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Two dimensional fermions in four dimensional YM

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ABSTRACT: Dirac fermions in the fundamental representation of $SU(N)$ live on a two dimensional torus flatly embedded in R^4 . They interact with a four dimensional $SU(N)$ Yang Mills vector potential preserving a global chiral symmetry at finite N . As the size of the torus in units of $\frac{1}{\Lambda_{SU(N)}}$ is varied from small to large, the chiral symmetry gets spontaneously broken in the infinite N limit.

KEYWORDS: Lattice Gauge Field Theories, 1/N Expansion

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1 Introduction

When the local description of a physical system includes strong non-linearities, one tries to trade the original variables for “better” variables, in terms of which nonlinear effects become weaker. A goal more likely within reach is to find “good” observables whose behavior displays the essence of the nonlinearity. In the context of relativistic field theory these observables are non-local in terms of the original variables. As such, their definition requires specific study of their renormalization.

In this paper we introduce a non-local operator in four dimensional gauge theory which generalizes the Wilson loop associated with a closed one dimensional curve in spacetime to an object associated with a closed two dimensional surface embedded in spacetime.

2 Wilson loop operators

2.1 General properties

The nonlinear dynamics of pure SU(N) Yang Mills gauge theory on Euclidean R^4 is best captured by nonlocal observables. The Wilson loop operators are standard.

$$\text{Tr}_r \langle \mathcal{P} e^{i \oint_{\mathcal{C}} A \cdot dx} \rangle \equiv \text{Tr}_r \langle \Omega(\mathcal{C}) \rangle = W_r(\mathcal{C}) \quad (2.1)$$

Here r denotes an irreducible representation of SU(N), \mathcal{P} denotes path ordering round a closed, smooth, non-selfintersecting curve \mathcal{C} , and A is the hermitian Yang Mills connection of SU(N) Yang Mills theory in R^4 with ordinary action. The θ parameter is set to zero.

The number of boxes modulo N in the Young pattern describing r is the N -ality k . Assume that the curve \mathcal{C} has a unique minimal spanning area \mathcal{A} in the standard metric of R^4 . A major feature of the theory is confinement: for $k \neq 0$ one has

$$\log [W_r(\mathcal{C})] \underset{\mathcal{A} \rightarrow \infty}{\sim} -\sigma_k \mathcal{A} \quad (2.2)$$

Here the limit $\mathcal{A} \rightarrow \infty$ is taken at fixed loop shape, $\hat{\mathcal{C}}$, by uniformly dilating the loop until the required value of \mathcal{A} is achieved. $\sigma_k > 0$, the k -th string tension, does not depend on $\hat{\mathcal{C}}$. Since we have set the θ -parameter to zero, charge conjugation gives $\sigma_k = \sigma_{N-k}$. Thus, Wilson loop operators provide a precise definition of the property of confinement, quantitatively expressed by the σ_k .

\mathcal{C} can be parameterized by \mathcal{A} and $\hat{\mathcal{C}}$. In perturbation theory a dependence on \mathcal{A} comes in only through the quantum violation of classical scale invariance. The behavior of $W_r(\mathcal{C})$ as $\mathcal{A} \rightarrow 0$ is determined by perturbation theory in terms of the scale $\Lambda_{SU(N)}$ which sets a convention for a running coupling constant at short scales, common to all observables. The perturbative expansion of $W_r(\mathcal{C})$ is defined only after making some choices to fix free parameters one encounters in the process of renormalization. These choices may extend beyond perturbation theory, but should not affect σ_k .

The renormalization of $W_r(\mathcal{C})$ is well understood [1, 2]. In perturbation theory one needs to expand $\log[W_r(\mathcal{C})]$ and the needed ‘‘exponentiation’’ Feynman rules are known [3]. The conclusion is that for every r one has one operator dependent linear perimeter divergence and all the other divergences are the same as in pure gauge theory. Imposing symmetry under conjugation of the renormalized Wilson loops leaves one new arbitrary real finite part for distinct self-conjugate representations, $r \cong \bar{r}$, or pairs of distinct conjugate representations, $r \not\cong \bar{r}$.

2.2 Fermionic representation

We shall focus on the set of all single column (totally antisymmetric) representations. This set has then $\lfloor \frac{N}{2} \rfloor$ free real parameters. These parameters can be counted also by simultaneously dealing with all the $W_r(\mathcal{C})$, assembled into the average characteristic polynomial:

$$P(z, \mathcal{C}) = \langle \det [z + \Omega(\mathcal{C})] \rangle = \langle \det [1 + z\Omega^\dagger(\mathcal{C})] \rangle \quad (2.3)$$

The diagrammatic expansion rules for $\log P(z, \mathcal{C})$ are obtained from a fermionic representation. Let $X_1, X_2 \dots X_n$ be n arbitrary $N \times N$ matrices and let $X = X_1 X_2 \dots X_n$ be their matrix product. We then have the following identity [4, 5]:

$$\int \prod_{j=1}^n [d\bar{\psi}_j d\psi_j] e^{\sum_{j=1}^n [\bar{\psi}_j X_j \psi_{j+1} - \bar{\psi}_j \psi_j]} = \det(1 + X) \quad (2.4)$$

Here $\psi_{n+1} \equiv -\psi_1$ and each ψ_j has N components. Taking a formal continuum limit we get [10]:

$$\det [1 + z\Omega^\dagger(\mathcal{C})] = \int [d\bar{\psi}d\psi] e^{\int_0^l d\sigma \bar{\psi}(\sigma) [\partial_\sigma - \mu - ia(\sigma)] \psi(\sigma)} \quad (2.5)$$

Here,

$$z = e^{-\mu l} \quad (2.6)$$

σ parametrizes \mathcal{C} by $x(\sigma)$. The parametrization $[\partial_\sigma x_\mu(\sigma)]^2 = 1$ identifies l as the length of the loop. The continuum one dimensional fermions $\bar{\psi}(\sigma), \psi(\sigma)$ obey anti-periodic boundary conditions round the curve. The one dimensional vector potential a is defined by the four dimensional one:

$$a(\sigma) = A_\mu(x(\sigma)) \frac{\partial x_\mu(\sigma)}{d\sigma} \quad (2.7)$$

The diagrams are read off from the following path integral:

$$P(z, \mathcal{C}) = \int [dA_\mu] [d\bar{\psi}d\psi] \frac{e^{-\frac{1}{4g^2} \int d^4x \text{Tr} F^2 + \int_0^l d\sigma \bar{\psi}(\sigma) [\partial_\sigma - \mu - ia(\sigma)] \psi(\sigma)}}{\int [dA_\mu] e^{-\frac{1}{4g^2} \int d^4x \text{Tr} F^2}} \quad (2.8)$$

g is the gauge coupling and $F_{\mu\nu}$ is the field strength made out of A_μ . One needs to choose a gauge fixing convention and add the needed ghosts. This is done without reference to the fermions.

The Feynman diagram expansion for $\log [P(z, \mathcal{C})]$ consists of connected diagrams which look exactly like those for the fermion contribution to the free energy of ordinary QCD. Only the mathematical expressions for the $\bar{\psi}\psi A$ vertex and for the $\bar{\psi}\bar{\psi}$ propagator differ.

If the diagrams are viewed in Euclidean configuration space the expression for the $\bar{\psi}\psi A$ vertex is simple. The fermion propagator, $S(\sigma)$, requires the inversion of the kernel taking into account the boundary conditions. With $|\sigma| < l$ (σ stands for $\sigma_1 - \sigma_2$ with $0 < \sigma_j < l$) it is given by [6]:

$$S(\sigma) = e^{-\mu\sigma} \left[\theta(\sigma) - \frac{1}{1 + e^{\mu l}} \right] \quad (2.9)$$

$S(\sigma)$ is diagonal in its suppressed $SU(N)$ indices. The dependence on \mathcal{C} comes from connected vacuum diagrams containing at least one fermion loop and makes a contribution of at most order N to the free energy if $g^2 N$ is kept fixed.

W_r for $r = N$ (the fundamental representation) can be extracted by taking $\mu \rightarrow \infty$ and picking out the term linear in z . For a given fermion loop the product of the prefactors of $S(\sigma)$ is one. All propagators are now θ -functions except one, which is $-z$. The fermion loop gives another overall -1 . The dropped θ function is equal to zero if all other θ functions are equal to one. This recovers the pure gauge Feynman rules for the fundamental Wilson

loop operator. To the same end, one could have also used the limit $\mu \rightarrow -\infty$ ($z \rightarrow \infty$) giving W_r with $r = \bar{N}$.

Standard power counting applies, producing all counter-terms potentially needed to eliminate all ultraviolet divergence in perturbation theory. With a standard gauge fixing method that is independent of \mathcal{C} , BRST symmetry is preserved and the counter-terms have to be BRST invariant.

σ has dimension length and therefore the fermions have zero dimension. Requiring gauge invariance leaves, in addition to the standard counter-terms, $N - 1$ new dimensionless counter-terms $[\bar{\psi}\psi]^k$, $k = 1, \dots, N$ with dimension one coefficients reflecting linear perimeter divergences in the Wilson loops underlying our observable. Setting $\mu = 0$, conjugation invariance is preserved, acting as follows:

$$\psi \rightarrow \bar{\psi}, \quad \bar{\psi} \rightarrow \psi, \quad A_\mu \rightarrow A_\mu^* \quad (2.10)$$

For odd k , $[\bar{\psi}\psi]^k$ changes sign. Therefore, these counter-terms can be set to zero. The parameter μ can be turned on perturbatively. By power counting, this could induce a new ultraviolet divergence, but it is at most logarithmic and can be absorbed in μ . This leaves $[\frac{N}{2}]$ independent counter-terms, corresponding to the perimeter divergences in W_r for r given by a k -box column; conjugation symmetry exchanges k with $N - k$ and we require that it be obeyed by the counter-terms.

More new counter-terms, of the form $\int \bar{\psi} \partial \psi [\bar{\psi}\psi]^k$ also are possible. They would correspond to new types of logarithmic divergences. These counter-terms can be made gauge invariant by minimal substitution. (There also is another set where the derivative acts on $\bar{\psi}$.) These terms can be ignored, since they can be eliminated by a field redefinition:

$$\psi = \chi + \alpha_1 \chi (\bar{\psi}\chi) + \alpha_2 \chi (\bar{\psi}\chi)^2 + \dots + \alpha_{N-1} \chi (\bar{\psi}\chi)^{N-1} \equiv \chi F(\bar{\psi}\chi) \quad (2.11)$$

One can change the integration variables ψ to χ and the Jacobian will have the form of the counter-terms already taken into account above. The inverse transformation is:

$$\chi = \psi - \alpha_1 \psi (\bar{\psi}\psi) + \beta_2 \psi (\bar{\psi}\psi)^2 + \dots + \beta_{N-1} \psi (\bar{\psi}\psi)^{N-1} \equiv \psi \tilde{F}(\bar{\psi}\psi) \quad (2.12)$$

with $\beta_2, \dots, \beta_{N-1}$ determined by the α_j . One has then

$$\bar{\psi} \partial \psi = \bar{\psi} \partial [\chi F(\bar{\psi}\chi)] = \bar{\psi} [\partial \chi] F(\bar{\psi}\chi) + \bar{\psi} \chi [\partial (\bar{\psi}\chi)] F'(\bar{\psi}\chi) \quad (2.13)$$

The last term is a total derivative, and can be dropped due to the boundary conditions.

There is a $[\frac{N}{2}]$ real dimensional space in which to choose the finite parts of the divergent quantities made finite by the counter-terms. Once this is done, one can calculate to arbitrary order in perturbation theory. From the result, by taking derivatives with respect to z at $z = 0$ one gets perturbative expressions for the N coefficients of z in $P(z, \mathcal{C})$. The expressions for each coefficient can be re-exponentiated, producing the final perturbative expression for $P(z, \mathcal{C})$ as a rank N polynomial in z . Alternatively, one could have computed each coefficient independently in exponentiated form and never employed the fermions.

On the $[\frac{N}{2}]$ dimensional submanifold of all possible finite part choices one has enough freedom to ensure that the so obtained $P(z, \mathcal{C})$ will have all its zeros on the unit circle.

This is so because preserving charge conjugation guarantees that the polynomial in z will be palindromic with real coefficients for any parameter choice [7]. Outside this submanifold $P(z, \mathcal{C})$ will have some roots off the unit circle for small enough \mathcal{A} .

Comparing with the known results from diagrammatic analysis, the fermionic representation is seen to produce correct free parameter counting in this case.

2.3 A large N phase transition

Using a lattice regularization one can define $P(z, \mathcal{C})$ even beyond perturbation theory. One can make sure that this fully defined $P(z, \mathcal{C})$ has all its zeros on the unit circle for any \mathcal{A} [4, 8].

There is numerical evidence for the following: $P(z, \mathcal{C})$ exhibits a nonanalytic behavior as the loop \mathcal{C} is kept at fixed shape $\hat{\mathcal{C}}$ and scaled. We first take the large N limit in the 't Hooft prescription at fixed z . The nonanalyticity occurs first at $z = 1$ as the loop is scaled down from a large size. Wilson loops in $SU(N)$ gauge theory in two, three and four dimensions all exhibit this infinite N phase transition as they are shrunk from a large size to a small one; in the course of this scaling down, the support of the eigenvalue distribution of the untraced Wilson loop unitary matrix contracts from encompassing the entire unit circle, to a small arc centered at $z = -1$ on the unit circle. An analogous effect takes place in the two dimensional principal chiral model for $SU(N)$ [9].

At finite ultraviolet cutoff, before the addition of the multi-fermion counter-terms, we observe that the large N transition occurs when the the spectrum of $D_1(\mathcal{C}) \equiv \partial_\sigma - ia(\sigma)$, which was gap-less for large \mathcal{A} , opens a gap as \mathcal{A} decreases through \mathcal{A}_c . The spectrum of $\Omega(\mathcal{C})$ directly determines the spectrum of $D_1(\mathcal{C})$, which can be viewed as a Dirac operator in one Euclidean dimension [10]. This observation will partially motivate the generalization in the next section.

The universality class of this transition is that of a random multiplicative ensemble of unitary matrices. The transition was discovered by Durhuus and Olesen [11] when they solved the Makeenko-Migdal [12] loop equations in two dimensional planar QCD. The nature of the transition is now better understood. The authors of [13] proposed a relation to Burgers' turbulence [14] and there are exact results in 2D supporting their view [15].

For finite N , in two dimensions, $\Omega(\mathcal{C})$ is an $SU(N)$ matrix diffusing on the $SU(N)$ group manifold. This holds approximatively also in four dimensions: see [16]. The "diffusion" is imagined at fixed $\hat{\mathcal{C}}$ and \mathcal{A} plays the role of diffusion time. At infinite N the transition occurs at $\mathcal{A} = \mathcal{A}_c$. For $N \gg 1$, $z \sim 1$ and at $\mathcal{A} \sim \mathcal{A}_c$, $P(z, \mathcal{C})$ [with $\hat{\mathcal{C}}$ fixed] is described by a universal function, with one, or perhaps two, non-universal, $\hat{\mathcal{C}}$ dependent constants. One constant sets the scale of $\mathcal{A} - \mathcal{A}_c$ and the other sets the scale of N itself. There is a possibility that the latter is actually always equal to unity on account of the discrete nature of N , even though in the $\frac{1}{N}$ expansion this does not come in.

For $N \gg 1$ there is a sharp crossover in the critical region. This crossover separates small loops of shape $\hat{\mathcal{C}}$ from large ones with identical shape. The crossover connects two distinct regimes. In the perturbative regime, one has, up to terms depending on the choices

made for the finite parts

$$P(z, \mathcal{C}) - (1+z)^N \underset{\mathcal{A} \rightarrow 0}{\sim} \frac{h_0(\hat{\mathcal{C}}, N, z)}{\log[\mathcal{A}\Lambda_{SU(N)}^2]}, \quad (2.14)$$

while in the nonperturbative regime we have

$$P(z, \mathcal{C}) - (1+z^N) \underset{\mathcal{A} \rightarrow \infty}{\sim} (z+z^{N-1}) h_\infty(\hat{\mathcal{C}}, N) e^{-\sigma_1 \mathcal{A}} \quad (2.15)$$

If we had an effective string algorithm to compute in the nonperturbative regime and managed to carry out a perturbative calculation for small loops the two would be connected at $N \gg 1$ by a known universal function and, by matched asymptotics, we would obtain the ratio $\frac{\sigma_1}{\Lambda_{SU(N)}^2}$. A renormalization scheme that effectively fattens the Wilson loop introduces [8] an additional scale which is finite in units of $\Lambda_{SU(N)}$ and which would alter in a calculable way the asymptotic behavior as $\mathcal{A} \rightarrow 0$.

To ensure that $P(z, \mathcal{C})$ has all its zeros on the unit circle for any area, a necessary condition is

$$\frac{1}{d_r} W_r(\mathcal{C}) < 1 \quad (2.16)$$

for all r (d_r is the dimension of r) and all $\mathcal{A} > 0$ at fixed $\hat{\mathcal{C}}$. However, many reasonable renormalization prescriptions would violate the above inequality for small enough \mathcal{A} .

3 The new observable

3.1 Fermionic representation and ultraviolet divergences

One would prefer an observable with no linear divergences and less free parameters after renormalization than $P(z, \mathcal{C})$.

As mentioned already, the Feynman diagrams themselves are those of QCD. We decide to restrict now the fermion lines to a two dimensional embedded closed smooth manifold, Σ , and add the required Dirac indices. We ensure that gauge invariance is preserved. To keep matters simple, we want the two dimensional manifold to be flat in the induced metric from R^4 . We take Σ to be a torus of sides $l_{1,2}$. The boundary conditions on the fermions are chosen as antiperiodic in both directions. Now the fermions will have dimension 1/2 and renormalization will not require terms with more than 4 fermions. Thus, the number of counter-terms will no longer grow with N . Moreover, if we keep the fermions massless there will be a continuous chiral symmetry eliminating the last potential linear divergence. This will hold also when the fermion mass is reinstated.

Formally, if we take $l_2 \rightarrow 0$ we get two copies of the fermion system used for $P(z, \mathcal{C})$. Thus, the new observable ought to still carry the essential information carried by the old one.

Consider now a uniform scaling of the embedded torus: $l_{1,2} \rightarrow \rho l_{1,2}$ and introduce a shape parameter $0 < \tau = \frac{l_1}{l_2} < \infty$ which is left invariant by this scaling. At fixed τ , as ρ is varied, maximally separated fermions will feel confining forces for $\rho \gg 1$ and $\frac{1}{\text{distance}}$

forces for $\rho \ll 1$. We guess therefore that the two regimes will be separated by a crossover which will become a phase transition at infinite N . As ρ decreases through some critical value, the spontaneously broken chiral symmetry will be restored. To some extent, this is a reincarnation of an old idea in [17], which in turn was motivated by [18].

At finite cutoff, before the addition of the 4 fermion counter-terms, this large N transition will be reflected in the eigenvalue spectrum of $D_2(\Sigma)$, the Dirac operator acting on the torus fermions. This is analogous to an observation we made in the previous section. However, it is well known that on the broken side the portion of the spectrum of $D_2(\Sigma)$ that is close to zero is described by a simple random matrix model [19]. This fact has become a potent tool for numerically determining that chiral symmetry is spontaneously broken. The random matrix description ceases to hold when chiral symmetry is restored.

Here are some equations summarizing the above. The embedded torus Σ is defined by $x(\sigma)$, σ is short for σ_α , with $\alpha = 1, 2$.¹

$$x_1(\sigma) = \frac{l_1}{2\pi} \cos \frac{2\pi\sigma_1}{l_1}; \quad x_2(\sigma) = \frac{l_1}{2\pi} \sin \frac{2\pi\sigma_1}{l_1}; \quad x_3(\sigma) = \frac{l_2}{2\pi} \cos \frac{2\pi\sigma_2}{l_2}; \quad x_4(\sigma) = \frac{l_2}{2\pi} \sin \frac{2\pi\sigma_2}{l_2} \tag{3.1}$$

The induced metric on the torus is

$$d\sigma^2 = d\sigma_1^2 + d\sigma_2^2 \tag{3.2}$$

We define a two component gauge potential a_α on the torus by

$$a_\alpha = A_\mu(x(\sigma)) \frac{\partial x_\mu}{\partial \sigma_\alpha} \tag{3.3}$$

The Dirac matrices on the torus are

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.4}$$

The new observable is:

$$Q(\mu, \Sigma) = \frac{\int [dA_\mu] [d\bar{\psi}d\psi] e^{-\frac{1}{4g^2} \int d^4x \text{Tr} F^2 + \int_\Sigma d^2\sigma \bar{\psi}(\sigma) [\gamma_\alpha \partial_{\sigma_\alpha} - \mu - i\gamma_\alpha a_\alpha(\sigma)] \psi(\sigma)}}{\int [dA_\mu] e^{-\frac{1}{4g^2} \int d^4x \text{Tr} F^2}} \tag{3.5}$$

The two dimensional massless Dirac operator is:

$$D_2(\Sigma) = \gamma_\alpha \partial_{\sigma_\alpha} - i\gamma_\alpha a_\alpha(\sigma) \tag{3.6}$$

Denoting the Hermitian generators of $su(N)$ in the fundamental representation by T^j , $j = 1, \dots, N - 1$ the nonabelian fermion current coupled to a is given by

$$J_\alpha^j(\sigma) = \bar{\psi}(\sigma) \gamma_\alpha T^j \psi(\sigma) \tag{3.7}$$

¹From the context it should be clear when $\sigma_{1,2}$ refers to these coordinates and when to the first two string tensions among the σ_k .

The abelian vector current is

$$J_\alpha = \bar{\psi}(\sigma)\gamma_\alpha\psi(\sigma) \tag{3.8}$$

Power counting and symmetries allow for two local four-fermion counter-terms at $\mu = 0$:

$$\mathcal{L}_1 = J_\alpha^j J_\alpha^j, \quad \mathcal{L}_2 = J_\alpha J_\alpha \tag{3.9}$$

\mathcal{L}_1 can be replaced by a chiral invariant linear combination of terms made out of the product of two terms bilinear in the fermions, each a $SU(N)$ singlet. Suppose one integrates out the Yang Mills and ghost fields first. Now, one keeps the four fermion terms of order g^2 , but drops all higher order terms in g^2 . We leave aside the question how valid this approximation is, but intuitively it should capture correctly some of the physics if Σ is small in units of $\Lambda_{SU(N)}^{-1}$. This makes it possible to use standard methods [20] to solve the fermion model exactly in the large N -limit, even though the induced four fermion term is non-local [21]. The required \mathcal{L}_1 counter-term can also be exactly taken into account in the large N limit. As is well known, this provides expressions for some observables that are non-perturbative in the 't Hooft coupling $g^2 N$.

Both counter-terms are dimensionless, so there are no ultraviolet divergences worse than logarithmic. The Thirring term \mathcal{L}_2 is probably not strictly needed, and would add a free parameter to the theory if included [22]. \mathcal{L}_1 is certainly needed, as indicated above. To identify the ultraviolet divergences it is enough to consider the case where our closed Σ is replaced by an infinite two dimensional plane. The A_μ Feynman gauge propagator in four dimensional Fourier space induces an effective a_α propagator in two dimensions:

$$\int_{p_\parallel^2 + p_\perp^2 \leq \Lambda^2} \frac{d^2 p_\parallel d^2 p_\perp}{(2\pi)^4} \frac{f(p_\parallel^2)}{p_\parallel^2 + p_\perp^2} = \int_{p_\parallel^2 \leq \Lambda^2} \frac{d^2 p_\parallel}{(2\pi)^3} f(p_\parallel^2) \log [\Lambda/|p_\parallel|] \tag{3.10}$$

We have not yet carried out a detailed direct diagrammatic analysis of all ultraviolet divergences.

3.2 General properties

The new observable thus achieves the main objectives of reducing all ultraviolet divergences to logarithmic and making the number of needed counter-terms N -independent. The new observable is more amenable to perturbation theory, as infinite sets of diagrams can be summed using results from large N vector models and non-analytic dependencies in the Yang Mills coupling can be generated [21]. A calculation of the relevant β -functions for this system is left for the future. Perturbation theory should work for small $l_{1,2}$, but for large $l_{1,2}$ we need something different. Possibly, an effective two dimensional field theory could be found and then we would need to match it to the four dimensional underlying YM theory. Alternatively, although we do not have a clear candidate for an effective string description, we could guess one. If we have an effective description of the Wilson operators for large loops by using open strings whose ends are restricted to the curve \mathcal{C} , maybe a similar effective description employing open strings now restricted to end on Σ (a sort of D-brane) would work for large Σ .

One issue left to address is how confinement is reflected by the new observable, qualitatively and quantitatively. We already noted that the limit $\tau \rightarrow 0$ takes the new observable into the old one, so confinement will be seen. There are several new ways in which the new observable would reflect confinement. It is well known [12] that one can write a formal expression for the chiral condensate in terms of a sum over Wilson loops. Here, the chiral condensate would be a 2D one, but the Wilson loops, although restricted to Σ will have four dimensional values.² Large loops will exhibit the area law because of four dimensional confinement and those loops would make a contribution to the condensate which is similar to the contribution made in two dimensional gauge theory if the two fundamental string tensions are matched. Note however that the minimal area may not lie on Σ and therefore even for large loops one cannot expect an asymptotically perfect match. In addition, in two dimensional gauge theory there is no shape dependence, while in four dimensions there is. Nevertheless, from the point of view of the fermionic condensate the four dimensional character of the ambient YM theory reflects itself mostly at small Wilson loops. It is unclear by how much the relation between the σ_1 and the condensate would differ in our system from an effective two dimensional one. Note that two dimensional Yang Mills theory with the inclusion of $\mathcal{L}_{1,2}$ is still exactly solvable at infinite N [23].

The finite nature of the torus reflects itself in the free fermion propagator. If the gauge fields were truly two dimensional, at infinite N the dependence on the size of the torus would disappear from intensive quantities [24]. The mechanism behind this relies on the extra finite-temperature-like $Z^2(N)$ symmetry of the gauge sector which acts non-trivially on Polyakov loop operators on the torus. These symmetries are absent when the gauge fields are induced from the four dimensional world. However, for a large enough torus the four dimensional Wilson loop operators interacting with the fermions as Polyakov loop operators have eigenvalue distributions that are very flat. The deviation from total flatness is exponentially small in the area because of four dimensional confinement. Thus, up to a small correction, the fermions free energy per unit torus area will become independent of the volume at very large N . More precisely,

$$\lim_{N \rightarrow \infty} \frac{1}{N l_1 l_2} \log[Q(\mu, \Sigma)] \tag{3.11}$$

would approach its infinite $l_{1,2}$ limit (at fixed $\tau \neq 0, \infty$) with a correction that goes as $\exp[-\sigma_1 \min^2(l_1, l_2)]$. We see that in the large N limit the fundamental string tension controls the large scale behavior of the new observable, just as that of the old one.

A last issue of comparison is that in the old observable the fermions can be easily integrated out (this is how they were introduced in the first place) leaving one with explicit expressions in terms of Wilson loop operators. The fermions of the new observable can also be integrated out, leaving a more complicated observable behind. If Σ were a two sphere the answer has an explicit form [25]. For a torus some care is required with loops wrapping the torus and we are not sure whether a closed expression can be written down. Recall that a local form of the answer requires the introduction of a third dimension along which the gauge field is smoothly deformed. In our application one needs not invent such

²We are assuming $N \gg 1$ where the feedback of the fermions can be neglected.

a deformation: Σ can be “filled” in two topologically distinct ways inside the R^4 and many possible deformations are available.

3.3 Hamiltonian version

One can imagine a Hamiltonian formulation, in which we choose $l_2 = \infty$. Then the infinitely heavy finitely separated source and sink picture associated with an ordinary rectangular Wilson loop, and used for the extraction of the heavy quark-antiquark potential, has a simple generalization. Instead of a nailed down pair of fundamentals we have a circle on which any number of fundamentals can travel and many types of states will contribute. The locations of the charges are still restricted in four dimensions, but not to two separated points, but rather to a common circle. So, in particular, as time evolves, we guess that states made out of diametrically opposite quark-anti-quark pairs, rotating around each other, would develop. This is closer to the semi-classical picture we usually invoke when extracting the experimental string tension from meson Regge trajectories in QCD than the static heavy quark potential we get from traditional Wilson loops. Moreover, since there is no fundamental matter in the bulk, the rotating string along the diameter cannot just decay by pair formation, as it would in QCD [26]. To be sure, while our observable lets the pair members rotate round each other, it still constrains the radius of their trajectory to a fixed number, so the four dimensional argument cannot be fully carried over. In short, the new observable provides an opportunity to look at mesonic states of definite angular momentum in some direction.

Replacing the finite torus Σ by an infinite cylinder does not eliminate the large N phase transition. The Hamiltonian system is expected to undergo a large N phase transition of spontaneous chiral symmetry breaking as the compact direction is shrunk, in analogy with the finite temperature Gross-Neveu model [27].

4 Numerical results for large N

4.1 General observations

One cannot put a smooth closed loop on a hypercubic lattice. One has to allow corners, and there are extra logarithmic divergences associated with these. When dealing with Wilson loops the definition of the observable we used in our numerical work [4, 8, 9] also eliminated these divergences.

Similarly, one cannot put a smoothly embedded torus on the lattice. However, the worst new ultraviolet divergences induced by the extra folds one requires are weaker than the leading ones, which now are just logarithmic. Thus we can use a torus embedded into a four dimensional hypercubic lattice in which the circle of circumference l_1 in Σ is replaced by a square of perimeter L_1 (in lattice units and divisible by 4) and similarly the circle of circumference l_2 in Σ is replaced by a square of perimeter L_2 . It is now straightforward to adapt the overlap [28] action to the lattice Σ so constructed. This allows us to have exact chiral symmetry on the lattice.

Since the fermions only contribute at order N to the free energy, to determine what happens at $N = \infty$ it suffices to keep them quenched, that is their feedback on the dis-

tribution of the gauge fields can be ignored [29]. This simplifies the numerical work by a significant amount.

Equation (3.10) tells us that in order to go to the continuum limit we need to add a 4 fermion term on the embedded surface. Its sign will be the opposite of the standard sign; after a Fierz transformation and a Hubbard-Stratonovich decoupling, the kernel of the quadratic fermion action no longer obeys a property under hermitian conjugation that simplifies the numerical computation of its eigenvalues. Our numerical tests below did not include 4 fermion terms. We looked for numerical evidence for spontaneous chiral symmetry breaking at infinite N in the lattice regularized theory. More work would be needed to go to the continuum limit.

Even as a pure lattice study, our numerical work is just exploratory; our main objective was to find examples of parameters for which we can be reasonably sure that chiral symmetry is spontaneously broken at $N = \infty$ and other examples where we can be reasonably sure that chiral symmetry is preserved by the vacuum of the fermionic system even at $N = \infty$. This provides evidence for the existence of at least one large N phase transition. The simplest assumption is that there exists only one such transition, but we have certainly not ruled out numerically the existence of more transitions than one. Our extrapolations to infinite N are simplistic and would need to be refined in future work.

4.2 Parameter ranges

The four dimensional gauge field at fixed lattice coupling, $b = \frac{1}{g^2 N}$ and given N , is generated in the standard manner on lattices of sizes 6^4 and 10^4 . We looked at the range of b -values 0.348–0.367. The highest b value is determined by the requirement that the 10^4 torus still be in the confining phase at $N = \infty$. The lowest b value is determined by the requirement that we stay out of the infinite N strong coupling phase, at least by metastability. The smaller volume can be used for smaller b -s in the range.

A two dimensional finite flat torus of size $4L \times 4L$ is constructed by the Cartesian product of a closed $L \times L$ loop in the (1,2) plane by another closed $L \times L$ loop in the (3,4) plane. We looked at sizes $4L = 8, 16$.

A “semi-infinite” two dimensional torus of size $4L$ in its finite direction is embedded in the four dimensional lattice forming one closed $L \times L$ loop in the (1,2) plane and another loop in the 3 (or 4) direction closed by winding around the lattice. At $N = \infty$, because of continuum reduction, the torus may be viewed as infinite in the winding direction so long as we are in the confined phase. Again, we looked at $4L = 8, 16$.

We construct the two dimensional overlap operator with anti-periodic boundary conditions on the two dimensional torus, using the four dimensional gauge fields residing on the links of the embedded torus. We compute the two lowest positive eigenvalues of the overlap Dirac operator for 100 random distinct translations of the two dimensional torus on the four dimensional lattice. In this manner we get an average value for the two lowest eigenvalues, $\bar{\lambda}_i$, $i = 1, 2$. We also obtain the distribution of $r = \frac{\lambda_1}{\lambda_2}$, denoted by $P(r)$, for each four dimensional gauge field configuration. We obtain an estimate for $\langle \bar{\lambda}_i \rangle$, $i = 1, 2$, $\langle \bar{r} \rangle$ and $\langle P(r) \rangle$ by averaging over several configurations.

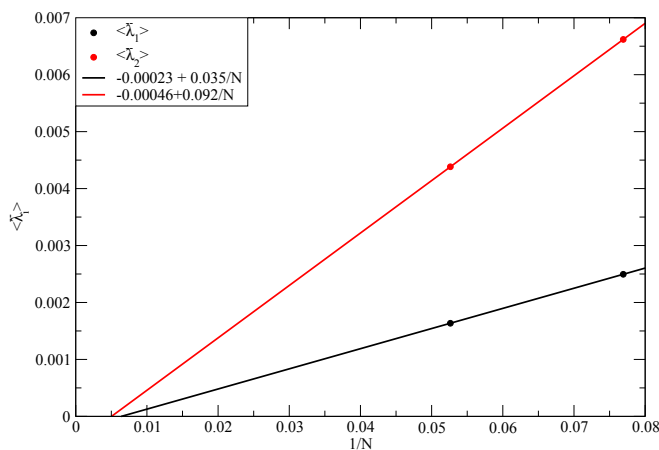


Figure 1. Lowest two eigenvalues as a function of $1/N$ at $b = 0.350$ for a 16×16 torus embedded in a 10^4 lattice.

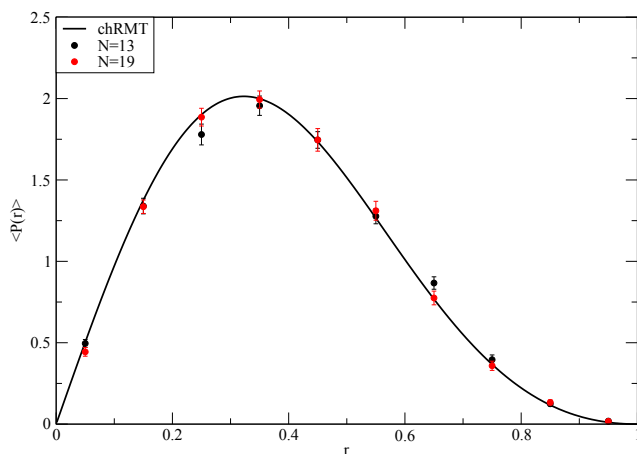


Figure 2. Distribution of the ratio of the two lowest eigenvalues for $N = 13$ and $N = 19$ at $b = 0.350$ for a 16×16 torus embedded in a 10^4 lattice.

4.3 Finite torus

4.3.1 Broken chirality

We present one example in the phase of broken chirality ($L = 16$, $b = 0.350$ on a 10^4 lattice). Figures 1 and 2 show data for $N = 13$ and $N = 19$ and provide evidence for the absence of a spectral gap (figure 1) and for chiral random matrix behavior of the eigenvalue ratio distribution (figure 2). The two lowest eigenvalues in figure 1 extrapolate at $N = \infty$ to small negative numbers which we interpret as saying that both eigenvalues go to zero. One can also calculate the average eigenvalue ratio and extrapolate it to $N = \infty$; one obtains a number comfortably close to the chiral random matrix theory prediction, $0.37674227\dots$

The absence of a gap and the agreement with chiral random matrix theory are strong indicators that chiral symmetry is spontaneously broken in the $N \rightarrow \infty$ limit at this choice of the parameters L and b .

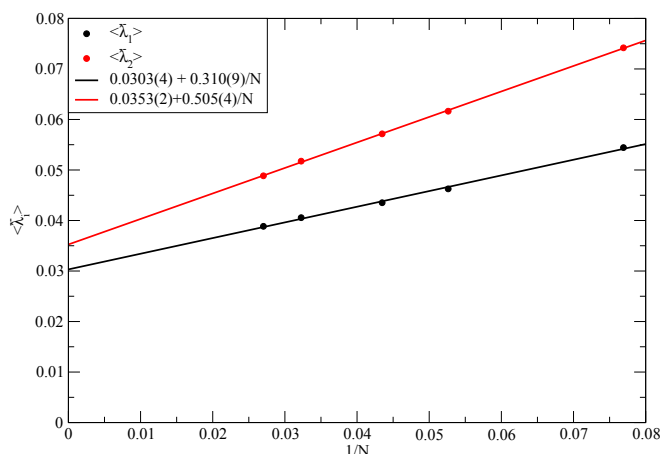


Figure 3. Lowest two eigenvalues as a function of $1/N$ at $b = 0.348$ for a 8×8 torus embedded in a 6^4 lattice.

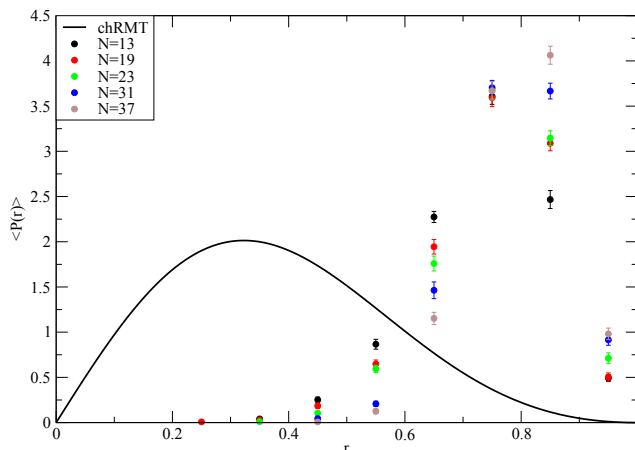


Figure 4. Distribution of the ratio of the two lowest eigenvalues for $N = 13, 19, 23, 31, 37$ at $b = 0.348$ for a 8×8 torus embedded in a 6^4 lattice.

4.3.2 Preserved chirality

Our other example ($L = 8$, $b = 0.348$ on a 6^4 lattice) is in the phase of preserved chirality. We consider a larger range of N values, $N = 13, 19, 23, 31, 37$ because this case requires a more thorough investigation of the N -dependence: chiral random matrix behavior might set in at only high values of N , and we want to rule this out. The results are shown in figure 3. Figure 4 shows that the distribution of the ratio of the two lowest eigenvalues gets more peaked closer to unity as N is increased, becoming more and more different from the chiral random matrix theory result.

The existence of a gap separating the lowest eigenvalue from zero at infinite N implies that chiral symmetry is preserved at this set of $L = 8$, $b = 0.348$ values.

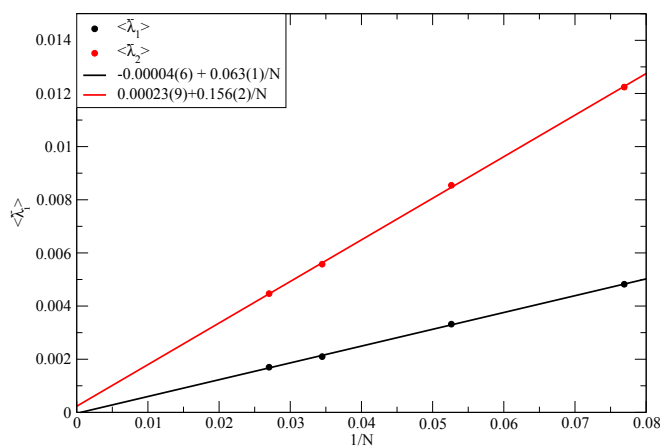


Figure 5. Lowest two eigenvalues as a function of $1/N$ at $b = 0.363$ for a 16×10 torus embedded in a 10^4 lattice.

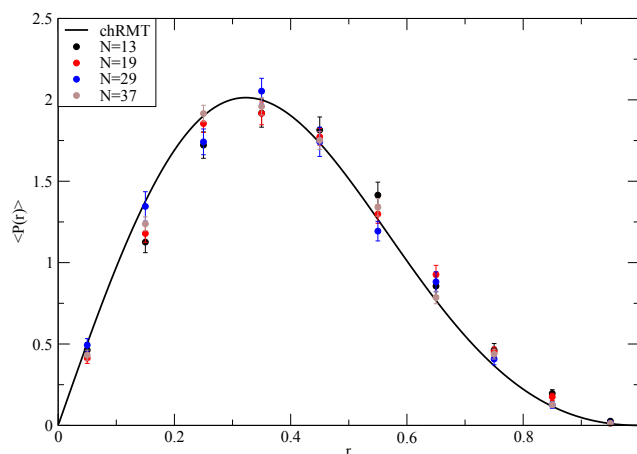


Figure 6. Distribution of the ratio of the two lowest eigenvalues for $N = 13, 19, 29, 37$ at $b = 0.363$ for a 16×10 torus embedded in a 10^4 lattice.

4.4 Torus with one infinite extent

The behavior of the lowest two eigenvalues on a torus with one infinite extent follows the same pattern as on the finite torus.

4.4.1 Broken chirality

We consider a 16×10 torus on a 10^4 lattice and set $b = 0.363$. The N values we used are $N = 13, 19, 28, 27$. The fits to the two lowest eigenvalues in figure 5 are again interpreted as saying that both eigenvalues extrapolate to zero at $N = \infty$.

Figure 6 shows that the distribution of the ratio of the two lowest eigenvalues agrees with the prediction from chiral random matrix theory when extrapolated to infinite N . The average ratio again approaches the chiral random matrix theory number.

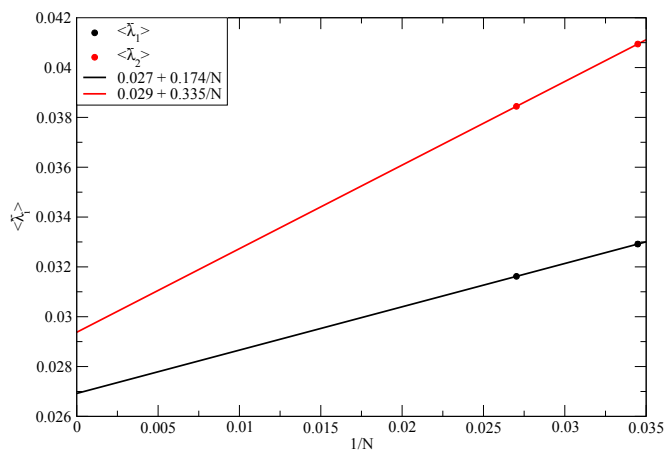


Figure 7. Lowest two eigenvalues as a function of $1/N$ at $b = 0.363$ for a 4×10 torus embedded in a 10^4 lattice.

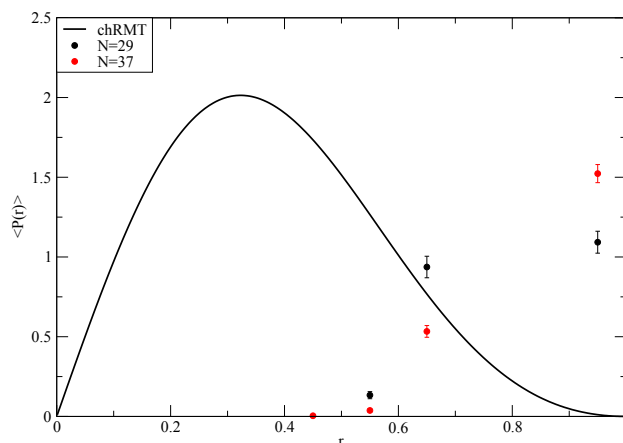


Figure 8. Distribution of the ratio of the two lowest eigenvalues for $N = 29$ and $N = 37$ at $b = 0.363$ for a 4×10 torus embedded in a 10^4 lattice.

We conclude that chiral symmetry is spontaneously broken at $N = \infty$ at this point in parameter space.

4.4.2 Preserved chirality

We now take an 8×10 torus on a 10^4 lattice at $b = 0.363$ for $N = 29$ and $N = 37$. The plot of the two lowest eigenvalues as a function of $1/N$ in figure 7 indicates the existence of a gap at $N = \infty$. The two lowest eigenvalues in figure 7 extrapolate to values which show the presence of a gap at $N = \infty$. Figure 8 shows that the distribution of the ratio of the two lowest eigenvalues gets more peaked closer to unity as N is increased, departing from the prediction of chiral random matrix theory.

We conclude that chiral symmetry is preserved also at $N = \infty$ in this example.

4.5 Comments on numerical work

We have carried out more types of fits in the above selected examples and also simulations at other parameters sets. These studies showed that the simplistic extrapolations of $\frac{1}{N}$ to zero have systematic errors that can overcome the statistical ones. We also found points in parameter space where we could not credibly make a determination about the status of chiral symmetry at infinite N . Therefore we cannot rule out the existence of some more esoteric phases in the large N limit. If more esoteric phases do turn out to exist, there will be more than one large N phase transitions and the large N structure of the crossover may be more complicated than we would like.

These issues need to be addressed. It is possible that the computational resources at our disposal at the moment won't be able to reach the values of N needed to settle these questions conclusively.

5 Summary and discussion

Our main point was to introduce a system coupling two dimensional fermions to four dimensional gauge fields. This provides new non-local observables in pure Yang Mills theory and we gave reasons why they are interesting. We hope that these observables in pure gauge theory will produce parameters expressible in terms of $\Lambda_{SU(N)}$ by an asymptotic matching of perturbation theory to a yet unknown, effective, systematic, description of long distance physics. We provided numerical evidence that these new observables vary with scale through a crossover where a nonanalytic behavior will set in at infinite N . We hypothesize that at $N = \infty$ the crossover becomes a phase transition governed by a well understood universality class.

If this hope pans out, we could match parameters between the two descriptions and use them in other circumstances. For example, one could consider fermions living on the infinite z, t plane while the gauge fields it couples to live on infinite Minkowski space x, y, z, t . It is now convenient to pick the light cone gauge in the z, t directions and consider scattering events of objects made out of fermions in the z, t plane. Unlike in the cylinder or torus case, for a scattering event characterized by a single scale, one expects that even at infinite N the crossover as that scale is varied from short to long will be smooth. The infinite N phase transitions we are studying are observable dependent and so are their universality classes. The basic idea is then to use an observable which has a large N phase transition in order to relate the parameters of a long distance effective theory to those convenient at short distances, and then exploit the general applicability of the long distance theory to calculate other observables, which are smooth in scale even at $N = \infty$.

The infinite spacetime could be taken to be three dimensional, in which case the ultraviolet divergence (3.10) goes away. It is possible that in this case one can do without a four fermion counter-term in the continuum. One cannot embed a torus flatly in the ambient three dimensional spacetime, and therefore one might as well use a sphere and deal with its curvature. On the lattice the sphere would be replaced by the surface of a cube equipped with 8 singular corners. We leave further study of this case for the future.

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