I. SUMMARY OF GRAPHICAL CORRECTIONS

The various constituent elements of the Feynman diagrams are illustrated in Fig. 1 in the main text. The one loop graphical corrections to various coefficients in the model Eq. (1) in the main text are illustrated in Fig. 2 in the main text. Since we are only interested in the RG flows near the critical point, we have evaluated all of these graphs at $a = 0$, except for the graph (a) for $a$, the $a$-dependence of which we need to determine the “thermal” eigenvalue $y_a$. The corrections from graphs (a), (b), (c), and (d) are given respectively by:

$$
\delta a = \frac{(d - 1)(d + 2)}{d} \frac{S_d}{(2\pi)^2} \frac{D b}{\mu^2} \Lambda^{d-4} d\ell, \\
\delta b = - \left[ \frac{d + 1 + 6(d^2 - 2)}{d(d + 2)} \right] \frac{S_d}{(2\pi)^2} \frac{b^2 D}{\mu^2} \Lambda^{d-4} d\ell, \\
\delta \lambda = - \frac{2(d^2 - 2)}{d(d + 2)} \frac{S_d}{(2\pi)^2} \frac{D b \lambda}{\mu^2} \Lambda^{d-4} d\ell - \frac{(d - 2)}{d} \frac{S_d}{(2\pi)^2} \frac{D b \lambda}{\mu^2} \Lambda^{d-4} d\ell, \\
\delta \mu = \frac{(d - 2)}{2d} \frac{S_d}{(2\pi)^2} \frac{D \lambda^2}{\mu^2} \Lambda^{d-4} d\ell.
$$

Combining these corrections (including the zero correction to $D$ ) with the rescalings described in the main text, and setting dimension $d = 4$ in all of these expressions (which is sufficient to obtain results to first order in $\epsilon = 4 - d$), leads to the recursion relations (4–8) in the main text, (although in (4) we have in addition expanded to linear order in $a$, which is sufficient to determine the exponents to $O(\epsilon)$).
II. EVALUATION OF THE FEYNMAN DIAGRAMS

A. Graph (a)

This graph represents an additional contribution \( \delta (\partial_t v_1) \) to \( \partial_t v_1 \) given by:

\[
\delta (\partial_t v_1) = -bQ_{lnmp}(k)\nu_p(\vec{k}) \int_{q, \Omega} \frac{2DP_{mn}(q)}{(\mu q^2 + a)^2 + \Omega^2} = -bDQ_{lnmp}(k)\nu_p(\vec{k}) \int_q P_{mn}(q) \frac{\delta_{mn}(1 - \frac{1}{d})}{\mu q^2 + a} = -(d + 2)bDQ_{v1}(\vec{k}) \left(1 - \frac{1}{d}\right) \int_q \frac{1}{\mu q^2 + a} = -(d + 2)bDQ_{v1}(\vec{k}) \left(1 - \frac{1}{d}\right) \frac{S_d}{(2\pi)^d} \frac{\Lambda^d d\ell}{\mu \Lambda^2 + a},
\]

(5)

where \( P_{mn}(q) = \delta_{mn} - q_m q_n / q^2 \) is the transverse projection operator, and \( Q_{lnmp} \) is defined in the main text after Eq. (3). In going from the second to the third line above, we have used the well-known identity \( \langle \delta_{mn} \rangle_q = (1 - \frac{1}{d}) \delta_{mn} \), where \( \langle \cdot \rangle_q \) denotes the average over directions \( \hat{q} \) of \( q \) for fixed \( |q| \). This identity is derived in part (III) of these Supplemental Materials. Clearly this correction to \( \partial_t v_1(k, \omega) \) is exactly what one would obtain by adding to the parameter \( a \) (hiding in the "propagator") in Eq. (3) in the main text a correction \( \delta a \) given by the coefficient of \( v_1 \) in (5); i.e., Eq. (1).

B. Graph (b)

This graph represents an additional contribution \( \delta (\partial_t v_1) \) to \( \partial_t v_1 \) given by:

\[
\delta (\partial_t v_1) = 18 \left(\frac{b}{\delta}\right)^2 2DQ_{lmeno}(k) \int_{p, \hat{h}} v_j(\hat{p})v_o(\vec{k} - \hat{h})v_s(\hat{h} - \hat{p}) \int_{q, \Omega} \frac{P_{mn}(q)Q_{njks}(q - k)}{(\mu q^2 + \Omega^2)[\mu |h - q|^2 - i(\omega h - \Omega)]}.
\]

(6)

where the combinatoric prefactor of 18 arises because there are 3 ways to pick the leg with index \( o \) on the left. Then, once this choice has been made, there are 2 ways to pick the leg with index \( m \) on the left, and 3 ways to pick the leg with index \( i \) on the right.

To lowest order in the external momenta \( (k, p, \text{ and } h) \) and frequencies \( \omega, \omega_p, \text{ and } \omega_h \), we can set all of these external momenta and frequencies equal to zero in the integrand of the integral over \( \hat{q} \) in Eq. (6). This proves, as we shall see, to make this graph into a renormalization of the cubic non-linearity \( b \). Making this simplification, Eq. (6) becomes, after using

\[
\frac{1}{\mu q^2 + \Omega} = \frac{\mu q^2 - i\Omega}{(\mu q^2 + \Omega^2)},
\]

(7)

and dropping an integral of an odd function of \( \Omega \),

\[
\delta (\partial_t v_1) = 4b^2 DQ_{lmeno}(k) \int_{p, \hat{h}} v_j(\hat{p})v_o(\vec{k} - \hat{h})v_s(\hat{h} - \hat{p}) \int_{q, \Omega} \frac{\mu q^2 P_{mn}(q)Q_{njks}(q)}{(\mu q^2 + \Omega^2)^2}.
\]

(8)

Using the definition of \( Q_{njks}(q) \), and contracting indices, we obtain

\[
P_{mj}(q)Q_{njks}(q) = P_{mn}(q)\delta_{js} + P_{ms}(q)P_{jn}(q) + P_{jm}(q)P_{nk}(q).
\]

(9)

As we’ve done repeatedly throughout these Supplemental Materials, we’ll replace this tensor with its average over all directions of \( \hat{q} \). This is easily obtained from the known direction averages (39) and (53), and gives, after a little algebra,

\[
\langle P_{mj}(q)Q_{njks}(q) \rangle_q = \frac{d + 1}{d + 2} \delta_{mj}\delta_{js} + \frac{d^2 - 2}{d(d + 2)} \left( \delta_{ms}\delta_{jn} + \delta_{jm}\delta_{ns} \right).
\]

(10)
Inserting this into our earlier expression for $\delta (\partial_t v_l)$ and performing a few tensor index contractions gives
\[
\delta (\partial_t v_l) = 4 b^2 D \left( \frac{d+1}{d+2} \delta_{js} Q_{lmno}(k) + \frac{d^2 - 2}{d(d+2)} (Q_{lsjo}(k) + Q_{jisn}(k)) \right)
\times \int_{\mathbf{p}, \mathbf{h}} v_j(\mathbf{p}) v_o(\mathbf{k} - \mathbf{h}) v_s(\mathbf{h} - \mathbf{p}) \int_{q, \Omega} \frac{\mu q^2}{(\mu^2 q^4 + \Omega^2)^2} .
\] (11)

The trace in this expression can be evaluated as
\[
Q_{lmno}(k) = P_{lm}(k) \delta_{mo} + P_{lm}(k) \delta_{mo} + P_{lo}(k) \delta_{mn} = (d+2) P_{lo}(k) ,
\] (12)
and the integral over frequency $\Omega$ and $\mathbf{q}$ is readily evaluated, and is given by
\[
\int_{q, \Omega} \frac{\mu q^2}{(\mu^2 q^4 + \Omega^2)^2} = \frac{1}{4 \mu^2 (2\pi)^d} \Lambda^{d-4} d\ell .
\] (13)

Putting (12) and (13) into (11), and taking advantage of the complete symmetry of $\int_{\mathbf{p}, \mathbf{h}} v_j(\mathbf{p}) v_o(\mathbf{k} - \mathbf{h}) v_s(\mathbf{h} - \mathbf{p})$ under interchanges of the indices $j, o,$ and $s$ to symmetrize the tensor prefactor gives
\[
\delta (\partial_t v_l) = \frac{b^2 D}{3 \mu^2} \left( d + 1 + \frac{d(d-2)}{d+2} \right) \int_{q, \Omega} \frac{2 D P_{mi}(q)}{(\mu^2 q^4 + \Omega^2)^2} .
\] (14)

This is readily recognized as a contribution to $-\frac{2}{3} \lambda$ term in Eq. (3) in the main text; hence, the correction of $b$ is given by (2).

C. The First Graph in (c)

This graph represents an additional contribution $\delta (\partial_t v_l)$ to $\partial_t v_l$ given by:
\[
\delta (\partial_t v_l) = 6 \left( -\frac{i \lambda}{2} \right) \left( -\frac{b}{3} \right) P_{lmn}(k) \int_{\mathbf{h}} v_j(\mathbf{h}) v_o(\mathbf{k} - \mathbf{h}) \int_{q, \Omega} \frac{2 D P_{mi}(q)}{(\mu^2 q^4 + \Omega^2)^2} ,
\] (15)
where $P_{lmn}(k)$ is defined in the main text after Eq. (3), and the combinatoric prefactor of 6 arises because there are 2 ways to pick the leg with index $m$ on the left, and 3 ways to pick the one with index $i$ on the right.

The piece of $\delta (\partial_t v_l)$ linear in the external momentum $k$ is immediately recognized as a contribution to $-\frac{2}{3} \lambda$ term in Eq. (3) in the main text. Since there is already an implicit factor of $k$ in the $P_{lmn}(k)$ in this expression, we can evaluate this graph to linear order in $k$ by setting both $k$ and the external frequency $\omega$ to zero in the integrand of the integral over $\mathbf{q}$. Doing so, and in addition using (7) gives, after dropping an integral of an odd function of $\Omega$ that vanishes,
\[
\delta (\partial_t v_l) = 2i \lambda b D P_{lmn}(k) \int_{\mathbf{h}} v_j(\mathbf{h}) v_o(\mathbf{k} - \mathbf{h}) \int_{q, \Omega} \frac{\mu q^2 P_{mi}(q) Q_{nijs}(q)}{(\mu^2 q^4 + \Omega^2)^2} ,
\] (16)
where we’ve used the fact that $Q_{nijs}(q)$ is an even function of $q$. Using the definition of $Q_{nijs}(q)$, and performing the tensor index contractions, we can simplify the numerator of the integrand as follows:
\[
P_{mi}(q) Q_{nijs}(q) = P_{mn} \delta_{js} + P_{jn} P_{ms} + P_{ns} P_{mj} ,
\] (17)
where we’ve also used the fact that, e.g., $P_{mi}(q) P_{jn}(q) = P_{mn}(q)$ (which is a consequence of the definition of $P_{mi}(q)$ as a projection operator). Using this result (17), and taking the angle average of the numerator of (16) (which is the only factor in the integral that depends on the direction of $\mathbf{q}$) gives,
\[
\langle P_{mi}(q) Q_{nijs}(q) \rangle_q = \left( \frac{d+1}{d+2} \right) \delta_{mn} \delta_{js} + \left( \frac{d^2 - 2}{d(d+2)} \right) (\delta_{jn} \delta_{ms} + \delta_{ns} \delta_{mj}) .
\] (18)
In deriving this expression, we’ve made liberal use of the angle averages (53) and (39).

The first term on the right hand side of (18) contributes nothing, since it contracts two of the indices on the prefactor $P_{lmn}(k)$ together, which gives zero, as can be seen from the definition of $P_{lmn}(k)$:
\[
P_{lmn}(k) = P_{lm}(k) k_m + P_{lm}(k) k_m = 0 .
\] (19)
Keeping only the second term gives

\[ \delta (\partial_v v_l) = \frac{2i\lambda b D}{d(d+2)} \int_{\Omega} \frac{\mu q^2}{(\Omega^2 + \mu^2 q^4)^2} \left[ \delta_{jn}\delta_{ms} + \delta_{ns}\delta_{mj} \right] v_j(\tilde{h})v_m(\tilde{k} - \tilde{h}) + v_m(\tilde{h})v_n(\tilde{k} - \tilde{h}) \]

where in the last step we have used the symmetry of $P_{mn}(k)$ under interchange of its last two indices.

This is immediately recognized as a contribution to $-\frac{i}{2}\lambda$ term in Eq. (3) in the main text, which implies a correction to $\lambda$ given by

\[ \delta \lambda = -\frac{2\lambda b D \Lambda^d - \text{d} \nu}{\mu^2} \left( \frac{\text{d}^2 - \text{d} n}{\text{d}^\nu} \right) \]

which is just the first term on the RHS of Eq. (3).

D. The Second Graph in (c)

This graph represents an additional contribution $\delta (\partial_v v_l)$ to $\partial_v v_l$ given by

\[ \delta (\partial_v v_l) = 12 \left( -\frac{i\lambda}{2} \right) \left( -\frac{b}{3} \right) 2DQ_{l\omega\omega}(k) \int_{\tilde{h}} v_j(\tilde{h})v_\omega(\tilde{k} - \tilde{h}) \int_{\Omega^2} \frac{P_{nl}(q)P_{nj}(h - q)}{(\mu^2 q^2 + \mu^2 h - q^2)^2 - i(\omega - \Omega)} \]

where the combinatoric prefactor of 12 arises because there are 3 ways to pick the leg with index $\omega$ on the left. Then, once this choice has been made, there are 2 ways to pick the leg with index $n$ on the left, and 2 ways to pick the one with index $i$ on the right.

To extract from (22) the contribution to $-\frac{i}{2}\lambda$ term in Eq. (3) in the main text, we need the piece of $\partial_v v_l$ that is linear in either the external momentum $k$ or $h$. We notice that the integral over $q$ vanishes if we set $h = 0$ in its integrand. This implies the entire term is at least of order $h$. Thus, to obtain a correction to $\lambda$ we can simply set the external frequency $\omega = 0$ in the integrand; doing so, and integrating over $\Omega$, we obtain

\[ \delta (\partial_v v_l) = \frac{2i\lambda b D Q_{l\omega\omega}(k)}{\mu^2} \int_{\tilde{h}} v_j(\tilde{h})v_\omega(\tilde{k} - \tilde{h}) \int_{\Omega^2} \frac{P_{nl}(q)}{\mu^2 q^2 + \mu^2 h - q^2} \left[ P_{nl}(h - q)(h_j - q_j) + P_{nj}(h - q)(h_i - q_i) \right] \]

This can be rewritten as

\[ \delta (\partial_v v_l) = \frac{2i\lambda b D Q_{l\omega\omega}(k)}{\mu^2} \int_{\tilde{h}} v_j(\tilde{h})v_\omega(\tilde{k} - \tilde{h}) \left[ I_{l\omega\omega}^{(2)}(h) + I_{l\omega\omega}^{(3)}(h) + I_{l\omega\omega}^{(4)}(h) + I_{l\omega\omega}^{(5)}(h) \right] \]

where $I_{l\omega\omega}^{(2)}(h)$ is given by equation (48) (with the obvious substitution $k \rightarrow h$), and the other integrals are defined as:

\[ I_{l\omega\omega}^{(3)}(h) \equiv h_j \int_{\Omega^2} \frac{P_{nl}(q)P_{nj}(h - q)}{\mu^2 q^2 + \mu^2 h - q^2}, \]

\[ I_{l\omega\omega}^{(4)}(h) \equiv h_i \int_{\Omega^2} \frac{P_{nl}(q)P_{nj}(h - q)}{\mu^2 q^2 + \mu^2 h - q^2}, \]

\[ I_{l\omega\omega}^{(5)}(h) \equiv -h_i \int_{\Omega^2} \frac{P_{nl}(q)P_{nj}(h - q)q_i}{\mu^2 q^2 + \mu^2 h - q^2} \]

Immediately, by the properties of the projection operator we get

\[ I_{l\omega\omega}^{(5)}(h) = 0. \]
We notice that both \( I_{jmn}^{(3)}(h) \) and \( I_{jmn}^{(4)}(h) \) are already proportional to \( h \), so we can simply set \( h = 0 \) inside the integral. Thus, we obtain

\[
I_{jmn}^{(3)}(h) = h_j \int_q \frac{P_m(q)P_n(q)}{2\mu^2q^4} = h_j \int_q \frac{P_{mn}(q)}{2\mu^2q^4} = \frac{d-1}{2d} S_d \mu^{d-4} \Lambda^d \delta_m h_n.
\]

\[
I_{jmn}^{(4)}(h) = h_i \int_q \frac{P_m(q)P_{nj}(q)}{2\mu^2q^4} = h_i \int_q \frac{P_{mn}(q)P_{nj}(q)}{2\mu^2q^4} = \frac{1}{d(d+2)} S_d \mu^{d-4} \Lambda^d \left[ (d^2 - 3) \delta_m h_n + \delta_{mn} h_j + \delta_{mj} h_n \right].
\]

Plugging the values \((48), (29), (30), \) and \((28)\) of the various integrals into Eq. (24), we obtain

\[
\delta(\partial_k v_l) = \frac{i \lambda b D}{2\mu^2} \frac{S_d}{(2\pi)^d} \Lambda^d \delta_m h_n.
\]

The third piece vanishes due to the incompressibility condition \( h_j v_j(\tilde{h}) = 0 \). The first and the second pieces can be grouped together since \( Q_{lmno} \) is invariant under the interchange of \( m \) and \( n \). Therefore, the above expression can be simplified as

\[
\delta(\partial_k v_l) = \frac{i (d-2) \lambda b D}{d} \frac{S_d}{\mu^2} \frac{1}{(2\pi)^d} \Lambda^d \delta_m h_n.
\]

where in the second equality we have dropped the second and third pieces. The second piece can be dropped because it vanishes, as can be seen by simply changing variables of integration from \( h \) to \( k - \tilde{h} \); this gives

\[
\int_h P_{en}(k)h_n v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) = \int_h P_{en}(k)h_n v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) = \int_h P_{en}(k)(k_n - h_n) v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}).
\]

Adding the left and the right hand side of this equation, and dividing by 2, implies

\[
\int_h P_{en}(k)h_n v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) = \frac{1}{2} \left( \int_h P_{en}(k)h_n v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) + \int_h P_{en}(k)(k_n - h_n) v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) \right) = \frac{1}{2} \int_h P_{en}(k)k_n v_o(\tilde{k} - \tilde{h})v_o(\tilde{h}) = 0.
\]

with the last equality following from \( P_{en}(k)k_n = 0 \). The third piece can be dropped due to the incompressibility condition \( h_j v_j(\tilde{h}) = 0 \). The remaining piece of \( (32) \) is readily recognized as a contribution to \(-\frac{\lambda}{2} \) term in Eq. (3) in the main text. This implies a correction to \( \lambda \) given by the second piece on the RHS of Eq. (3).

E. Graph (d)

This graph represents an additional contribution \( \delta(\partial_k v_l) \) to \( \partial_k v_l \) given by:

\[
\delta(\partial_k v_l) = -\lambda^2 P_{lmn}(k) v_j(k) \int_q \frac{2D P_m(q) P_{nj}(k - q)}{\left[ \mu^2 q^4 + \Omega^2 \right] \left[ \mu^2 q^2 - i(\omega - \Omega) \right]} = -D \lambda^2 P_{lmn}(k) v_j(k) \int_q \frac{P_m(q) P_{nj}(k - q)}{\mu^2 q^2 - i\omega + \mu q^2}.\]
The integral in this expression is readily seen to vanish when \( \mathbf{k} \to 0 \), since the integrand then becomes odd in \( \mathbf{q} \). Hence, the integral is at least of order \( \mathbf{k} \), so the entire term (include the implicit first power of \( \mathbf{k} \) coming from the \( P_{\mathbf{mn}}(\mathbf{k}) \) in front), is \( \mathcal{O}(k^3) \). Since we do not need to keep any terms in the equations of motion higher order in \( \mathbf{k} \) and \( \omega \) than \( \mathcal{O}(k^3) \), this means that we can safely set \( \omega = 0 \) inside the integral. Keeping just the \( \mathcal{O}(k) \) piece of the integral then gives us a modification to the equation of motion of \( \mathcal{O}(k^2 \mathbf{v}) \), which is clearly a renormalization of the diffusion constant \( \mu \). So setting \( \omega = 0 \) in the integrand for the reasons just discussed, and then writing \( P_{\mathbf{mn}}(\mathbf{q} - \mathbf{k}) \) using its definition as given in the main text after Eq. (3); this gives for the integral in (34):

\[
\int \frac{P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{k} - \mathbf{q})}{\mu q^2 [\mu |\mathbf{k} - \mathbf{q}|^2 + \omega^2]} = \int \frac{P_m(\mathbf{q}) [P_{\mathbf{mn}}(\mathbf{q} - \mathbf{k})(k_i - q_i) + P_m(\mathbf{q} - \mathbf{k})(k_j - q_j)]}{\mu q^2 [\mu |\mathbf{k} - \mathbf{q}|^2 + \omega^2]}.
\]  

(35)

The term proportional to \( k_j \) in this expression can be dropped, since \( k_j v_j = 0 \) (this is just the incompressibility condition \( \nabla \cdot \mathbf{v} = 0 \) written in Fourier space). The term proportional to \( q_i \) can be dropped since \( P_m(\mathbf{q}) q_i = 0 \) by the properties of the transverse projection operator \( P_m(\mathbf{q}) \). This leaves two terms in the integral, which can be written as

\[
I_{\mathbf{jm}}^{(1)}(\mathbf{k}) \equiv k_i \int q_j \frac{P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{q} - \mathbf{k})}{q^4}.
\]  

(36)

and

\[
I_{\mathbf{jm}}^{(2)}(\mathbf{k}) \equiv - k_i \int q_j \frac{P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{q} - \mathbf{k})q_j}{q^4}.
\]  

(37)

Since \( I_{\mathbf{jm}}^{(1)}(\mathbf{k}) \) already has an explicit factor of \( k \) in front, we can evaluate it to linear order in \( k \) by setting \( \mathbf{k} = 0 \) inside the integral. Doing so gives

\[
I_{\mathbf{jm}}^{(1)}(\mathbf{k}) = \frac{1}{2 \mu^2} k_i \int q_j \frac{P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{q})}{q^4}.
\]  

(38)

The integral in this expression can now be evaluated by replacing the only piece that depends on the direction \( \hat{\mathbf{q}} \) of \( \mathbf{q} \), namely, the factor \( P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{q}) \), with its angle average. As shown in (III), this average is given by

\[
\langle P_m(\mathbf{q})P_{\mathbf{mn}}(\mathbf{q}) \rangle = \left( \frac{d^2 - 3}{d(d + 2)} \right) \delta_{mn}\delta_{jn} + \frac{1}{d(d + 2)} (\delta_{mn}\delta_{ij} + \delta_{mj}\delta_{ni}).
\]  

(39)

Inserting this into (38) gives

\[
I_{\mathbf{jm}}^{(1)}(\mathbf{k}) = \frac{1}{2 \mu^2} \left[ \left( \frac{d^2 - 3}{d(d + 2)} \right) k_i \delta_{jn} + \frac{1}{d(d + 2)} (k_j \delta_{mn} + k_m \delta_{nj}) \right] \frac{S_d}{(2\pi)^d} \Lambda^{d-4} d \ell.
\]  

(40)

Now let us expand \( I_{\mathbf{jm}}^{(2)}(\mathbf{k}) \) to linear order in \( k \). Changing variables of integration from \( \mathbf{q} \) to a shifted variable \( \mathbf{p} \) defined by:

\[
\mathbf{q} = \mathbf{p} + \frac{\mathbf{k}}{2}
\]  

(41)

gives

\[
I_{\mathbf{jm}}^{(2)}(\mathbf{k}) = - \int_{\mathbf{p}} \frac{P_m(\mathbf{p}+\mathbf{k})P_m(\mathbf{p}-\mathbf{k})}{\Gamma(\mathbf{p}+\mathbf{k}) + \Gamma(\mathbf{p}-\mathbf{k})} k_i \int_{\mathbf{p}} \frac{P_m(\mathbf{p}+\mathbf{k})P_m(\mathbf{p})}{\Gamma(\mathbf{p}+\mathbf{k}) + \Gamma(\mathbf{p})} - \frac{k_j}{2} \int_{\mathbf{p}} \frac{P_m(\mathbf{p}+\mathbf{k})P_m(\mathbf{p})}{\Gamma(\mathbf{p}+\mathbf{k}) + \Gamma(\mathbf{p})}
\]

\[
\equiv I_{\mathbf{jm}}^{(2,1)}(\mathbf{k}) + I_{\mathbf{jm}}^{(2,2)}(\mathbf{k}),
\]  

(42)

where we’ve defined

\[
\mathbf{p} + \equiv \mathbf{p} + \frac{\mathbf{k}}{2}, \quad \mathbf{p} - \equiv \mathbf{p} - \frac{\mathbf{k}}{2},
\]  

(43)

and \( I_{\mathbf{jm}}^{(2,1)}(\mathbf{k}) \) and \( I_{\mathbf{jm}}^{(2,2)}(\mathbf{k}) \) to be the first and second terms in the expression for \( I_{\mathbf{jm}}^{(2)}(\mathbf{k}) \). Since \( I_{\mathbf{jm}}^{(2,2)}(\mathbf{k}) \) has an explicit factor of \( k \) in front, we can evaluate this term to linear order in \( \mathbf{k} \) by setting \( \mathbf{k} = 0 \) inside the integral. This leads to

\[
I_{\mathbf{jm}}^{(2,2)}(\mathbf{k}) = - \frac{k_j}{2} \int_{\mathbf{p}} \frac{P_m(\mathbf{p})P_m(\mathbf{p})}{\Gamma(\mathbf{p}+\mathbf{k}) + \Gamma(\mathbf{p})} = - \frac{d - 1}{4d} \frac{S_d}{(2\pi)^d} \mu^2 \Lambda^{d-4} d \ell k_j \delta_{mn}.
\]  

(44)
The calculation of \( I_{jmn}^{(2,1)}(k) \) requires more effort. Expanding the numerator we get

\[
I_{jmn}^{(2,1)}(k) = - \int_p \frac{p_j}{\Gamma(p_+)} \left[ \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \right] \left( \delta_{mn} - \frac{p_m p_n}{p_+^2} \right) \left( \delta_{si} - \frac{p_s p_i}{p_-^2} \right)
\]

\[
= - \int_p \frac{p_j}{\Gamma(p_+)} \left[ \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \right] \int_p \frac{p_j p_m p_n}{\Gamma(p_+)} \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \frac{p_m p_n}{p_+^2} \frac{p_n p_m}{p_-^2} p^2.
\]

(Note that we have purposely written each term on the RHS of the second equality in Eq. (45) as an even function of \( k \) multiplied by a non-even function. We can simply set \( k = 0 \) inside the even part since it cannot be expanded to a linear piece in \( k \). Therefore, Eq. (45) can be simplified as)

\[
I_{jmn}^{(2,1)}(k) = - \int_p \frac{p_j}{\Gamma(p_+)} \left[ \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \right] \left( \delta_{mn} - \frac{p_m p_n}{2p_+} \right) \left( \delta_{si} - \frac{p_s p_i}{2p_-} \right)
\]

\[
= - \int_p \frac{p_j}{\Gamma(p_+)} \left[ \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \right] \int_p \frac{p_j p_m p_n}{\Gamma(p_+)} \frac{1}{\Gamma(p_+) + \Gamma(p_-)} \left( \frac{p_m p_n}{2p_+} + \frac{p_n p_m}{2p_-} - \frac{k_m k_n}{4} \right)
\]

\[
= \frac{1}{2p^2} \int_p \frac{1}{p^2} \left( - \frac{p_j p_m p_n}{2} + \frac{p_n p_m k_s}{2} - \delta_{mn} p_j p_s + \frac{p_j p_m p_n k_s}{p^2} \right) + O(k^2).
\]

The integral over \( p \) in this expression can now be evaluated by replacing \( p_j p_m p_n, p_j p_m, p_j p_s, \) and \( p_j p_s p_m p_n \) with their angular averages over all directions of \( p \) for fixed \( |p| \), as given by equations (52) and (60) of section (III). This gives

\[
I_{jmn}^{(2,1)}(k) = \frac{1}{2 \mu^2 (2\pi)^d} \Lambda^{d-4} d \left( - \frac{\delta_{mj} k_n}{d} + \frac{\delta_{jn} k_m}{d} - \frac{\delta_{mn} k_j}{d} + \frac{\delta_{mn} k_j + \delta_{nj} k_m + \delta_{mj} k_n}{d} \right) + O(k^2)
\]

\[
= \frac{1}{4d(d+2) (2\pi)^d} \mu^{-2} \Lambda^{d-4} d \left( d \delta_{mn} k_n - (d+4) \delta_{nj} k_m + 2(d+1) \delta_{mn} k_j \right) + O(k^2).
\]

Plugging Eqs. (44,47) into Eq. (42) we get

\[
I_{jmn}^{(2)}(k) = \frac{1}{4d(d+2) (2\pi)^d} \mu^{-2} \Lambda^{d-4} d \left[ d \delta_{mn} k_n - (d+4) \delta_{nj} k_m + (d^2 + d + 4) \delta_{mn} k_j \right].
\]

The terms in \( I_{jmn}^{(1)}(k) \) (40) and \( I_{jmn}^{(2)}(k) \) (48) that are proportional to \( k_j \) may be dropped, since they multiply \( v_j(k) \), and, hence, vanish by the incompressibility condition \( k_j v_j = 0 \). Dropping them, and adding these two integrals \( I_{jmn}^{(1)}(k) \) and \( I_{jmn}^{(2)}(k) \) makes the entire correction to the equation of motion coming from graph II become:

\[
\delta \left( \partial_t v_i \right) = - D \lambda^2 P_{mn}(k) v_j(\tilde{k}) \left( I_{jmn}^{(1)}(k) + I_{jmn}^{(2)}(k) \right) = - \frac{D \lambda^2 S_d}{2 \mu^2 (2\pi)^d} \Lambda^{d-4} d \left[ P_{mn}(k) v_j(\tilde{k}) \right]
\]

\[
\times \left[ \left( 1 - \frac{2}{d} + \frac{1}{d(d+2)} \right) k_m \delta_{nj} + \left( \frac{1}{d(d+2)} - \frac{d+4}{2d(d+2)} \right) k_n \delta_{mj} \right].
\]

Simplifying, and performing the tensor index contractions, gives

\[
\delta \left( \partial_t v_i \right) = - \frac{D \lambda^2 S_d}{2 \mu^2 (2\pi)^d} \Lambda^{d-4} d \left[ P_{mj}(k) k_m \left( 1 - \frac{5}{2d} \right) \right. \left. + P_{jn}(k) k_n \frac{1}{2d} \right] v_j(\tilde{k})
\]

\[
= - \frac{D \lambda^2 S_d}{2 \mu^2 (2\pi)^d} \Lambda^{d-4} d \left[ P_{mj}(k) k_m \left( 1 - \frac{2}{d} \right) \right] v_j(\tilde{k}).
\]
where in the second step we have used the symmetry of $P_{ljn}(k)$ to write $P_{ljn}(k)k_n = P_{mlj}(k)k_m$. Now from the definition of $P_{lmj}(k)$, we have $P_{lmj}(k)k_m = P_{m}(k)k_j k_m + P_{lj}(k)k_m$. The first term in this expression vanishes by the fundamental property of the transverse projection operator, while the second is just $k^2 P_{lj}(k)$. Thus, we finally obtain

$$\delta (\partial_t v_l) = - \left( \left( \frac{d-2}{d} \right) \frac{D \Lambda^2}{2 \mu^2} \frac{S_d}{(2\pi)^d} \Lambda^{d-4} d \ell \right) k^2 P_{lj}(k)v_j(\tilde{k}),$$

(51)

which is exactly what one would get by adding to $\mu$ (hiding in the “propagator”) in Eq. (3) in the main text a correction $\delta \mu$ given by Eq. (4).

\section*{F. Vanishing diagrams}

Besides the five non-vanishing one-loop diagrams, there are also two sets of one-loop diagrams that cancel exactly, giving zero contribution to the corrections (Fig. 1).

\section*{III. ANGULAR AVERAGES}

In this section we derive the various angular averages used in the previous sections. We begin by deriving the identity

$$\frac{i q_m q_n}{q^2} I_q = \frac{1}{d} \delta_{mn},$$

(52)

FIG. 1: Vanishing sets of diagrams. \textit{Left}: The three diagrams consist of three three-point vertices. They cancel each other, leading to zero net contribution to $\lambda$. \textit{Right}: The three diagrams consist of one four-point vertex and two three-point vertices. They again cancel each other, leading to zero net contribution to $b$. 
where $\langle \rangle_a$ denotes the average over directions $\hat{q}$ of $q$ for fixed $|q|$. The identity (52) follows by symmetry: the average in question clearly must vanish when $m \neq n$, since then the quantity being averaged is odd in $q$. Furthermore, when $m = n$, the average must be independent of the value that $m$ and $n$ both equal. Hence, this average must be proportional to $\delta_{mn}$.

The constant of proportionality in (52) is easily determined by noting that the trace of this average over $mn$ is

$$\frac{\langle q_m q_n \rangle_a}{q^4} \frac{\langle q \rangle_a}{q} = \frac{\langle q \rangle_a}{q} = \frac{1}{q} \frac{\langle q \rangle_a}{q} = 1.$$  

This forces the prefactor of $\frac{1}{q}$ in (52).

From (52), it obviously follows that

$$\langle P_{mn} \rangle_a = (\delta_{mn} - \frac{q_m q_n}{q^2})_a = (1 - \frac{1}{d})\delta_{mn},$$  

(53)

which is the identity we used in equation (5).

We now consider the average of two projection operators $\langle P_{mi}(q)P_{jn}(q) \rangle_a$ that appears in (39). Using the definition of the projection operator, this can be written as follows:

$$\langle P_{mi}(q)P_{jn}(q) \rangle_a = \langle (\delta_{mi} - \frac{q_m q_i}{q^2})(\delta_{jn} - \frac{q_j q_n}{q^2}) \rangle_a = \delta_{mn}\delta_{jn} - \delta_{mi}\langle \frac{q_i q_n}{q^2} \rangle_a - \delta_{jn}\langle \frac{q_m q_n}{q^2} \rangle_a + \langle \frac{q_i q_j q_m q_n}{q^4} \rangle_a.$$  

(54)

The first two angular averages on the right hand side of this expression can be read off from (52). The last is new, and can be evaluated as follows:

First, note that by symmetry, this average vanishes unless the four indices $i, j, k, l$, are equal in pairs. Furthermore, if they are equal in pairs, but the pairs are different (e.g., if $i = j = x$ and $m = n = z$), then the average will have one value, independent of what the values of the two pairs of indices are (e.g., if $i = j = y$ and $m = n = x$, the average would be the same as in the example just cited. The only other non-zero possibility is that all four indices are equal, in which case the average is the same no matter which index all four are equal to (i.e., the average when $i = j = m = n = x$ is the same as that when $i = j = m = n = z$). Furthermore, this average must be completely symmetric under any interchange of its indices. This can be summarized by saying that the average must take the form:

$$\langle \frac{q_i q_j q_m q_n}{q^4} \rangle_a = A\Upsilon_{ijmn} + B(\delta_{mi}\delta_{jn} + \delta_{mj}\delta_{in} + \delta_{im}\delta_{jn}),$$  

(55)

where $\Upsilon_{ijmn} = 1$ if and only if $i = j = m = n$, and is zero otherwise, and $A$ and $B$ are unknown, dimension $(d)$ dependent constants that we'll now determine.

We can derive one condition on $A$ and $B$ by taking the trace of (55) over any two indices (say, $i$ and $j$). This gives

$$\langle \frac{q_m q_n}{q^2} \rangle_a = (A + (d + 2)B)\delta_{mn}.$$  

(56)

Comparing this with (52) gives

$$A + (d + 2)B = \frac{1}{d}.$$  

(57)

A second condition can be derived by explicitly evaluating the angle average when all four indices are equal. Since it doesn’t matter what value they all equal, we’ll choose it to be $z$. Defining $\theta$ to be the angle between the $z$-axis and $q$, we can obtain the needed average in $d$-dimensions by integrating in hyperspherical coordinates:

$$\langle \frac{q^4}{q^4} \rangle_a = \int_0^\pi d\theta \cos^4 \theta \sin^{d-2} \theta = \frac{3}{d(d + 2)}. $$  

(58)

Comparing this with (55) evaluated for $i = j = m = n$ gives

$$A + 3B = \frac{3}{d(d + 2)}. $$  

(59)

The simultaneous solution for $A$ and $B$ of equations (57) and (59) is $A = 0$ and $B = \frac{1}{4(d + 2)}$. Using these in (55) gives

$$\langle \frac{q_i q_j q_m q_n}{q^4} \rangle_a = \frac{1}{d(d + 2)}(\delta_{mi}\delta_{jn} + \delta_{mj}\delta_{in} + \delta_{im}\delta_{jn}).$$  

(60)
Using this and (52) in (54) gives (39).