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Renormalization group in Lifshitz-type theories

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Abstract: We study the one-loop renormalization and evolution of the couplings in scalar field theories of the Lifshitz type, i.e. with different scaling in space and time. These theories are unitary and renormalizable, thanks to higher spatial derivative terms that modify the particle propagator at high energies, but at the expense of explicitly breaking Lorentz symmetry. We study if and under what conditions the Lorentz symmetry can be considered as emergent at low energies by studying the RG evolution of the "speed of light" coupling c_ϕ^2 and, for more than one field, of $\delta c^2 \equiv c_{\phi_1}^2 - c_{\phi_2}^2$ in simple models. We find that in the UV both c_ϕ^2 and δc^2 generally flow logarithmically with the energy scale. A logarithmic running of c^2 persists also at low-energies, if $\delta c^2 \neq 0$ in the UV. As a result, Lorentz symmetry is not recovered at low energies with the accuracy needed to withstand basic experimental constraints, unless all the Lorentz breaking terms, including δc^2, are unnaturally fine-tuned to extremely small values in the UV. We expect that the considerations of this paper will apply to any generic theory of Lifshitz type, including a recently proposed quantum theory of gravity by Hořava.

Keywords: Models of Quantum Gravity, Renormalization Group

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1 Introduction

Recently, there has been an increasing interest in non-relativistic quantum field theories where Lorentz invariance is explicitly broken at high energies and hopefully recovered at low energies. In particular, in [1–3] (see also [4]), general gauge theories, including non-relativistic extensions of the Standard Model, were proposed and investigated, while in [5, 6] similar constructions were implemented in Yang-Mills theories in 4+1 space-time dimensions and membrane theory. The same type of construction was then extended to four-dimensional quantum gravity in [7],[1] where it was suggested that the resulting theory may provide a candidate for a renormalizable and unitary quantum theory of gravity which flows in the infrared (IR) to Einstein theory.

The ultraviolet (UV) behavior of all these theories is substantially ameliorated by the presence of higher derivative (in the spatial directions only) quadratic terms that improve the UV behavior of the particle propagator, without introducing ghost-like degrees of freedom that in Lorentz invariant higher derivative theories typically spoil the unitarity of the theory. The proposed theories are of Lifshitz type [10]. In the UV, they exhibit, at the classical level, an anisotropic scaling symmetry under which time and space scale differently:

[1] See also [8, 9] for a (partial) list of further works that investigate the proposal of [7].

$t = \lambda^z t'$, $\vec{x} = \lambda \vec{x}'$, where z is the critical exponent, equal to one in a Lorentz invariant theory. The renormalizability properties of these theories have been extensively studied in [1, 2] for scalar, fermion and gauge theories. The usual power-counting argument for the renormalizability of a theory does not strictly hold anymore, but it is essentially still valid, provided one substitutes the standard scaling dimensions of the operators by their "weighted scaling dimensions" [1], i.e. by the dimensions implied by the assignment $[x]_w = -1$ and $[t]_w = -z$. Lifshitz-type theories exhibit at least two qualitative different energy regimes, set by the scale (denoted by Λ) of the higher derivative operators. We will generally denote as UV regime the energy range $E \gg \Lambda$, where the theory is manifestly non-Lorentz invariant. This is the proper "Lifshitz" regime, where the effective scaling dimensions of the operators are the weighted ones. We denote as IR the range $E \ll \Lambda$, where the theory is expected to smoothly reach the "standard" regime, where the operators are classified by their standard scaling dimensions. Weighted relevant operators break the anisotropic scaling symmetry explicitly and, at low energies, the theory is expected to flow to the Lorentz invariant theory with $z = 1$. This is, however, a non-trivial (and obviously crucial) point, since there is no dynamical principle for which Lorentz symmetry should emerge in the IR. As a matter of fact, we find that Lorentz invariance is recovered in the IR only if an unnatural fine tuning of the parameters of the theory ensure that all sources of Lorentz violation in the theory are tiny and below the rather stringent, presently known experimental bounds. These issues, expected on general grounds [11], are explored by performing a one-loop calculation in concrete, simple, models, which also clarify the IR-UV structure of Lifshitz-type theories.

More specifically, the purpose of this paper is to study the Renormalization Group (RG) evolution at one-loop level of simple scalar field theories of Lifshitz type. In particular, we will calculate the one-loop beta-functions of the weighted marginal operators in the theory and solve the corresponding equations to study the evolution of the associated couplings g_i. Once this is performed, we will focus our attention on a particular weighted relevant operator, $c_\phi^2(\vec{\partial}\phi)^2$, and study the RG evolution of the "speed of light" parameter c_ϕ for ϕ (physically, in the low energy theory c_ϕ represents the maximal speed for ϕ particles). In order to keep the technical analysis as simple as possible, we will mostly consider scalar field theories in higher dimensions with $z = 2$, namely a ϕ^6 theory in $D = 4$ spatial dimensions, with quartic derivative couplings as well, and a ϕ^3 theory in $D = 10$ spatial dimensions. These two theories are among the simplest theories which are i) of Lifshitz type, ii) their β-functions are respectively positive and negative in the UV, iii) c_ϕ^2 has a non-trivial running already at one-loop level. These theories are obviously toy laboratories (in particular, the ϕ^3 theory is not even stable), yet they manifest, in a simple context, the main features that more complicated and "realistic" models of this sort should exhibit. In both theories, c_ϕ^2 typically shows a logarithmic running in the range $E \gg \Lambda$, where Λ is the high-energy scale where the theory is anisotropic:

$$c_\phi^2(E) = c_\phi^2(E_0)\left[1 + f \log\left(\frac{E}{E_0}\right)\right]^{n_\phi}, \tag{1.1}$$

here n_ϕ an $\mathcal{O}(1)$ particle-dependent constant and $E_0 \gg \Lambda$ a reference scale in the UV range. In eq. (1.1), we schematically denote by f the radiative coefficient governing the RG flow, which depends on the coupling constants of the weighted marginal operators. It is also important to investigate what happens in the presence of more than one field, and particularly if and under what conditions their "speed of light" parameters converge to the same value. To address this question, we have also studied the RG evolution of the difference $\delta c^2 = c^2_{\phi_1} - c^2_{\phi_2}$. Under the assumption that $\delta c^2 \ll 1$, one schematically finds, for $E \gg \Lambda$,

$$\delta c^2(E) = \delta c^2(E_0)\left[1 + f' \log\left(\frac{E}{E_0}\right)\right]^{n_\delta} + \delta g(E_0)\left\{\left[1 + f'' \log\left(\frac{E}{E_0}\right)\right]^{n_g} - 1\right\}, \qquad (1.2)$$

where δg are small perturbations around some fixed-point solutions of the RG evolution. Eqs. (1.1) and (1.2) summarize the quantum evolution of c^2 and δc^2 in the UV regime, as given by the marginal couplings.

After having studied the UV, we move on to analyze the IR regime $E \ll \Lambda$. We will see that Λ is the characteristic scale below which Lifshitz theories turn into "standard theories". More precisely, we will explicitly show that the RG evolution of all weighted marginal couplings is essentially frozen below Λ, in complete analogy to the decoupling of a massive particle in a standard quantum field theory. The key point is, of course, that in the IR the relevant propagator term is the usual one, quadratic in the momentum, while the higher derivative terms can be neglected (it is however a delicate point, given that the theory with the usual quadratic propagator is non-renormalizable). Taking into account only the effect of weighted marginal couplings, for $E \ll \Lambda$ we find

$$c^2_\phi(E) = c^2_\phi(\Lambda)\left[1 + \mathcal{O}\left(\frac{E^2}{\Lambda^2}\right)\right], \qquad \delta c^2(E) = \delta c^2(\Lambda)\left[1 + \mathcal{O}\left(\frac{E^2}{\Lambda^2}\right)\right], \qquad (1.3)$$

which shows that, for sufficiently high Λ, the IR effect of the weighted marginal couplings can be neglected. This is expected, since in the IR the usual classification of operators in terms of canonical rather than weighted dimensions holds. What is marginal in the UV becomes then irrelevant in the IR. We will explicitly show how the β-functions smoothly change their behavior going from the UV to the IR by computing them in a momentum subtraction renormalization scheme, where all the decoupling effects are manifest.

However, care has now to be paid for the weighted relevant operators which become standard marginal in the IR, since they can efficiently mediate the UV Lorentz violation to the low-energy theory. Indeed, we will show, by explicitly working out a toy example in 3+1 space-time dimensions, that a logarithmic running like eq. (1.1) (with E_0 replaced now by Λ) still holds in the IR, with f depending now on the (standard) relevant couplings and being proportional to any Lorentz symmetry breaking coefficient of the low-energy effective theory, remnant of the Lifshitz-like nature of the UV completion.

In general, then, Lorentz symmetry is *not* emergent in the IR in theories of Lifshitz type. Recovering Lorentz symmetry would require some dynamical principle keeping all sources of Lorentz violation sufficiently small. The experimental bounds on δc^2 for ordinary particles are of the order of $10^{-(21 \div 23)}$ [12], which give an idea of the order of magnitude of the fine-tuning that is needed. An indirect bound on δc^2 for any charged particle is implicitly given by eq. (1.1). In the case of photons, for instance, an experimental constraint

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on the energy dependence of c_γ^2 by the FERMI experiment [13] gives, taking $n_\gamma \sim 1$ in eq. (1.1), the following rough constraint on f:

$$|f| \lesssim 10^{-16} \,. \tag{1.4}$$

Modulo a loop-factor and coefficients of order one, the bound (1.4) can be seen as a bound on δc^2 for any charged particle. We expect that these fine-tuning problems will affect all generic quantum field theories of Lifshitz type, in particular, the non-relativistic standard model of [3] and the proposed quantum gravity theory of [7]. In the latter case, after coupling the theory to matter, the problems mentioned above will reappear for the Standard Model particles, where parameters like δc^2 have tight experimental constraints.

The plan of this paper is as follows. In section 2 we briefly review the main properties of the Lifshitz-like theories, using, for the sake of illustration, a scalar field theory in 3+1 dimensions. In section 3.1 we study the one-loop renormalization of a single scalar field ϕ^6 theory with derivative ϕ^4 interactions in $D = 4$ spatial dimensions; in section 3.2 this analysis is extended to the case of two coupled fields. In section 4.1 we study the one-loop renormalization of a single scalar field ϕ^3 in $D = 10$; the two-field case is dealt with in section 4.2. The analysis in sections 3 and 4 are performed using the minimal subtraction (MS) scheme, suitable for finding the β-functions in the UV regime ($E \gg \Lambda$). In section 5 we study the IR behavior ($E \ll \Lambda$) of Lifshitz-like theories by using the momentum subtraction scheme. After reviewing in section 5.1 the IR behavior of the β function in conventional ϕ^4- theory, in section 5.2 we show that the β-functions produced by weighted marginal couplings go to zero for $E \ll \Lambda$. In section 5.3 we discuss the contribution of the weighted relevant operators (marginal in the standard sense) on Lorentz symmetry breaking effects in the IR. We will see that the presence of a non-zero δc, inherited from the UV, induces a running in c^2 at low-energies. The effect is general and we expect that it should apply to any low-energy field theory (i.e. not only to theories of Lifshitz type) perturbed by Lorentz symmetry breaking terms. In section 6 we conclude.

2 Renormalizable Lifshitz-like scalar field theories

Unconventional scalar field theories of the Lifshitz type, with higher derivative interactions and higher derivative quadratic terms, have been extensively studied in [1]. Here we briefly review some aspects of the construction that will be useful in what follows and refer the reader to [1] for more details. As mentioned in the introduction, the key point of the whole construction is to break Lorentz invariance, so that one is allowed to introduce higher derivative terms in the spatial derivatives and quadratic in the fields, without necessarily introducing the dangerous higher time derivative terms that would lead to violations of unitarity. In doing so, the UV behavior of the propagator is improved and theories otherwise non-renormalizable become effectively renormalizable. A useful guiding principle to easily classify and identify the renormalizable theories in this enlarged set-up is achieved by

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demanding an invariance under "anisotropic"[2] scale transformations:

$$t = \lambda^z t', \qquad x^i = \lambda x^{i\prime}, \qquad \phi(x^i, t) = \lambda^{\frac{z-D}{2}} \phi'(x^{i\prime}, t'), \tag{2.1}$$

where $i = 1, \ldots, D$ parametrizes the spatial directions. The parameter z is known as "critical exponent" and, when it equals one, the transformations (2.1) reduce to the usual Lorentz invariant scale transformations. According to eq. (2.1), we can assign to the coordinates and to the fields a "weighted" scaling dimension as follows:

$$[t]_w = -z, \qquad [x^i]_w = -1, \qquad [\phi]_w = \frac{D-z}{2}. \tag{2.2}$$

It is straightforward to see that at the quadratic level, modulo total derivative terms, the Lagrangian for a single scalar field, invariant under (2.1), reads

$$\mathcal{L}_{\text{quad}} = \frac{1}{2}\dot{\phi}^2 - \frac{a^2}{2\Lambda^{2(z-1)}}(\partial_i^z \phi)^2, \tag{2.3}$$

with a^2 being a dimensionless coupling and Λ a high-energy scale parametrizing the strength of the higher derivative operator. Due to the improved UV behavior of the propagator resulting from (2.3) when $z > 1$, the usual power-counting argument for the renormalizability of a theory is no longer applicable. The required modification is obtained by substituting the scaling dimensions of the operators by their "weighted scaling dimensions" [1].[3] In other words, a theory is renormalizable if all the operators $\mathcal{O}_i$ appearing in the Lagrangian have weighted scaling dimensions $[\mathcal{O}_i]_w$ (not to be confused with the standard scaling dimensions $[\mathcal{O}_i]$) which are not greater than $z + D$. Thus, the second term in (2.3), although manifestly irrelevant in the standard sense, behaves (and should be considered) as a marginal operator in this theory.

It is useful to illustrate this construction with a specific simple example, namely a scalar field in $3+1$ space-time dimensions ($D = 3$) and $z = 2$. For simplicity, we also impose a $\mathbf{Z}_2$ discrete symmetry $\phi \to -\phi$. The most general renormalizable Lagrangian, invariant under the transformations (2.1), is given by

$$\mathcal{L}_r = \frac{1}{2}\dot{\phi}^2 - \frac{a^2}{2\Lambda^2}(\Delta\phi)^2 - \frac{h_2}{48\Lambda^4}(\partial_i\phi)^2\phi^4 - \frac{g_4}{10!\Lambda^6}\phi^{10}, \tag{2.4}$$

where $\Delta = \partial_i\partial_i$ is the Laplace operator in the spatial directions. All the operators appearing in (2.4) are weighted marginal. The renormalizability properties of the theory are not changed if the scaling symmetry (2.1) is softly broken by adding weighted relevant operators. They are given by

$$\mathcal{L}_{\text{sr}} = -\frac{m^2}{2}\phi^2 - \frac{c^2}{2}(\partial_i\phi)^2 + \sum_{n=1}^{3} \frac{g_n}{(2n+2)!\Lambda^{2(n-1)}}\phi^{2n+2} + \frac{h_1}{4\Lambda^2}(\partial_i\phi)^2\phi^2, \tag{2.5}$$

[2]The word "anisotropic" arises from condensed-matter physics. In all the instances we consider, we assume to be in a so-called "preferred frame" [12] where spatial SO(3) symmetry, translations and time-reversal symmetry will always be assumed.

[3]Sometimes in the literature the weighted scaling dimension is introduced as the standard scaling dimension in some non-standard natural units. Although there is nothing wrong in doing so, we prefer to distinguish between $[\mathcal{O}_i]_w$ from $[\mathcal{O}_i]$ and use the usual natural units.

so that the final Lagrangian is the sum of eqs. (2.4) and (2.5). In conventional scalar field theories in $3+1$ dimensions, the interactions appearing in (2.4) would be non-renormalizable. What renders the theory renormalizable — and the interactions in (2.4) weighted marginal — is the modification of the propagator, which in momentum space now reads

$$i\left(k_0^2 - c^2 k^2 - \frac{a^2}{\Lambda^2}k^4 - m^2\right)^{-1} \tag{2.6}$$

with $k^2 = k_i k_i$. The $1/k^4$ high-energy behavior of the propagator leads to an improvement of the ultraviolet behavior of the theory. As a result, if no coupling of higher dimension is added, the theory is power-counting renormalizable. In the UV the RG evolution of the weighted relevant parameters, such as c^2 or g_1 in eq. (2.5), will be governed by the RG evolution of a combination of the weighted marginal couplings h_2, g_4 and a^2. As we will see, in the IR the Lifshitz-type theory turns into a low-energy effective theory, where the weighted marginal couplings turn back into standard irrelevant ones and do not effectively run anymore. In this regime, if no Lorentz violating parameter appears in the Lagrangian terms with standard dimension ≤ 4, then we effectively recover Lorentz symmetry in the IR, which protects c^2 from any possible running. On the other hand, if some Lorentz violating parameter is left (like e.g. $\delta c = c_{\phi_1} - c_{\phi_2} \neq 0$ in a two-field model), it will still generically induce a running of c^2, governed now by standard marginal couplings.

For the 3+1 dimensional model defined by eqs. (2.4) and (2.5), there is no renormalization of the couplings at one-loop level, due to the fact that the vertices in (2.4) involve at least six fields. For this reason, in what follows we will consider higher dimensional scalar field theories, for which the weighted marginal vertices contain less fields and, as we will see, there is a non-trivial renormalization of the couplings already at one-loop level.

3 UV behavior: a model with $z = 2$, $D = 4$

3.1 One scalar particle

We look for a weighted renormalizable scalar field theory with a non-trivial renormalization of the $(\partial_i\phi)^2$ operator at one-loop level. A simple quantum field theory of this sort is obtained in $4+1$ space-time dimensions with anisotropic scaling $z = 2$. The most general renormalizable Lagrangian, up to total derivative terms and including all possible weighted marginal and relevant operators, is

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{a^2}{2\Lambda^2}(\Delta\phi)^2 - \frac{c^2}{2}(\partial_i\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!\Lambda}\phi^4 - \frac{g}{4\Lambda^3}(\phi\partial_i\phi)^2 - \frac{k}{6!\Lambda^4}\phi^6 \,. \tag{3.1}$$

All couplings appearing in $\mathcal{L}$, but the mass m and Λ, are dimensionless. In order to reduce the number of operators, we have imposed on $\mathcal{L}$ a discrete $\mathbf{Z}_2$ symmetry under which $\phi \to -\phi$. For simplicity, in the following we set $\Lambda = 1$.

Our first aim is to renormalize the theory at one-loop level and study the RG flows of the weighted marginal couplings g and k in the UV. The coupling a^2, although weighted marginal, is one-loop finite, since there is no way to extract four powers of external spatial momentum from the tadpole graph given by the quartic couplings appearing in $\mathcal{L}$. Similarly,

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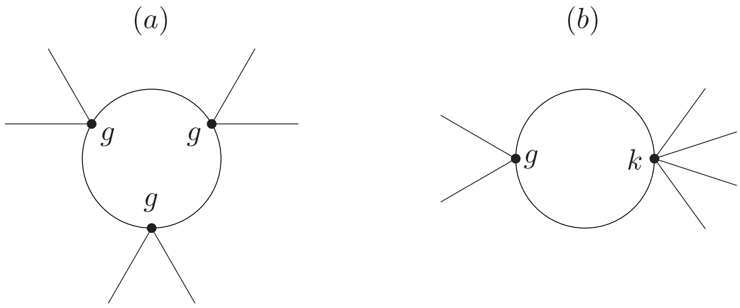

Figure 1. One-loop graphs contributing to the renormalization of the ϕ^6 vertex. All external momenta are vanishing.

the wave function renormalization of ϕ is trivial at one-loop level, $Z = 1+\mathcal{O}(\text{2-loops})$. Once the RG flows for g and k are solved, we will study the evolution of the weighted relevant coupling c^2.[4] We regularize the theory using a variant of dimensional regularization applied only to the spatial directions ($D = 4-\epsilon$) and thus renormalize using a minimal subtraction scheme where only the poles in $1/\epsilon$ are subtracted, with no finite term.

The superficial degree of divergence of a graph is easily computed by looking at the weighted scaling of a graph. The one-loop corrections to the coupling k come from a one-loop graph with 3 insertions of the coupling g and from another graph with one insertion of k and g. See figure 1. Due to some unusual properties of these theories, we report, in detail, the computation of the divergence term of the graph (a) in figure 1. A divergence can only arise when all the momenta of the vertex are taken in the internal lines, so that we can set to zero all external momenta p_i:

$$(a) = (-ig\mu^\epsilon)^3 \frac{15}{2} \int \frac{dq^0 d^D q}{(2\pi)^{D+1}} q^6 G(q^0, q^i)^3 \,, \tag{3.2}$$

where 15/2 is a geometrical factor taking into account all possible channels and

$$G(q^0, q^i) = \frac{i}{q_0^2 - a^2 q^4 - c^2 q^2 - m^2} \tag{3.3}$$

is the propagator for ϕ. Here and in the following $q^2 \equiv q_i q_i$. After Wick rotating q^0 ($q^0 = iq^5$), we can rewrite (a) as

$$\begin{aligned} (a) &= -\frac{15ig^3\mu^{3\epsilon}}{4} \frac{d^2}{(dm^2)^2} \int_0^\infty d\alpha \int \frac{dq_5 d^D q}{(2\pi)^{D+1}} q^6 e^{-\alpha(q_5^2 + a^2 q^4 + c^2 q^2 + m^2)} \\ &= -\frac{15ig^3}{4} I_{3,1} + \text{finite} \,, \end{aligned} \tag{3.4}$$

[4]Strictly speaking, the RG evolution of c^2, as determined in a physical scheme, starts at two-loop order, even in presence of the quartic derivative interaction, since the momentum-dependence of the one-loop graph (which is a tadpole) is trivial. However, we can still define a running c^2 coupling by adding a fictitious momentum in the loop, seen as the momentum carried by the composite operator $(\partial_i\phi)^2$. This is a standard trick. See e.g. [14] for the completely analogous case of the one-loop RG evolution of the mass parameter in the usual ϕ^4 theory.

where

$$I_{n,j} \equiv \int_0^\infty d\alpha\, \alpha^{j+1} \int \frac{dq_5 d^D q}{(2\pi)^{D+1}}\, q^{2n}\, e^{-\alpha(q_5^2+a^2q^4+m^2)}\,. \tag{3.5}$$

In writing the last equality of eq. (3.4), we have expanded the $\alpha c^2 q^2$ term in the exponential, since insertions of these terms lower the divergence of the graph. It is straightforward to check that the divergence arises only from the leading, c^2-independent term. The integral (3.5) is easily done by going to radial coordinates and changing variables $\alpha a^2 q^4 = r$. Performing the integrals we get

$$I_{n,j} = \frac{\Omega_D\sqrt{\pi}}{4(2\pi)^{D+1}}(m^2)^{\frac{D-6+2n-4j}{4}}(a^2)^{-\frac{D+2n}{4}}\Gamma\left(\frac{D+2n}{4}\right)\Gamma\left(\frac{6+4j-D-2n}{4}\right), \tag{3.6}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the area of the S^D sphere. Using the same techniques, we can compute the graph (b) as well. By denoting $-\delta_k\phi^6/6!$ the Lagrangian counterterm canceling the divergences coming from the graphs $(a)+(b)$, we then get

$$\delta_k = \left(-\frac{45g^3l_4}{32a^5}+\frac{15gkl_4}{8a^3}\right)\frac{1}{\epsilon}\,, \tag{3.7}$$

where we find convenient to express the result in terms of the usual loop factor for Lorentz invariant theories in D space-time dimensions, $l_D \equiv \Omega_D/(2\pi)^D$. Similar manipulations allow to compute δ_g, the coefficient of the counterterm $-\delta_g(\phi\partial_i\phi)^2/4$:

$$\delta_g = \frac{3g^2l_4}{8a^3}\frac{1}{\epsilon}\,. \tag{3.8}$$

From eqs. (3.7) and (3.8) we obtain the one-loop β functions for g and k:

$$\begin{aligned} \dot{k} &= \beta_k = \frac{15l_4}{8}\frac{gk}{a^3} - \frac{45l_4}{32}\frac{g^3}{a^5}\,, \\ \dot{g} &= \beta_g = \frac{3l_4}{8}\frac{g^2}{a^3}\,, \end{aligned} \tag{3.9}$$

where a dot stands for a derivative with respect to $t = \ln\mu/\mu_0$ and $\mu_0 \gg 1$ is a given UV reference scale. Note that the effective couplings of the theory are

$$\hat{g} \equiv \frac{g}{a^3}\,, \qquad \hat{k} = \frac{k}{a^4}\,. \tag{3.10}$$

The solutions of the RG equations (3.9) are

$$\begin{aligned} \hat{g}(t) &= \frac{\hat{g}_0}{1-\frac{3l_4\hat{g}_0}{8}t}\,, \\ \hat{k}(t) &= \hat{k}_0\left(\frac{\hat{g}(t)}{\hat{g}_0}\right)^5 + \frac{5\hat{g}_0^2}{4}\left(\frac{\hat{g}(t)}{\hat{g}_0}\right)^2\left[1-\left(\frac{\hat{g}(t)}{\hat{g}_0}\right)^3\right], \end{aligned} \tag{3.11}$$

with $\hat{g}_0 = \hat{g}(0)$, $\hat{k}_0 = \hat{k}(0)$. Since a Landau pole appears at the scale

$$E_{\rm pole} = \mu_0 e^{\frac{8}{3l_4\hat{g}_0}}\,, \tag{3.12}$$

the range in which eqs. (3.11) are reliable is $1 \ll E \ll E_{\rm pole}$.

Having found the RG evolution of the weighted marginal couplings k and g, we can now go on and study the evolution of the weighted relevant coupling c^2. Its β function reads

$$\frac{dc^2}{dt} = \beta_{c^2} = \frac{l_4 \hat{g}}{8} c^2 \,, \tag{3.13}$$

giving

$$c^2(t) = c_0^2 \left(\frac{\hat{g}(t)}{\hat{g}_0} \right)^{\frac{1}{3}} . \tag{3.14}$$

Equation (3.14) shows that in the UV regime c^2 has a logarithmic RG running, governed by the coupling g.

We expect that in any generic, weakly-coupled quantum field theory of Lifshitz type, including also theories with gauge fields and matter in $D = 3$, the running of c^2 will be qualitatively similar to (3.14), i.e. with a logarithmic dependence on the energy in the UV regime.

3.2 Two scalar particles

We will now show that theories with anisotropic scalings generically lead to different "speed of light" parameters (defined as coefficients of $(\vec{\partial}\phi_i)^2$) associated with different particles. More precisely, we will show that the RG evolution of the difference $\delta c^2 \equiv c_1^2 - c_2^2$, even in the most optimistic case when $\delta c^2 = 0$ is an attractive fixed point, is generally too slow to give $\delta c^2 \simeq 0$ with the needed accuracy. A severe fine-tuning in the UV for δc^2 seems to be inevitable.

In what follows we consider an extension of the single field model defined by the Lagrangian density (3.1) to two fields ϕ_1 and ϕ_2, imposing the $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry $\phi_i \to -\phi_i$, $i = 1, 2$. The Lagrangian is given by

$$\mathcal{L}_{2\phi} = \mathcal{L}_1 + \mathcal{L}_2 - g_{12}(\phi_1 \partial_i \phi_1)(\phi_2 \partial_i \phi_2) - \frac{h_1}{4}(\partial_i \phi_1)^2 \phi_2^2 - \frac{h_2}{4}(\partial_i \phi_2)^2 \phi_1^2 - V_{12}(\phi_1, \phi_2) \,, \tag{3.15}$$

where $\mathcal{L}_{1,2}$ are two copies of the Lagrangian appearing in (3.1) for the fields ϕ_1 and ϕ_2, and $V_{12}(\phi_1, \phi_2)$ is an additional potential:

$$V_{12}(\phi_1, \phi_2) = \frac{\lambda_{12}}{4} \phi_1^2 \phi_2^2 + \frac{k_{12}}{4!2} \phi_1^4 \phi_2^2 + \frac{k_{21}}{4!2} \phi_2^4 \phi_1^2 \,. \tag{3.16}$$

As can be seen, the Lagrangian contains a number of new interactions, which considerably complicate the analysis. In particular, the one-loop renormalization of the couplings k_1, k_2, k_{12} and k_{21} involves diagrams with all possible combinations of three insertions of the quartic couplings $g_{1,2}$, $h_{1,2}$ and g_{12} as well as diagrams with one insertion of any of the order six terms and one insertion of any of the quartic couplings. Fortunately, at one-loop level, as in the single field model considered before, the renormalization of $g_{1,2}$, $h_{1,2}$, g_{12}, c_1^2 and c_2^2 does not involve the k_1, k_2, k_{12}, k_{21} couplings and therefore we do not need to compute the associated Feynman diagrams. In analogy to eq. (3.11), there will always be a choice of boundary conditions for the couplings at $t = 0$ such that the model is stable all the way down to the far UV.

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Taking $a_1^2 = a_2^2 \equiv a^2$ for simplicity, after a straightforward but lengthy computation, we get

$$\begin{aligned}
\beta_{g_1} &= \frac{l_4}{8}\Big(3g_1^2 + 4g_{12}h_2 + h_1h_2 - 2h_2^2\Big)\,, \\
\beta_{g_2} &= \frac{l_4}{8}\Big(3g_2^2 + 4g_{12}h_1 + h_1h_2 - 2h_1^2\Big)\,, \\
\beta_{h_i} &= \frac{l_4}{8}\Big(g_{12}^2 + h_i(g_1+g_2) + h_i^2 + 2g_{12}h_i\Big)\,, \qquad i=1,2\,, \\
\beta_{g_{12}} &= \frac{l_4}{16}\Big[3g_{12}^2 + 2g_{12}(g_1+g_2) + 3g_{12}(h_1+h_2) - h_1h_2\Big]\,,
\end{aligned} \tag{3.17}$$

where all couplings have been rescaled by a factor $1/a^3$ while keeping the same notation for the couplings for simplicity (i.e. we omit hats). The β functions for the c_i^2 couplings are easily computed to be

$$\begin{aligned}
\beta_{c_1^2} &= \frac{l_4}{8}\Big(c_1^2 g_1 + c_2^2 h_1\Big)\,, \\
\beta_{c_2^2} &= \frac{l_4}{8}\Big(c_2^2 g_2 + c_1^2 h_2\Big)\,.
\end{aligned} \tag{3.18}$$

The RG equations (3.17) do not admit a simple analytic solution in general. A class of exact solutions is however obtained by substituting the ansatz

$$\begin{aligned}
&g_1(t) = g_2(t) = g(t) = \frac{g_0}{1 - x l_4 t}\,, \\
&h_1(t) = h_2(t) = h(t) = \frac{h_0}{1 - x l_4 t}\,, \\
&g_{12}(t) = \frac{g_{12,0}}{1 - x l_4 t}\,,
\end{aligned} \tag{3.19}$$

in eqs. (3.17) and solving the (now algebraic) equations for $g_0, h_0, g_{12,0}$ and x:

$$\begin{aligned}
8xg_0 - (3g_0^2 + 4g_{12,0}h_0 - h_0^2) &= 0\,, \\
8xh_0 - [g_{12,0}^2 + 2g_{12,0}h_0 + h_0(2g_0 + h_0)] &= 0\,, \\
16xg_{12,0} - [3g_{12,0}^2 + g_{12,0}(4g_0 + 6h_0) - h_0^2] &= 0\,.
\end{aligned} \tag{3.20}$$

In terms of g_0, h_0 and x, the running for $c^2 = (c_1^2 + c_2^2)/2$ and $\delta c^2 = (c_1^2 - c_2^2)/2$ is given by

$$c^2(t) = c_0^2\left(\frac{g(t)}{g_0}\right)^{\frac{g_0+h_0}{8x}}\,, \qquad \delta c^2(t) = \delta c_0^2\left(\frac{g(t)}{g_0}\right)^{\frac{g_0-h_0}{8x}}\,. \tag{3.21}$$

The system (3.20) is under-constrained (3 equations for 4 variables), so we fix one of the couplings, say $g_0 = 1$, and look for solutions for x and for the other couplings $h_0, g_{12,0}$. Taking $g_0 = r$ would simply rescale the solutions $x \to rx$, $h_0 \to rh_0$, $g_{12,0} \to rg_{12,0}$, so that the RG evolution of c^2 and δc^2 is unaffected. A sufficient condition to get a (semi)positive definite interaction requires $g_0, h_0 \geq 0$ and $|g_{12,0}| \leq (g_0 + h_0)/2$. We find seven solutions, one of which is unstable and is disregarded. The remaining six solutions are

$$\begin{aligned}
1)\ x &= \frac{3}{8}\,, & g_{12,0} &= 0\,, & h_0 &= 0\,;\\
2)\ x &= \frac{3}{4}\,, & g_{12,0} &= 1\,, & h_0 &= 1\,;\\
3)\ x &= \frac{5}{12}\,, & g_{12,0} &= \frac{1}{3}\,, & h_0 &= \frac{1}{3}\,;\\
4-5)\ x &= \frac{5}{16}(5\mp\sqrt{17}), & g_{12,0} &= \frac{1}{2}(3\mp\sqrt{17})\,, & h_0 &= \frac{1}{2}(13\mp 3\sqrt{17})\,;\\
6)\ x &= \frac{1}{256}(77+5\sqrt{17}), & g_{12,0} &= \frac{1}{16}(7-\sqrt{17})\,, & h_0 &= \frac{1}{8}(1+\sqrt{17})\,.
\end{aligned} \tag{3.22}$$

As can be seen from x, all six solutions correspond to couplings (and c^2) which grow in the UV in all cases. The deviation δc^2, instead, increases in the UV for 1), 3), 4) and 6), is constant in the case 2) and decreases in the UV in 5). Note that solution 1) reproduces the RG evolution (3.11) for a single field.

We can also perturb the solutions found above and study the RG evolution of the fluctuations at the linear level. We focus on the case 1), since it is the only one giving rise to simple analytical results. We look at linear perturbations around the solutions putting

$$\begin{aligned}
g_1 &= g+\delta g+\delta u\,, & g_2 &= g-\delta g+\delta u\,,\\
h_1 &= h+\delta h+\delta v\,, & h_2 &= h-\delta h+\delta v\,, & g_{12} &\to g_{12}+\delta g_{12}\,,
\end{aligned} \tag{3.23}$$

and consider δc^2 as a fluctuation. In this way, we get

$$\begin{aligned}
&\frac{\delta g(t)}{\delta g_0} = \frac{\delta u(t)}{\delta u_0} = \left(\frac{g(t)}{g_0}\right)^2,\\
&\frac{\delta h(t)}{\delta h_0} = \frac{\delta v(t)}{\delta v_0} = \frac{\delta g_{12}(t)}{\delta g_{12,0}} = \left(\frac{g(t)}{g_0}\right)^{\frac{2}{3}}.\\
&\delta c^2(t) = \delta c_0^2\left(\frac{g(t)}{g_0}\right)^{\frac{1}{3}} + \frac{c_0^2 l_4 t}{8}\left(\frac{g(t)}{g_0}\right)^{\frac{4}{3}}\delta g_0 + c_0^2\left[\left(\frac{g(t)}{g_0}\right)^{\frac{1}{3}}-1\right]\delta h_0\,.
\end{aligned} \tag{3.24}$$

From eqs. (3.24) we see that the fixed-point is stable, with all fluctuations decreasing towards the IR. A similar study can be done for the other solutions, which are not all IR stable. From eq. (3.24) we see that it is not enough to start with $\delta c_0^2 = 0$ at $\mu_0 \gg 1$ to ensure that $\delta c^2 = 0$ near $E \sim \Lambda$. In addition, one has to ensure that also other perturbations around some fixed point are fine-tuned to zero.

In section 5 we will argue that in the IR the RG evolution of δc^2, as given by the contributions of only weighted marginal couplings, essentially stops. Given the experimental bounds on δc^2 for ordinary particles mentioned before, recovering Lorentz invariance in the IR with enough accuracy requires fine-tuning of δc^2, δg and δh in the UV to extremely small values. We illustrate this point in figures 2(a) and 2(b), where the running of $\delta c^2(t)$ over 40 orders of magnitude for an arbitrary given choice of boundary conditions is shown.

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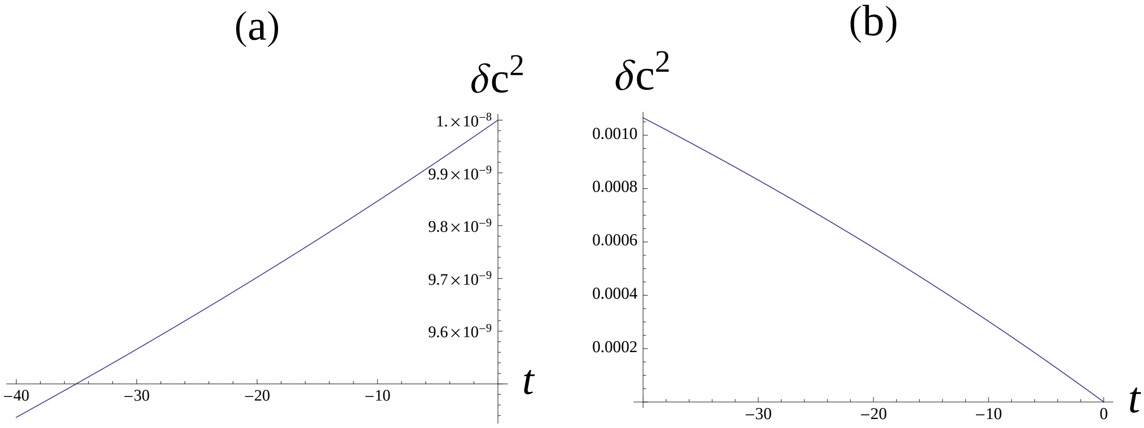

Figure 2. (a) RG evolution of δc^2 as given by eq. (3.24) for $c_0^2 = 1$, $\delta g_0 = \delta h_0 = 0$, $\delta c_0^2 = 10^{-8}$. (b) RG evolution of δc^2 as given by eq. (3.24) for $c_0^2 = 1$, $\delta g_0 = \delta h_0 = -10^{-2}$, $\delta c_0^2 = 0$.

3.3 Case of particle with no self-interactions

The two-scalar model can be adapted to the case where one of the particles, say ϕ_1, has no self-interaction terms, i.e. $g_1 = 0$. This case represents a situation of physical interest: in electrodynamics the photon has no self-interaction term and one may wonder if the speed of light will still significantly depend on the energy. Here we will show that the energy dependence of either speed-of-light parameters c_1 or c_2 does not rely on the presence or absence of self-interaction terms.

Consider the beta functions (3.17). An exact solution can be found by setting

$$g_1 = h_2 = g_{12} = 0 \ . \tag{3.25}$$

We are left with two equations

$$\dot{h}_1 = \frac{l_4}{8} h_1 p \,, \qquad \dot{p} = \frac{l_4}{8}(3p^2 - 5ph_1 + h_1^2) \,, \qquad p \equiv g_2 + h_1 \,. \tag{3.26}$$

Next, we write $p = fh_1$, so that the second equation takes the form

$$\dot{f} = \frac{l_4}{4} h_1 (f - f_+)(f - f_-) \,, \qquad f_\pm = \frac{1}{4}(5 \pm \sqrt{17}) \ . \tag{3.27}$$

One obvious solution is $\dot{f} = 0$ which requires $f = f_+$ or $f = f_-$. The solution with $f = f_-$ is not physical, since $f = 1 + g_2/h_1 > 1$. Setting $f = f_+$ leads to a solution similar to the class of solutions considered in (3.19),

$$g_2(t) = \frac{g_0}{1 - x l_4 t} \,, \qquad h_1(t) = \frac{h_0}{1 - x l_4 t} \,, \qquad x = \frac{f_+ h_0}{8} \ . \tag{3.28}$$

Now consider solutions where f is not constant. Writing

$$\dot{f} = \frac{df}{dh_1}\dot{h}_1 = \frac{df}{dh_1}\frac{l_4}{8} h_1^2 f \,, \tag{3.29}$$

eq. (3.27) becomes

$$\frac{f}{(f-f_+)(f-f_-)}\,\frac{df}{dh_1}=\frac{2}{h_1}\,. \tag{3.30}$$

We can now integrate this equation and obtain

$$h_1=k_0\,\frac{(f-f_+)^{\frac{f_+}{2(f_+-f_-)}}}{(f-f_-)^{\frac{f_-}{2(f_+-f_-)}}}\,, \tag{3.31}$$

where k_0 is an integration constant. Inserting eq. (3.31) into the first equation in (3.26) gives a first order differential equation for f that can be integrated. The result is

$$\frac{l_4k_0}{4}\,t=-\int_f^\infty \frac{df'}{(f'-f_+)^{1+\frac{f_+}{2(f_+-f_-)}}(f'-f_-)^{1-\frac{f_-}{2(f_+-f_-)}}}\,. \tag{3.32}$$

We have chosen μ_0 as the ultra-high energy scale at which $f\to\infty$, so that f is defined in the interval $-\infty<t<0$. The integral (3.32) defines $f=f(t)$ and hence $h_1(t)=h_1[f(t)]$ and $g_2(t)=(f(t)-1)h_1(t)$. It can be expressed in terms of hypergeometric functions, but its explicit expression is not needed in order to see the qualitative behavior of the couplings in the regime $t\ll 0$. In this limit, $f\to f_+$ and hence one approaches the constant f solution (3.28) discussed above. For large $(-t)$, one has the behavior,

$$(f-f_+)^{\frac{f_+}{2(f_+-f_-)}}\sim h_1\sim g_2\sim\frac{1}{|t|}\,. \tag{3.33}$$

Hence both h_1 and g_2 decrease towards the IR.

Next, we consider the RG equations (3.18) for c_1 and c_2, which simplify to

$$\frac{dc_1^2}{dt}=c_2^2h_1\,,\qquad \frac{dc_2^2}{dt}=c_2^2g_2\,. \tag{3.34}$$

Integrating these equations using the behavior of g_2 and h_1 given in eq. (3.33), one can see that c_2 goes to zero like $1/|t|^k$, with $k>0$, while c_1^2 approaches a constant value for $t\ll 0$.

4 UV behavior: a UV free model with $z=2$, $D=10$

In this section we look for a weighted renormalizable scalar field theory which is UV free. In order to simplify our analysis, we consider a ϕ^3 model which is weighted-counting renormalizable in $D=10$ spatial dimensions. Although the ϕ^3 model is manifestly an unphysical theory, having an unbounded potential, this instability does not affect the RG equations, which can then be formally studied and hopefully seen as a toy laboratory for more complicated stable UV free theories, such as Yang-Mills (YM) theories. As in section 3, we first study the one-loop RG evolution of a single field, which will enable us to compute the RG evolution of c^2. Then we shall consider a two-field model, where δc^2 will also be considered.

4.1 One scalar

The most general renormalizable Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{a^2}{2}(\Delta\phi)^2 - \frac{c^2}{2}(\partial_i\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3 \,. \tag{4.1}$$

As in section 2, we use a dimensional regularization in the spatial directions only. We will add counterterms to subtract the poles in $1/\epsilon$ only, with no finite term. We study the theory in the energy range $E \gg 1$, where the analysis is reliable.

The wave-function renormalization in the ϕ^3 theory is non-trivial and hence the relevant one-loop graphs to compute are those associated with the two and three point functions. These graphs are computed exactly along the same lines followed in section 3, so we will just report the results for the β functions and the anomalous dimension γ of ϕ. We find $\gamma = l_{10}\lambda^2/(64a^5)$ and

$$\beta_{a^2} = 2\omega_a \frac{\lambda^2}{a^5}a^2 \,, \qquad \beta_\lambda = -\omega_\lambda \frac{\lambda^2}{a^5}\lambda \,, \tag{4.2}$$

where $\omega_\lambda = 9l_{10}/64$, $\omega_a = 21l_{10}/640$. The effective coupling of the theory is

$$x = \frac{\lambda^2}{a^5} \,. \tag{4.3}$$

The RG equations are easily solved in terms of x. One has

$$\dot{x} = -\omega_x x^2 \,, \tag{4.4}$$

with $\omega_x = 2\omega_\lambda + 5\omega_a$, and thus

$$x(t) = \frac{x_0}{1 + x_0\omega_x t} \,. \tag{4.5}$$

Plugging eq. (4.5) into eqs. (4.2) give

$$\lambda(t) = \lambda_0 \left(\frac{x(t)}{x_0}\right)^{\frac{\omega_\lambda}{\omega_x}} \,, \quad a^2(t) = a_0^2 \left(\frac{x(t)}{x_0}\right)^{-\frac{2\omega_a}{\omega_x}} \,. \tag{4.6}$$

The coupling a^2 increases in the UV, while λ and the effective coupling x are UV free.

Let us now study the evolution of c^2. Its associated β is

$$\beta_{c^2} = \omega_c x c^2 \,, \tag{4.7}$$

where $\omega_c = 3l_{10}/16$. Hence

$$c^2(t) = c_0^2 \left(\frac{x(t)}{x_0}\right)^{-\frac{\omega_c}{\omega_x}} \,. \tag{4.8}$$

The RG evolution of c^2 is again governed by the weighted marginal couplings of the theory. Interestingly enough, c^2 logarithmically increases in the UV, as in the model of section 3.

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4.2 Two scalars

We can add a further scalar η to the ϕ^3 model and, for simplicity, we impose a discrete $\mathbf{Z}_2$ symmetry under which $\eta \to -\eta$. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{a^2}{2}(\Delta\phi)^2 - \frac{c^2}{2}(\partial_i\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3 + \frac{1}{2}\dot{\eta}^2 - \frac{\tilde{a}^2}{2}(\Delta\eta)^2 - \frac{\tilde{c}^2}{2}(\partial_i\eta)^2 - \frac{\tilde{m}^2}{2}\eta^2 - \frac{\tilde{\lambda}}{2}\eta^2\phi\,. \tag{4.9}$$

The computation of the β functions for λ, $\tilde{\lambda}$, a^2, $\tilde{a}^2$, c^2 and $\tilde{c}^2$ is straightforward, although a bit laborious, mainly because we keep $a^2 \neq \tilde{a}^2$ in general. We find

$$\begin{aligned}
\beta_\lambda &= \left[-\tilde{\lambda}^3 F(\tilde{a},\tilde{a}) + \lambda^3\left(-F(a,a) + \frac{3}{2}K(a,a)\right) + \frac{3}{2}\lambda\tilde{\lambda}^2 K(\tilde{a},\tilde{a})\right]\frac{l_{10}}{4}\,,\\
\beta_{\tilde{\lambda}} &= \left[-\lambda\tilde{\lambda}^2 F(a,\tilde{a}) + \frac{1}{2}\lambda^2\tilde{\lambda}K(a,a) + \tilde{\lambda}^3\left(-F(\tilde{a},a) + 2K(a,\tilde{a}) + \frac{1}{2}K(\tilde{a},\tilde{a})\right)\right]\frac{l_{10}}{4}\,,\\
\beta_{a^2} &= \left[\lambda^2\left(a^2K(a,a) + Q(a,a)\right) + \tilde{\lambda}^2\left(a^2K(\tilde{a},\tilde{a}) + Q(\tilde{a},\tilde{a})\right)\right]\frac{l_{10}}{4}\,,\\
\beta_{\tilde{a}^2} &= \tilde{\lambda}^2\left(\tilde{a}^2K(a,\tilde{a}) + Q(a,\tilde{a})\right)\frac{l_{10}}{2}\,,\\
\beta_{c^2} &= \left[\lambda^2\left(c^2K(a,a) + R(a,a,c,c)\right) + \tilde{\lambda}^2\left(c^2K(\tilde{a},\tilde{a}) + R(\tilde{a},\tilde{a},\tilde{c},\tilde{c})\right)\right]\frac{l_{10}}{4}\,,\\
\beta_{\tilde{c}^2} &= \tilde{\lambda}^2\left(\tilde{c}^2K(a,\tilde{a}) + R(a,\tilde{a},c,\tilde{c})\right)\frac{l_{10}}{2}\,,
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
F(a_1,a_2) &= \frac{2a_1 + a_2}{a_1^3 a_2(a_1+a_2)^2}\,,\\
K(a_1,a_2) &= \frac{1}{a_1 a_2(a_1+a_2)^3}\,,\\
Q(a_1,a_2) &= \frac{(a_1^4 + 5a_1^3a_2 + 10a_1^2a_2^2 + 5a_1a_2^3 + a_2^4)}{5a_1a_2(a_1+a_2)^5}\,,\\
R(a_1,a_2,c_1,c_2) &= \frac{c_1^2a_2^2(3a_1^3 + 12a_1^2a_2 + 8a_1a_2^2 + 2a_2^3) + (1\leftrightarrow 2)}{5a_1^3a_2^3(a_1+a_2)^4}\,.
\end{aligned} \tag{4.11}$$

When $\tilde{\lambda} = 0$, eqs. (4.10) collapse to eqs. (4.2) and (4.7), as expected. Finding the general, analytic solutions to eqs. (4.10) is complicated. Interestingly, eqs. (4.10) admit, nevertheless, a simple fixed-point solution given by $\lambda = \tilde{\lambda} \equiv \bar{\lambda}(t)$, $a^2 = \tilde{a}^2 \equiv \bar{a}^2(t)$, $c^2 = \tilde{c}^2 \equiv \bar{c}^2(t)$. The RG equations for $\bar{\lambda}$, $\bar{a}^2$ and $\bar{c}^2$ are precisely given by eqs. (4.2) and (4.7), provided one makes the substitution $\omega_\lambda \to 2\omega_\lambda$, $\omega_a \to 2\omega_a$, and $\omega_c \to 2\omega_c$. For completeness, below we report the corresponding solutions:

$$\bar{\lambda}(t) = \bar{\lambda}_0\left(\frac{\bar{x}(t)}{\bar{x}_0}\right)^{\frac{\omega_\lambda}{\omega_x}}, \quad \bar{a}^2(t) = \bar{a}_0^2\left(\frac{\bar{x}(t)}{\bar{x}_0}\right)^{-\frac{2\omega_a}{\omega_x}}, \quad \bar{c}^2(t) = \bar{c}_0^2\left(\frac{\bar{x}(t)}{\bar{x}_0}\right)^{-\frac{\omega_c}{\omega_x}}. \tag{4.12}$$

where

$$\bar{x}(t) \equiv \frac{\bar{\lambda}^2(t)}{\bar{a}^5(t)} = \frac{\bar{x}_0}{1 + 2\bar{x}_0\omega_x t}\,. \tag{4.13}$$

The fixed-point solution (4.12) is unstable under small deformations. In order to show that, we consider the following "symmetric" perturbations

$$\begin{aligned} \lambda &= \bar\lambda + \delta\lambda\,, & \tilde\lambda &= \bar\lambda - \delta\lambda\,, \\ a^2 &= \bar a^2 + \delta a^2\,, & \tilde a^2 &= \bar a^2 - \delta a^2\,, \\ c^2 &= \bar c^2 + \delta c^2\,, & \tilde c^2 &= \bar c^2 - \delta c^2\,, \end{aligned} \tag{4.14}$$

keeping up to linear terms in the perturbations. We get

$$\begin{aligned} \beta_{\delta\lambda} &= -2\omega_{\delta\lambda}\frac{\bar\lambda^2}{\bar a^5}\delta\lambda\,, \\ \beta_{\delta a^2} &= \frac{l_{10}}{160}\frac{\bar\lambda}{\bar a^5}(21\bar a^2\,\delta\lambda + 10\delta a^2\,\bar\lambda)\,, \\ \beta_{\delta c^2} &= \frac{l_{10}}{16}\frac{\bar\lambda}{\bar a^5}(6\bar c^2\,\delta\lambda + \delta c^2\,\bar\lambda)\,, \end{aligned} \tag{4.15}$$

where $\omega_{\delta\lambda} = 7l_{10}/64$. The UV evolution of the couplings is therefore given by

$$\begin{aligned} \delta\lambda(t) &= \delta\lambda_0\left(\frac{\bar x(t)}{\bar x_0}\right)^{\frac{\omega_{\delta\lambda}}{\omega_x}}\,, \\ \delta a^2(t) &= \left(\delta a_0^2 - \frac{a_0^2\delta\lambda_0}{\bar\lambda_0}\right)\left(\frac{\bar x(t)}{\bar x_0}\right)^{-\frac{4}{57}} + \frac{a_0^2\delta\lambda_0}{\bar\lambda_0}\left(\frac{\bar x(t)}{\bar x_0}\right)^{-\frac{62}{285}}\,, \\ \delta c^2(t) &= \left(\delta c_0^2 - \frac{c_0^2\delta\lambda_0}{\bar\lambda_0}\right)\left(\frac{\bar x(t)}{\bar x_0}\right)^{-\frac{4}{57}} + \frac{c_0^2\delta\lambda_0}{\bar\lambda_0}\left(\frac{\bar x(t)}{\bar x_0}\right)^{-\frac{28}{57}}\,. \end{aligned} \tag{4.16}$$

The fixed-point is stable in the UV only along $\delta\lambda$. The situation is similar to the one encountered in the model of section 3. The RG evolution of δc^2 in the UV does not help in alleviating the fine-tuning needed to get δc^2 small enough for energies near Λ.

We expect that the general lessons for the behavior of the different couplings under the Renormalization Group learned in this section should be qualitatively shared by a large class of UV free quantum field theories, including the relevant case of perturbative YM theories coupled to matter.

5 IR behavior of Lifshitz-type theories

As well known, in conventional Lorentz invariant theories, the MS scheme has to be used with care in studying the evolution of the couplings in presence of mass terms when $E \ll m$, since the decoupling of massive particles is not manifest. Indeed, the MS β-functions, being mass-independent, are formally the same for any E, while for $E \ll m$ the physical coupling does not effectively run anymore. In conventional theories, a reasonable approximation to overcome this problem and still use the MS scheme is to take an effective approach, where for $E \ll m$ the heavy particle is integrated out and its contribution to the running neglected. One could also make a more refined study of the transient region around $E \sim m$ (we will not do it in our similar case below). Since we are investigating here unconventional theories of the Lifshitz type, it is instead preferable to use a more physical renormalization scheme, such as the momentum subtraction scheme, where the β-functions are mass dependent and the decoupling is manifest, even if the associated one-loop computations become necessarily more involved. In the present case, because the propagator is given by

$$i\left(k_0^2 - c^2 k^2 - \frac{a^2}{\Lambda^2} k^4 - m^2\right)^{-1}, \tag{5.1}$$

IR effects are expected as soon as $c^2\Lambda^2 k^2 \sim a^2 k^4$, i.e. at momentum scales of order $c\Lambda/a$. The scale $c\Lambda/a$ plays now the role of m. As explained below, the underlying reason is that the term $c^2 k^2$ in the propagator itself provides an IR regularization of the amplitudes even when $m = 0$.

5.1 β-function in the IR in conventional ϕ^4 theory

Let us begin by briefly reviewing the standard computation of the one-loop β-function of the Lorentz-invariant ϕ^4 theory in $3+1$ dimensions in the momentum subtraction scheme. Let $\Gamma^{(4)}$ be the tree+one-loop+counterterms 1PI four-point function. The physical coupling λ can be defined as the value of $\Gamma^{(4)}$ at some given energy scale. In terms of the Mandelstam variables s, t, u, we can define

$$\Gamma^{(4)}(s = t = u = -\mu^2) \equiv \Gamma^{(4)}(\mu) = -\lambda \,. \tag{5.2}$$

Here we used the standard trick of introducing the renormalization group scale at some euclidean value of the kinematical invariants to circumvent threshold singularities. Eq. (5.2) fixes the value of the counterterms, so that

$$\Gamma^{(4)}(s,t,u) = -\lambda + \Gamma_l^{(4)}(s,t,u) - \Gamma_l^{(4)}(\mu)\,, \tag{5.3}$$

where $\Gamma_l^{(4)}$ is purely the 1-loop contribution to $\Gamma^{(4)}$. Since β_{m^2} and the anomalous dimension of ϕ vanish at one-loop order, being $\Gamma_l^{(2)}$ momentum-independent, inserting eq. (5.3) in the Callan-Symanzik (CS) equation satisfied by $\Gamma^{(4)}$ gives directly $\beta_\lambda = \mu d\lambda/d\mu$:

$$\beta_\lambda = -\mu \frac{\partial \Gamma_l^{(4)}}{\partial \mu}\,. \tag{5.4}$$

Using standard techniques, we get

$$\beta_\lambda = 6\lambda^2 \int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{\mu^2 x(1-x)}{[k_E^2 + m^2 + \mu^2 x(1-x)]^3} = \frac{3\lambda^2}{16\pi^2} \int_0^1 dx \frac{\mu^2 x(1-x)}{m^2 + \mu^2 x(1-x)}\,. \tag{5.5}$$

The integral in x in eq. (5.5) can analytically be performed, but it is not necessary to do so in order to see the UV and IR behavior of β_λ. For $\mu^2 \gg m^2$, $\beta_\lambda \simeq 3\lambda^2/(16\pi^2)$, in agreement with what one would have obtained with, say, a computation in the MS scheme. For $\mu^2 \ll m^2$, $\beta_\lambda \simeq \lambda^2/(32\pi^2)(\mu^2/m^2)$. The behavior of λ below m, its freezing, can essentially be understood by noticing that m acts as an IR regulator to the one-loop graph, which would otherwise have an IR divergence when $\mu \to 0$.

5.2 Computation of β-functions of Lifshitz-type theories in the IR

As we have already mentioned at the beginning of section 5, even in the absence of mass terms, the RG evolution of coupling constants presents at least two regimes, characterized by $\mu \gg c/a$ (UV) and $\mu \ll c/a$ (IR).[5] Here we consider the ϕ^3 model of section 4 in detail as an illustrative case. As it will shortly be clear, the important qualitative aspects of the results should be completely general and apply to any Lifshitz-like theory, including the model in section 3. We define the renormalized field ϕ and the couplings λ, a^2 and c^2 as follows:

$$\frac{\partial \Gamma^{(2)}}{\partial (p^0)^2}\bigg|_0 = 1\,; \quad \frac{1}{4!}\frac{\partial^4 \Gamma^{(2)}}{\partial p^4}\bigg|_0 = -a^2\,; \quad \frac{1}{2}\frac{\partial^2 \Gamma^{(2)}}{\partial p^2}\bigg|_0 = -c^2\,,$$
$$\Gamma^{(3)}[(p_1^0)^2 = -\omega(\mu^2), p_{2,3}^0 = \vec{p}_{1,2,3} = 0] \equiv \Gamma^{(3)}(\mu) = -\lambda, \tag{5.6}$$

where $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are the tree+one-loop+counterterms 1PI two- and three-point functions and 0 stands for the subtraction point $(p^0)^2 = -\omega(\mu^2) = -a^2\mu^4 - c^2\mu^2$, $p = |\vec{p}| = 0$. We define the subtraction point at vanishing spatial momenta for technical reasons, since the computation is greatly simplified in this way. Due to the Lifshitz nature of the theory, the energy p^0 should depend quadratically (for $z = 2$) on the sliding scale μ, so in the UV we get $\omega(\mu^2) \simeq a^2\mu^4$. The factor a^2 here has been inserted just to slightly simplify the analysis that will follow, but it is by no means necessary. In the IR, we get $\omega(\mu^2) \simeq c^2\mu^2$. In both cases, we have chosen the subtraction point at an euclidean energy scale to circumvent threshold singularities; this is a non-relativistic analog of the more familiar condition (5.2). Using the CS equations satisfied by $\Gamma^{(2)}$ and $\Gamma^{(3)}$, we can derive β_λ, β_{a^2}, β_{c^2} and the anomalous dimension γ of ϕ in terms of the purely one-loop contributions (denoted by $\Gamma_l^{(2)}$ and $\Gamma_l^{(3)}$) to $\Gamma^{(2)}$ and $\Gamma^{(3)}$. We have

$$\gamma = -\frac{1}{2}\mu\frac{\partial \dot{\Gamma}_{l,0}^{(2)}}{\partial\mu}\,, \qquad \beta_\lambda = -\mu\frac{\partial \Gamma_{l,0}^{(3)}}{\partial \mu} + 3\lambda\gamma\,,$$
$$\beta_{a^2} = -\frac{1}{4!}\mu\frac{\partial \Gamma_{l,0}^{(2)''''}}{\partial\mu} + 2\gamma a^2\,, \quad \beta_{c^2} = -\frac{1}{2}\mu\frac{\partial \Gamma_{l,0}^{(2)''}}{\partial\mu} + 2\gamma c^2\,. \tag{5.7}$$

To simplify the notation, here we have denoted by a dot and a prime a derivative with respect to $(p^0)^2$ and p, respectively. We now show the computation of γ in some detail. One has

$$\gamma = 3\lambda^2 \int \frac{d^{10}k dk_E}{(2\pi)^{11}} \int_0^1 dx \frac{\mu^2\omega'(\mu^2)x^2(1-x)^2}{[k_E^2 + a^2k^4 + c^2k^2 + \omega(\mu^2)x(1-x)]^4}$$
$$= \frac{15 l_{10}\lambda^2}{32}\int_0^1 dx \int_0^\infty dk \frac{k^9\mu^2\omega'(\mu^2)x^2(1-x)^2}{[a^2k^4 + c^2k^2 + \omega(\mu^2)x(1-x)]^{7/2}}\,. \tag{5.8}$$

Computing the integral in x, we find

$$\gamma = \lambda^2 l_{10}\int_0^\infty dk \frac{k^8\mu^2\omega'(\mu^2)}{\sqrt{a^2k^2+c^2}[4(a^2k^4+c^2k^2)+\omega(\mu^2)]^3}\,. \tag{5.9}$$

[5] In units of Λ. In the following, Λ will be restored in some formulas when convenient.

Note that the integrand in eq. (5.9) is UV *and* IR finite for any value of $\mu^2 > 0$. In the UV, $\mu \gg c\Lambda/a$, the c^2 terms can be neglected, in which case the integral is easily performed giving

$$\gamma^{(\mathrm{UV})} \simeq \frac{l_{10}\lambda^2}{64a^5}\,, \qquad \mu \gg \frac{c}{a}\,\Lambda\,. \tag{5.10}$$

In the IR, one can neglect the μ^2 term appearing in the denominator of the integrand of eq. (5.9), in which case we get

$$\gamma^{(\mathrm{IR})} \simeq \frac{l_{10}\lambda^2}{480a^3c^2}\frac{\mu^2}{\Lambda^2}\,, \qquad \mu \ll \frac{c}{a}\,\Lambda\,. \tag{5.11}$$

Equation (5.11) implies that the RG evolution of γ in the IR is essentially frozen, in complete analogy to what happens in standard Lorentz invariant theories below the mass scale. Equation (5.11) is best understood by noticing that eq. (5.8) is well-behaved in the IR, due to the c^2k^2 term that acts as an IR regulator and forbids the presence of any IR singularity for $k \to 0$. Like in the usual Lorentz-invariant ϕ^4 theory considered before, the IR finiteness is responsible for the freezing of the coupling. The essential aspects of the above results should be general and apply to any weighted marginal operator in Lifshitz-type theories.

A computation very similar to the one above (but algebraically more involved) gives

$$\beta_\lambda^{(\mathrm{IR})} \simeq k_\lambda \frac{\lambda^2}{a^3c^2}\frac{\mu^2}{\Lambda^2}\lambda\,, \qquad \beta_{a^2}^{(\mathrm{IR})} \simeq \frac{2k_a\lambda^2}{a^3c^2}\frac{\mu^2}{\Lambda^2}a^2\,, \qquad \beta_{c^2}^{(\mathrm{IR})} \simeq \frac{2k_c\lambda^2}{a^3c^2}\frac{\mu^2}{\Lambda^2}c^2\,, \tag{5.12}$$

where $k_\lambda = -7l_{10}/480$, $k_a = 183l_{10}/22400$ and $k_c = -3l_{10}/400$. We have checked that $\beta_\lambda^{(\mathrm{UV})}$, $\beta_{a^2}^{(\mathrm{UV})}$ and $\beta_{c^2}^{(\mathrm{UV})}$ completely agree with those found in the MS scheme in section 4, eqs. (4.2) and (4.7). Notice that it is crucial to take $p^0 \propto \mu^2$ in the UV regime to get agreement with the MS scheme. By taking $p^0 \propto \mu$, we would have obtained a mismatch by a factor 1/2 in the $\beta^{(UV)}$-functions. The effective IR coupling is

$$y \equiv \frac{\lambda^2}{a^3c^2}\,, \tag{5.13}$$

to be compared with the UV effective coupling $x = \lambda^2/a^5$. Contrary to x, however, y is not well-defined, in the sense that it sensitively depends on the choice of subtraction point, i.e. the choice of μ. This is obvious, considering that in the IR the β-functions present an explicit dependence on μ. It is straightforward to solve the RG equations in the IR. Restoring Λ, we get

$$\begin{aligned} y(\mu) &= \frac{y(\Lambda)}{1+k_y y(\Lambda)(1-\frac{\mu^2}{\Lambda^2})/2}\,, \\ \lambda(\mu) &= \lambda(\Lambda)\left(\frac{y(\mu)}{y(\Lambda)}\right)^{\frac{k_\lambda}{k_y}}\,, \\ a^2(\mu) &= a^2(\Lambda)\left(\frac{y(\mu)}{y(\Lambda)}\right)^{\frac{2k_a}{k_y}}\,, \\ c^2(\mu) &= c^2(\Lambda)\left(\frac{y(\mu)}{y(\Lambda)}\right)^{\frac{2k_c}{k_y}}\,, \end{aligned} \tag{5.14}$$

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where $k_y = 2k_\lambda - 2k_c - 3k_a$. Eqs. (5.14) exhibit the fact that in the IR regime the RG evolution of a^2, c^2 and λ, as induced by λ itself, freezes. More precisely, for $\mu \ll c\Lambda/a$ — modulo a small threshold effect shifting the IR values of the couplings from the UV ones — the energy-dependence of the low-energy couplings is proportional to $(\mu/\Lambda)^2$.

5.3 IR effects of relevant couplings

In the previous subsection we have shown that weighted marginal operators become inoperative for $\mu \ll c\Lambda/a$. This is in some sense expected, since in the IR one should recover the usual classification of operators in terms of canonical rather than weighted dimensions. What is weighted marginal in the UV becomes standard irrelevant in the IR. However, care has to be paid to the weighted relevant operators which become standard marginal in the IR. These operators deserve a separate discussion. They are obviously negligible in the UV but they can efficiently transmit the UV Lorentz violation to the IR theory. Indeed, as we will see, Lorentz-symmetry-breaking weighted relevant operators which become standard marginal in the IR will in general mediate Lorentz violation effects to the whole IR Lagrangian.

In order to illustrate the effect, we consider a simple IR toy model in 3+1 dimensions, consisting of a fermion interacting with a scalar by means of a Yukawa interaction. We imagine that, say, the scalar is described at high energy by a Lifshitz-like dynamics with some $z > 1$. The Yukawa coupling is standard marginal in the usual sense, but weighted relevant. We want to study its effect in the IR where, as we have just shown, the weighted marginal couplings of the Lifshitz theory have a negligible beta function. We assume that the only remnant of the non-Lorentz invariance of the UV completed theory is a tiny difference in the speed of light of the fermion and scalar: $c_\psi = 1$, $c_\phi = 1 + \delta c$, with $\delta c \ll 1$. The Lagrangian is simply

$$\mathcal{L} = \frac{1}{2}(\dot\phi)^2 - \frac{c_\phi^2}{2}(\partial_i\phi)^2 + \bar\psi(\partial\!\!\!/_0 - c_\psi \vec{\partial\!\!\!/}\psi) - g\bar\psi\psi\phi\,. \tag{5.15}$$

We will now show how, due to a non-vanishing value of δc, a logarithmic running in c_ψ is induced. We work at linear order in δc and in the MS scheme. We omit details, since the one-loop graphs of this model are straightforward and can be found in standard textbooks. We denote by Z_ϕ and Z_ψ the wave-function renormalization constants of ϕ and ψ, associated with the renormalization of $(\dot\phi)^2$ and $\bar\psi\,\partial\!\!\!/_0\psi$, by $Z_{c\phi}$ and $Z_{c\psi}$ the wave-function renormalization constants associated with $(\partial_i\phi)^2$ and $\bar\psi\vec{\partial\!\!\!/}\psi$ and by Z_g the vertex renormalization constant. We get

$$\begin{aligned}
Z_{c\psi}Z_\psi^{-1} &= 1 - \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\Big(\frac{2}{3}\delta c + \mathcal{O}(\delta c^2)\Big) \\
Z_{c\phi}Z_\phi^{-1} &= 1 - \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\Big(8\delta c + \mathcal{O}(\delta c^2)\Big)\,, \\
Z_g &= 1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\Big(2 + \mathcal{O}(\delta c)\Big)\,.
\end{aligned} \tag{5.16}$$

From eqs. (5.16) we get

$$\beta_{c_\psi} = -\frac{g^2}{24\pi^2}\delta c\,, \qquad \beta_{\delta c} = \frac{5g^2}{24\pi^2}\delta c\,, \qquad \beta_g = \frac{5g^3}{16\pi^2}\,. \tag{5.17}$$

Defining $\alpha \equiv g^2/(4\pi)$, we finally get the following RG evolution for $c_\psi(t)$, $\delta c(t)$ and $\alpha(t)$:

$$\begin{aligned}
\alpha(t) &= \frac{\alpha_0}{1-\frac{5}{2\pi}\alpha_0 t}\,,\\
\delta c(t) &= \delta c_0\left(\frac{\alpha(t)}{\alpha_0}\right)^{\frac{7}{15}}\,,\\
c_\psi(t) &= c_{\psi,0} - \frac{\delta c_0}{7}\left[\left(\frac{\alpha(t)}{\alpha_0}\right)^{\frac{7}{15}} - 1\right].
\end{aligned} \tag{5.18}$$

Equations (5.18) shows that any small Lorentz symmetry breaking term in the IR theory (coming from the underlying UV theory) induces, by quantum effects, an energy-dependent speed of light for all particles sensible to the breaking term. From this example it should also be clear that the effect is general.

6 Conclusions

In this work we have carried out the analysis of the one-loop RG evolution of Lifshitz-like theories, by mainly focusing on two specific Lorentz violating scalar field theories. Our primary interest was to look for possible simple mechanisms (see also [15]) which would alleviate the fine-tuning otherwise needed in these theories to recover Lorentz symmetry at low energies with the needed accuracy. We have focused our attention on two particular operators, the spatial kinetic terms $c_{1,2}^2(\vec{\partial}\phi_{1,2})^2$. There are essentially two regimes of interest. In the UV Lifshitz-regime, c^2 and δc^2 logarithmically run with the energy, and the running is governed by weighted marginal operators. In the IR, if all sources of Lorentz symmetry breaking are vanishing, one recovers Lorentz invariance, which clearly forbids any scale-dependence for c^2. However, due to the effects of standard marginal couplings, any small Lorentz symmetry breaking term in the theory leads to phenomenologically unacceptable logarithmic dependence of c^2 on the energy scale. As it has already been pointed out, our considerations seem to be very general and independent of additional structure and/or symmetry that would be present in more realistic theories, such as gauge symmetries. It should be clear that, although we have been focusing on the spatial kinetic term operator, our considerations may be equally applied to other Lorentz violating parameters associated with other operators in a theory (this could lead to the need of additional fine-tuning).

In section 5.3 we computed the evolution of c in the IR in a simple low-energy scalar-fermion theory, assuming the presence of a small δc, coming from the non-Lorentz invariant UV theory. In this example δc is driven to small values at low energies, see eq. (5.18). A logical possibility is that in the Standard Model the RG flow could similarly drive the $\delta c_{ij} \equiv c_i - c_j$ corresponding to any i, j pair of particles to sufficiently small values, below the existing phenomenological bounds. Although not completely excluded by our analysis (since we are not computing RG in a Lifshitz-type Standard Model), this possibility seems unlikely, since the logarithmic running of δc at low energies is too slow to drive a δc of order

1 at the scale Λ to values compatible with the stringent existing bounds $\delta c < 10^{-(21\div 23)}$. Moreover, in a UV free theory, the same mechanism of section 5.3 would give a δc that increases towards the IR.

An order of magnitude of other potential phenomenological problems can also be obtained by applying our considerations to the photon in a weighted renormalizable version of QED. At low energies we can neglect higher orders terms in p (always present in Lifshitz type theories), so that the photon dispersion relation will be typically of the form

$$\omega^2 = c_\gamma^2(t)p^2\,, \tag{6.1}$$

where ω is the energy and

$$c_\gamma^2(t) = c_0^2\big(1-ft\big)^r\,, \tag{6.2}$$

with some unspecified constants f and r. The behavior eq. (6.2) should be induced by one loop effects as in section 5.3 once a small δc between the photon and a charged particle is introduced in the Largrangian. This behavior can be compared with known experimental constraints. An interesting constraint on the energy dependence of c_γ^2 at moderately high energies has been recently given by the FERMI experiment, detecting the photon spectrum of the gamma ray burst GRB 080916C at red-shift $z = 4.35$ [13].[6] By measuring the time delay (~ 10 sec) between the "low-energy" ($\lesssim 1\,\mathrm{MeV}$) component of the burst with respect to the high energy one ($\sim 10\,\mathrm{GeV}$), we get the following rough experimental bounds[7]

$$|c^2(1\mathrm{MeV}) - c^2(10\mathrm{GeV})| \lesssim 10^{-17}\,. \tag{6.3}$$

By taking $r \sim 1$ in eq. (6.2), eq. (6.3) approximately gives the constraint on f described in eq. (1.4). Similar (although milder) bounds exist for other particles as well.

One might think that, in the context of the Hořava theory of gravity, the bounds we are finding can be evaded for all the Standard Model (SM) particles by simply not introducing the higher derivative spatial interactions responsible for the effects studied in this paper. Being the SM renormalizable, there is no real need of introducing them. Aside from the lack of a clear principle besides this choice, radiative effects induced by graviton loops should nevertheless generate these Lorentz-violating couplings in the SM sector at some loop order.

Finally, let us conclude by noticing that the one-loop corrections to c^2 we considered are quadratically divergent. If new physics above the scale Λ is assumed, this can give rise to a naturalness problem, similar to the standard one affecting the Higgs boson mass.

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[6] We are grateful to David Mattingly for bringing to our attention [13] and providing us the estimate that follows.

[7] In [16], the same time delay observed by FERMI in the photon spectrum emitted by GRB 080916C was "explained" in the context of the Hořava theory of gravity, by means of the higher derivative terms appearing in the photon dispersion relation. They did not consider the one-loop effect studied here.

References

[1] D. Anselmi and M. Halat, *Renormalization of Lorentz violating theories*, *Phys. Rev.* **D 76** (2007) 125011 [arXiv:0707.2480] [SPIRES].

[2] D. Anselmi, *Weighted power counting and Lorentz violating gauge theories. I: General properties*, *Annals Phys.* **324** (2009) 874 [arXiv:0808.3470] [SPIRES]; *Weighted power counting and Lorentz violating gauge theories. II: Classification*, *Annals Phys.* **324** (2009) 1058 [arXiv:0808.3474] [SPIRES].

[3] D. Anselmi, *Weighted power counting, neutrino masses and Lorentz violating extensions of the Standard Model*, *Phys. Rev.* **D 79** (2009) 025017 [arXiv:0808.3475] [SPIRES]; *Standard Model Without Elementary Scalars And High Energy Lorentz Violation*, arXiv:0904.1849 [SPIRES].

[4] A. Dhar, G. Mandal and S.R. Wadia, *Asymptotically free four-fermi theory in 4 dimensions at the z=3 Lifshitz-like fixed point*, arXiv:0905.2928 [SPIRES].

[5] P. Hořava, *Quantum Criticality and Yang-Mills Gauge Theory*, arXiv:0811.2217 [SPIRES].

[6] P. Hořava, *Membranes at Quantum Criticality*, *JHEP* **03** (2009) 020 [arXiv:0812.4287] [SPIRES].

[7] P. Hořava, *Quantum Gravity at a Lifshitz Point*, *Phys. Rev.* **D 79** (2009) 084008 [arXiv:0901.3775] [SPIRES].

[8] T. Takahashi and J. Soda, *Chiral Primordial Gravitational Waves from a Lifshitz Point*, *Phys. Rev. Lett.* **102** (2009) 231301 [arXiv:0904.0554] [SPIRES];
G. Calcagni, *Cosmology of the Lifshitz universe*, *JHEP* **09** (2009) 112 [arXiv:0904.0829] [SPIRES];
E. Kiritsis and G. Kofinas, *Hořava-Lifshitz Cosmology*, *Nucl. Phys.* **B 821** (2009) 467 [arXiv:0904.1334] [SPIRES];
H. Lü, J. Mei and C.N. Pope, *Solutions to Hořava Gravity*, *Phys. Rev. Lett.* **103** (2009) 091301 [arXiv:0904.1595] [SPIRES];
H. Nastase, *On IR solutions in Hořava gravity theories*, arXiv:0904.3604 [SPIRES];
D. Orlando and S. Reffert, *On the Renormalizability of Hořava-Lifshitz-type Gravities*, *Class. Quant. Grav.* **26** (2009) 155021 [arXiv:0905.0301] [SPIRES];
R.-G. Cai, B. Hu and H.-B. Zhang, *Dynamical Scalar Degree of Freedom in Hořava-Lifshitz Gravity*, *Phys. Rev.* **D 80** (2009) 041501 [arXiv:0905.0255] [SPIRES];
T.P. Sotiriou, M. Visser and S. Weinfurtner, *Quantum gravity without Lorentz invariance*, *JHEP* **10** (2009) 033 [arXiv:0905.2798] [SPIRES];
C. Germani, A. Kehagias and K. Sfetsos, *Relativistic Quantum Gravity at a Lifshitz Point*, *JHEP* **09** (2009) 060 [arXiv:0906.1201] [SPIRES].

[9] C. Charmousis, G. Niz, A. Padilla and P.M. Saffin, *Strong coupling in Hořava gravity*, *JHEP* **08** (2009) 070 [arXiv:0905.2579] [SPIRES];
S. Mukohyama, *Dark matter as integration constant in Hořava-Lifshitz gravity*, *Phys. Rev.* **D 80** (2009) 064005 [arXiv:0905.3563] [SPIRES].

[10] E.M. Lifshitz, *On the theory of second-order phase transitions I*, *Zh. Eksp. Teor. Fiz.* **11** (1941) 255; *On the theory of second-order phase transitions II*, *Zh. Eksp. Teor. Fiz.* **11** (1941) 269.

[11] J. Collins, A. Perez, D. Sudarsky, L. Urrutia and H. Vucetich, *Lorentz invariance and quantum gravity: an additional fine-tuning problem?*, *Phys. Rev. Lett.* **93** (2004) 191301 [gr-qc/0403053].

[12] S.R. Coleman and S.L. Glashow, *High-Energy Tests of Lorentz Invariance*, *Phys. Rev.* **D 59** (1999) 116008 [hep-ph/9812418] [SPIRES].

[13] Fermi LAT and Fermi GBM collaboration, A.A. Abdo et al., *Fermi Observations of High-Energy Gamma-Ray Emission from GRB 080916C*, *Science* **323** (2009) 1688 [SPIRES].

[14] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, chap. 12, Westview Press (1995).

[15] S. Chadha and H.B. Nielsen, *Lorentz invariance as a low-energy phenomenon*, *Nucl. Phys.* **B 217** (1983) 125 [SPIRES].

[16] B. Chen and Q.-G. Huang, *Field Theory at a Lifshitz Point*, arXiv:0904.4565 [SPIRES].

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