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Non-Abelian Chern–Simons models with discrete gauge groups on a lattice

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New Journal of Physics 7 (2005) 187

Received 3 May 2005

Published 26 August 2005

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/7/1/187

Abstract. We construct the local Hamiltonian description of the Chern–Simons theory with discrete non-Abelian gauge group on a lattice. We show that the theory is fully determined by the phase factors associated with gauge transformations and classify all possible non-equivalent phase factors. We also construct the gauge invariant electric-field operators that move fluxons around and create/annihilate them. We compute the resulting braiding properties of the fluxons. We apply our general results to the simplest class of non-Abelian groups, the dihedral groups D_n .

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1. Introduction

Realization of a quantum computer would be very important for the solution of hard problems in classical mathematics [1] and even more important for the solution of many quantum problems in field theory, quantum chemistry, material physics, etc [2]–[4]. However, despite a large effort of many groups, this realization remains distant because of the conflicting requirements imposed by scalability and decoupling from the environment. This dichotomy can be naturally resolved by the error-free computation in the topologically protected space [5, 6] if the physical systems realizing such spaces are identified and implemented. Though difficult, this seems more realistic than attempts to decouple simple physical systems implementing individual bits from the environment. Thus, the challenge of the error-free quantum computation has resulted in a surge of interest in physical systems and mathematical models that were previously considered very exotic.

The main idea of the whole approach, due to Kitaev [9], is that the conditions of long decoherence rate and scalability can be in principle satisfied if elementary bits are represented by anyons, the particles that undergo non-trivial transformations when moved adiabatically around each other (braided) [9]–[11] and that cannot be distinguished by local operators. One of the most famous examples of such excitations is provided by the fractional quantum Hall effect [17, 18]. The difficult part is, of course, to identify a realistic physical system that has such excitations and allows their manipulations. To make it more manageable, this problem should be separated into different layers. The bottom layer is the physical system itself, the second is the theoretical model that identifies the low-energy processes, the third is the mathematical model that starts with the most relevant low-energy degrees of freedom and gives the properties of anyons while the fourth deals with construction of the set of rules on how to move the anyons in order to achieve a set of universal quantum gates (further lies the layer of algorithms and so on).

Ideally, the study of each layer should provide the one below it with a few simple alternatives and the one above the resulting properties of the remaining low-energy degrees of freedom.

In this paper, we focus on the third layer: we discuss a particular set of mathematical models that provides anyon excitations, namely the Chern–Simons gauge theories with discrete gauge groups. Generally, an appealing realization of the anyons is provided by the fluxes in non-Abelian gauge theories [10, 11]. The idea is to encode individual bits in the value of fluxes belonging to the same conjugacy class of the gauge group: such fluxes cannot be distinguished locally because they are transformed into one another by a global gauge transformation and would be even completely indistinguishable in the absence of other fluxes in the system. Alternatively, one can protect anyons from the adverse effect of the local operators by spreading the fluxes over a large region of space. In this case, one does not need to employ a non-Abelian group: the individual bits can even be encoded by the presence/absence of flux in a given area, for instance a large hole in the lattice. Such models [12]–[16] are much easier to implement in solid-state devices but they do not provide a large set of manipulations that would be sufficient for universal quantum computation. Thus, these models provide a perfect quantum memory but not a quantum processor. On the other hand, the difficulty with the flux representation is that universal computation can be achieved only by large non-Abelian groups (such as A_5) that are rather difficult to implement, or if one adds charge motion to the allowed set of manipulations. Because the charges form a non-trivial representation of the local gauge group, it is difficult to protect their coherence in the physical system which makes the latter alternative also difficult to realize in physical systems. The last alternative is to realize a Chern–Simons model where on the top of the conjugacy transformations characteristic of non-Abelian theories, the fluxes acquire non-trivial phase factors when moved around each other. We explore this possibility in this paper.

Chern–Simons theories with finite non-Abelian gauge groups have been extensively studied for a continuous $(2 + 1)$ -dimensional space-time. Unlike continuous gauge group, discrete groups on a continuous space allow non-trivial fluxes only around cycles which cannot be contracted to a single point. So in a path-integral approach, and by contrast to the case of continuous groups, the integration domain is restricted to gauge-field configurations for which the local flux density vanishes everywhere. Such path integrals were introduced, analysed, and classified in an original paper by Dijkgraaf and Witten [20]. They showed that for a given finite gauge group G , all possible Chern–Simons actions in $2 + 1$ dimensions are in one-to-one correspondence with elements in the third cohomology group $H^3(G, U(1))$ of G with coefficients in $U(1)$. They also provided a description in terms of a $2 + 1$ lattice gauge theory, where space-time is tiled by tetrahedra whose internal states are described by just three independent group elements (g, h, k) because fluxes through all triangular plaquettes vanish. Elements of $H^3(G, U(1))$ are then identified with functions $\alpha(h, k, l)$ that play the role of the elementary action for a tetrahedron. This description turns out to be rather cumbersome because $\alpha(h, k, l)$ does not have specific symmetry properties, so that the definition of the total action requires to choose an ordering for all the lattice sites, which cannot be done in a natural way. As a result, it seems very difficult to take the limit of a continuous time that is necessary for our purposes because physical implementations always require an explicit Hamiltonian form.

In principle, the knowledge of an elementary action α in $H^3(G, U(1))$ allows one to derive all braiding properties of anyonic excitations (i.e. local charges and fluxes). This has been done directly from the original path-integral formulation [21, 22], using as an intermediate step the representation theory of the so-called quasi-quantum double associated to the group G [23].

This mathematical structure is completely defined by the group G and a choice of any element α in $H^3(G, U(1))$. Using this latter formalism, which bypasses the need to construct microscopic Hamiltonians, many detailed descriptions of anyonic properties for various finite groups, such as direct products of cyclic groups, or dihedral groups, have been given by de Wild Propitius [24]. Unfortunately, these general results do not provide a recipe of how to construct a microscopic Hamiltonian with prescribed braiding properties.

To summarize, in spite of a rather large body of theoretical knowledge, we still do not know which Chern–Simons models with a finite gauge group can ‘in principle’ be realized in a physical system with a local Hamiltonian, that is which one can appear as low-energy theory. To answer this question is the purpose of this paper.

The main ingredients of our construction are the following. We shall be mostly interested in a Hilbert space spanned by ‘dilute’ classical configurations of fluxes because it is only these configurations that are relevant for quantum computation that involves flux braiding and fusion. Furthermore, we expect that a finite spatial separation between fluxons may facilitate various manipulations and measurements. Note in this respect that in theories with discrete group symmetry in a continuous space [20]–[22], non-trivial fluxes appear only thanks to a non-trivial topology of the ambient space [25] and thus are restricted to large well-separated holes in a flat two-dimensional (2D) space. The second feature of our construction is that gauge generators are modified by phase factors which depend on the actual flux configuration in a local way. This is a natural generalization of the procedure we have used recently for the construction of Chern–Simons models with \mathbb{Z}_N symmetry group [19, 27]. Note that in the most general situations, the representation of the local gauge group in a discrete Chern–Simons theory becomes ‘projective’ [20]–[24]. This would be inappropriate for a robust implementation because projective representations lead to degenerate multiplets that are strongly coupled to local operators, and therefore become very fragile in the presence of external noise. We shall, therefore, restrict ourselves to the models where no projective representations occur. In practice, they are associated with the non-trivial elements of the group $H^2(G, U(1))$, and it turns out that for some interesting classes of groups such as the dihedral groups D_N with N odd, $H^2(G, U(1))$ is trivial [24], so this restriction is not too important. As we show in section 3, these assumptions allow us to find all the possible phase factors associated with the gauge transformations in terms of homomorphisms from the subgroups of G that leave invariant a fixed element of G under conjugacy into $U(1)$. The last step is to construct a set of local gauge-invariant operators corresponding to the following elementary processes: moving a single fluxon, creating or annihilating a pair of fluxons with vacuum quantum numbers, and branching processes where two nearby fluxons merge into a single one (or conversely). We shall see that the requirement of local gauge invariance leads to a relatively mild constraint for the possible phase factors, leaving a rather large set of solutions.

The main result of this work is two-fold. First, we provide an explicit computation of holonomy properties of fluxes in a set of models based on dihedral groups. Of special interest is the simplest of them, D_3 which is also the permutation group of three elements S_3 . This group belongs to a class which is in principle capable of universal quantum computation [11]. This part is therefore connected to the upper layer (designing a set of universal gates from properties of anyons) in the classification outlined above. But our construction of a local Hamiltonian version for a Chern–Simons model on a lattice provides some guidelines for possible desirable physical implementations.

The plan of the paper is the following. Section 2 is mostly pedagogical, providing the motivation for our general construction through the simplest example of a Chern–Simons gauge theory, namely the non-compact $U(1)$ model. In section 3, we formulate general conditions on local Chern–Simons phase factors that satisfy gauge invariance conditions. In section 4, we discuss the construction of the electric-field operator and derive condition on the phase factor that allows one a gauge invariant fluxon dynamics. In section 5, we discuss the fluxon braiding properties and derive the Chern–Simons phase factors associated with the braiding. In section 6, we apply our results to the simplest non-Abelian groups D_n . Finally, section 7 gives conclusions. Some technical points relevant for the general construction have been relegated to appendix A, and appendix B discusses some interesting features of the torus geometry. Although this geometry is not easy to implement in experiments, it is conceptually very interesting since it is the simplest example of a 2D space without boundary and with topologically non-trivial closed loops.

2. Overview on Abelian Chern–Simons models

To motivate the construction of this paper, it is useful to discuss some important features of Chern–Simons gauge theories in the simpler case of Abelian groups. For this purpose, we shall consider here as an illustration the model based on the ‘continuous’ Abelian group with one generator in its non-compact version. Of course, our main purpose here is to address ‘finite’ groups, but as we shall discuss, this non-compact $U(1)$ model contains already the key ingredients. On a $(2 + 1)$ -dimensional space-time, it is defined from the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\lambda(\dot{A}_x^2 + \dot{A}_y^2) - \frac{1}{2}\mu B^2 + \nu(\dot{A}_x A_y - \dot{A}_y A_x), \quad (1)$$

where $B = \partial_x A_y - \partial_y A_x$ is the local magnetic field (a pseudo-scalar in $2 + 1$ dimensions) and dots stand for time-derivatives. We have used the gauge in which the time component A_0 of the vector potential is zero. Because of this, we shall consider only invariance under time-independent gauge transformations in this discussion. These are defined as usual by $A_\rho \rightarrow A_\rho + \partial_\rho f$, where $f(x, y)$ is any time-independent smooth scalar function of spatial position. Under such a gauge transformation, the action associated to the system evolution on a 2D space manifold M during the time interval $[t_1, t_2]$ varies by the amount $\Delta\mathcal{S}$ where

$$\Delta\mathcal{S} = \nu \int_M d^2\mathbf{r} \int_{t_1}^{t_2} dt \left(\dot{A}_x \frac{\partial f}{\partial y} - \dot{A}_y \frac{\partial f}{\partial x} \right). \quad (2)$$

Because f is time-independent, the integrand is a total time-derivative, so we may write $\Delta\mathcal{S} = \nu(I(A_2, f) - I(A_1, f))$, where A_i denotes the field configuration at time t_i , $i = 1, 2$, and

$$I(A, f) = \int_M d^2\mathbf{r} \left(A_x \frac{\partial f}{\partial y} - A_y \frac{\partial f}{\partial x} \right). \quad (3)$$

In the case where M has no boundary (and in particular no hole), we may integrate by parts and get

$$I(A, f) = \int_M d^2\mathbf{r} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) f = \int_M d^2\mathbf{r} B f. \quad (4)$$

When $\nu \neq 0$, this modifies in a non-trivial way the behaviour of the corresponding quantized model under time-independent gauge transformations. One way to see this is to consider the time-evolution of the system's wave-functional $\Psi(A, t)$. In a path-integral approach, the probability amplitude to go from an initial configuration $A_1(\mathbf{r})$ at time t_1 to a final $A_2(\mathbf{r})$ at time t_2 is given by

$$\mathcal{A}_{21} = \int \mathcal{D}A(\mathbf{r}, t) \exp \left\{ \frac{i}{\hbar} \mathcal{S}(A) \right\},$$

where fields $A(\mathbf{r}, t)$ are required to satisfy boundary conditions $A(\mathbf{r}, t_j) = A_j(\mathbf{r})$ for $j = 1, 2$. After the gauge transformation $\mathbf{A}' = \mathbf{A} + \nabla f$, and using the above expression for $\Delta \mathcal{S}$, we see that the probability amplitude connecting the transformed field configurations A'_1 and A'_2 is

$$\mathcal{A}_{2'1'} = \mathcal{A}_{21} \exp \left\{ i \frac{\nu}{\hbar} (I(A_2, f) - I(A_1, f)) \right\}. \quad (5)$$

It is therefore natural to define the gauge-transformed wavefunctional $\tilde{\Psi}$ by

$$\tilde{\Psi}(A') = \Psi(A) \exp \left(i \frac{\nu}{\hbar} I(A, f) \right). \quad (6)$$

This definition ensures that $\Psi(A, t)$ and $\tilde{\Psi}(A, t)$ evolve according to the same set of probability amplitudes.

In a Hamiltonian formulation, we associate to any classical field configuration $A(\mathbf{r})$ a basis state $|A\rangle$. The gauge transformation corresponding to f is now represented by a unitary operator $U(f)$ defined by

$$U(f)|A\rangle = \exp \left(i \frac{\nu}{\hbar} I(A, f) \right) |A'\rangle. \quad (7)$$

The presence of the phase factor is one of the essential features of the Chern–Simons term (i.e. the term proportional to ν) added to the action. Note that when f varies, the family of operators $U(f)$ gives rise to a representation of the full group of local gauge transformations. Indeed, at the classical level, the composition law in this group is given by the addition of the associated f functions, and because $I(A, f)$ given in equation (4) is itself gauge invariant, we have $U(f)U(g) = U(f + g)$. It is interesting to give an explicit expression for $U(f)$. It reads:

$$U(f) = U_{\nu=0}(f) \exp \left\{ i \frac{\nu}{\hbar} \int_M d^2\mathbf{r} Bf \right\}, \quad U_{\nu=0}(f) = \exp \left\{ \frac{i}{\hbar} \int_M d^2\mathbf{r} (\partial_x \Pi_x + \partial_y \Pi_y) f \right\}, \quad (8)$$

where Π_x and Π_y are the canonically conjugated variables to A_x and A_y . Note that this no longer holds in the case of a manifold M with a boundary, as will be discussed in a few paragraphs.

In the Hamiltonian quantization, a very important role is played by the gauge-invariant electric operators E_x and E_y . In the absence of Chern–Simons term, they are simply equal to Π_x and Π_y . When $\nu \neq 0$, because of equation (8), the transformation law for Π_x and Π_y becomes

$$\Pi_x \rightarrow \Pi_x + \nu \partial_y f, \quad \Pi_y \rightarrow \Pi_y - \nu \partial_x f.$$

To compensate for this new gauge sensitivity of conjugated momenta, the gauge-invariant electric field becomes

$$E_x = \Pi_x - \nu A_y, \quad E_y = \Pi_y + \nu A_x.$$

Any classical gauge-invariant Lagrangian gives rise, after Legendre transformation, to a Hamiltonian which is a functional of E_x , E_y and B fields. If we add the Chern–Simons term to the original Lagrangian and perform the Legendre transformation, we get a new Hamiltonian which is expressed in terms of the new gauge-invariant E_x , E_y and B fields through the ‘same’ functional as without the Chern–Simons term. For the special example of the Maxwell–Chern–Simons Lagrangian (1), this functional reads

$$H = \int_M d^2\mathbf{r} \left(\frac{1}{2\lambda} \mathbf{E}^2 + \frac{\mu}{2} B^2 \right).$$

But although the Chern–Simons term preserves the Hamiltonian functional, it does modify the dynamical properties of the system through a modification of the basic commutation rules between E_x and E_y . More precisely, we have

$$[E_x(\mathbf{r}), E_y(\mathbf{r}')] = -i\hbar(2\nu)\delta(\mathbf{r} - \mathbf{r}'). \quad (9)$$

So it appears that finding the appropriate deformations of electric-field operators plays a crucial role in constructing the Hamiltonian version of a Chern–Simons theory. We have also seen that such deformations are strongly constrained by the requirement of invariance under local gauge transformations. This discussion shows that most of the relevant information is implicitly encoded in the additional phase factor $I(A, f)$ involved in quantum gauge transformations, as in equations (6) and (7).

Let us now briefly discuss what happens when the model is defined on a 2D space manifold M with a boundary ∂M . Using Stokes’ formula, we may recast the phase factor $I(A, f)$ attached to a gauge transformation as

$$I(A, f) = \int_M d^2\mathbf{r} Bf - \int_{\partial M} f\mathbf{A}\cdot d\mathbf{l}. \quad (10)$$

In this situation, the phase factor $I(A, f)$ is no longer gauge invariant, and this implies that two gauge transformations attached to functions f and g do not commute because

$$(I(A, f) + I(A + \nabla f, g)) - (I(A, g) + I(A + \nabla g, f)) = \int_{\partial M} (f\nabla g - g\nabla f)\cdot d\mathbf{l}. \quad (11)$$

In more mathematical terms, this reflects the fact that the phase factor $I(A, f)$, as used in equations (6) and (7) defines only a projective representation of the classical gauge group, that is

$$U(f)U(g) = \exp\left(-i\frac{\nu}{\hbar} \int_{\partial M} f\nabla g\cdot d\mathbf{l}\right) U(f+g). \quad (12)$$

As first shown by Witten [26], this may be understood in terms of a chiral matter field attached to the boundary of M . An explicit example of boundary degrees of freedom induced by a Chern–Simons term has been discussed recently in the case of a \mathbb{Z}_2 model on a triangular lattice [27].

To close this preliminary section, it is useful to discuss the case of a finite cyclic group \mathbb{Z}_N . In the $U(1)$ case, for a pair of points \mathbf{r} and \mathbf{r}' , we have a natural group element defined by $\exp(i\frac{2\pi}{\Phi_0} \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A} \cdot d\mathbf{l})$, where the integral is taken along the segment joining \mathbf{r} and \mathbf{r}' , and Φ_0 is the flux quantum in the model. For a finite group G , the notion of a Lie algebra is no longer available, so it is natural to define the model on a lattice. In a classical gauge theory, each oriented link ij carries a group element $g_{ij} \in G$. We have the important constraint $g_{ij}g_{ji} = e$, where e is the neutral element of the group G . In the quantum version, the Hilbert space attached to link ij is the finite dimensional space generated by the orthogonal basis $|g_{ij}\rangle$ where g_{ij} runs over all elements of G . For a lattice, the corresponding Hilbert space is obtained by taking the tensor product of all these finite dimensional spaces associated to links. In the \mathbb{Z}_N model, g_{ij} becomes an integer modulo N , p_{ij} . The connection with the continuous case is obtained through the identification $\int_i^j \mathbf{A} \cdot d\mathbf{l} = \frac{\Phi_0}{N} p_{ij}$. On each link ij , we introduce the unitary operator π_{ij}^+ which sends $|p_{ij}\rangle$ into $|p_{ij} + 1\rangle$. In the absence of a Chern–Simons term, the generator of the gauge transformation based at site i (which turns p_{jk} into $p_{jk} + \delta_{ji} - \delta_{ki}$) is $U_i = \prod_j^{(i)} \pi_{ij}^+$, where the product involves all the nearest neighbours of site i . By analogy with the continuous case, the presence of a Chern–Simons term is manifested by an additional phase factor whose precise value depends on the lattice geometry and is to some extent arbitrary, since fluxes are defined on plaquettes, not on lattice sites. On a square lattice, a natural choice is to define U_i according to [19]:

$$U_i = \prod_j^{(i)} \pi_{ij}^+ \exp \left(-i \frac{v}{4\hbar} \left(\frac{2\pi}{N} \right)^2 \sum_{(jk) \in \mathcal{L}(i)} p_{jk} \right), \quad (13)$$

where $\mathcal{L}(i)$ is the oriented loop defined by the outer boundary of the four elementary plaquettes adjacent to site i . This expression has exactly the same structure as equation (8), but somehow, the local magnetic field at site i is replaced by a smeared field on the smallest available loop centred around i . It has been shown [19] that a consistent quantum theory can be constructed only when v/\hbar is an integer multiple of N/π .

3. Generators of local gauge transformations

As discussed in the previous section, the most important step is to construct a non-trivial phase factor which appears in the definition of unitary operators associated to local gauge transformations, generalizing equation (7). For this, let us first define the operator $L_{ij}(g)$ which is the left multiplication of g_{ij} by g , namely: $L_{ij}(g)|g_{ij}\rangle = |gg_{ij}\rangle$. For any site i and group element g , we choose the generator of a local gauge transformation based at i to be of the following form

$$U_i(g) = \prod_j^{(i)} L_{ij}(g) \prod_{\mathbf{r}}^{(i)} \chi_{\Phi(i,\mathbf{r})}(g), \quad (14)$$

where j denotes any nearest neighbour of i and $\Phi(i, \mathbf{r})$ is the flux around any of the four square plaquettes, centred at \mathbf{r} , adjacent to i . Here, and throughout this paper, we shall focus on the square lattice geometry, to simplify the presentation. But adaptations of the basic construction to other lattices are clearly possible. Since, we are dealing with a non-Abelian group, we have to specify an origin in order to define these fluxes, and it is natural to choose the same site i ,

which is expressed through the notation $\Phi(i, \mathbf{r})$. Since, we wish $U_i(g)$ to be unitary, we require $|\chi_\Phi(g)| = 1$. It is clear from this construction that two generators $U_i(g)$ and $U_j(h)$ based on different sites commute, since the phase factors $\chi_\Phi(g)$ are gauge invariant.

This form is a simple generalization of the lattice Chern–Simons models for the cyclic groups \mathbb{Z}_N discussed in the previous section. In this example, for a square plaquette $ijkl$, the flux Φ is equal to $p_{ij} + p_{jk} + p_{kl} + p_{li}$ modulo N , and g is simply any integer modulo N . Equation (13) above may be interpreted as

$$\chi_\Phi(g) = \exp\left(-i\frac{\nu}{4\hbar}\left(\frac{2\pi}{N}\right)^2\Phi g\right). \quad (15)$$

This is a well-defined function for Φ and g modulo N only if ν/\hbar is an integer multiple of $2N/\pi$. We have not succeeded in casting odd integer multiples of N/π for ν/\hbar in the framework of the general construction to be presented now. This is not too surprising since these models were obtained by imposing special periodicity conditions on an infinite-dimensional Hilbert space where p_{ij} can take any integer value [19]. Our goal here is not to write down all possible Chern–Simons theories with a finite group, but to easily construct a large number of them, therefore allowing for non-trivial phase factors when two localized flux excitations are exchanged.

As discussed in section 1, a very desirable property, at least for the sake of finding possible physical implementations, is that these deformed generators define a unitary representation of the group G . So we wish to impose

$$U_i(g)U_i(h) = U_i(gh) \quad (16)$$

or equivalently

$$\chi_{h\Phi h^{-1}}(g)\chi_\Phi(h) = \chi_\Phi(gh). \quad (17)$$

To solve these equations let us first choose a group element Φ . Let us denote by H_Φ the stabilizer of Φ under the operation of conjugacy, namely h belongs to H_Φ whenever $h\Phi h^{-1} = \Phi$. This notion is useful to describe the elements of the conjugacy class of Φ . Indeed, we note that $gh\Phi(gh)^{-1} = g\Phi g^{-1}$ if h belongs to H_Φ . Therefore, the elements in the conjugacy class of Φ are in one to one correspondence with the left cosets of the form gH_Φ . Let us pick one representative g_n in each of these cosets. We shall now find all the functions $\chi_{g_n\Phi g_n^{-1}}(g)$. First, we may specialize equation (17) to the case where h belongs to H_Φ , giving

$$\chi_\Phi(g)\chi_\Phi(h) = \chi_\Phi(gh). \quad (18)$$

In particular, it shows that the function $h \rightarrow \chi_\Phi(h)$ defines a group homomorphism from H_Φ to $U(1)$. Once this homomorphism is known, we can specify completely $\chi_\Phi(g)$ for any group element g once the values $\chi_\Phi(g_n)$ for the coset representatives are known. More explicitly, we have

$$\chi_\Phi(g_n h) = \chi_\Phi(g_n)\chi_\Phi(h), \quad (19)$$

where $h \in H_\Phi$. Finally, equation (17) yields

$$\chi_{g_n\Phi g_n^{-1}}(g) = \frac{\chi_\Phi(g g_n)}{\chi_\Phi(g_n)}. \quad (20)$$

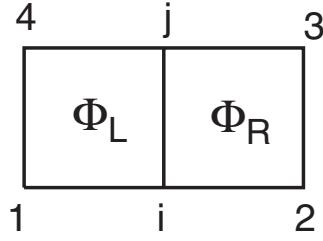


Figure 1. The site-labelling convention. For any bond (ij) (shown here as middle vertical bond) the surrounding sites are labelled 1, 2, 3, 4 as indicated here. The fluxes $\Phi(i, L)$, $\Phi(i, R)$, $\Phi(j, L)$, $\Phi(j, R)$ are counted counterclockwise, starting from site i , (j) , e.g. $\Phi(i, L) = g_{ij}g_{j4}g_{41}g_{1i}$.

Let us now show that for any choice of homomorphism $h \rightarrow \chi_\Phi(h)$, $h \in H_\Phi$, and unit complex numbers for $\chi_\Phi(g_n)$, equations (19) and (20) reconstruct a function $\chi_\Phi(g)$ which satisfies the condition (17). Any element g' in G may be written as $g' = g_n h$, with $h \in H_\Phi$. We have

$$\chi_{g'\Phi g'^{-1}}(g) = \chi_{g_n \Phi g_n^{-1}}(g) \stackrel{(20)}{=} \frac{\chi_\Phi(g g_n)}{\chi_\Phi(g_n)} \stackrel{(19)}{=} \frac{\chi_\Phi(g g_n h)}{\chi_\Phi(g_n h)} = \frac{\chi_\Phi(g g')}{\chi_\Phi(g')}, \quad (21)$$

which is exactly equation (17).

Note that there are many equivalent ways to choose these functions $\chi_\Phi(g)$. Let us multiply the system wavefunction by a phase factor of the form $\prod_i \prod_{\mathbf{r}}^{(i)} \epsilon(\Phi(i, \mathbf{r}))$, where $|\epsilon(\Phi)| = 1$. Under this unitary transformation, $\chi_\Phi(g)$ is changed into $\tilde{\chi}_\Phi(g)$ given by

$$\tilde{\chi}_\Phi(g) = \epsilon(g \Phi g^{-1}) \chi_\Phi(g) \epsilon(\Phi)^{-1}. \quad (22)$$

In particular, it is possible to choose the values of $\epsilon(g_n \Phi g_n^{-1})$ so that $\tilde{\chi}_\Phi(g_n) = 1$. Although this does not seem to be required at this stage of the construction, it is also necessary to assume that when the flux Φ is equal to the neutral element e , $\chi_e(g) = 1$. This will play an important role later in ensuring that the phase factor accumulated by the system wavefunction as a fluxon winds around another is well defined.

4. Basic processes for fluxon dynamics

4.1. General description

Our goal here is to construct local gauge invariant operations for basic fluxon processes: fluxon motion, creation of a pair with vacuum quantum numbers, branching of a single fluxon into two fluxons and their time reversions. These three types of elementary processes can all be derived from a single operation, the electric-field operator that at the level of classical configurations, is simply a left multiplication $L_{ij}(g)$ attached to any link ij . To show this, let us consider a pair of two adjacent plaquettes as shown on figure 1. We denote by $\Phi(i, L)$ (resp. $\Phi(i, R)$) the local flux through the left (resp. right) plaquette, with site i chosen as origin. Similarly, we define fluxes $\Phi(j, L)$ and $\Phi(j, R)$. Changing the origin from i to j simply conjugates fluxes, according to $\Phi(j, L(R)) = g_{ji} \Phi(i, L(R)) g_{ij}$. The left multiplication $L_{ij}(g)$ changes g_{ij} into $g'_{ij} = g g_{ij}$.

Therefore, it changes simultaneously both fluxes on the left and right plaquettes adjacent to link ij . More specifically, we have

$$\Phi'(i, L) = g\Phi(i, L), \quad (23)$$

$$\Phi'(i, R) = \Phi(i, R)g^{-1}. \quad (24)$$

In particular, this implies

$$\Phi'(i, R)\Phi'(i, L) = \Phi(i, R)\Phi(i, L), \quad (25)$$

$$\Phi'(i, L)\Phi'(i, R) = g\Phi(i, L)\Phi(i, R)g^{-1}. \quad (26)$$

Note that transformation laws for fluxes based at site j are slightly more complicated since they read

$$\Phi'(j, L) = \Phi(j, L)(g_{ji}gg_{ij}), \quad (27)$$

$$\Phi'(j, R) = (g_{ji}g^{-1}g_{ij})\Phi(j, R). \quad (28)$$

This asymmetry between i and j arises because $g'_{ji} = g_{ji}g^{-1}$, so we have

$$L_{ij}(g) = R_{ji}(g^{-1}), \quad (29)$$

where $R_{ji}(h)$ denotes the right multiplication of g_{ji} by the group element h . In the absence of Chern–Simons term, $L_{ij}(g)$ commutes with all local gauge generators with the exception of $U_i(h)$ since

$$U_i(h)L_{ij}(g)U_i(h)^{-1} = L_{ij}(hgh^{-1}). \quad (30)$$

We now apply these general formulas to the elementary processes involving fluxes. Suppose that initially a flux Φ was localized on the left plaquette, and that the right plaquette is fluxless. Applying $L_{ij}(g = \Phi^{-1})$ on such configuration gives

$$\Phi'(i, L) = e, \quad (31)$$

$$\Phi'(i, R) = \Phi, \quad (32)$$

which shows that a Φ -fluxon has moved from the left to the right plaquette. A second interesting situation occurs when both plaquettes are initially fluxless. The action of $L_{ij}(\Phi^{-1})$ on such state produces a new configuration characterized by

$$\Phi'(i, L) = \Phi^{-1}, \quad (33)$$

$$\Phi'(i, R) = \Phi. \quad (34)$$

So, we have simply created a fluxon and antifluxon pair from the vacuum. Of course, applying $L_{ij}(\Phi)$ on the final state annihilates this pair. Finally, a single flux $\Phi = \Phi_2\Phi_1$ originally located

on the left plaquette may split into a pair Φ_1 on the left and Φ_2 on the right. This is achieved simply by applying $L_{ij}(\Phi_2^{-1})$.

In order to incorporate these elementary processes into a Hamiltonian Chern–Simons theory, we have to modify $L_{ij}(g)$ into an electric-field operator $\mathcal{E}_{ij}(g)$ by introducing phase factors so that it commutes for all generators $U_k(h)$ if $k \neq i$ and that

$$U_i(h)\mathcal{E}_{ij}(g)U_i(h)^{-1} = \mathcal{E}_{ij}(hgh^{-1}). \quad (35)$$

As explained in the introduction, we shall need only to construct $\mathcal{E}_{ij}(g)$ for special types of configurations for which at least one of the four fluxes $\Phi(i, L)$, $\Phi(i, R)$, $\Phi'(i, L)$, $\Phi'(i, R)$ vanishes. Nevertheless, it is useful to first construct $\mathcal{E}_{ij}(g)$ for an arbitrary flux background. The requirement of local gauge symmetry induces strong constraints on the phase factors $\chi_\Phi(h)$ as we shall now show. These constraints are less stringent when we restrict the action of $\mathcal{E}_{ij}(g)$ to the limited set of configurations just discussed.

4.2. Construction of gauge-invariant electric-field operators

We now construct the electric-field operators $\mathcal{E}_{ij}(g)$ attached to links. In the absence of the Chern–Simons term the electric-field operator is equivalent to the left multiplication of the link variable

$$\mathcal{E}_{ij}(g) = L_{ij}(g). \quad (36)$$

In this case, $\mathcal{E}_{ij}(g)$ commutes with all local gauge generators with the exception of $U_i(h)$, and equation (35) is satisfied. We have also the group property, namely

$$\mathcal{E}_{ij}(g)\mathcal{E}_{ij}(h) = \mathcal{E}_{ij}(gh). \quad (37)$$

The Chern–Simons term gives phase factors $\chi_\Phi(g)$ to the gauge generators, so we expect some phase factor, Υ_{ij} , to appear in the electric-field operators as well: $\mathcal{E} = L_{ij}\Upsilon_{ij}$. The gauge invariance condition allows us to relate the phase factors associated with different field configurations to each other. Specifically, we introduce the reference states (shown on the top of figure 2) in which only two bonds carry non-trivial group elements. We define the transition amplitude induced by the electric field between these reference states by $A(\Phi_L, \Phi_R, g)$. In order to find the phases Υ_{ij} for arbitrary field configuration, we first transform both the initial and final state by $U(\{h\}) = U_1(h_1)U_2(h_2)U_3(h_3)U_4(h_4)U_j(h_j)$. This relates the amplitude of the generic process to the amplitude, $a(\Phi_L, \Phi_R, g)$, of the process that starts with the reference state but leads to the special final state shown in figure 2 (middle right). Collecting the phase factors associated with the gauge transformation $U(\{h\})$ before and after the electric field moved the flux we get

$$\Upsilon_{ij} = \frac{\chi_{\Phi'(i,L)}(h_1) \chi_{\Phi'(i,R)}(h_2) \chi_{\Phi'(j,R)}(h_3) \chi_{\Phi'(j,L)}(h_4) \chi_{\Phi'(j,L)}(h_j) \chi_{\Phi'(j,R)}(h_j)}{\chi_{\Phi(i,L)}(h_1) \chi_{\Phi(i,R)}(h_2) \chi_{\Phi(j,R)}(h_3) \chi_{\Phi(j,L)}(h_4) \chi_{\Phi(j,L)}(h_j) \chi_{\Phi(j,R)}(h_j)} a(\Phi_L, \Phi_R, g), \quad (38)$$

where $\Phi(i, L)$ denotes the flux in the left plaquette counted from site i , $\Phi(j, R)$ denotes flux in the right plaquette counted from site j , and prime refers to the flux configuration after the action of the electric field. Finally, we employ the gauge transformation $W(g) = U_4(g)U_j(g)U_3(g)$ to

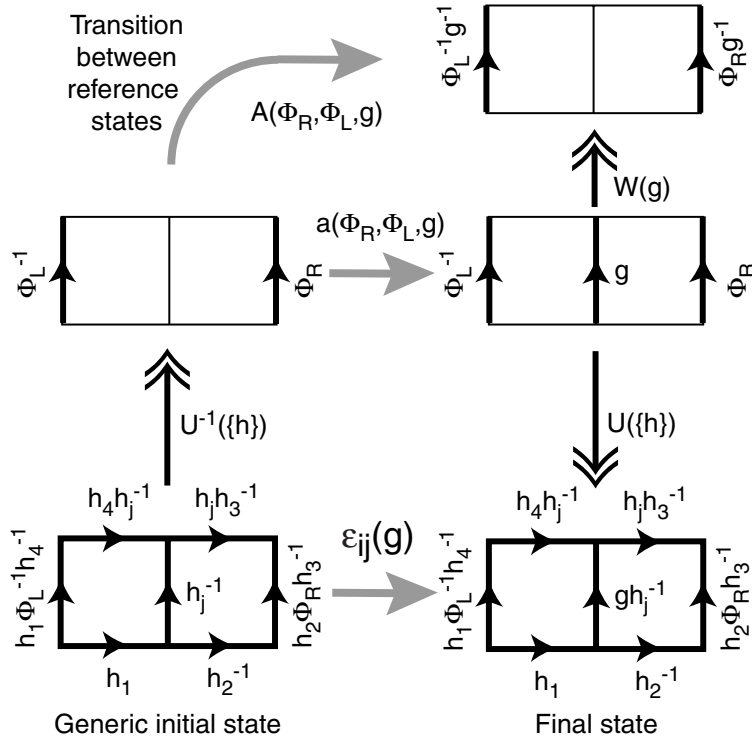


Figure 2. Construction of the gauge-invariant electric-field operator $\mathcal{E}_{ij}(g)$. This operator transforms the pair of fluxes (Φ_L, Φ_R) of the left and right plaquettes into the pair $(g\Phi_L, \Phi_R g^{-1})$. We define the amplitude of the transition between two reference states shown at the top of the figure by $A(\Phi_L, \Phi_R, g)$. The amplitude of the process starting from a generic initial state shown in the lower left can be related to $A(\Phi_L, \Phi_R, g)$ using the gauge invariance. This is done in two steps. Firstly, the gauge transformation $U(\{h\}) = U_1(h_1)U_2(h_2)U_3(h_3)U_4(h_4)U_j(h_j)$ is used to relate the amplitude of the process starting from the generic state to the amplitude of the transition $a(\Phi_L, \Phi_R, g)$ between reference state and a special state shown in the middle right. Secondly, we use gauge transformation $W(g) = U_4(g)U_j(g)U_3(g)$ to relate the special state to the reference state on the upper right. Site labelling is the same as in figure 1.

relate this special final state to the reference state. The phase factor associated with this gauge transformation is $(\chi_{\Phi_L g}(g)\chi_{g^{-1}\Phi_R}(g))^2$ so

$$a(\Phi_L, \Phi_R, g) = \frac{A(\Phi_L, \Phi_R, g)}{(\chi_{\Phi_L g}(g)\chi_{g^{-1}\Phi_R}(g))^2}. \quad (39)$$

In order to express the phase factors, Υ_{ij} , through the initial field configuration we relate the parameters, h_k , of the gauge transformations to the bond variables g_k by

$$h_1 = g_{1i}, \quad h_2 = g_{2i}, \quad h_3 = g_{3j}g_{ji}, \quad h_4 = g_{4j}g_{ji}, \quad h_j = g_{ji}$$

and the fluxes in the left and right plaquettes before and after electric-field operator has changed them. Before the electric-field operator has acted the fluxes were

$$\Phi(i, L) = \Phi_L, \quad \Phi(j, L) = g_{ji} \Phi_L g_{ij}, \quad \Phi(i, R) = \Phi_R, \quad \Phi(j, R) = g_{ji} \Phi_R g_{ij},$$

while afterwards they become

$$\Phi'(i, L) = g \Phi_L, \quad \Phi'(j, L) = g_{ji} \Phi_L g g_{ij}, \quad \Phi'(i, R) = \Phi_R g^{-1}, \quad \Phi'(j, R) = g_{ji} g^{-1} \Phi_R g_{ij}.$$

Combining the preceding equations and using the relation (17) a few times, we get the final expression for the phase factor,

$$\Upsilon_{ij} = \frac{\chi_{\Phi'(i,L)}(g_{1i}) \chi_{\Phi'(i,R)}(g_{2i}) \chi_{\Phi'(j,L)}(g_{4j}) \chi_{\Phi'(j,R)}(g_{3j}) \chi_{\Phi'(i,L)}(g'_{ji})^2 \chi_{\Phi'(i,R)}(g'_{ji})^2}{\chi_{\Phi(i,L)}(g_{1i}) \chi_{\Phi(i,R)}(g_{2i}) \chi_{\Phi(j,L)}(g_{4j}) \chi_{\Phi(j,R)}(g_{3j}) \chi_{\Phi(i,L)}(g_{ji})^2 \chi_{\Phi(i,R)}(g_{ji})^2} A(\Phi_L, \Phi_R, g), \quad (40)$$

where we have used the definition $g'_{ij} = g g_{ij}$ ($g'_{ji} = g_{ji} g^{-1}$) to make it more symmetric.

Commutation of $\mathcal{E}_{ij}(g)$ with $U_1(h_1)$, $U_2(h_2)$, $U_3(h_3)$, $U_4(h_4)$, and $U_j(h_j)$ follows directly from this construction. It can also be checked directly from (40), using the condition (17) on the elementary phase factors $\chi_\Phi(g)$. Note that sites i and j play different roles, which is expected because $\mathcal{E}_{ij}(g)$ acts by ‘left’ multiplication on g_{ij} , whereas $\mathcal{E}_{ji}(g)$ acts by ‘right’ multiplication on the same quantity.

Although the electric-field operator commutes with $U_1(h_1)U_2(h_2)$, $U_3(h_3)$, $U_4(h_4)$ and $U_j(h_j)$, it does not necessarily commute with $U_i(h)$ even if $hgh^{-1} = g$. In fact, the requirement of this commutation leads to an important constraint on possible choices of phases $\chi_\Phi(g)$. The appearance of the new constraints becomes clear when one considers a special field configuration shown in figure 3. Two identical field configurations shown on the top and bottom left of this figure can be obtained by two different gauge transformations from the reference state if both Φ_L and Φ_R commute with h : in one case one applies gauge transformation ‘only’ at site i , in the other one applies gauge transformation on all sites ‘except’ i . Provided that the resulting states are the same, i.e. $gh = hg$, the total phase factor Υ_{ij} obtained by these two different ways should be the same.

The phase factors associated with these gauge transformations are the following:

$$U_i(h) \rightarrow \chi_{\Phi_L}(h) \chi_{\Phi_R}(h), \quad U_i^{-1}(h) \rightarrow \frac{1}{\chi_{g\Phi_L}(h) \chi_{\Phi_R g^{-1}}(h)},$$

$$V_i^{-1}(h) \rightarrow \chi_{\Phi_L}^3(h^{-1}) \chi_{\Phi_R}^3(h^{-1}), \quad V_i(h) \rightarrow \frac{1}{\chi_{\Phi_L g}^2(h^{-1}) \chi_{g\Phi_L}(h^{-1}) \chi_{g^{-1}\Phi_R}^2(h^{-1}) \chi_{\Phi_R g^{-1}}(h^{-1})}.$$

Putting all these factors together, we conclude that the gauge invariance of the electric-field operator implies that

$$\frac{\chi_{\Phi_L}(h) \chi_{\Phi_R}(h)}{\chi_{g\Phi_L}(h) \chi_{\Phi_R g^{-1}}(h)} = \frac{\chi_{\Phi_L}^3(h^{-1}) \chi_{\Phi_R}^3(h^{-1})}{\chi_{\Phi_L g}^2(h^{-1}) \chi_{g\Phi_L}(h^{-1}) \chi_{g^{-1}\Phi_R}^2(h^{-1}) \chi_{\Phi_R g^{-1}}(h^{-1})}. \quad (41)$$

This condition can be further simplified by using the main phase factor equation (17). We start by noting that because h commutes with Φ_L , Φ_R and g , $\chi_{\Phi_{R,L}}(h) \chi_{\Phi_{R,L}}(h^{-1}) \equiv 1$ and $\chi_{g\Phi_{R,L}}(h) \chi_{g\Phi_{R,L}}(h^{-1}) \equiv 1$. This gives

$$\chi_{\Phi_L}^4(h) \chi_{\Phi_R}^4(h) = \chi_{\Phi_L g}^2(h) \chi_{g\Phi_L}^2(h) \chi_{g^{-1}\Phi_R}^2(h) \chi_{\Phi_R g^{-1}}^2(h). \quad (42)$$

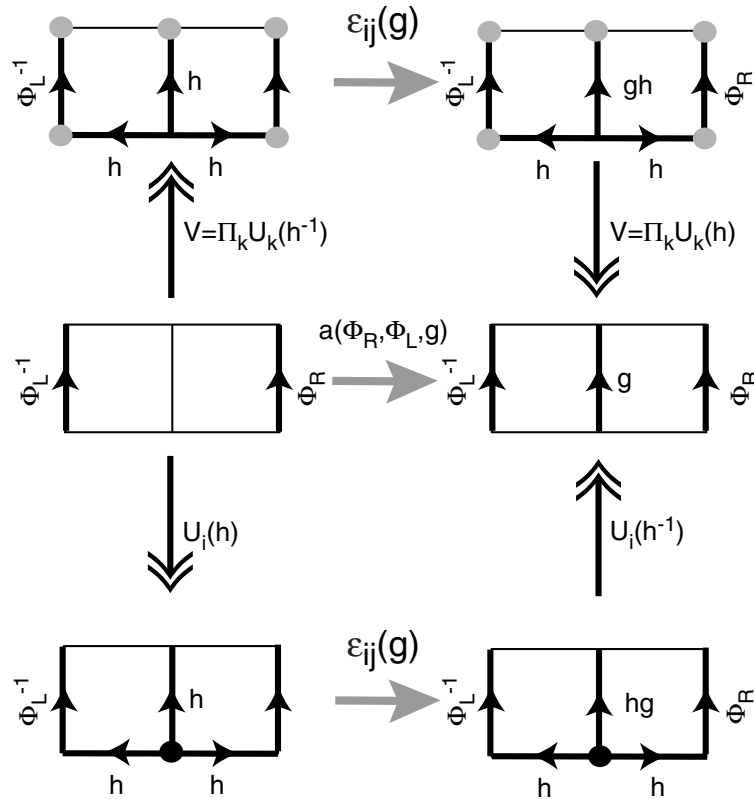


Figure 3. Gauge invariance of electric-field operator implies condition on the phase factors $\chi_\Phi(g)$. The state shown at the top and bottom left of the figure can be transformed to the reference state (middle) in two ways: by applying a gauge transformation $V = \prod_{k \neq i} U_k(h)$ on all shaded sites or by applying transformation $U_i(h^{-1})$ on site i . If $\Phi_L h = h \Phi_L$ and $\Phi_R h = h \Phi_R$ these transformations lead to the same reference state shown in the middle. Furthermore, the action of electric-field on these states can be found by making a gauge transformation of the final state shown on the middle right if $gh = hg$. The phases of electric-field operator obtained in these two ways should be equal giving additional condition on the phase associated with the gauge-field transformation.

Furthermore, combining the identities

$$\chi_{g\Phi_L}(h) = \chi_{g\Phi_L}(ghg^{-1}) = \chi_{\Phi_L g}(g)\chi_{\Phi_L g}(h)\chi_{g\Phi_L}(g^{-1})$$

and

$$1 = \chi_{g\Phi_L}(gg^{-1}) = \chi_{\Phi_L g}(g)\chi_{g\Phi_L}(g^{-1}),$$

we get

$$\chi_{g\Phi_L}(h) = \chi_{\Phi_L g}(h), \quad (43)$$

$$\chi_{\Phi_R g^{-1}}(h) = \chi_{g^{-1}\Phi_R}(h). \quad (44)$$

This reduces the condition (42) to a much simpler final form

$$\left(\frac{\chi_{g\Phi_L}(h) \chi_{\Phi_R g^{-1}}(h)}{\chi_{\Phi_L}(h) \chi_{\Phi_R}(h)} \right)^4 = 1. \quad (45)$$

We emphasize that constraint (45) on the phase factors has to be satisfied only for fluxes satisfying the condition $(h\Phi_L h^{-1}, h\Phi_R h^{-1}, hgh^{-1}) = (\Phi_L, \Phi_R, g)$. Although, we have derived this condition imposing only the gauge invariance of the electrical field acting on a very special field configuration, a more lengthy analysis shows that it is sufficient to ensure that in a general case

$$U_i(h) \mathcal{E}_{ij}(g) |\Psi\rangle = \mathcal{E}_{ij}(hgh^{-1}) U_i(h) |\Psi\rangle. \quad (46)$$

The details of the proof are presented in appendix A.

Unlike equation (17), the constraint (45) relates functions $\chi_\Phi(g)$ and $\chi_{\Phi'}(g)$ for Φ and Φ' belonging to ‘different’ conjugacy classes. As shown in section 6.2 this constraint strongly reduces the number of possible Chern–Simons theories. Note that if both $\chi_\Phi^{(1)}(g)$ and $\chi_\Phi^{(2)}(g)$ are solutions of the fundamental relations (17) and (45), their product $\chi_\Phi^{(1)}(g)\chi_\Phi^{(2)}(g)$ is also a solution. So there is a natural group structure on the set of possible Chern–Simons models based on the group G , which is transparent in the path-integral description: this means that the sum of two Chern–Simons action is also a valid Chern-Simons action.

Is this construction also compatible with the group property (37)? From equation (45), we obtain

$$\mathcal{E}_{ij}(g') \mathcal{E}_{ij}(g) |\Psi\rangle = \beta(\Phi_L, \Phi_R, g) \mathcal{E}_{ij}(g') |\Psi\rangle, \quad (47)$$

where

$$\beta(\Phi_L, \Phi_R, g) = \frac{A(\Phi_L, \Phi_R, g) A(g\Phi_L, \Phi_R g^{-1}, g')}{A(\Phi_L, \Phi_R, g'g)}.$$

It does not seem that $\beta(\Phi_L, \Phi_R, g)$ are always equal to unity for any choice of $\chi_\Phi(g)$ that in turn determines the amplitudes $A(s)$ (see appendix A). But this is not really a problem because this group property does not play much role in the construction of gauge-invariant Hamiltonians.

We now specialize the most general constraint arising from local gauge invariance to the various physical processes which are required for fluxon dynamics. For the single fluxon moving operation, we have $\Phi_L = \Phi$, $\Phi_R = e$ and $g = \Phi^{-1}$, so the condition equation (45) is always satisfied. For the pair creation process, we have $\Phi_L = \Phi_R = e$ and $g = \Phi^{-1}$. The constraint becomes

$$(\chi_{\Phi^{-1}}(h) \chi_\Phi(h))^4 = 1 \quad \text{if} \quad h\Phi h^{-1} = \Phi. \quad (48)$$

Finally, let us consider the splitting of an isolated fluxon into two nearby ones. This is described by $\Phi_L = \Phi_2 \Phi_1$, $\Phi_R = e$ and $g = \Phi_2^{-1}$. We need then to impose

$$\left(\frac{\chi_{\Phi_1}(h) \chi_{\Phi_2}(h)}{\chi_{\Phi_1 \Phi_2}(h)} \right)^4 = 1 \quad \text{if} \quad (h\Phi_1 h^{-1}, h\Phi_2 h^{-1}) = (\Phi_1, \Phi_2). \quad (49)$$

It is clear that condition (48) is a special case of the stronger condition (49). Furthermore, multiplying the conditions (49) for pairs of fluxes (Φ_L, Φ_R) and $(g\Phi_L, \Phi_R g^{-1})$, we get the most general condition (45); this shows that constraint (49) is necessary and sufficient condition for the gauge-invariant definition of the electric-field operator acting on any flux configuration.

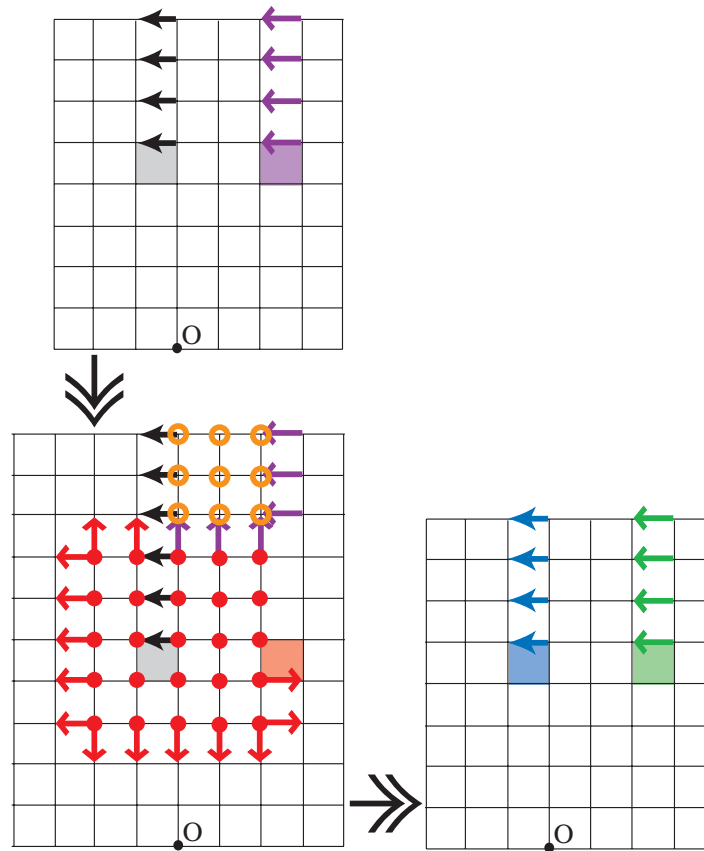


Figure 4. Braiding of two fluxes yields in a non-trivial transformation of their values and a phase factor. We start with the flux configuration shown in upper pane with fluxes g_1 (right) and g_2 (left) connected to the edge by the strings of g_1 (purple arrow) or g_2 (black) group elements. Moving the right flux around the left leaves behind a string of g_1 elements until its path crosses the vertical string of g_2 . Upon crossing the string changes to the string of $g'_1 = g_2^{-1}g_1g_2$ (red arrows). Performing the gauge transformations with $h = (g'_1)$ on all sites indicated by full (red) dots and with $h = g_1h$ on sites indicated by empty (orange) dots reduces the configuration to the template shown in the last pane with the new fluxes g'_1 and g'_2 (see text, equation (52)).

5. General expression for holonomy of fluxons

Let us consider two isolated fluxes carrying group elements g_1 and g_2 , and move the first one counterclockwise around the second one, as shown in figure 4. This can be done by successive applications of local gauge-invariant electric-field operators as discussed in the previous section. Although we wish to work in the gauge-invariant subspace, it is very convenient to use special configurations of link variables to illustrate what happens in such a process. We simply have to project all special states on the gauge-invariant subspace, which is straightforward since the fluxon moving operator commutes with this projector. The initial fluxes are described by two vertical strings of links carrying the group elements g_1 and g_2 , see figure 4. When several other fluxes are present, besides the two to be exchanged, it is necessary to choose the location of the strings in

such a way that no other flux is present in the vertical band delimited by the strings attached to the two fluxons. During the exchange process, the first fluxon collides with the second string, and after this event, it no longer carries the group element g_1 , but its conjugate $g'_1 = g_2^{-1} g_1 g_2$. After the process is completed, the first flux has recovered its original position, but the configuration of group elements has changed. If we measure the second flux using a path starting at point O shown on figure 4, we find that it has also changed into $g'_2 = g_2^{-1} g_1^{-1} g_2 g_1 g_2$. The final state can be reduced to its template state built from two strings carrying group elements g'_1 and g'_2 by the gauge transformation $\prod_i U_i(h_i)$, where h_i is locally constant in the two following regions: the core region inside the circuit followed by the first fluxon ($h_i = h_{\text{core}}$), the intermediate region delimited by the two initial vertical strings and the upper part of the circuit ($h_i = h_{\text{int}}$). Note that because we do not wish to modify external fluxes, we cannot perform gauge transformations in the bulk outside of these regions.

Group elements h_{core} and h_{int} have to satisfy the following conditions, which may be obtained readily upon inspection of figure 4

$$\begin{aligned} h_{\text{core}} &= g'_1, & h_{\text{core}} g_1^{-1} &= e, & h_{\text{int}} g'_2 &= g_2, \\ h_{\text{core}} h_{\text{int}}^{-1} &= g_1, & g'_1 h_{\text{int}}^{-1} &= g_1, & h_{\text{core}} g'_2 h_{\text{core}}^{-1} &= g_2. \end{aligned}$$

These equations are mutually compatible, and we get $h_{\text{int}} = g_1^{-1} g_2^{-1} g_1 g_2$. Since fluxes are present in the core region, they will contribute a phase factor $f(g_1, g_2)$ when the gauge transformation from the template to the actual final state is performed. The final result may be summarized as follows:

$$|g_1, g_2\rangle \longrightarrow f(g_1, g_2) |g'_1, g'_2\rangle, \quad (50)$$

$$g'_1 = g_2^{-1} g_1 g_2, \quad (51)$$

$$g'_2 = g_2^{-1} g_1^{-1} g_2 g_1 g_2, \quad (52)$$

$$f(g_1, g_2) = \chi_{g'_1}(g'_1)^2 \chi_{g'_2}(g'_1)^4. \quad (53)$$

The new phase factor $f(g_1, g_2)$ appears not to depend on the detailed path taken by the first fluxon, but just on the fact that it winds exactly once around the second. In this sense, our construction really implements a topological theory.

6. Application to dihedral groups

6.1. General properties of dihedral groups

The dihedral groups D_N are among the most natural to consider, since they contain a normal cyclic group \mathbb{Z}_N which is simply extended by a \mathbb{Z}_2 factor to form a semi-direct product. In this respect, one may view this family as the most weakly non-Abelian groups. D_N can be described as the isometry group of a planar regular polygon with N vertices. The \mathbb{Z}_N subgroup corresponds to rotations with angles in multiples of $2\pi/N$. We shall denote by \mathcal{C} the generator of this subgroup, so \mathcal{C} may be identified with the $2\pi/N$ rotation. D_N contains also N reflections, of

the form τC^n . The two elements C and τ generate D_N , and they are subjected to the following minimal set of relations:

$$C^N = e, \quad (54)$$

$$\tau^2 = e, \quad (55)$$

$$\tau C \tau = C^{-1}. \quad (56)$$

This last relation shows that indeed D_N is non-Abelian.

The next useful information about these groups is the list of conjugacy classes. If $N = 2M + 1$, D_N contains $M + 2$ classes which are: $\{e\}$, $\{C, C^{-1}\}$, \dots , $\{C^M, C^{-M}\}$, $\{\tau, \tau C, \dots, \tau C^{N-1}\}$. If $N = 2M$, there are $M + 3$ classes whose list is: $\{e\}$, $\{C^M\}$, $\{C, C^{-1}\}$, \dots , $\{C^{M-1}, C^{-M+1}\}$, $\{\tau, \tau C^2, \dots, \tau C^{N-2}\}$, $\{\tau C, \tau C^3, \dots, \tau C^{N-1}\}$.

As shown in section 3, in order to construct possible phase factors associated to gauge transformations, we need to know the stabilizers of group elements for the conjugacy operation of D_N acting on itself. Here is a list of these stabilizers:

For N odd

$$e \rightarrow D_N, \quad C^p \rightarrow \mathbb{Z}_N, \quad 1 \leq p \leq N - 1, \quad \tau C^p \rightarrow \{e, \tau C^p\}, \quad 0 \leq p \leq N - 1.$$

For N even

$$e \rightarrow D_N, \quad C^{N/2} \rightarrow D_N, \quad C^p \rightarrow \mathbb{Z}_N, \quad 1 \leq p \leq N - 1, \quad p \neq \frac{N}{2}$$

$$\tau C^p \rightarrow \{e, C^{N/2}, \tau C^p, \tau C^{p+N/2}\}, \quad 0 \leq p \leq N - 1.$$

Finally, we need to choose homomorphisms from these stabilizers into $U(1)$. In the case of a cyclic group \mathbb{Z}_N generated by C , homomorphisms χ are completely determined by $\chi(C)$, so that $\chi(C^p) = \chi(C)^p$, with the constraint: $\chi(C)^N = 1$. For the group D_N itself, we have $\chi(\tau^r C^p) = \chi(\tau)^r \chi(C)^p$, with the following constraints

$$\chi(C)^N = 1, \quad (57)$$

$$\chi(\tau)^2 = 1, \quad (58)$$

$$\chi(C)^2 = 1. \quad (59)$$

These are direct consequences of generator relations (54)–(56). Again, the parity of N is relevant. For N odd, $\chi(C) = 1$, which leaves only two homomorphisms from D_N into $U(1)$. For N even, $\chi(C) = \pm 1$, and there are four such homomorphisms. The last possible stabilizer to consider is the four element subgroup of D_{2M} , $S = \{e, C^M, \tau C^p, \tau C^{p+M}\}$. This Abelian group has four possible homomorphisms into $U(1)$, which are characterized as follows:

$$\chi(C^M) = \pm 1, \quad (60)$$

$$\chi(\tau C^p) = \pm 1, \quad (61)$$

$$\chi(\tau C^{p+M}) = \chi(\tau C^p) \chi(C^M). \quad (62)$$

6.2. Classification of possible models

6.2.1. *N odd.* Let us first consider conjugacy classes of the form $\{\mathcal{C}^p, \mathcal{C}^{-p}\}$. Since the stabilizer of \mathcal{C}^p for the conjugacy action of D_N is \mathbb{Z}_N , we have $\chi_{\mathcal{C}^p}(\mathcal{C}^q) = \omega_p^q$, where $\omega_p^N = 1$. Choosing $\chi_{\mathcal{C}^p}(\tau) = 1$, we have for fluxes in the cyclic group generated by \mathcal{C}

$$\chi_{\mathcal{C}^p}(\mathcal{C}^q) = \omega_p^q, \quad (63)$$

$$\chi_{\mathcal{C}^p}(\tau\mathcal{C}^q) = \omega_p^q, \quad (64)$$

$$\omega_p^N = 1, \quad (65)$$

$$\omega_p\omega_{-p} = 1. \quad (66)$$

For the remaining conjugacy class, we have $\chi_\tau(\tau) = \eta$, with $\eta = \pm 1$. Choosing $\chi_\tau(\mathcal{C}^p) = 1$, and using equations (19) and (20), we obtain

$$\chi_{\tau\mathcal{C}^p}(\mathcal{C}^q) = 1, \quad (67)$$

$$\chi_{\tau\mathcal{C}^p}(\tau\mathcal{C}^q) = \eta. \quad (68)$$

All these possible phase factors satisfy the following property

$$\chi_{\Phi^{-1}}(h)\chi_\Phi(h) = 1. \quad (69)$$

so that equation (48) is always satisfied. So no new constraint is imposed by the requirement to create or annihilate a pair of fluxons. What about the stronger condition equation (49)? Its form suggests that we should first determine pairs of fluxes (Φ_1, Φ_2) such that their stabilizers H_{Φ_1} and H_{Φ_2} have a non-trivial intersection. This occurs if both Φ_1 and Φ_2 are in the \mathbb{Z}_N normal subgroup generated by \mathcal{C} , or if $\Phi_1 = \Phi_2 = \tau\mathcal{C}^p$. The second case simply implies $\chi_\tau(\tau)^2 = 1$, which is not a new condition. In the first case, choosing h in \mathbb{Z}_N as well shows that $\chi_\Phi(h)^4$ is a homomorphism from \mathbb{Z}_N to $U(1)$ with respect to both Φ and h . This is satisfied in particular if $\chi_\Phi(h)$ itself is a group homomorphism in both arguments. This sufficient (but possibly not necessary) condition simplifies algebraic considerations; it can be also justified from physical argument that the theory should allow for the sites with different number of neighbours, Z , which would change $\chi_\Phi(h)^4 \rightarrow \chi_\Phi(h)^Z$.

Replacing this the constraint on $\chi_\Phi(h)^4$ by the constraint on $\chi_\Phi(h)$, we get

$$\chi_{\mathcal{C}^p}(\mathcal{C}^q) = \omega_p^q, \quad (70)$$

$$\omega_p^N = 1, \quad (71)$$

$$\omega_p = \omega^p. \quad (72)$$

Therefore, the class of possible phase factors (which is stable under multiplication) is isomorphic to the group $\mathbb{Z}_N \times \mathbb{Z}_2$. This group of phase factors is identical to the group of possible Chern–Simons actions since in this case, $H^3(D_N, U(1)) = \mathbb{Z}_N \times \mathbb{Z}_2$ [24]. Very likely, the coincidence of these two results can be traced to the absence of projective representations for D_N for N odd [24] as explained above, the projective representations are not allowed in our construction but are allowed in the classification of all possible Chern–Simons actions [24].

6.2.2. *N even.* The conjugacy classes of the form $\{C^p, C^{-p}\}$ behave in the same way as for *N* odd. So writing $N = 2M$, we have

$$\chi_{C^p}(C^q) = \omega_p^q, \quad (73)$$

$$\chi_{C^p}(\tau C^q) = \omega_p^q, \quad (74)$$

$$\omega_p^N = 1, \quad (75)$$

$$\omega_p \omega_{-p} = 1, \quad (76)$$

$$p \notin \{0, M\}, \text{ mod } 2N. \quad (77)$$

The conjugacy class $\{C^M\}$ is special since its stabilizer is D_N itself. As discussed in section 6.1 above, there are four homomorphisms from D_N into $U(1)$ that we denote by

$$\chi_{C^M}(\tau^q C^p) = \tilde{\omega}^q \omega_M^p, \quad (78)$$

$$\tilde{\omega}, \omega_M \in \{1, -1\}. \quad (79)$$

Let us now turn to $\chi_\tau(g)$ with the corresponding stabilizer equal to $\{e, C^M, \tau, \tau C^M\}$. As seen in section 6.1, the four possible homomorphisms may be written as

$$\chi_\tau(\tau) = \eta_0 \in \{\pm 1\}, \quad (80)$$

$$\chi_\tau(C^M) = \zeta_0 \in \{\pm 1\}, \quad (81)$$

$$\chi_\tau(\tau C^M) = \eta_0 \zeta_0. \quad (82)$$

From this, we derive the expression of $\chi_\tau(g)$, in the following form ($0 \leq p \leq M - 1$)

$$\chi_\tau(C^p) = 1, \quad (83)$$

$$\chi_\tau(C^{p+M}) = \zeta_0, \quad (84)$$

$$\chi_\tau(\tau C^{-p}) = \eta_0, \quad (85)$$

$$\chi_\tau(\tau C^{-p+M}) = \eta_0 \zeta_0. \quad (86)$$

Furthermore, because $C^p \tau C^{-p} = \tau C^{-2p}$, equation (20) implies

$$\chi_{\tau C^{-2p}}(g) = \chi_{\tau}(g C^p). \quad (87)$$

The last conjugacy class to consider contains τC , with the stabilizer $\{e, C^M, \tau C, \tau C^{1+M}\}$. In this case, we may set $(0 \leq p \leq M - 1)$

$$\chi_{\tau C}(C^p) = 1, \quad (88)$$

$$\chi_{\tau C}(C^{p+M}) = \zeta_1 \in \{\pm 1\}, \quad (89)$$

$$\chi_{\tau C}(\tau C^{1-p}) = \eta_1 \in \{\pm 1\}, \quad (90)$$

$$\chi_{\tau C}(\tau C^{1-p+M}) = \eta_1 \zeta_1, \quad (91)$$

$$\chi_{\tau C^{1-2p}}(g) = \chi_{\tau C}(g C^p). \quad (92)$$

Here again, the constraint equation (48) is always satisfied. To impose equation (49), we have to consider pairs of fluxes (Φ_1, Φ_2) such that $H_{\Phi_1} \cap H_{\Phi_2}$ is non-trivial. As before, choosing Φ_1 and Φ_2 in the \mathbb{Z}_N subgroup imposes $\omega_p = \omega^p$, with $\omega^N = 1$. A new constraint arises by choosing $\Phi_1 = C^p$ and $\Phi_2 = \tau C^{p'}$. In this case, C^M belongs to their common stabilizer. Equation (49) implies

$$\chi_{C^p}(C^M) \chi_{\tau C^{p'}}(C^M) = \chi_{C^p \tau C^{p'}}(C^M) = \chi_{\tau C^{p'+p}}(C^M). \quad (93)$$

But using equation (17) this yields

$$\chi_{\tau C^{p+p'}}(C^{M+p}) = \chi_{\tau C^{p+p'}}(C^M) \chi_{\tau C^{p+p'}}(C^p). \quad (94)$$

Therefore, we have the constraint $\zeta_0 = \zeta_1 = 1$, which enables us to simplify drastically the above expression for phase factors

$$\chi_{\tau C^{2p}}(C^q) = 1, \quad (95)$$

$$\chi_{\tau C^{2p}}(\tau C^q) = \eta_0 \quad (96)$$

and

$$\chi_{\tau C^{2p+1}}(C^q) = 1, \quad (97)$$

$$\chi_{\tau C^{2p+1}}(\tau C^q) = \eta_1. \quad (98)$$

Table 1. Adiabatic phase $f(g_1, g_2)$ generated in the process where fluxon g_1 winds once around a fluxon g_2 . Values of phase-factors $\chi_{g'_1}(g'_1)$ and $\chi_{g'_2}(g'_1)$ are given for dihedral groups D_N with odd N . For even N , expressions for $\chi_{g'_1}(g'_1)$ and $\chi_{g'_2}(g'_1)$ are slightly more complicated, but interestingly, the main result for $f(g_1, g_2)$ is the same as for odd N , namely it involves only the complex number ω .

g_1	g_2	g'_1	g'_2	$\chi_{g'_1}(g'_1)$	$\chi_{g'_2}(g'_1)$	$f(g_1, g_2)$
C^p	C^q	C^p	C^q	ω^{p^2}	ω^{pq}	$\omega^{2p(p+2q)}$
τC^p	C^q	τC^{p+2q}	C^{-q}	η	$\omega^{-q(p+2q)}$	$\omega^{-4q(p+2q)}$
C^p	τC^q	C^{-p}	τC^{q-2p}	ω^{p^2}	1	ω^{2p^2}
τC^p	τC^q	τC^{-p+2q}	τC^{3q-2p}	η	η	1

But equation (93) now implies that $\chi_{C^p}(C^M) = 1$ for any p , which is satisfied only when $\omega^M = 1$. Specializing to $p = M$, we see that the common stabilizer contains now two more elements, namely $\tau C^{p'}$ and $\tau C^{p'+M}$. Equation (93) now requires that

$$M \text{ even} \rightarrow \tilde{\omega} = 1, \quad (99)$$

$$M \text{ odd} \rightarrow \tilde{\omega} \eta_0 \eta_1 = 1. \quad (100)$$

It is then easy to check that considering the common stabilizer of τC^p and $\tau C^{p'}$, which may contain two or four elements, does not bring any new constraint. Finally, ω belongs to the \mathbb{Z}_M group and among the three binary variables η_0 , η_1 , and $\tilde{\omega}$, only two are independent. Therefore, the set of all possible phase factors for the D_{2M} group is identical to $\mathbb{Z}_M \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This contains only half of $H^3(D_{2M}, U(1))$ which is equal to $\mathbb{Z}_{2M} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [24]. But D_{2M} admits non-trivial projective representations since $H^2(D_{2M}, U(1)) = \mathbb{Z}_2$ which cannot appear in our construction. The important result is that in spite of this restriction, we get a non-trivial subset of the possible theories also in this case.

6.3. Holonomy properties

Using the general expression (53), and the above description of possible phase factors, we may compute the adiabatic phase induced by a process where a fluxon g_1 winds once around another fluxon g_2 . The results are listed in the last column of table 1, and they are valid for ‘both’ parities of N .

7. Conclusion

Generally, in order to appear as a low-energy sector of some physical Hamiltonian, the Chern–Simons gauge theory has to involve gauge transformations that depend only on a local flux configuration. Furthermore, to be interesting from the view point of quantum computation, the theory should allow for a local gauge-invariant electric-field operator that moves a flux or fuses two fluxes together. Here, we have analysed non-Abelian gauge models that satisfy these general

conditions; our main result is the equation (49) for the phase factor χ associated with a gauge transformation. Furthermore, we have computed the flux braiding properties for a given phase factor that satisfies these conditions. Finally, we have applied our general results to the simplest class of non-Abelian groups, the dihedral groups D_n . The fluxon braiding properties in these groups are summarized in table 1.

Inspection of table 1 shows that even for the smallest groups, the Chern–Simons term modifies braiding properties in a very non-trivial manner and gives phases to the braiding that were trivial in the absence of the Chern–Simons term. In the scheme [11] where the pair of compensating fluxes ($\tau\mathcal{C}^p, \tau\mathcal{C}^p$) are used to encode one bit, the transformations allowed in the absence of the Chern–Simons term are limited to conjugation. In the presence of the Chern–Simons term the braiding of such bit with the controlled flux results in a richer set of transformations that involve both conjugation by group elements and phase factors (see table 1) but does not change the state of the flux as it should. We hope that this will make it possible to construct the universal quantum computation with the simplest group D_3 that does not involve operations that are difficult to protect (such as charge doublets).

The implementation of the microscopic Hamiltonians discussed in this paper in a realistic physical system is a challenging problem. The straightforward implementation would mean that the dominant term in the microscopic Hamiltonian is $H = -t \sum_{i,g} U_i(g)$ so that all low-energy states are gauge invariant. This is not easy to realize in a physical system because the operator $U(g)$ involves a significant number of surrounding bonds. We hope, however, this can be achieved by a mapping to the appropriate spin model as was the case for Abelian Chern–Simons theories; this is the subject of the future research.

Acknowledgments

We are thankful to M Görbig and R Moessner for useful discussions. LBI is thankful to LPTHE, Jussieu for their hospitality while BD has enjoyed the hospitality of the Physics Department at Rutgers University. This work was made possible by support from NATO CLG grant 979979, NSF DMR 0210575.

Appendix A. Electric field gauge invariance: general criterion

In the absence of additional constraints on the gauge transformation phase factors, $\chi_\phi(g)$, the definition of the electric-field operator implies that it is gauge invariant under gauge transformations on all sites except i while the the gauge transformation on the latter gives

$$U_i(h)\mathcal{E}_{ij}(g)|\Psi\rangle = \alpha(\Phi_L, \Phi_R, g|h)\mathcal{E}_{ij}(hgh^{-1})U_i(h)|\Psi\rangle, \quad (\text{A.1})$$

with

$$\alpha(\Phi_L, \Phi_R, g|h) = \left(\frac{\chi_{g\Phi_L}(h)}{\chi_{\Phi_L}(h)} \frac{\chi_{\Phi_R g^{-1}}(h)}{\chi_{\Phi_R}(h)} \right)^4 \times \frac{A(\Phi_L, \Phi_R, g)}{A(h\Phi_L h^{-1}, h\Phi_R h^{-1}, hgh^{-1})}. \quad (\text{A.2})$$

Here, the notations are those of figure 2, where state $|\Psi\rangle$ corresponds to the initial state on the figure. The computation leading to this equation is depicted in figure A.1. The question we have to address now is whether it is possible to choose amplitudes $A(\Phi_L, \Phi_R, g)$ so that the function $\alpha(\Phi_L, \Phi_R, g|h)$ is equal to unity.

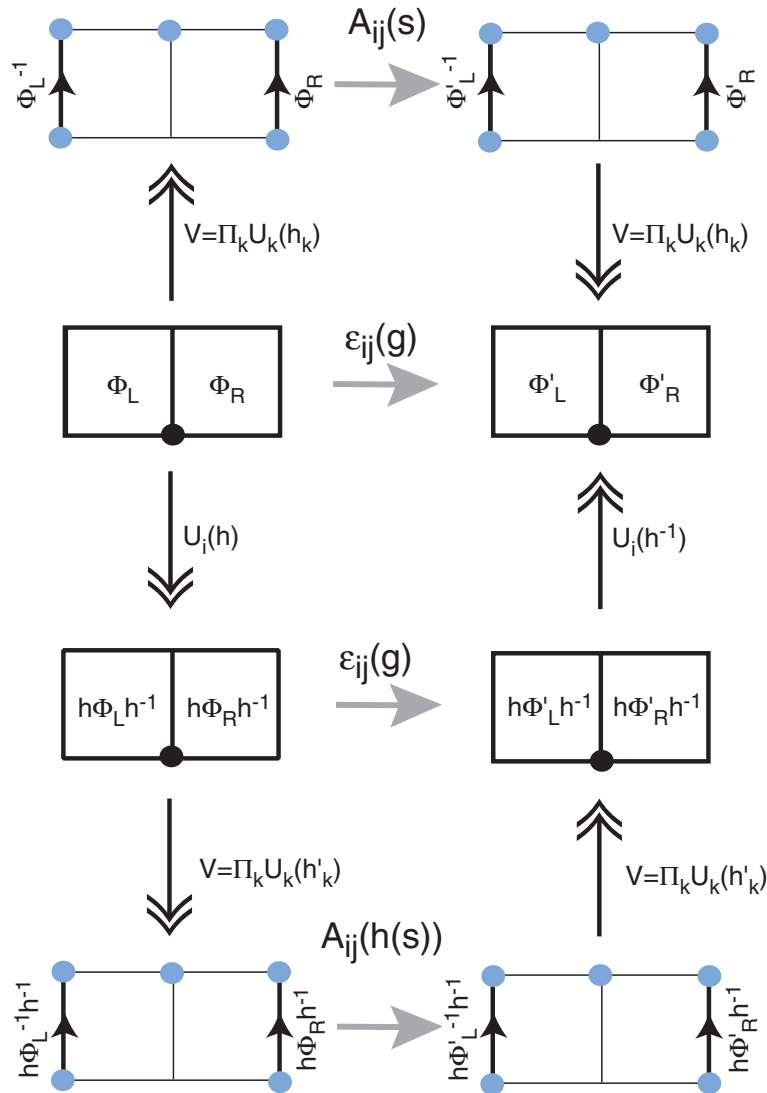


Figure A.1. Derivation of the general condition on the phase factors from the requirement of electric-gauge-field invariance. The action of the electric-field on arbitrary states (middle of the figure) related to each other by the gauge transformation at site i (shown as a black dot) can be expressed via the phase factors $A(s)$ and $A(h(s))$ characterizing the transition between reference states (bottom and top). This involves gauge transformations on all grey (blue) sites. The gauge invariance implies that the total phase factor accumulated when moving around the diagram is unity which implies the condition on the ratio $A(s)/A(h(s))$. As in the text, s stands for a triple (Φ_L, Φ_R, g) , and $h(s)$ is obtained by conjugating the components of s by h .

To simplify notations, let us denote triples (Φ_L, Φ_R, g) by a single label s . The group G acts on these triples according to

$$s \equiv (\Phi_L, \Phi_R, g) \xrightarrow{h} (h\Phi_L h^{-1}, h\Phi_R h^{-1}, hgh^{-1}) \equiv h(s). \quad (\text{A.3})$$

Let us also introduce the function $f(s|h)$ defined as

$$f(s|h) = \left(\frac{\chi_{g\Phi_L}(h)}{\chi_{\Phi_L}(h)} \frac{\chi_{\Phi_R g^{-1}}(h)}{\chi_{\Phi_R}(h)} \right)^4. \quad (\text{A.4})$$

The condition $\alpha(s|h) = 1$ now reads

$$\frac{A(h(s))}{A(s)} = f(s|h). \quad (\text{A.5})$$

Now, we show that solutions $A(s)$ exist if and only if the function $f(s|h)$ satisfies the following conditions:

$$f(s|h) = 1 \quad \text{if} \quad h(s) = s, \quad (\text{A.6})$$

$$f(s|h_1 h_2) = f(h_2(s)|h_1) f(s|h_2). \quad (\text{A.7})$$

Note that the second condition is very similar to the fundamental property (17) which enforces the existence of a local gauge symmetry group. In fact, using (17), we can check that property (A.7) always holds. Property (A.6) is a new and quite strong condition imposed on gauge phase factors $\chi_\Phi(g)$; it is equivalent to the condition (45) which in the main text has been derived as ‘necessary’. We now show that this condition is also ‘sufficient’.

Let us consider the equation

$$\frac{A(g(s))}{A(s)} = f(s|g), \quad (\text{A.8})$$

where $f(s|g)$ and $A(s)$ are complex numbers with unit modulus, s belongs to a set S on which the group G acts, and $g \in G$. Here, the function $f(s|g)$ is supposed to be known, and we are looking for ‘amplitudes’ $A(s)$. Let us denote by H_s the stabilizer of s , namely it is composed of all the group elements h such that $h(s) = s$. From equation (A.8), we get the two conditions:

$$f(s|h) = 1 \quad \text{if} \quad h(s) = s \quad (\text{A.9})$$

$$f(s|g_1 g_2) = f(g_2(s)|g_1) f(s|g_2). \quad (\text{A.10})$$

The second equation is obtained from

$$\frac{A(g_1 g_2(s))}{A(s)} = \frac{A(g_1 g_2(s))}{A(g_2(s))} \frac{A(g_2(s))}{A(s)}. \quad (\text{A.11})$$

Note that these two conditions imply that

$$f(s|gh) = f(s|g) \quad \text{when} \quad h \in H_s. \quad (\text{A.12})$$

Let us now show that when these two conditions are satisfied, we can always reconstruct a system of amplitudes solving equation (A.8). From the form of this equation, we see that the

various orbits in S for the action of G remain uncoupled. Let us then concentrate on one orbit generated by a fixed given element s . Elements of this orbit are in one-to-one correspondence with the left cosets gH_s of G , since $gh(s) = g(s)$ when $h \in H_s$. Let us choose in each coset a representative element g_n . Our problem is to find amplitudes $A(g_n(s))$, knowing $A(s)$. Let us define

$$A(g_n(s)) \equiv A(s)f(s|g_n). \quad (\text{A.13})$$

From equation (A.12), we have $A(g(s)) = A(s)f(s|g)$ for any g in G , since g may always be written as g_nh with $h \in H_s$. Now equation (A.8) is an immediate consequence of the condition (A.10).

To conclude, when conditions (A.9) and (A.10) are satisfied, solutions of equation (A.8) can be constructed independently on each orbit of S for the action of G . On a given orbit, the solution is unique, up to the choice of $A(s)$ for one particular element of this orbit.

Appendix B. Structure of the ground state on a torus

The distinguishing feature of a torus is the appearance of non-trivial closed loops that are classified according to their winding numbers associated with two fundamental cycles γ_x and γ_y , chosen with a common origin O . As a consequence, even if local fluxes vanish on all plaquettes, the fluxes Φ_x and Φ_y associated to γ_x and γ_y may not vanish. These global degrees of freedom are the origin of the topological degeneracies exhibited by a large class of gauge-invariant models on a closed space with non-trivial topology [28]. How are these degeneracies affected by the presence of a Chern–Simons term? Precise formulas for the Hilbert-space dimension of the ‘pure’ Chern–Simons theory with a finite gauge group G have been given [20] in terms of the action via functions $\alpha(h, k, l)$ in $H^3(G, U(1))$. In this paper, we consider a much larger Hilbert space induced by all allowed classical gauge configurations, $\{|g_{ij}\}$. We may expect to recover the pure Chern–Simons Hilbert space (which dimension is independent of the system size) by projecting the full Hilbert space onto the gauge-invariant ground state of some local gauge-invariant Hamiltonian. This goes beyond the scope of this paper because our goal here is classification and basic properties of such a local gauge-invariant Hamiltonian. However, it is instructive to understand how, in the present formalism, a Chern–Simons term affects the gauge dynamics in the topological fluxless sector on a torus.

Starting from the trivial configuration $g_{ij} = e$, we may induce a new fluxless state in the bulk but with a non-trivial Φ_x along γ_x by creating a string of parallel horizontal links carrying each the group element Φ_x as shown in figure B.1. This string may be viewed as the result of a succession of elementary processes discussed in section 4: first the creation of a Φ_x, Φ_x^{-1} fluxon pair, then the motion of one of them along γ_y and finally annihilation of the two fluxons. As we have seen, each of these processes may be described by a gauge-invariant operator. Let us denote by $\mathcal{C}_x(\Phi_x)$ the product of all these operators involved in the creation of the string. Similarly, we define $\mathcal{C}_y(\Phi_y)$. More generally, we may start from a state already characterized by a pair of fluxes (Φ_x, Φ_y) . Since γ_x and γ_y commute up to homotopy, we require that Φ_x and Φ_y commute in G . The string picture allows us to construct operators $\mathcal{C}_x(g_x)$ and $\mathcal{C}_y(g_y)$ such that

$$(\Phi_x, \Phi_y) \xrightarrow{\mathcal{C}_x(g_x)} (g_x \Phi_x, \Phi_y), \quad (\text{B.1})$$

$$(\Phi_x, \Phi_y) \xrightarrow{\mathcal{C}_y(g_y)} (\Phi_x, g_y \Phi_y). \quad (\text{B.2})$$

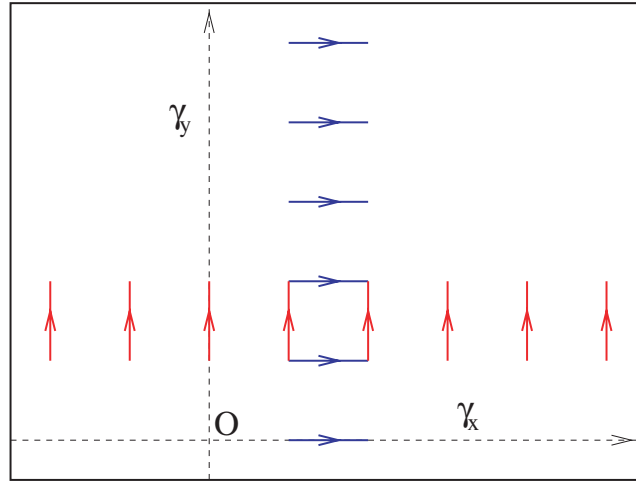


Figure B.1. For a square lattice on a torus, fundamental oriented cycles γ_x and γ_y are depicted as black dashed lines. The vertical string of horizontal blue lines represents links carrying the group element Φ_x . Similarly, the horizontal string of vertical red lines represents links carrying Φ_y . On all other links ij on the square lattice $g_{ij} = e$, so we have not represented them in order not to overcrowd the figure.

Denoting by $|\Phi_x, \Phi_y\rangle$ any state with the pair of fluxes (Φ_x, Φ_y) , we show below that $\mathcal{C}_x(g_x)$ and $\mathcal{C}_y(g_y)$ do not commute, but their actions on a state $|\Phi_x, \Phi_y\rangle$ are related by phase-factors

$$\mathcal{C}_y(g_y)\mathcal{C}_x(g_x)|\Phi_x, \Phi_y\rangle = \lambda(\Phi_x, \Phi_y|g_x, g_y)\mathcal{C}_x(g_x)\mathcal{C}_y(g_y)|\Phi_x, \Phi_y\rangle, \quad (\text{B.3})$$

$$\lambda(\Phi_x, \Phi_y|g_x, g_y) = \frac{\chi_{g_x}(g_y)^2 \chi_{\Phi_x^{-1}g_x\Phi_x}(g_y)^2}{\chi_{g_y^{-1}}(g_x)^2 \chi_{\Phi_y^{-1}g_y^{-1}\Phi_y}(g_x)^2}. \quad (\text{B.4})$$

Note that in order for all four states $|\Phi_x, \Phi_y\rangle, |g_x\Phi_x, \Phi_y\rangle, |\Phi_x, g_y\Phi_y\rangle, |g_x\Phi_x, g_y\Phi_y\rangle$ to be defined, the following constraints have to be imposed:

$$\Phi_x\Phi_y = \Phi_y\Phi_x, \quad (\text{B.5})$$

$$\Phi_x g_y = g_y \Phi_x, \quad (\text{B.6})$$

$$g_x \Phi_y = \Phi_y g_x, \quad (\text{B.7})$$

$$g_x g_y = g_y g_x. \quad (\text{B.8})$$

This relation (B.4) is reminiscent of the phase factors involved in permuting electrical-field operators attached to adjacent links for Abelian models with a Chern–Simons term [19, 27].

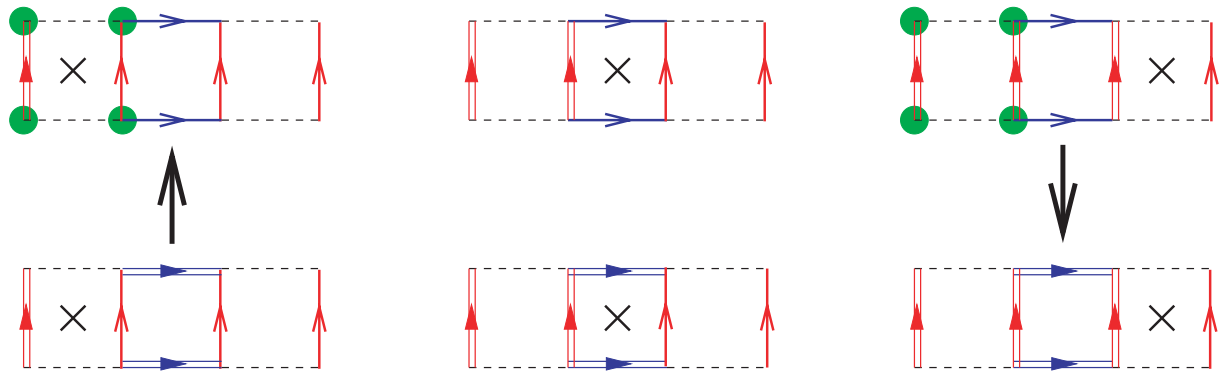


Figure B.2. The upper line illustrates intermediate steps in the action of $\mathcal{C}_y(g_y)$ on the template $|\Phi_x, \Phi_y\rangle$. The plain (double) vertical red lines carry the element Φ_y ($g_y\Phi_y$), whereas the plain horizontal blue lines carry Φ_x . Successive locations of the fluxon moving from left to right are represented by a black cross. The lower line describes the action of the same string operator on the new state $\mathcal{C}_x(g_x)|\Phi_x, \Phi_y\rangle$ and the double horizontal blue lines now carry $g_x\Phi_x$. One goes from the upper to the lower line by a gauge transformation attached to the sites indicated by a green circle. Since this transformation commutes with the fluxon motion operators, these processes generate a contribution to $\lambda(\Phi_x, \Phi_y|g_x, g_y)$ defined in appendix B equal to the product of the phase factors due to the two gauge transformations shown by thick black arrows. This factor is equal to $(\chi_{g_y^{-1}}(g_x)\chi_{\Phi_y^{-1}g_y^{-1}\Phi_y}(g_x))^{-2}$. A second contribution to $\lambda(\Phi_x, \Phi_y|g_x, g_y)$ comes from the comparison between the actions of $\mathcal{C}_x(g_x)$ on $|\Phi_x, \Phi_y\rangle$ and $\mathcal{C}_y(g_y)|\Phi_x, \Phi_y\rangle$. This process is not shown on the figure, and yields a term equal to $(\chi_{g_x}(g_y)\chi_{\Phi_x^{-1}g_x\Phi_x}(g_y))^2$.

Let us now briefly explain how to derive the relations (B.3) and (B.4) expressing the commutation rules between $\mathcal{C}_x(g_x)$ and $\mathcal{C}_y(g_y)$. The first step is to choose template states for a pair of fluxes (Φ_x, Φ_y) along elementary cycles γ_x and γ_y . These are depicted in figure B.1. Let us now compare the action of $\mathcal{C}_x(g_x)\mathcal{C}_y(g_y)$ and $\mathcal{C}_y(g_y)\mathcal{C}_x(g_x)$ on this template $|\Phi_x, \Phi_y\rangle$. The situation is illustrated in figure B.2. Note that we do not need explicitly the values of the fluxon hopping amplitudes $A(\Phi_L, \Phi_R, g)$, but simply the fact they are local and that fluxon moving operators commute with gauge transformation. The value of $\lambda(\Phi_x, \Phi_y|g_x, g_y)$ obtained in this way (see equation (B.4)) is valid not only for our template state $|\Phi_x, \Phi_y\rangle$ but for any state deduced from it by a gauge transformation.

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