtain equation (A.1), we perform a Born–Markov approximation [15] and hence obtain an equation of motion for the resonator alone,

$$\begin{aligned} \dot{\tilde{\rho}}_{rI}(t) = & -\frac{1}{\hbar^2} \int_0^t \text{Tr} B_2(t) B_2(t') P_{SET}(0) [(a^\dagger + a)(t), [(a^\dagger + a)(t'), \rho_I(t)]] dt' \\ & -\frac{i}{\hbar} \int_0^t \text{Tr} B_2(t) B_1(t') P_{SET}(0) [(a^\dagger + a)(t), \{(a^\dagger + a)(t'), \rho_I(t)\}] dt' \\ & -\frac{i}{\hbar} \int_0^t \text{Tr} B_1(t) B_2(t') P_{SET}(0) \{(a^\dagger + a)(t), [(a^\dagger + a)(t'), \rho_I(t)]\} dt' \\ & + \int_0^t \text{Tr} B_1(t) B_1(t') P_{SET}(0) \{(a^\dagger + a)(t), \{(a^\dagger + a)(t'), \rho_I(t)\}\} dt', \end{aligned} \quad (\text{B.7})$$

where P_{SET} is a vector that describes the occupation of the SET. The Born approximation is based on the assumption of weak electromechanical coupling which from equation (B.1) implies $\hbar\omega_0\lambda \ll eV$ (i.e. $\chi \ll eV$). The Markov approximation assumes that the SET correlation functions decay very rapidly on the timescale of the rate of change of the interaction picture density matrix [22].

We now need to evaluate the SET correlation functions. The only non-zero terms are,

$$\text{Tr}[B_2(t)B_1(t')P_{SET}(0)] = -\frac{g(\hbar\omega_0)^2\lambda^2}{2}e^{-geV(t-t')}, \quad (\text{B.8})$$

$$\text{Tr}[B_2(t)B_2(t')P_{SET}(0)] = (\hbar\omega_0)^2\lambda^2\frac{E'_DE'_S}{eV}e^{-geV(t-t')} \quad (\text{B.9})$$

and, in line with our Markov approximation, we discard terms that decay like e^{-geVt} .⁷ Noting that $(a^\dagger + a)(t) = (a^\dagger + a)\cos(\omega_0t) + i(a^\dagger - a)\sin(\omega_0t)/\omega_0$, we can insert the correlation functions into equation (B.7) and do the integrals over t' exactly. Converting back to the Schrödinger picture,

$$\begin{aligned} \dot{\rho}_r(t) = & -\frac{i}{\hbar}[H_r, \rho_r(t)] - \frac{1}{\hbar^2}\left(\lambda^2\hbar^2\omega_0\frac{E'_DE'_S}{(eV)^2}\right)\frac{[(a^\dagger + a), [\Gamma(a^\dagger + a) - i(a^\dagger - a), \rho_r(t)]]}{\Gamma^2 + 1} \\ & + \frac{i}{\hbar}\frac{g\lambda^2\hbar^2\omega_0}{2}\frac{[(a^\dagger + a), \{\Gamma(a^\dagger + a) - i(a^\dagger - a), \rho_r(t)\}]]}{\Gamma^2 + 1}, \end{aligned} \quad (\text{B.10})$$

we find a master equation that can be easily rearranged into a more familiar form, which is very close to the Caldeira–Leggett equation,

$$\begin{aligned} \dot{\rho}(t) = & -\frac{i}{\hbar}[H_r, \rho_r(t)] - \frac{i}{\hbar}\left(\frac{g\lambda^2\hbar^2\omega_0}{2x_q^2}\frac{\Gamma}{1 + \Gamma^2}\right)[x^2, \rho_r(t)] - \frac{i}{\hbar}\frac{\gamma_{CL}}{2}[x, \{p, \rho_r(t)\}] \\ & - D[x, [x, \rho_r(t)]] - \frac{f}{\hbar}[x, [p, \rho_r(t)]]. \end{aligned} \quad (\text{B.11})$$

The second and first terms can be combined, leading to a renormalization of the resonator frequency,

$$\omega_R = \omega_0\left(1 - \kappa\frac{\Gamma^2}{1 + \Gamma^2}\right)^{\frac{1}{2}}$$

and the damping, diffusion and anomalous diffusion constant are given by

$$\begin{aligned} \gamma_{CL} &= \frac{2\lambda^2g\hbar\omega_0}{\Gamma^2 + 1} = \omega_0\frac{\kappa\Gamma}{1 + \Gamma^2}, \\ D &= \frac{m\gamma_{CL}E'_DE'_S}{eV\hbar^2}, \\ \hbar f &= -\frac{E'_DE'_S\kappa}{eV(1 + \Gamma^2)}, \end{aligned}$$

respectively. These values are consistent with previous (classical) calculations in the limit $\Gamma \gg 1$ [7, 15].

⁷ It is therefore no surprise that the equation we eventually obtain shows transient deviations from the ‘true’ resonator behaviour on times $\sim (geV)^{-1}$.

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