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Gravitation and electromagnetic wave propagation with negative phase velocity

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Abstract. Gravitation has interesting consequences for electromagnetic wave propagation in vacuum. The propagation of plane waves with phase velocity directed opposite to the time-averaged Poynting vector is investigated for a generally curved spacetime. Conditions for such negative-phase-velocity (NPV) propagation are established in terms of the spacetime metric components for general and special cases. Implications of the negative energy density of NPV propagation are discussed.

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Contents

1. Introduction

Negative refraction of a plane wave at the planar interface of two linear, isotropic, homogeneous materials is said to occur when the projections of the real parts of the wave vectors of the incident and the refracted plane waves on the interface normal are oppositely directed. Then the real part of the wave vector and the time-averaged Poynting vector are parallel in one material, but antiparallel in the other [1]. We call the latter kind negative-phase-velocity (NPV) materials, but at least two other names have common currency too: left-handed materials, and negative-index materials. In order to extend the phenomenon of negative refraction to anisotropic materials, NPV materials are characterized by the negativity of the projection of the real part of the wave vector on the time-averaged Poynting vector.

Since the beginning of the year 2000 [2], NPV materials have excited much theoretical as well as experimental interest. Initial disbelief and criticism in some sections of the electromagnetics research community [3, 4] eventually gave way to widespread, but perhaps still not universal, acceptance of NPV materials with unequivocal demonstrations by several independent groups [5]–[7]. A simplistic expression of the (monochromatic) electromagnetic energy density turns out to yield negative values [8], which are generally held as impossible in the electromagnetics research community, but more sophisticated investigations indicate that the electromagnetic energy density in NPV materials is indeed positive when account is taken of the frequency-dependent constitutive properties [9].

Perhaps potentially the most useful application of NPV materials is for the so-called perfect lenses [10]. Once satisfactorily designed and fabricated, such lenses—although not really perfect [11, 12]—could find widespread use in modern optics, for communications, entertainment, and data storage and retrieval. More uses would emerge with ongoing research on anisotropic NPV materials, particularly with negligibly small dissipation in certain frequency ranges.

Instead of concentrating on devices, we turned our attention to the marriage of the special and the general theories of relativity (STR and GTR) and NPV propagation of electromagnetic fields. We found, a few months ago, that materials that appear to be of the non-NPV type to relatively stationary observers can appear to be of the NPV type to observers moving with uniform velocity [13]. This result permitted us to envisage STR negative refraction being exploited in astronomical scenarios [14] such as for the remote sensing of planetary and asteroidal surfaces from space stations. Application to remotely guided, extraterrestrial mining and manufacturing industries can also be envisioned. Furthermore, many unusual astronomical phenomena would be discovered and/or explained via STR negative refraction to interpret data collected via telescopes.

Ordinary vacuum (i.e., matter-free space) appears the same to all observers moving at constant relative velocities. Therefore, NPV propagation in vacuum cannot be observed by such observers. This could lead one to believe that NPV propagation is impossible in huge expanses of interstellar space. However, gravitational fields from nearby massive objects will certainly distort electromagnetic propagation, which is a principal tenet of the GTR and is indeed used nowadays in GPS systems, so that NPV propagation under the influence of a gravitational field required investigation. In a short communication [15], we derived a condition for NPV propagation to occur along a specific direction in a region of spacetime, with the assumption of a piecewise uniform but otherwise general spacetime metric. As the consequences of such a possibility are highly relevant to further exploration of outer space as well as for industrial operations therein, we undertook a more general study, the results of which are being reported here.

The plan of this paper is as follows: in section 2, electromagnetism in generally curved spacetime is transformed from a covariant to a noncovariant formalism, wherein vacuum resembles a bianisotropic 'medium' which enables planewave propagation to be examined using standard techniques. A piecewise uniform approximation of the spacetime metric is then implemented in section 3, and thereby a condition for NPV propagation is derived. Section 4 is devoted to a discussion of energy density, and the paper concludes with a summary in section 5.

2. Electromagnetism in gravitationally affected vacuum

The effect of a gravitational field is captured by the metric $g_{\alpha\beta}$ which is a function of spacetime x^α and carries the signature $(+, -, -, -)$.[5] In the absence of charges and currents, electromagnetic fields obey the covariant Maxwell equations

$$f_{\alpha\beta;\nu} + f_{\beta\nu;\alpha} + f_{\nu\alpha;\beta} = 0, \qquad h^{\alpha\beta}{}_{;\beta} = 0, \tag{1}$$

where $f_{\alpha\beta}$ and $h^{\alpha\beta}$ are, respectively, the covariant and the contravariant electromagnetic field tensors whereas the subscript $_{;\nu}$ indicates the covariant derivative with respect to the νth spacetime coordinate.

2.1. Noncovariant equations for vacuum

Following common practice [16]–[18], the Maxwell equations (1) may be expressed in noncovariant form in vacuum as

$$f_{\alpha\beta,\nu} + f_{\beta\nu,\alpha} + f_{\nu\alpha,\beta} = 0, \qquad [(-g)^{1/2} h^{\alpha\beta}]_{,\beta} = 0, \tag{2}$$

[5] Greek indices take the values 0, 1, 2 and 3; roman indices take the values 1, 2 and 3; $x^0 = ct$, where c is the speed of light in vacuum in the absence of all gravitational fields, whereas $x^{1,2,3}$ are the three spatial coordinates.

wherein $g = \det[g_{\alpha\beta}]$ and the subscript $_{,\nu}$ denotes ordinary differentiation with respect to the νth spacetime coordinate. Although the generalization of the Maxwell equations from noncovariant to covariant formulations is not completely unambiguous [19], we adopt the standard generalization (1) in the absence of experimental resolution of the ambiguity.

Introduction of the electromagnetic field vectors

$$\begin{aligned} E_\ell &= f_{\ell 0}, \\ B_\ell &= (1/2)\varepsilon_{\ell mn} f_{mn}, \\ D_\ell &= (-g)^{1/2} h^{\ell 0}, \\ H_\ell &= (1/2)\varepsilon_{\ell mn}(-g)^{1/2} h^{mn}, \end{aligned} \tag{3}$$

with $\varepsilon_{\ell mn}$ being the three-dimensional Levi-Civita symbol, allows us to state the Maxwell equations in the familiar form

$$\begin{aligned} B_{\ell,\ell} &= 0, \qquad B_{\ell,0} + \varepsilon_{\ell mn} E_{m,n} = 0, \\ D_{\ell,\ell} &= 0, \qquad -D_{\ell,0} + \varepsilon_{\ell mn} H_{m,n} = 0. \end{aligned} \tag{4}$$

The accompanying constitutive relations of vacuum can be written for the electromagnetic field vectors as

$$D_\ell = \gamma_{\ell m} E_m + \varepsilon_{\ell mn} \Gamma_m H_n, \qquad B_\ell = \gamma_{\ell m} H_m - \varepsilon_{\ell mn} \Gamma_m E_n, \tag{5}$$

where

$$\gamma_{\ell m} = -(-g)^{1/2} \frac{g^{\ell m}}{g_{00}}, \qquad \Gamma_m = \frac{g_{0m}}{g_{00}}. \tag{6}$$

The most important of the foregoing equations can be expressed in SI units as

$$\nabla \times \underline{E}(ct, \underline{r}) + \frac{\partial}{\partial t} \underline{B}(ct, \underline{r}) = \underline{0}, \tag{7}$$

$$\nabla \times \underline{H}(ct, \underline{r}) - \frac{\partial}{\partial t} \underline{D}(ct, \underline{r}) = \underline{0}, \tag{8}$$

$$\underline{D}(ct, \underline{r}) = \epsilon_0 \underline{\underline{\gamma}}(ct, \underline{r}) \cdot \underline{E}(ct, \underline{r}) - \frac{1}{c} \underline{\Gamma}(ct, \underline{r}) \times \underline{H}(ct, \underline{r}), \tag{9}$$

$$\underline{B}(ct, \underline{r}) = \mu_0 \underline{\underline{\gamma}}(ct, \underline{r}) \cdot \underline{H}(ct, \underline{r}) + \frac{1}{c} \underline{\Gamma}(ct, \underline{r}) \times \underline{E}(ct, \underline{r}), \tag{10}$$

where space $\underline{r}$ has been separated from time t, the scalar constants ϵ_0 and μ_0 denote the permittivity and permeability of vacuum in the absence of a gravitational field; $\underline{\underline{\gamma}}(ct, \underline{r})$ is the dyadic-equivalent of $\gamma_{\ell m}$, and $\underline{\Gamma}(ct, \underline{r})$ is the vector-equivalent of Γ_m. These four equations are stated in the usual style of three-dimensional vectors and dyadics for convenience, but the spacetime is still curved.

2.2. Partitioning of spacetime

Let the spacetime region of interest $\mathcal{X}$ be partitioned into an appropriate number of subregions ${}^{(m)}\mathcal{X}$ ($m = 1, 2, 3, \ldots$). In the mth subregion, the nonuniform metric $g_{\alpha\beta}$ is written as the sum of the uniform metric ${}^{(m)}\tilde{g}_{\alpha\beta}$ and the nonuniform residual metric ${}^{(m)}d_{\alpha\beta}$ as follows:

$$g_{\alpha\beta} = {}^{(m)}\tilde{g}_{\alpha\beta} + {}^{(m)}d_{\alpha\beta}. \tag{11}$$

Notice that, whereas the curved spacetime metric $g_{\alpha\beta}$ is transformable into the Lorentzian metric $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ of flat spacetime at every point in $\mathcal{X}$, in accordance with the Einstein equivalence principle, the transformation is not universal [20]. That is to say, since $g_{\alpha\beta}$ represents a generally curved spacetime it cannot be replaced by the flat spacetime metric $\eta_{\alpha\beta}$ everywhere in $\mathcal{X}$ by using a spacetime-independent transformation. Furthermore, there is no reason for ${}^{(m)}\tilde{g}_{\alpha\beta}$ to be transformable into $\eta_{\alpha\beta}$ at even one point in ${}^{(m)}\mathcal{X}$.

The Maxwell curl postulates read as follows in ${}^{(m)}\mathcal{X}$:

$$\begin{aligned}\nabla \times \underline{E}(ct, \underline{r}) = &-\left[\mu_0 {}^{(m)}\underline{\underline{\tilde{\gamma}}} \cdot \frac{\partial}{\partial t}\underline{H}(ct, \underline{r}) + \frac{1}{c} {}^{(m)}\underline{\tilde{\Gamma}} \times \frac{\partial}{\partial t}\underline{E}(ct, \underline{r})\right] \\ &- \frac{\partial}{\partial t}\left[\mu_0 {}^{(m)}\underline{\underline{\phi}}(ct, \underline{r}) \cdot \underline{H}(ct, \underline{r}) + \frac{1}{c} {}^{(m)}\underline{\Phi}(ct, \underline{r}) \times \underline{E}(ct, \underline{r})\right],\end{aligned} \tag{12}$$

$$\begin{aligned}\nabla \times \underline{H}(ct, \underline{r}) = &\left[\epsilon_0 {}^{(m)}\underline{\underline{\tilde{\gamma}}} \cdot \frac{\partial}{\partial t}\underline{E}(ct, \underline{r}) - \frac{1}{c} {}^{(m)}\underline{\tilde{\Gamma}} \times \frac{\partial}{\partial t}\underline{H}(ct, \underline{r})\right] \\ &+ \frac{\partial}{\partial t}\left[\epsilon_0 {}^{(m)}\underline{\underline{\phi}}(ct, \underline{r}) \cdot \underline{E}(ct, \underline{r}) - \frac{1}{c} {}^{(m)}\underline{\Phi}(ct, \underline{r}) \times \underline{H}(ct, \underline{r})\right].\end{aligned} \tag{13}$$

Here, ${}^{(m)}\underline{\underline{\tilde{\gamma}}}$ and ${}^{(m)}\underline{\tilde{\Gamma}}$ are related to ${}^{(m)}\tilde{g}_{\alpha\beta}$, and ${}^{(m)}\underline{\underline{\phi}}$ and ${}^{(m)}\underline{\Phi}$ to ${}^{(m)}d_{\alpha\beta}$, in the same way that $\underline{\underline{\gamma}}$ and $\underline{\Gamma}$ are related to $g_{\alpha\beta}$.

2.3. Piecewise uniform approximation

Equations (12) and (13) are complicated. Therefore, for preliminary analysis, ${}^{(m)}\tilde{g}_{\alpha\beta}$ can be selected appropriately for ${}^{(m)}\mathcal{X}$ and ${}^{(m)}d_{\alpha\beta}$ can be ignored. This piecewise uniform approximation leads to the simpler equations

$$\nabla \times \underline{E}(ct, \underline{r}) = -\left[\mu_0 {}^{(m)}\underline{\underline{\tilde{\gamma}}} \cdot \frac{\partial}{\partial t}\underline{H}(ct, \underline{r}) + \frac{1}{c} {}^{(m)}\underline{\tilde{\Gamma}} \times \frac{\partial}{\partial t}\underline{E}(ct, \underline{r})\right], \tag{14}$$

$$\nabla \times \underline{H}(ct, \underline{r}) = \left[\epsilon_0 {}^{(m)}\underline{\underline{\tilde{\gamma}}} \cdot \frac{\partial}{\partial t}\underline{E}(ct, \underline{r}) - \frac{1}{c} {}^{(m)}\underline{\tilde{\Gamma}} \times \frac{\partial}{\partial t}\underline{H}(ct, \underline{r})\right], \tag{15}$$

for electromagnetic fields in ${}^{(m)}\mathcal{X}$.

The nature of the 'medium' implicit in equations (14) and (15) is worth stating: this medium is spatially homogeneous and local, it does not age, and it reacts purely instantaneously—just like vacuum in the absence of a gravitational field. However, it is bianisotropic. As $\underline{\underline{\tilde{\gamma}}}$ is real symmetric, both the permittivity and the permeability dyadics (i.e., $\epsilon_0\, {}^{(m)}\underline{\underline{\tilde{\gamma}}}$ and $\mu_0\, {}^{(m)}\underline{\underline{\tilde{\gamma}}}$, respectively) are orthorhombic and have the same eigenvectors. Furthermore, the gyrotropic-like magnetoelectric terms on the right sides of the two equations can be removed in the temporal-frequency domain by a simple transform [21], so that this medium is unirefringent despite its anisotropy. Unless ${}^{(m)}\underline{\tilde{\Gamma}}$ is a null vector, this medium is not reciprocal in the Lorentz sense [22]; despite its nonreciprocity in general, the medium satisfies the Post constraint [23]. Finally, the medium is nondissipative [24, p 71].

3. Planewave analysis

In this section, we turn to the propagation of plane waves in a medium characterized by the constitutive relations (9) and (10), within the subregion ${}^{(m)}\mathcal{X}$ of $\mathcal{X}$. The presuperscript (m) is not generally used henceforth in this section to avoid cluttering up of the equations. It is to be understood that all equations in this section hold in ${}^{(m)}\mathcal{X}$ with $\underline{\underline{\tilde{\gamma}}} \equiv {}^{(m)}\underline{\underline{\tilde{\gamma}}}$ and $\underline{\tilde{\Gamma}} \equiv {}^{(m)}\underline{\tilde{\Gamma}}$.

We derive an expression for the time-averaged Poynting vector $\langle \underline{\mathsf{P}} \rangle_t$ associated with the plane wave with wave vector $\underline{k}$, and thereby establish sufficient conditions for NPV propagation, as signalled by

$$\underline{k} \cdot \langle \underline{\mathsf{P}} \rangle_t < 0. \tag{16}$$

3.1. Fourier representation

The following three-dimensional Fourier representation of the electromagnetic fields is appropriate for further analysis:

$$\underline{E}(ct, \underline{r}) = \frac{1}{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{\mathsf{E}}(\omega/c, \underline{k}) \exp\left[\mathrm{i}(\underline{k} \cdot \underline{r} - \omega t)\right] \mathrm{d}\omega\, \mathrm{d}k_1\, \mathrm{d}k_2, \tag{17}$$

$$\underline{H}(ct, \underline{r}) = \frac{1}{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{\mathsf{H}}(\omega/c, \underline{k}) \exp\left[\mathrm{i}(\underline{k} \cdot \underline{r} - \omega t)\right] \mathrm{d}\omega\, \mathrm{d}k_1\, \mathrm{d}k_2. \tag{18}$$

Here, $\mathrm{i} = \sqrt{-1}$, $\underline{k}$ is the wave vector with components $\underline{k}_1$, $\underline{k}_2$ and $\underline{k}_3$ in an appropriate cartesian coordinate system, ω is the angular frequency, and

$$\underline{\mathsf{E}}(\omega/c, \underline{k}) = A_a(\omega/c, \underline{k})\underline{\mathsf{e}}_a(\omega/c, \underline{k}) + A_b(\omega/c, \underline{k})\underline{\mathsf{e}}_b(\omega/c, \underline{k}), \tag{19}$$

$$\underline{\mathsf{H}}(\omega/c, \underline{k}) = A_a(\omega/c, \underline{k})\underline{\mathsf{h}}_a(\omega/c, \underline{k}) + A_b(\omega/c, \underline{k})\underline{\mathsf{h}}_b(\omega/c, \underline{k}). \tag{20}$$

The complex-valued scalars $A_{a,b}$ are functions of unknown amplitude that can be determined from initial and boundary conditions, and the planewave functions $\underline{\mathsf{e}}_{a,b}$ and $\underline{\mathsf{h}}_{a,b}$ are as yet unspecified. Clearly, the electromagnetic fields are thus represented in terms of an ensemble of propagating plane waves, both propagating (real-valued k_3) and evanescent (complex-valued k_3).

3.2. Propagating plane waves

Further interest being only in propagating waves, we set $k_3 \in \mathbb{R}$. The determination of $\underline{\mathsf{e}}_{a,b}$ and $\underline{\mathsf{h}}_{a,b}$ follows the same path as for propagation in a simply moving medium that is isotropic dielectric-magnetic at rest [24, ch 8].

Substituting (17) and (18) in (14) and (15), we find that

$$\underline{p} \times \underline{\mathsf{e}}_{a,b}(\omega/c, \underline{k}) = \omega\mu_0 \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{h}}_{a,b}(\omega/c, \underline{k}), \tag{21}$$

$$\underline{p} \times \underline{\mathsf{h}}_{a,b}(\omega/c, \underline{k}) = -\omega\epsilon_0 \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_{a,b}(\omega/c, \underline{k}), \tag{22}$$

where

$$\underline{p} = \underline{k} - \frac{\omega}{c}\underline{\tilde{\Gamma}}. \tag{23}$$

Hence,

$$\underline{\mathsf{h}}_{a,b}(\omega/c, \underline{k}) = \frac{1}{\omega\mu_0}\underline{\underline{\tilde{\gamma}}}^{-1} \cdot [\underline{p} \times \underline{\mathsf{e}}_{a,b}(\omega/c, \underline{k})], \tag{24}$$

while

$$\left\{(\underline{p} \times \underline{\underline{I}}) \cdot (\mathrm{adj}\, \underline{\underline{\tilde{\gamma}}}) \cdot (\underline{p} \times \underline{\underline{I}}) + \left(\frac{\omega}{c}\right)^2 \underline{\underline{\tilde{\gamma}}}|\underline{\underline{\tilde{\gamma}}}|\right\} \cdot \underline{\mathsf{e}}_{a,b} = \underline{0}, \tag{25}$$

where 'adj' stands for the adjoint, $\underline{\underline{I}}$ is the identity dyadic, and $|\underline{\underline{\tilde{\gamma}}}|$ is the determinant of $\underline{\underline{\tilde{\gamma}}}$. As $\underline{\underline{\tilde{\gamma}}}$ is symmetric, the foregoing equation can be further simplified to

$$\left\{\left[\left(\frac{\omega}{c}\right)^2 |\underline{\underline{\tilde{\gamma}}}| - \underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p}\right] \underline{\underline{I}} + \underline{p}\,\underline{p} \cdot \underline{\underline{\tilde{\gamma}}}\right\} \cdot \underline{\mathsf{e}}_{a,b} = \underline{0}. \tag{26}$$

Equation (26) has nontrivial solutions if

$$\left|\left[\left(\frac{\omega}{c}\right)^2 |\underline{\underline{\tilde{\gamma}}}| - \underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p}\right] \underline{\underline{I}} + \underline{p}\,\underline{p} \cdot \underline{\underline{\tilde{\gamma}}}\right| = 0. \tag{27}$$

From this condition, the dispersion relation

$$\left[\underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p} - \left(\frac{\omega}{c}\right)^2 |\underline{\underline{\tilde{\gamma}}}|\right]^2 = 0 \tag{28}$$

to determine k_3 for a specific $\{\omega,\ k_1,\ k_2\}$ emerges. In the three-dimensional p-space, this relation represents the surface of an ellipsoid [25, section 3.5.4].

Substituting (28) in (26), we obtain

$$\underline{p}\,\underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_{a,b} = \underline{0}, \tag{29}$$

whence

$$\underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_{a,b} = 0. \tag{30}$$

Thus, both $\underline{\mathsf{e}}_a$ and $\underline{\mathsf{e}}_b$ must be orthogonal to $\underline{p} \cdot \underline{\underline{\tilde{\gamma}}}$, but neither of the two is generally orthogonal to $\underline{p}$. A similar exercise yields

$$\underline{p} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{h}}_{a,b} = \underline{0}, \tag{31}$$

so that both $\underline{\mathsf{h}}_a$ and $\underline{\mathsf{h}}_b$ must be orthogonal to $\underline{p} \cdot \underline{\underline{\tilde{\gamma}}}$ but not necessarily to $\underline{p}$.

The selection of $\underline{\mathsf{e}}_{a,b}$ to satisfy (30) produces a trilemma, which can be explained as follows: we can always choose two unit vectors $\underline{w}$ and $\underline{y}$ that are orthogonal to each other as well as to $\underline{p}$. Then, without loss of generality, (30) is satisfied by

$$\underline{\mathsf{e}}_a = \frac{\underline{\underline{\tilde{\gamma}}}^{-1} \cdot \underline{w}}{|\underline{\underline{\tilde{\gamma}}}^{-1} \cdot \underline{w}|}, \qquad \underline{\mathsf{e}}_b = \frac{\underline{\underline{\tilde{\gamma}}}^{-1} \cdot (\underline{w} + q\underline{y})}{|\underline{\underline{\tilde{\gamma}}}^{-1} \cdot (\underline{w} + q\underline{y})|}, \tag{32}$$

where $q \in \mathbb{R}$, while $\underline{\mathsf{h}}_{a,b}$ can be obtained from (24). The following three conditions appear reasonable in order to fix q:

(i) $\underline{\mathsf{e}}_a \cdot \underline{\mathsf{e}}_b = 0$,

(ii) $\underline{\mathsf{h}}_a \cdot \underline{\mathsf{h}}_b = 0$, and

(iii) $\underline{\mathsf{e}}_a \times \underline{\mathsf{h}}_b = \underline{0}$ (or, equivalently, $\underline{\mathsf{e}}_b \times \underline{\mathsf{h}}_a = \underline{0}$).

In general, the three conditions turn out to be mutually exclusive, i.e., only one of the three can be fulfilled. We choose

$$\underline{w} + q\underline{y} = \underline{p} \times \underline{\mathsf{e}}_a \tag{33}$$

in order to fulfil the third condition; thus,

$$\underline{\mathsf{e}}_a = \frac{\underline{\underline{\tilde{\gamma}}}^{-1} \cdot \underline{w}}{|\underline{\underline{\tilde{\gamma}}}^{-1} \cdot \underline{w}|}, \qquad \underline{\mathsf{e}}_b = \frac{\underline{\underline{\tilde{\gamma}}}^{-1} \cdot (\underline{p} \times \underline{\mathsf{e}}_a)}{|\underline{\underline{\tilde{\gamma}}}^{-1} \cdot (\underline{p} \times \underline{\mathsf{e}}_a)|}. \tag{34}$$

3.3. NPV propagation

The rate of energy flow for a specific $\{\omega, \underline{k}\}$ in ${}^{(m)}\mathcal{X}$ is obtained by averaging the Poynting vector over one cycle in time; thus, using (24) and (34) we find that

$$\langle \underline{\mathsf{P}} \rangle_t = \frac{1}{2\omega\mu_0|\underline{\underline{\tilde{\gamma}}}|}[|A_a|^2\underline{\mathsf{e}}_a \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_a + |A_b|^2\underline{\mathsf{e}}_b \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_b]\underline{\underline{\tilde{\gamma}}} \cdot \underline{p}. \tag{35}$$

The important result therefore emerges: the NPV condition (16) is satisfied provided that

$$\frac{1}{|\underline{\underline{\tilde{\gamma}}}|}[|A_a|^2\underline{\mathsf{e}}_a \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_a + |A_b|^2\underline{\mathsf{e}}_b \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{e}}_b]\underline{k} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p} < 0. \tag{36}$$

Let us elaborate on the NPV inequality (36) for positive- and negative-definite $\underline{\underline{\tilde{\gamma}}}$ and indefinite $\underline{\underline{\tilde{\gamma}}}$ [27].

3.3.1. Positive- and negative-definite $\underline{\underline{\tilde{\gamma}}}$. Since $\underline{\underline{\tilde{\gamma}}}$ is a 3×3 dyadic, we have $(\underline{\mathrm{e}}_a \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathrm{e}}_a)/|\underline{\underline{\tilde{\gamma}}}| > 0$ and $(\underline{\mathrm{e}}_b \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathrm{e}}_b)/|\underline{\underline{\tilde{\gamma}}}| > 0$ when $\underline{\underline{\tilde{\gamma}}}$ is either positive-definite or negative-definite. Thus, the NPV inequality (36) immediately reduces to

$$\underline{k} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p} < 0. \tag{37}$$

We therefore observe that NPV propagation arises for negative-definite $\underline{\underline{\tilde{\gamma}}}$ if the gyrotropic-like magnetoelectric term $\underline{\tilde{\Gamma}} = \underline{0}$, whereas NPV propagation is not possible for positive-definite $\underline{\underline{\tilde{\gamma}}}$ if $\underline{\tilde{\Gamma}} = \underline{0}$.

In order to establish the sign of $\underline{k} \cdot \langle \underline{\mathrm{P}} \rangle_t$, let us introduce the mutually orthogonal basis vectors $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$, where $\underline{b}_3$ is parallel to $\underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}}$ but $\underline{b}_1$ and $\underline{b}_2$ lie in the plane perpendicular to $\underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}}$. With respect to these basis vectors, we express $\underline{k}$ and $\underline{\tilde{\Gamma}}$ as

$$\underline{k} = \kappa_1 \underline{b}_1 + \kappa_2 \underline{b}_2 + \kappa_3 \underline{b}_3, \qquad \underline{\tilde{\Gamma}} = \tilde{G}_1 \underline{b}_1 + \tilde{G}_2 \underline{b}_2 + \tilde{G}_3 \underline{b}_3. \tag{38}$$

The definiteness of $\underline{\underline{\tilde{\gamma}}}$ ensures that $\underline{\tilde{\Gamma}}$ is not perpendicular to $\underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}}$. Thus, $\tilde{G}_3 \neq 0$ and we have

$$\underline{k} = \frac{\kappa_3}{\tilde{G}_3} \underline{\tilde{\Gamma}} + \left(\kappa_1 - \frac{\tilde{G}_1}{\tilde{G}_3} \kappa_3 \right) \underline{b}_1 + \left(\kappa_2 - \frac{\tilde{G}_2}{\tilde{G}_3} \kappa_3 \right) \underline{b}_2. \tag{39}$$

Equivalently, the wave vector $\underline{k}$ may be written in the form

$$\underline{k} = \rho_1 \frac{\omega}{c} \underline{\tilde{\Gamma}} + \rho_2 \underline{z}, \tag{40}$$

with $\rho_{1,2} \in \mathbb{R}$ being scalar constants and $\underline{z}$ being a unit vector in the plane perpendicular to $\underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}}$. It follows that

$$\underline{k} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p} = \left(\frac{\omega}{c} \right)^2 \rho_1 (\rho_1 - 1) \underline{\tilde{\Gamma}} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}} + \rho_2^2 \, \underline{z} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{z}. \tag{41}$$

Hence, NPV propagation is a consequence of the inequality

$$\rho_2^2 \, \underline{z} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{z} < \left(\frac{\omega}{c} \right)^2 \rho_1 (1 - \rho_1) \underline{\tilde{\Gamma}} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}} \tag{42}$$

being satisfied.

We emphasize that the NPV condition (42) applies for an arbitrarily oriented wave vector $\underline{k}$. Two particular cases are worthy of special mention. Firstly, if $\underline{k}$ lies in the plane perpendicular to $\underline{\underline{\tilde{\gamma}}} \cdot \underline{\tilde{\Gamma}}$ (i.e., $\rho_1 = 0$), then NPV propagation cannot occur regardless of the value of ρ_2 or orientation of $\underline{z}$. Secondly, suppose the wave vector $\underline{k}$ is aligned with $\underline{\tilde{\Gamma}}$ (i.e., $\rho_2 = 0$). Then the NPV inequality (42) is satisfied for all $\rho_1 \in (0, 1)$.

3.3.2. Indefinite $\underline{\underline{\tilde{\gamma}}}$. The dyadic $\underline{\underline{\tilde{\gamma}}}$ need not be positive- or negative-definite. For example, the dyadic $\underline{\underline{\tilde{\gamma}}}$ corresponding to the Kerr metric of a rotating black hole is indefinite in the ergosphere region [20]. Where $\underline{\underline{\tilde{\gamma}}}$ is indefinite, the sufficient conditions

$$\frac{1}{|\underline{\underline{\tilde{\gamma}}}|}(\underline{e}_a \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{e}_a)(\underline{k} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p}) < 0 \quad \text{and} \quad \frac{1}{|\underline{\underline{\tilde{\gamma}}}|}(\underline{e}_b \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{e}_b)(\underline{k} \cdot \underline{\underline{\tilde{\gamma}}} \cdot \underline{p}) < 0 \tag{43}$$

for NPV propagation emerge from (36).

3.4. Summary of main results

Let us summarize our main results in this section: for the subregion of generally curved spacetime ${}^{(m)}\mathcal{X} \in \mathcal{X}$, with curvature specified by the uniform metric ${}^{(m)}\tilde{g}_{\alpha\beta}$, plane waves propagate with phase velocity directed opposite to the direction of the time-averaged Poynting vector provided that the inequality (36) holds. When ${}^{(m)}\underline{\underline{\tilde{\gamma}}}$ is either positive- or negative-definite, the NPV inequality (36) reduces to the simpler relation (42), whereas the sufficient conditions (43) apply when ${}^{(m)}\underline{\underline{\tilde{\gamma}}}$ is indefinite.

4. Energy density

When dealing with plane waves in linear, homogeneous materials, it is common to define the time-averaged electric and magnetic energy densities as

$$\begin{aligned} \langle \mathsf{W}_e(\omega/c, \underline{k}, \underline{r})\rangle_t &= \tfrac{1}{4}\mathrm{Re}[\underline{\mathsf{e}}(\omega/c, \underline{k}) \cdot \underline{\mathsf{D}}^*(\omega/c, \underline{k})] \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})], \\ \langle \mathsf{W}_m(\omega/c, \underline{k}, \underline{r})\rangle_t &= \tfrac{1}{4}\mathrm{Re}[\underline{\mathsf{H}}^*(\omega/c, \underline{k}) \cdot \underline{\mathsf{B}}(\omega/c, \underline{k})] \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})], \end{aligned} \tag{44}$$

where the asterisk indicates the complex conjugate, while $\underline{\mathsf{D}}(\omega/c, \underline{k})$ and $\underline{\mathsf{B}}(\omega/c, \underline{k})$ are defined similarly to $\underline{\mathsf{E}}(\omega/c, \underline{k})$ in (17). According to the Maxwell curl equations,

$$\underline{k} \times \underline{\mathsf{E}}(\omega/c, \underline{k}) = \omega\,\underline{\mathsf{B}}(\omega/c, \underline{k}), \qquad \underline{k} \times \underline{\mathsf{H}}(\omega/c, \underline{k}) = -\omega\,\underline{\mathsf{D}}(\omega/c, \underline{k}), \tag{45}$$

therefore,

$$\begin{aligned} \langle \mathsf{W}_e(\omega/c, \underline{k}, \underline{r})\rangle_t &= \tfrac{1}{4\omega}\mathrm{Re}\{\underline{k}^* \cdot [\underline{\mathsf{E}}(\omega/c, \underline{k}) \times \underline{\mathsf{H}}^*(\omega/c, \underline{k})]\} \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})], \\ \langle \mathsf{W}_m(\omega/c, \underline{k}, \underline{r})\rangle_t &= \tfrac{1}{4\omega}\mathrm{Re}\{\underline{k} \cdot [\underline{\mathsf{E}}(\omega/c, \underline{k}) \times \underline{\mathsf{H}}^*(\omega/c, \underline{k})]\} \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})]. \end{aligned} \tag{46}$$

The total time-averaged electromagnetic energy density $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$ is the sum

$$\begin{aligned} \langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t &= \langle \mathsf{W}_e(\omega/c, \underline{k}, \underline{r})\rangle_t + \langle \mathsf{W}_m(\omega/c, \underline{k}, \underline{r})\rangle_t \\ &= \frac{1}{2\omega}\mathrm{Re}(\underline{k}) \cdot \mathrm{Re}[\underline{\mathsf{E}}(\omega/c, \underline{k}) \times \underline{\mathsf{H}}^*(\omega/c, \underline{k})] \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})]. \end{aligned} \tag{47}$$

As the time-averaged Poynting vector

$$\langle \underline{\mathsf{P}}(\omega/c, \underline{k}, \underline{r})\rangle_t = \tfrac{1}{2}[\underline{\mathsf{E}}(\omega/c, \underline{k}) \times \underline{\mathsf{H}}^*(\omega/c, \underline{k})] \exp[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})], \tag{48}$$

it follows that

$$\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t = \frac{1}{\omega}\,\mathrm{Re}(\underline{k}) \cdot \langle \underline{\mathsf{P}}(\omega/c, \underline{k}, \underline{r})\rangle_t \exp\left[-2\,\mathrm{Im}(\underline{k} \cdot \underline{r})\right]. \tag{49}$$

Thus, the electromagnetic energy density (as calculated in this paragraph) associated with a NPV plane wave must be negative.

Relevant to section 3,

$$\begin{aligned} \underline{\mathsf{D}}(\omega/c, \underline{k}) &= \epsilon_0 \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{E}}(\omega/c, \underline{k}) - \frac{1}{c}\underline{\tilde{\Gamma}} \times \underline{\mathsf{H}}(\omega/c, \underline{k}), \\ \underline{\mathsf{B}}(\omega/c, \underline{k}) &= \mu_0 \underline{\underline{\tilde{\gamma}}} \cdot \underline{\mathsf{H}}(\omega/c, \underline{k}) + \frac{1}{c}\underline{\tilde{\Gamma}} \times \underline{\mathsf{E}}(\omega/c, \underline{k}), \end{aligned} \tag{50}$$

and the possibility of negative $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$ for propagating plane waves in gravitationally affected vacuum emerges.

The possibility of a negative electromagnetic energy density requires discussion. In the research on isotropic, homogeneous, dielectric-magnetic NPV materials, the negative value has been noted [8]. Equally important is the fact that such materials have been artificially fabricated as composite materials comprising various types of electrically small inclusions, and their planewave response characteristics (over limited ω-ranges) are substantially as predicted [7]. This implies the aforementioned procedure to compute $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$ may not be always correct. Indeed it is not, because it applies only to nondissipative and nondispersive mediums. When account is taken of the dissipative and the dispersive nature of the NPV materials [26], $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$ does turn out to be positive [9].

However, the medium in section 3 is nondissipative and nondispersive, so that the foregoing paragraph does not apply—but it does provide the basis for the following argument. Electromagnetic energy densities for plane waves, howsoever computed, are not necessarily indicative of the true picture. This is because an electromagnetic signal is of finite spatiotemporal extent, while plane waves are infinitely extended over the entire spacetime; indeed, it can be argued that a plane wave has infinite energy! Therefore, the energy density of a signal is meaningful, but the time-averaged energy density of a plane wave may not be. In computing the energy density of a signal, one must consider the bandwidth in the $\omega \oplus \underline{k}$ domain. Since the NPV conditions in section 3 appear unaffected by ω but not by the direction of propagation, NPV plane waves could appear in gravitationally affected vacuum as part of a pulsed electromagnetic beam (of finite cross-section) which has positive and finite energy density.

A proposal to overcome the negative value of $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$ in NPV materials is to fabricate them out of active inclusions [28]. Whereas passive inclusions are modelled in terms of resistances, capacitances and inductances, the modelling of active inclusions invokes amplifiers as well. In other words, there is a source of energy to offset negative $\langle \mathsf{W}(\omega/c, \underline{k}, \underline{r})\rangle_t$.

Reverting to (11), we see that ${}^{(m)}d_{\alpha\beta}$ could be considered as a spatiotemporally nonhomogeneous source term. The effect of this term must be included in all energy density calculations, in addition to the effect of the finite spatiotemporal extent of any electromagnetic signal. In other words, the spatiotemporal fluctuations of gravitation can act as a source term. Thus, one must consider the total energy density, not just the electromagnetic part of it.

Astrophysics researchers have formulated several different energy conditions for classical (i.e., nonquantum) GTR: all are just conjectures lacking rigorous proofs from fundamental principles and were set up simply to prove certain theorems [29]. Violations of these energy conditions are known [30, 31], and negative energy densities are invoked for the formation of certain black holes [32] as well as for the phenomenon of superradiant scattering of electromagnetic, gravitational and scalar waves [33, section 12.4]. In fact, two astrophysicists have recently written [34]:

> It is often (mistakenly) believed that every kind of matter, on scales for which we do not need to consider its quantum features, has an energy density that is everywhere positive.

This situation arises because the local energy density of a gravitational field cannot be defined uniquely in GTR. However, the notion of the total gravitational energy-momentum of an isolated system—such as ADM energy-momentum [35]—is available in an asymptotically flat spacetime; see also [36, 37]. But there is no guarantee that the ADM total energy should be positive. The condition of positivity of energy can only be expected to hold if the spacetime is nonsingular and this condition is imposed on matter distribution [38], and specifically for isolated systems [39, 40].

Under certain circumstances, many exotic solutions of general relativity have been shown to have negative energy densities. Such studies have exploited the use of quantum fields as possible sources of negative energy densities [41]–[43]. Unlike classical physics, quantum physics does not restrict energy density to have unboundedly negative values (although there are some bounds that constrain their duration and magnitude [41, 42]), which then enable the quantum fields to be used to produce macroscopic effects.

In summary, the issue of energy density remains to be carefully investigated for electromagnetic fields in gravitationally affected vacuum, regardless of the satisfaction of the NPV condition (36). This will require numerical studies with specific spacetime metrics. A similar resolution is needed for the Casimir effect [44, 45].

5. Concluding remarks

We have investigated the propagation of electromagnetic plane waves in a generally curved spacetime. Sufficient conditions (36), (42) and (43) for NPV propagation are established in terms of the spacetime metric components. We conjecture that research on new phenomena encountered during space exploration—e.g., the anomalous acceleration of Pioneer 10 currently being observed [46]—may benefit from NPV considerations. The negative energy density implications of NPV propagation require further investigation.

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