Supplementary Information:
Antipolar ordering of topological defects in active liquid crystals

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ADVECTION TERM IN GOVERNING EQUATION

We here motivate the form of the advection term \(\nabla \cdot (vQ)\) in Eq. (1) of the Main Text. Let \(P = P(t, r, p)\) denote the probability density function of active liquid crystal positions \(r\) and orientations \(p\), the latter being a unit vector. The conservation law for \(P\) assumes the form

\[
\frac{\partial}{\partial t} P = -\nabla_r \cdot j_r - \nabla_p \cdot j_p,
\]

(1)

where \(j_r\) and \(j_p\) are the position and orientation fluxes, respectively. Since the ALCs are advected with the fluid, the position flux has the form

\[
j_r = vP + \ldots,
\]

(2)

where \(v = v(t, r)\) is the local fluid velocity and the ‘\(\ldots\)’ stand for additional translational diffusive and interaction contributions. Multiplying both sides of (1) by \(pp\) and integrating with respect to \(p\), we obtain an equation of the form

\[
\frac{\partial}{\partial t} pp = -\nabla_r \cdot (vpp) + \ldots,
\]

(3)

where

\[
pp(t, r) = \int_{|p|=1} dp P(x, n, t)pp
\]

(4)

is the second moment tensor of the local ALC orientations. Projecting on the traceless part of Eq. (3) and moving the advective term to the lhs. then gives

\[
\frac{\partial}{\partial t} Q + \nabla_r \cdot (vQ) = \ldots
\]

(5)

for the nematic order parameter tensor \(Q\).

As discussed in the Main Text, the simplification \(\nabla_r \cdot (vQ) = v \cdot \nabla_r Q\) is not valid for compressible flow, for which \(\nabla_r \cdot v \neq 0\) in general. Note that the contributions from the orientation flux \(j_p\) lead to the free energy contributions, and possibly to additional non-potential terms depending on the details of the microscopic model, as has been shown for dry active matter [2]. However, they will not affect the advection term, as \(j_p\) couples only through \(\nabla_p\) but not \(\nabla_x\). We have thus shown that the advection term \(\nabla \cdot (vQ)\) emerges naturally from a generic microscopic model of particles that are confined to a 2D surface (e.g., by surfactants imposing an effective trapping potential as in the experiments [1]) and transported by a compressible interfacial flow.

PARTIAL DIFFERENTIAL EQUATION AND NONDIMENSIONALIZATION

In this paper, we study the following partial differential equation for the nematic order tensor \(Q(t, r)\):

\[
\frac{\partial}{\partial t} Q + \nabla_r \cdot (vQ) - \kappa[Q, \omega] = -\frac{\delta F}{\delta Q},
\]

(6)

where \(\omega = (\nabla v - (\nabla v)^T)/2\) is the vorticity tensor and \(v = -D\nabla \cdot Q\) the fluid velocity. The effective free energy \(F[Q] = \int d^2r F(Q)\) has density

\[
F(Q) = Tr \left\{ -\frac{a}{2}Q^2 + \frac{b}{4}Q^4 - \frac{\gamma_2}{2}(\nabla Q)^2 + \frac{\gamma_4}{2}(\nabla \nabla Q)^2 \right\},
\]

(7)
where $\text{Tr}\{(\nabla Q)^2\} = (\partial_i Q_{ij})(\partial_i Q_{ij})$ and $\text{Tr}\{(\nabla^2 Q)^2\} = (\partial_i \partial_j Q_{ij})(\partial_i \partial_j Q_{ij})$, summation over repeated indices being implied. Abbreviating $\Delta = \nabla^2$, the variational derivative of $F$ with respect to $Q$ has the form

$$\frac{\delta F}{\delta Q} = -aQ + bQ^3 + \frac{\gamma_1}{2} \nabla \cdot \frac{\partial F}{\partial (\nabla Q)} + \frac{\gamma_4}{2} \nabla \cdot \left[ \nabla \cdot \frac{\partial F}{\partial (\nabla^2 Q)} \right] = -aQ + bQ^3 + \gamma_2 \Delta Q + \gamma_4 \Delta^2 Q$$

(8)

from which we obtain the governing equation

$$\partial_t Q - D \nabla \cdot ((\nabla \cdot Q)Q) - \kappa Q_{ij} = aQ - bQ^3 - \gamma_2 \Delta Q - \gamma_4 \Delta^2 Q, \quad \mathbf{r} \in [0, L]^2.$$  

(9)

In component form, we have $\omega_{ij} = (\partial_i v_j - \partial_j v_i)/2$ and $v_i = -D\partial_i Q_{ij}$ and so obtain

$$\partial_t Q_{ij} - D\partial_k ((\partial_k Q_{ik}) Q_{ij}) + \frac{\kappa D}{2} (Q_{ik}\partial_l (\partial_k Q_{lj} - \partial_l Q_{ik}) + Q_{kj}\partial_l (\partial_k Q_{li} - \partial_l Q_{ki})) = aQ_{ij} - b(Q_{ij})^3 - \gamma_2 \Delta Q_{ij} - \gamma_4 \Delta^2 Q_{ij}.$$  

(10)

We rescale $Q$ according to $Q \rightarrow 2\sqrt{a/b}Q$, and define a characteristic length scale $L_c = (4a/\gamma_4)^{1/4}$ and time scale $T_c = 1/4a$. The dimensionless equation is thus

$$\partial_t Q - \bar{D} \nabla \cdot ((\nabla \cdot Q)Q) - \bar{\kappa} Q_{ij} = Q \left( \frac{1}{4} - Q^2 \right) - \bar{\gamma}_2 \Delta Q - \Delta^2 Q, \quad \mathbf{r} \in [0, L/L_c]^2,$$

(11)

where $\bar{D} = D/\sqrt{b/a}$ and $\bar{\gamma}_2 = \gamma_2/\sqrt{a/b}$. In our simulations, we use $L/L_c = 2\pi N_1/3 \leq N_\gamma \leq 9$. Dropping the tildes and writing $Q = \left( \begin{array}{c} \lambda \\ \mu \end{array} \right)$, we obtain the following equations for $\lambda(t, \mathbf{r})$ and $\mu(t, \mathbf{r})$:

$$\partial_t \lambda = D \left( \lambda_x - \lambda^2 + \lambda_x \mu_y + \lambda_y \mu_x \right) + D \left( \lambda_{xx} + 2\mu_{xy} - \lambda_{yy} \right) \lambda - \kappa D \left( \mu_{yy} + 2\lambda_{xy} - \mu_{xx} \right) \mu + \lambda \left( \frac{1}{4} - \lambda^2 - \mu^2 \right) - \gamma_2 \Delta \lambda - \Delta^2 \lambda$$

$$\partial_t \mu = D \left( \lambda_x \mu_y - \lambda_y \mu_x + 2\mu_{xy} \right) + D \left( \lambda_{xx} + 2\mu_{xy} - \lambda_{yy} \right) \mu + \kappa D \left( \mu_{yy} + 2\lambda_{xy} - \mu_{xx} \right) \lambda + \mu \left( \frac{1}{4} - \lambda^2 - \mu^2 \right) - \gamma_2 \Delta \mu - \Delta^2 \mu$$

(12)

In this paper, we restrict our attention to the regime $D, \kappa \geq 0$, corresponding to an extensile active liquid crystal. All variables are henceforth to be dimensionless unless otherwise stated.

The nematic order parameter $S(t, \mathbf{r})$ and director $\mathbf{n}(t, \mathbf{r})$ may be written in terms of $\lambda$ and $\mu$ as follows:

$$S = 2\sqrt{\lambda^2 + \mu^2}, \quad \mathbf{n} = \frac{1}{\sqrt{(\lambda + \frac{\gamma_2}{2})^2 + \mu^2}} \left( \begin{array}{c} \lambda + \frac{\gamma_2}{2} \\ \frac{\mu}{\sqrt{(\lambda + \frac{\gamma_2}{2})^2 + \mu^2}} \end{array} \right).$$

(13)

Note that $S$ is twice the larger eigenvalue of $Q$, and $\mathbf{n}$ is the corresponding eigenvector.

**LINEAR STABILITY ANALYSIS**

We here consider the linear stability of the uniform state $\lambda = \frac{1}{2} \cos 2\theta$ and $\mu = \frac{1}{2} \sin 2\theta$, in which the nematic director field is uniformly aligned with fixed angle $\theta$. Specifically, we substitute $\lambda = \frac{1}{2} \cos 2\theta + \epsilon \hat{\lambda}(t)e^{ik \cdot \mathbf{r}}$ and $\mu = \frac{1}{2} \sin 2\theta + \epsilon \hat{\mu}(t)e^{ik \cdot \mathbf{r}}$ into (12) and retain terms at order $\epsilon$ (dropping the hats):

$$\hat{\lambda}_t = \frac{D}{2} \cos 2\theta \left[ -\lambda \cos 2\phi \cdot \mu \sin 2\phi \right] k^2 - \frac{\kappa D}{2} \sin 2\theta \left[ \mu \cos 2\phi - \lambda \sin 2\phi \right] k^2,$$

$$- \frac{1}{2} \cos 2\theta \left( \lambda \cos 2\theta + \mu \sin 2\theta \right) + (\gamma_2 k^2 - k^4) \lambda$$

$$\hat{\mu}_t = \frac{D}{2} \sin 2\theta \left[ -\lambda \cos 2\phi \cdot \mu \sin 2\phi \right] k^2 + \frac{\kappa D}{2} \cos 2\theta \left[ \mu \cos 2\phi - \lambda \sin 2\phi \right] k^2,$$

$$- \frac{1}{2} \sin 2\theta \left( \lambda \cos 2\theta + \mu \sin 2\theta \right) + (\gamma_2 k^2 - k^4) \mu,$$

(14)
where \( k = k(\cos \phi, \sin \phi) \). This system of linear equations may be written in the form \( \dot{\Psi} = M \Psi \) where \( \Psi = (\lambda, \mu) \). The eigenvalues \( \sigma_{\pm} \) of \( M \) are

\[
\sigma_{\pm}(k, u) = \gamma_2 k^2 - k^4 - \frac{1}{4} \left[ 1 + (1 - \kappa)Dk^2 \pm \sqrt{[1 + (1 + \kappa)Dk^2]^2 + 4\kappa D^2 k^4(1 - u^2)} \right]
\]  

where \( u = \cos[2(\phi - \theta)] \). Note that \( \sigma_{\pm} \in \mathbb{R} \) and \( \sigma_- > \sigma_+ \), so the stability of the uniform state is determined by the wavenumber \( k = k^* \) and angle \( \phi = \phi^* \) for which \( \sigma_-(k, u) \) is largest.

We begin by finding the value of \( u \) (or equivalently, \( \phi \)) for which \( \sigma_-(k, u) \) is largest for any fixed \( k \). One can show that \( d\sigma_- / du \neq 0 \) on the interval \(-1 \leq u \leq 1 \) for \( \kappa > 0 \), so \( \sigma_- \) must attain its maximum at \( u = \pm 1 \) for every \( k \). Note that \( \sigma_-(k, u = 1) = \gamma_2 k^2 - k^4 - \frac{1}{4} \left[ 1 + (1 - \kappa)Dk^2 - |1 + (1 + \kappa)Dk^2| \right] \) and \( \sigma_-(k, u = -1) = \gamma_2 k^2 - k^4 - \frac{1}{4} \left[ 1 - (1 - \kappa)Dk^2 - |1 - (1 + \kappa)Dk^2| \right] \). So the difference is

\[
\sigma_-(k, u = 1) - \sigma_-(k, u = -1) = \frac{1}{2} \cdot \begin{cases} 2\kappa Dk^2 & \text{if } k \leq [(1 + \kappa)D]^{-1/2} \\ 1 + (\kappa - 1)Dk^2 & \text{if } k > [(1 + \kappa)D]^{-1/2} \end{cases}
\]  

This quantity is nonnegative if \( 0 \leq \kappa < 1 \) and \( k \leq k_c \equiv [(1 - \kappa)D]^{-1/2} \), or if \( \kappa \geq 1 \). We thus obtain

\[
\sigma^*(k) \equiv \max_{-1 \leq u \leq 1} \sigma_-(k, u) = \gamma_2 k^2 - k^4 + \frac{1}{2} \begin{cases} \kappa Dk^2 & \text{for } u = 1 \text{ if } \kappa \geq 1 \\ 1 + (\kappa - 1)Dk^2 & \text{for } u = u^* \text{ if } 0 \leq \kappa < 1 \text{ and } k \leq k_c \\ Dk^2 - 1 & \text{for } u = -1 \text{ if } 0 \leq \kappa < 1 \text{ and } k > k_c \end{cases}
\]  

where \( u^* = 1 \) for \( 0 < \kappa < 1 \), and is arbitrary for \( \kappa = 0 \); indeed, \( \sigma_-(k, u) \) is independent of \( u \) for \( \kappa = 0 \) and \( k \leq k_c \). We now need to find the wavenumber \( k \) for which \( \sigma^*(k) \) is largest. To this end, we consider separately the cases \( \gamma_2 > 0 \) and \( \gamma_2 < 0 \).

Case 1: \( \gamma_2 > 0 \)

For \( \gamma_2 > 0 \), note that the maxima of the functions obtained in (18) are

\[
r_1 \equiv \max_{k \geq 0} \left( \gamma_2 + \frac{\kappa D}{2} \right)^2 - k^4 = \frac{1}{4} \left( \gamma_2 + \frac{\kappa D}{2} \right)^2 \quad \text{for } k = k_1 \equiv \left[ \frac{1}{2} \left( \gamma_2 + \frac{\kappa D}{2} \right) \right]^{1/2},
\]

\[
r_2 \equiv \max_{k \geq 0} \left( \frac{1}{2} + \gamma_2 + \frac{D}{2} \right) - k^4 = \frac{1}{2} \left( \gamma_2 + \frac{D}{2} \right)^2 \quad \text{for } k = k_2 \equiv \left[ \frac{1}{2} \left( \gamma_2 + \frac{D}{2} \right) \right]^{1/2}.
\]

The system may thus undergo either of three instabilities: an isotropic instability in which the dominant instability is independent of the direction \( \phi \), a longitudinal instability in which the most unstable mode points along the nematic director field, or a transverse instability in which it points perpendicular to the nematic director field. The isotropic and longitudinal instabilities have growth rate \( r_1 \) corresponding to wavenumber \( k^* = k_1 \), the longitudinal mode having angle \( \phi^* = \theta \). The transverse instability has growth rate \( r_2 \) corresponding to wavenumber \( k^* = k_2 \) and angle \( \phi^* = \theta + \pi/2 \). From (18), we conclude that the system undergoes a longitudinal instability for \( \kappa \geq 1 \).

The behavior for \( 0 < \kappa < 1 \) is more complicated, as there are four possibilities to consider: (i) \( k_2 < k_c \), (ii) \( k_c < k_1 \), (iii) \( k_1 < k_c < k_2 \) and \( r_1 > r_2 \), and (iv) \( k_1 < k_c < k_2 \) and \( r_1 < r_2 \). A longitudinal instability occurs in cases (i) and (iii), whereas a transverse instability occurs in cases (ii) and (iv). Note that

\[
r_1 - r_2 = \frac{1}{2} \left( \gamma_2 + \frac{D}{2} \right) \left( \gamma_2 + \frac{(1 + \kappa)D}{4} \right) - \frac{1}{2} \left( \gamma_2 + \frac{(1 - \kappa)D}{4} \right) \left( \gamma_2 + \frac{(1 + \kappa)D}{4} \right) = \frac{1}{2} \left( \gamma_2 + \frac{(1 + \kappa)D}{4} \right) \left( \gamma_2 + \frac{(1 - \kappa)D}{4} \right).
\]
so \( r_1 > r_2 \) is equivalent to \( k_1^2 + k_2^2 < 2k_c^2 \). Since \( k_1 < k_2 \) we conclude that we obtain a longitudinal instability in the parameter regime

\[
\frac{(1 - \kappa)D}{2} \left( \gamma_2 + \frac{(1 + \kappa)D}{4} \right) < 1,
\]

or equivalently

\[
D < D_c \equiv \frac{2}{1 + \kappa} \left( -\gamma_2 + \sqrt{\gamma_2^2 + \frac{2(1 + \kappa)}{1 - \kappa}} \right),
\]

(21)

whereas we obtain a transverse instability in the regime \( D > D_c \). If \( D = D_c \), both instabilities have the same growth rate and thus are equally responsible for the destabilization of the uniformly aligned state.

For the special case \( \kappa = 0 \) considered in the main text, the system undergoes an isotropic instability for \( D < D_c \), and a transverse instability otherwise. We may also simplify the above formulae: \( D_c = 2 \left( -\gamma_2 + \sqrt{\gamma_2^2 + \frac{2}{1}} \right), r_1 = \gamma_2^2/4 \) and \( k_1 = \sqrt{\gamma_2/2} \). The formulae for \( r_2 \) and \( k_2 \) remain unchanged.

**Case 2: \( \gamma_2 < 0 \)**

An extension of the analysis in the previous section allows us to determine the stability properties of the uniformly aligned state in the parameter regime \( \gamma_2 < 0 \). The system may now exhibit one of four behaviors: an isotropic instability, a longitudinal instability, a transverse instability, or no instability at all, in which perturbations to the uniformly aligned state do not grow in time.

For \( \kappa \geq 1 \), we find that the uniform state undergoes a longitudinal instability if \( D > 2|\gamma_2|/\kappa \) and is stable otherwise. For \( 0 \leq \kappa < 1 \), the behavior is more complicated, but it is straightforward to use our analysis from the previous section, keeping in mind that the unstable wavenumbers \( k_1 \) and \( k_2 \) must be real. We find that the uniform state undergoes a longitudinal instability if \( 2|\gamma_2|/\kappa < D < D_c \), a transverse instability if \( D > \max\left( 2 \left( |\gamma_2| + \sqrt{2} \right), D_c \right) \), and is stable otherwise.

It is interesting to note that, even in the parameter regime \( \gamma_2 < 0 \), the effect of hydrodynamics is sufficient to destabilize the uniform state, provided that the velocity coefficient \( D \) and vorticity coefficient \( \kappa \) are sufficiently large so as to overcome the stabilizing influence of the elastic coefficient \( \gamma_2 \). Specifically, for \( \kappa \geq 1 \), the uniform state undergoes an instability for \( \frac{\kappa D}{2} > |\gamma_2| \). For \( 0 \leq \kappa < 1 \), it destabilizes for \( D > \min\left( 2 \left( |\gamma_2| + \sqrt{2} \right), 2|\gamma_2|/\kappa \right) \). The system dynamics for \( \gamma_2 < 0 \) will be considered in future work.

The results of the stability analysis are summarized in Fig. S1.

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**Fig. S1:** Results of the linear stability analysis of the uniformly aligned state in the \((\gamma_2, D)\) plane, for (a) \( \kappa = 0 \), (b) \( 0 < \kappa < 1 \) and (c) \( \kappa \geq 1 \). The uniform state is stable in the white regions; it undergoes an isotropic instability in the pink region, a longitudinal instability along the director field in the red regions, and a transverse instability perpendicular to the director field in the blue regions. The dashed curve corresponds to \( D = 2 \left( |\gamma_2| + \sqrt{2} \right) \), the dotted curve to \( D = D_c \equiv \frac{2}{1 + \kappa} \left( -\gamma_2 + \sqrt{\gamma_2^2 + \frac{2(1 + \kappa)}{1 - \kappa}} \right) \), and the dashed-dotted curve to \( D = 2|\gamma_2|/\kappa \).
TOPOLOGICAL DEFECT IDENTIFICATION AND TRACKING

We here present our algorithm for locating topological defects, identifying their sign, and tracking their motion. To locate the defects at a given time \( t \), we compute the zero-contours of \( \lambda(t, \mathbf{r}) \) and \( \mu(t, \mathbf{r}) \) in Matlab. Since defects are located at points for which \( \lambda = \mu = 0 \), we then find the points \((x_i, y_i)\) at which the zero-contours intersect. For each defect, its charge \( q \) is determined by computing the rotation of the director angle \( \theta \) around a closed counterclockwise contour \( C \) enclosing the defect. Specifically,

\[
q = \frac{1}{2\pi} \oint_C d\theta, \quad \text{where} \quad \theta = \text{angle}(\mathbf{n}) = \tan^{-1}\left( \frac{\mu}{\lambda + \sqrt{\lambda^2 + \mu^2}} \right). \tag{22}
\]

We use a square contour and determine its size \( s \) as follows. We first construct a \( m \times m \) matrix \( R \) with entries

\[
R_{ij} = \max[d_L(x_i, x_j), d_L(y_i, y_j)], \tag{23}
\]

where \( d_L(x, y) = \min(|x - y|, L - |x - y|) \) for \( x, y \in \mathbb{R} \) is the periodic Euclidean metric in 1D and \( L \) is the size of the simulation box. We then define the size \( s_i \) of the contour around the \( i \)th defect as

\[
s_i = \frac{1}{2} \left( \min_{j<i} R_{ij} \right). \tag{24}
\]

We then compute the nematic director angle \( \theta \) at each grid point on \( C \) and thus determine its rotation around \( C \). A rotation by \( \pi \) radians corresponds to a +1/2-defect, by \( -\pi \) radians to a −1/2-defect. We observe that this method works quite well, but occasionally fails to accurately locate and identify defects near the locations of creation and annihilation. During such instances, the net charge is nonzero, and the algorithm detects some spurious defects (indexed by \( i = i_1, \ldots, i_r \)) for which \( s_i \) is less than the simulation grid size. Since we initialize the simulations in configurations with zero total charge, we remedy this by placing a single defect of the appropriate sign at the location \((x_k, y_k)\) where

\[
k = \arg \min_{t=i_1, \ldots, i_r} \sum_{n=1}^r R_{t i_n}. \tag{25}
\]

Once the defect locations at every time step are identified, we track their trajectories using the James Munkres’ variant of the Hungarian assignment algorithm in Matlab \([3]\). Specifically, let \((x_i^+, y_i^+)\) be the +1/2-defect locations at a time step and \((\tilde{x}_j^+, \tilde{y}_j^+)\) those at the next time step, where \( 1 \leq i \leq I \) and \( 1 \leq j \leq J \). Let \( A \) be the \( I \times J \) matrix with entries

\[
A_{ij} = \sqrt{d_L(x_i, \tilde{x}_j)^2 + d_L(y_i, \tilde{y}_j)^2}, \tag{26}
\]

which corresponds to the periodic Euclidean distance between pairs of defects. An assignment of \( K \) pairs is a set of indices \( \{(i_k, j_k)\}_{k=1}^K \) in the intervals \( 1 \leq i_k \leq I \) and \( 1 \leq j_k \leq J \) that minimizes the cost measure

\[
W = \sum_{k=1}^K A_{i_k j_k} + \Upsilon \tag{27}
\]

subject to the constraints \( i_a \neq i_b \) and \( j_a \neq j_b \) for \( a \neq b \), where \( \Upsilon \) is the cost of unassigned defects. That is, the defects \((x_{i_k}, y_{i_k})\) are assigned to \((\tilde{x}_{j_k}, \tilde{y}_{j_k})\) for \( 1 \leq k \leq K \), and the rest are unassigned. To ensure that \( K = \min(I, J) \), or that all available defects are assigned to each other, we take

\[
\Upsilon = \max_{i,j} A_{ij} + 1. \tag{28}
\]

The same is done for the −1/2-defects, and the defect tracks are thus constructed iteratively. New tracks are created from the unassigned defects when \( J > I \) or ended when \( J < I \). Note that this method assumes that defect pair creation and annihilation events do not occur simultaneously, for which one track should be created and one ended despite \( I = J \). Figure S2 shows an example of the computed defect paths for a particular set of simulation parameters.
MODELING ALIGNMENT NEAR BOUNDARIES

It was found that high-density solutions of actin filaments in rectangular chambers are sensitive to changes in the size and aspect ratio of the confining chamber, as the filaments typically align parallel to the long walls [4]. We here include an effective short-range alignment potential in our model for ALCs, with a view to determining whether the observed orientational order of +1/2-defects is affected by an alignment interaction with the boundaries.

In a square simulation box of size $L$, we impose an alignment potential on the edges $y = 0$ and $y = L$, so that the nematic vector field is preferentially oriented nearby in the horizontal direction $\hat{x} = (1, 0)$. To do this, we append the terms $-f(y)\partial G/\partial \lambda$ and $-f(y)\partial G/\partial \mu$ to the right hand sides of the governing equations (12) for $\lambda(t, r)$ and $\mu(t, r)$, respectively, where

$$f(y) = \sum_{n=-M}^{M} e^{-(y-Ln)^2/\sigma^2} \quad \text{and} \quad G(\lambda, \mu) = c(\mathbf{n} \cdot \hat{x})^2((\mathbf{n} \cdot \hat{x})^2 - 2), \quad c > 0, \quad 0 < \sigma \ll 1. \quad (29)$$

The expression for the nematic vector field $\mathbf{n}$ is given in (13). Note that $f(y) \approx 1$ in small horizontal strips around $y = 0$ and $y = L$ and is nearly zero otherwise. The function $G(\lambda, \mu)$ may be interpreted as a potential with minima at $\mathbf{n} \cdot \hat{x} = \pm 1$, which causes the vector field to align horizontally. While this approach is rather ad hoc, it provides a simple theoretical model for alignment interactions near solid boundaries. A more thorough investigation of boundary interactions should be conducted in the future, perhaps through incorporating a boundary condition directly on the nematic order tensor $Q(t, \mathbf{r})$.

Figure S5a shows the local pair orientation probability distribution of +1/2-defect orientations in a simulation with the alignment potential. Figures S5b,c compare the corresponding polar $P(r)$ and nematic $N(r)$ correlation functions with those in a simulation without an alignment potential, as shown in Figure 6c,d in the Main Text. We note that both sets of simulations exhibit antipolar ordering of +1/2-defect orientations, and that the degree of orientational order is not significantly altered by including the alignment potential.
Fig. S3: Quantitative comparison of defect statistics between theoretical predictions (with $\kappa = 0$) and experiment [1], using the parameter estimation procedure described in the Main Text. The simulation domain is nine times smaller in area compared to that in Figure 7 in the Main Text, the other parameters being equal. Top row: For $-\frac{1}{2}$-defects, both speed distribution and lifetime distribution agree well. Bottom row: For $+\frac{1}{2}$-defects, experimentally measured speed values are slightly larger, as our model assumes a strongly overdamped limit. The numbers $n_\pm$ reflect the total number of $\pm\frac{1}{2}$-defects tracked. Note that the defect lifetime and speed statistics closely resemble those in Figure 7 of the Main Text, indicating that they are not very sensitive to the size of the periodic simulation domain.

SPATIAL RESOLUTION OF NUMERICAL SIMULATIONS

The numerical simulations were performed with $N = 256$ points in each spatial direction. Fig. S6 shows the power spectra of the functions $\lambda(t_f, r)$ and $\mu(t_f, r)$ at the final time $t_f$ of a sample simulation run. Note that the power decays to a level close to machine precision at high wavenumbers, indicating that $N = 256$ is sufficient to resolve the numerical solution in space.

Fig. S4: Quantitative comparison of defect statistics between theoretical predictions (with $\kappa = 1$) and experiment [1], using the parameter estimation procedure described in the Main Text. Top row: For $-\frac{1}{2}$-defects, both speed distribution and lifetime distribution agree well. Bottom row: For $+\frac{1}{2}$-defects, experimentally measured speed values are slightly larger, as our model assumes a strongly overdamped limit. Dimensionless simulation parameters $D = 1.75$ and $\gamma_2 = 1$ translate into the following dimensional values: $a = 0.02 \text{s}^{-1}$, $b = 0.09 \text{s}^{-1}$, $D = 869 \mu\text{m}^2/\text{s}$, $\gamma_2 = 497 \mu\text{m}^2/\text{s}$, $\gamma_4 = 2.7 \times 10^6 \mu\text{m}^4/\text{s}$. The numbers $n_\pm$ reflect the total number of $\pm\frac{1}{2}$-defects tracked.

Fig. S5: Orientational order of $+\frac{1}{2}$-defect orientations in a simulation with an alignment potential imposed at the horizontal edges of the simulation box, as described in the Supplementary Information. The dimensionless simulation parameters are $D = 1.5$ and $\gamma_2 = 2$, and the parameters for the functions defined in eq. (29) are $c = 5$, $M = 2$ and $\sigma = 0.1$. (a) Local pair orientation PDF $p(\theta_{ij}|r)$ signals antipolar ordering of $+\frac{1}{2}$-defects, and reflects the typical defect-lattice spacing in units of the mean nearest-neighbor distance $r_0$ between $+\frac{1}{2}$-defects. (b) Polar correlation function $P(r)$, both with the alignment potential (blue) and without (red, Fig. 6c of Main Text). (c) The nematic correlation function $N(r)$, both with the alignment potential (blue) and without (red, Fig. 6d of Main Text). The plots show that the degree of orientational order is not significantly altered by the alignment potential.
Fig. S6: Power spectra of $\lambda(t_f, r)$ and $\mu(t_f, r)$ for a simulation with dimensionless parameters $D = 1.5$, $\kappa = 0$ and $\gamma_2 = 2$, where $t_f$ is the maximum simulation time. The simulation is performed with $N = 256$ points in each spatial direction. The high frequency modes have power close to machine precision, indicating that the numerical solution is sufficiently resolved in space.