Appendices: trapped ion scaling with pulsed fast gates

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We present three appendices to consider in detail the effect of symmetry on the motional conditions, the details of deriving our fidelity equations, and the adapted equations for treating ions with momentum kicks applied to only one internal state (‘asymmetric’ kicks).

1 Symmetry and the motional conditions

Equation (25) in the main text gives the general condition for restoring motional mode $p$:

$$C_p = -i\sum_{k=1}^{N} c_ke^{i\nu_pt_k} = 0,$$

where the final displacement of the motional mode is given by

$$C_p \equiv -2i\eta_p(b_1^{(p)}\sigma^z_1 + b_2^{(p)}\sigma^z_2)\sum_{k=1}^{N} z_ke^{i\nu_pt_k},$$

$$= is_p\sum_{k=1}^{N} z_ke^{i\nu_pt_k}, \text{ for}$$

$$s_p \equiv -2\eta_p(b_1^{(p)}\sigma^z_1 + b_2^{(p)}\sigma^z_2).$$

Expanding the exponential as a Taylor series in $C_p$,

$$C_p = is_p\sum_{k=1}^{N} z_k(1 + i\nu_pt_k + \frac{\nu_p^2t_k^2}{2} + \frac{\nu_p^3t_k^3}{6} + ...),$$

and for $C_p = 0$, we require that

$$\text{Re}(C_p) = s_p\sum_{k=1}^{N} z_k(-\nu_p t_k + \frac{\nu_p^2t_k^2}{2} + \frac{\nu_p^3t_k^3}{6} + ...) = 0,$$

$$0 = \sum_{k=1}^{N} z_k(t_k - \frac{\nu_p^2t_k^2}{2} - ...),$$

$$\text{Im}(C_p) = s_p\sum_{k=1}^{N} z_k(1 - \frac{\nu_p^2t_k^2}{2} + ...) = 0,$$

$$0 = \sum_{k=1}^{N} z_k(1 - \frac{\nu_p^2t_k^2}{2} + ...).$$

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For $1 \gg \nu^2 \tau_k^2$ for each $t_k$, the motional conditions reduce to

$$0 = \sum_{k=1}^{N} z_k \tau_k,$$

(10)

$$0 = \sum_{k=1}^{N} z_k,$$

(11)

with no mode dependence. This regime and these conditions correspond to free evolution, before the trap evolution has any significant effect. The required regime may be difficult to reach since the dominant term in our phase condition contains the third order of the pulse times, indicating that phase accrual has strong dependence on the trap evolution time. Strong momentum kicks are required to accommodate shorter gate times, as explored in the main text. Close to this ideal regime, error in the scheme ($C_p \neq 0$) will be mode dependent, and the dominant error term is from the sum in equation (8), proportional to $z_k \nu^2 \tau_k^2$ for each kick $k$. We can broaden the mode-independent regime by imposing constraints on the scheme pulse directions and timings. This provides increasing motional robustness.

A simple example is to impose a reflected antisymmetry, such that the momentum kicks $z$ and their timings $\tau$ obey

$$z = (a_1, \ a_2, \ ..., \ a_m, \ |\ -a_m, \ -a_{m-1}, \ ..., \ -a_1)$$

and

$$\tau = (-\tau_1, \ -\tau_2, \ ..., \ -\tau_m, \ |\ \tau_m, \ \tau_{m-1}, \ ..., \ \tau_1),$$

This sets $\text{Im}(C_p)$ to zero by virtue of its even powers of $t_k$, while we still require

$$\text{Re}(C_p) = 2s_p \sum_{k=1}^{m} a_k \sin(\nu_p \tau_k) = 0,$$

(12)

$$0 = \sum_{k=1}^{m} a_k \tau_k - a_k \frac{\nu^2 \tau_k^3}{6} + \ldots$$

(13)

The highest order term in our new motional condition equation corresponds to equation (10), while we have subsumed equation (11) into the symmetry condition. Assuming mode independence means that the dominant error term is proportional to $a_k \nu^3 \tau_k^3$, which is on the order of the cube of the pulse times. The asymmetry decreases the error size from a quadratic power of pulse times above, providing motional stability.

We can introduce a more complex symmetry to further expand our region of mode-independent total gate times:

$$z = (a_1, \ ..., \ a_l, \ |\ -a_l, \ ..., \ -a_1, \ |\ -a_1, \ ..., \ -a_l, \ |\ a_l, \ ..., \ a_1)$$

and

$$\tau = (-f - \tau_1, \ ..., \ -f - \tau_l, \ |\ -f + \tau_1, \ ..., \ -f + \tau_l, \ |\ f - \tau_1, \ ..., \ f - \tau_l, \ |\ f + \tau_1, \ ..., \ f + \tau_l),$$

for $f > \tau_1 > \tau_2 > \ldots > \tau_l$. Here $\text{Re}(C_p)$ is zero, and we require

$$\text{Im}(C_p) = 2s_p \sum_{k=1}^{l} a_k \cos(\nu_p (f + \tau_k)) - a_k \cos(\nu_p (f - \tau_k)) = 0,$$

(14)

$$0 = \sum_{k=1}^{l} a_k \left[ \nu^2_p 2 f \tau_k - \frac{\nu^4_p}{24} ((f + \tau_k)^4 - (f - \tau_k)^4) + \ldots \right],$$

(15)

where if the first term dominates the sum, we have a mode-independent condition equivalent to equation (10). The largest error term in the sum is on the order of $a_k \nu^4_p f^3 \tau_k$, on the fourth
order of the pulse times for \( f \sim \tau_1 \). Again, the added symmetry conditions increase the motional stability region for the scheme.

Adding ions reduces the mode-independent regime, since the mode frequency plays a part in the terms we require to be negligible, and these frequencies increase with ions added. Extra degrees of freedom found in the GZC and FRAG schemes relative to the Duan single degree of freedom also provide extra motional stability. Note also that while motional conditions can be satisfied independent of ion numbers for sufficiently fast gates, the phase equation still has mode dependence. However, only local phonon modes can be excited as discussed in the main text.

2 Fidelity measure derivation

The computational fidelity before state averaging is given by

\[
F_1 = \text{Tr}_m[\langle \psi_0 | U_{id}^\dagger U_{re} | \psi_0 \rangle \otimes \rho_m U_{re}^\dagger U_{id} | \psi_0 \rangle],
\]

where \( U_{re} \) and \( U_{id} \) denote the real and ideal gate operations, including motional displacement for the real gate as performed by our gate schemes to follow. The motional trace \( \text{Tr}_m \) is taken along with the inner product of the ideal and real operations with respect to the internal ion states. The initial internal state of the ions is \( |\psi_0\rangle \), and the initial motional state is given by the density operator \( \rho_m \). In the computational basis \( \{ |gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle \} \), the ideal unitary is given by

\[
U_I = \begin{pmatrix}
e^{i\pi/4} & 0 & 0 & 0 \\
0 & e^{-i\pi/4} & 0 & 0 \\
0 & 0 & e^{-i\pi/4} & 0 \\
0 & 0 & 0 & e^{i\pi/4}
\end{pmatrix},
\]

up to global phase.

The real operator, representing our approximation to the ideal unitary with some error, is given by

\[
U_{re} = \begin{pmatrix}
e^{i\phi_{gg}} \hat{D}_{gg} & 0 & 0 & 0 \\
0 & e^{i\phi_{ge}} \hat{D}_{ge} & 0 & 0 \\
0 & 0 & e^{i\phi_{eg}} \hat{D}_{eg} & 0 \\
0 & 0 & 0 & e^{i\phi_{ee}} \hat{D}_{ee}
\end{pmatrix},
\]

representing phase changes \( \phi_{ij} \) as well as motional displacements \( \hat{D}_{ij} \) specific to internal states, \( i, j \in \{ g, e \} \). Note that there are no off-diagonal terms as the initial internal states are preserved through the real operation due to the assumed perfect \( \pi \)-pulse pairs providing momentum kicks while perfectly restoring state population.

Recall that the final displacement is given in equation (1) for each mode \( p \):

\[
C_p \equiv -i \sum_{k=1}^{N} c_{pk} e^{i\nu_p \Delta k}.
\]

The total displacement operator is the tensor product of each mode displacement. The internal state dependence comes from \( c_{pk} \):

\[
c_{pk} = 2z_k (b_1^{(p)} \sigma_1^\dagger + b_2^{(p)} \sigma_2^\dagger) \eta_p,
\]

\[
C_p = -2i (b_1^{(p)} \sigma_1^z + b_2^{(p)} \sigma_2^z) \eta_p \sum_{k=1}^{N} z_k e^{i\nu_p \Delta k}.
\]
We have defined the motional operators
\begin{equation}
\hat{D}^{(p)}_{ij} = \langle i_1 j_2 | \hat{D}_p(C_p) | i_1 j_2 \rangle,
\end{equation}
for \( i \) and \( j \) internal states of ions one and two. Each mode displacement is related; for both ions in the same or opposite internal states these relations are simple:
\begin{align}
\hat{D}^{(p)}_{gg} &= (\hat{D}^{(p)}_{ee})^\dagger \\
\hat{D}^{(p)}_{ge} &= (\hat{D}^{(p)}_{eg})^\dagger
\end{align}
due to the opposite direction of each displacement kick. The other relations are by a scalar displacement amount, such that the motional displacement \( C_p \) for internal states \( |ee\rangle \) and \( |eg\rangle \) are related by
\begin{equation}
C_p(ee) = \frac{b_1^{(p)} + b_2^{(p)}}{b_1^{(p)} - b_2^{(p)}} C_p(eg).
\end{equation}
The tensor product composite, mode independent, displacement operators for each state are related in the same way:
\begin{align}
\hat{D}_{gg} &= \prod_{p=1}^L \hat{D}^{(p)}_{gg} = \hat{D}^\dagger_{ee} \\
\hat{D}_{ge} &= \prod_{p=1}^L \hat{D}^{(p)}_{ge} = \hat{D}^\dagger_{eg}.
\end{align}

We derive the phase acquired for each state from equation (24) in the main text, the mode-dependent phase term:
\begin{equation}
\xi_p = \sum_{m=1}^N \sum_{k=1}^{m-1} c_{pm} c_{pk} \sin(\nu_p(t_m - t_k)) - \Re[\alpha \sum_{k=1}^N c_{pk} e^{-i \nu_p t_k}],
\end{equation}
for an initial motional coherent state \( \alpha \). This second component of the phase is zero when the motion is restored at the end of the gate. As discussed in the main text, when the displacement is nonzero and this term is significant, we can correct for it using single-ion rotations as it is not entangling. The total phase term, from equation (30) in the main text neglecting global phase, reduces to:
\begin{equation}
\sum_{p=1}^L \xi_p = 8 \sum_{p=1}^L \eta_p^2 \sigma_1^2 \sigma_2^2 b_1^{(p)} b_2^{(p)} \sum_{m=1}^N \sum_{k=1}^{m-1} z_m z_k \sin(\nu_p(t_m - t_k)).
\end{equation}
The presence of both \( U_{re} \) and its Hermitian conjugate in the fidelity equation ensure that global phase does cancel. Internal states determine the value of the sum, and we have defined
\begin{equation}
\phi_{ij} = \langle i_1 j_2 | \sum_{p=1}^L \xi_p | i_1 j_2 \rangle.
\end{equation}
for \( i, j \in \{g, e\} \).

Total phase acquired by particular internal states is related according to
\begin{equation}
\phi_{gg} = -\phi_{ge} = -\phi_{eg} = \phi_{ee},
\end{equation}
as each mode-dependent term in the sum changes sign for different internal states of ions one and two.
For a general initial state
\[ |\psi_0\rangle = a_{00}|gg\rangle + a_{01}|ge\rangle + a_{10}|eg\rangle + a_{11}|ee\rangle, \]
and
\[ A = \langle \psi_0|U_{n}^\dagger U_{n}|\psi_0\rangle \]
\[ = |a_{00}|^2 e^{i(\phi_{gg}-\pi/4)}\hat{D}_{gg} + |a_{01}|^2 e^{i(\phi_{ge}+\pi/4)}\hat{D}_{ge} + |a_{10}|^2 e^{i(\phi_{eg}+\pi/4)}\hat{D}_{eg} + |a_{11}|^2 e^{i(\phi_{ee}-\pi/4)}\hat{D}_{ee}. \]

Using the cyclic nature of the trace, we have
\[ F_1 = \text{Tr}_m[A^\dagger \rho_m]. \]

The trace becomes an expectation of pairs of motional displacement operators with respect to \( \rho_m \). Since operators on different motional modes commute, we can group the mode-dependent components, for example
\[ \hat{D}_{gg}\hat{D}_{ge} = \Pi_{p=1}^{L} \hat{D}_{gg}^{(p)} \hat{D}_{ge}^{(p)}. \]

For a single mode, products of displacement operators are determined by
\[ \hat{D}(a)\hat{D}(b) = e^{(ab^\ast-a^\ast b)/2} \hat{D}(a+b) \]
\[ = \hat{D}(a+b) \text{ if } b = ca \text{ for a scalar } c. \]

We have seen that the arguments of our displacement operators, \( C_p \) from equation (22), are indeed related by scalars for the same mode and different internal states.

If the initial motional state \( \rho_m \) is separable with respect to the motional modes, then the expectation values can be taken individually for separate modes. We assume an initial thermal product state as a typical motional distribution,
\[ \rho_m^{(p)} = (1 - e^{-\hbar\nu_p/kT}) \sum_{n=0}^{\infty} |n\rangle \langle n| e^{-n\hbar\nu_p/kT} \]
\[ \rho_m = \rho_m^{(1)} \otimes \rho_m^{(2)} \otimes \cdots \otimes \rho_m^{(L)}, \]
represented in the number basis, with temperature \( T \) and \( k \) the Boltzmann constant.

The expectation value for a single mode with a displacement \( z \) follows:
\[ \langle \hat{D}(z) \rangle_{\rho_m} = e^{-|z|^2(1/2+\bar{n}_p)} \]
\[ = e^{-|z|^2(1/2+\bar{n}_p)}, \]

where \( \bar{n}_p \) is the average phonon population of the \( p \)th motional mode.

Choosing a particular internal state has limited use as a figure of merit for a quantum gate, which takes inputs of different initial forms in practice, the fidelity of each being important. We consider the average fidelity, taking an even weighting over all initial states. This corresponds to integrating over \( F_1 \) for the possible values of the coefficients; an integral over the three-dimensional surface (given four basis states) of a hypersphere of radius one. We now have the formalism in place to calculate the average fidelity, which depends on the number of ions as this determines the phase and displacement equations. For two ions,
\[ F_{\text{ave}} = \frac{1}{12} \left( 6 + e^{-4m_1|C_1|^2} + e^{-4m_2|C_2|^2} + 4e^{-(m_1|C_1|^2+m_2|C_2|^2)} \sin(2\phi_{gg}) \right), \]
where \( m_i \equiv (1/2+\bar{n}_i) \) is a function of the motional mode mean occupation level, and \( C_p \) is the final displacement for mode \( p \) defined in equation (21) using equal and opposite internal states for nonzero \( C_1 \) and \( C_2 \) respectively.
For more ions, there are more complicated internal state dependencies in the $C_{p}$ final displacements, so we define

$$c_{p} = 2n_{p} \sum_{k=1}^{N} z_{k} e^{i\nu_{p} t_{k}},$$

(44)

which is independent of the internal state for a simpler fidelity definition. For three ions, the fidelity becomes

$$F_{3\text{ave}} = \frac{1}{12} (6 + e^{−2m_{2}|c_{2}|^{2}−6m_{3}|c_{3}|^{2}} + e^{−5.3\bar{m}_{1}|c_{1}|^{2}−2m_{2}|c_{2}|^{2}−0.6\bar{m}_{3}|c_{3}|^{2}}$$

$$+ 2(e^{−1.3\bar{m}_{1}|c_{1}|^{2}−2.66m_{3}|c_{3}|^{2}} + e^{−1.3\bar{m}_{1}|c_{1}|^{2}−2m_{2}|c_{2}|^{2}−0.66m_{3}|c_{3}|^{2}}) \sin 2\phi_{gg}).$$

(45)

The average fidelity for no relative phase and very large motional displacement is 0.5, a point to keep in mind as we consider the fidelity values for the schemes in the main text.

### 3 Asymmetric momentum kick equations

If the laser providing the momentum kicks addresses a transition from one computational state to an auxiliary state, then the other computational state is untouched by the ‘kick’. Without loss of generality, we choose the ground state to undergo the momentum kick, while the excited state is left invariant. The unitary kick operator, equation (12) in the main text, becomes

$$U_{\text{kick}} = e^{−2i\alpha_{1} z_{1} x_{1} + x_{2} \alpha_{2}^{2}}),$$

(46)

where $\alpha_{1}|0\rangle_{i} = 0$ and $\alpha_{1}|1\rangle_{i} = |1\rangle_{i}$. Thus $(\alpha_{2}^{2})^{2} = \alpha_{2}^{2}$ and $\alpha_{2}^{2} = (I_{i} + \sigma_{z}^{2})/2$ for identity operator $I_{i}$.

The mode basis transformation and the displacement and rotation operator expansion take place as for symmetric kicks in section 4.1 of the main article, with the adapted motional displacements

$$c_{p} = −2i\eta_{p} (b_{1}^{(p)} \alpha_{1}^{2} + b_{2}^{(p)} \alpha_{2}^{2}) \sum_{k=1}^{N} z_{k} e^{i\nu_{p} t_{k}},$$

(47)

(48)

for each mode. The phase acquired for a given scheme becomes

$$\sum_{p=1}^{L} \eta_{p} \sum_{m=2}^{N} z_{m} z_{k} \sin(\nu_{p}(t_{m} − t_{k}))$$

(49)

$$= 4 \sum_{p=1}^{L} \sum_{m=2}^{N} z_{m} z_{k} \sin(\nu_{p}(t_{m} − t_{k})),$$

(50)

which includes non-entangling state-dependent phase. This causes deviation from our target gate, which can be corrected by single-qubit gates.

We thus have the asymmetric phase condition equation

$$\frac{\pi}{4} = 2 \sum_{p=1}^{L} \sum_{m=2}^{N} z_{m} z_{k} \sin(\nu_{p}(t_{m} − t_{k})),$$

(51)

where the right hand side is one quarter the size of the symmetric kick phase equation (32). A symmetric scheme that solves the condition equations will thus provide only $\pi/16$ phase in equation (51), or $\pi/8$ relative phase between symmetric (|00⟩, |11⟩) and asymmetric (|10⟩, |01⟩) states. The adapted displacements and phase can be directly substituted into the fidelity derivation.