

## Real-time correlators and non-relativistic holography

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

JHEP11(2009)010

(<http://iopscience.iop.org/1126-6708/2009/11/010>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 38.107.179.211

The article was downloaded on 22/02/2012 at 01:01

Please note that [terms and conditions apply](#).

# Real-time correlators and non-relativistic holography

---

**Robert G. Leigh and Nam Nguyen Hoang**

*Department of physics, University of Illinois,  
1110 W. Green St., Urbana IL 61801-3080, U.S.A.*

*E-mail:* [rgleigh@illinois.edu](mailto:rgleigh@illinois.edu), [nnguyen2@illinois.edu](mailto:nnguyen2@illinois.edu)

**ABSTRACT:** We consider Lorentzian correlation functions in theories with non-relativistic Schrödinger symmetry. We employ the method developed by Skenderis and van Rees in which the contour in complex time defining a given correlation function is associated holographically with the gluing together of Euclidean and Lorentzian patches of spacetimes. This formalism extends appropriately to geometries with Schrödinger isometry.

**KEYWORDS:** Gauge-gravity correspondence, Space-Time Symmetries

**ARXIV EPRINT:** [0904.4270](https://arxiv.org/abs/0904.4270)

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The Schrödinger geometry and scalar fields</b>	<b>2</b>
2.1	Lorentzian signature	3
2.2	Euclidean signature	4
<b>3</b>	<b>Non-relativistic holography and correlators</b>	<b>5</b>
3.1	Matching conditions	5
3.2	Convergence and the choice of vacuum	6
<b>4</b>	<b>Correlation functions</b>	<b>7</b>
4.1	Bulk-Boundary Propagator and Time-ordered Correlator	7
4.2	Wightman function	9
4.3	Thermal correlator	10
<b>A</b>	<b>Useful formulas</b>	<b>11</b>

---

## 1 Introduction

Correlation functions of operators in strongly coupled conformal field theories can often be computed using the AdS/CFT correspondence. Euclidean correlators have a long history [1, 2] while the rich analytic structure of various Lorentzian signature correlators can also be obtained. The earliest proposal for the latter was by Son and Starinets [3], and there have also been several elaborations of that method (see for example [4, 5]). Recently, Skenderis and van Rees [6, 7] showed how the complex time contour of an arbitrary correlation function can systematically be accounted for by gluing together manifolds of various signatures, carefully matching fields at the interfaces. This method was used to calculate scalar two-point functions in AdS space, and in asymptotically AdS spaces.

The extension of gauge-gravity duality ideas to spacetimes of Galilean isometries and field theories with non-relativistic invariance [9, 10] has been of much interest in the recent literature. In particular, it is expected that such systems are of more direct relevance to condensed matter models. Correlation functions have recently been computed using standard holographic methods for scalars [9–11] and for fermions [12].

In this paper, we reconsider *Lorentzian* correlators of non-relativistic systems by directly calculating them using the techniques of refs. [6, 7] in Schrödinger geometries. We consider the time-ordered correlator and the Wightman function, as well as thermal correlators.

## 2 The Schrödinger geometry and scalar fields

We consider the  $d + 3$  dimensional Lorentzian geometry [9]

$$ds^2 = L^2 \left( -b^2 \frac{dt^2}{z^4} + \frac{2tdt\xi + d\vec{x}^2 + dz^2}{z^2} \right) \quad (2.1)$$

where  $z \geq 0$  and  $b, L$  are length scales. This geometry has Schrödinger isometry with dynamical exponent equal to two. The Killing vectors are of the form

$$N = \partial_\xi \quad (2.2)$$

$$D = z\partial_z + \vec{x} \cdot \vec{\partial} + 2t\partial_t \quad (2.3)$$

$$H = \partial_t \quad (2.4)$$

$$C = tz\partial_z + t\vec{x} \cdot \vec{\partial} + t^2\partial_t - \frac{1}{2}(\vec{x}^2 + z^2)\partial_\xi \quad (2.5)$$

$$\vec{K} = -t\vec{\partial} + \vec{x}\partial_\xi \quad (2.6)$$

$$\vec{P} = \vec{\partial} \quad (2.7)$$

$N$  is central, and  $D, H, C$  form an  $SL(2, \mathbb{R})$  algebra.

Consider a massive complex scalar propagating on the non-relativistic (Lorentzian) geometry with action

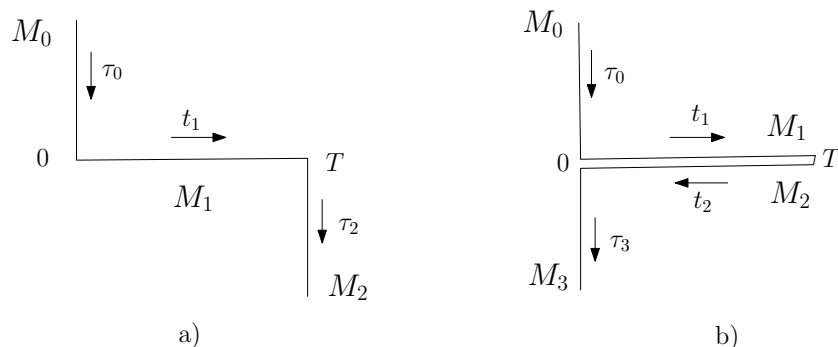
$$S = -\frac{1}{2} \int d^{d+3}x \sqrt{-g} (g^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi + m_0^2/L^2 |\phi|^2) \quad (2.8)$$

The usual interpretation is that the dual theory lives on  $\mathbb{R}^{1,d}$  at  $z = 0$  and is coordinatised by the  $(t, \vec{x})$  coordinates— $\xi$  is not geometric in the usual sense. The isometry  $N : \xi \mapsto \xi + a$  is central and thus  $N$  is strictly conserved. Each operator of the boundary theory can be taken to have a fixed momentum (‘particle number’) conjugate to  $\xi$ .  $\xi$  is usually taken compact (with circumference  $R$ ) so that the spectrum of possible momenta is discrete. In this case, the dimensionless ratio  $b/R$  is a parameter of the theory.

For example, the graviton mode coupling to the stress energy tensor of the boundary theory has particle number zero [13, 14]. Here, we will consider a complex scalar with definite but arbitrary particle number  $n$ . As we will see, it is very important that the scalar be complex. First, it carries a charge under  $N$  and so we should expect it to be complex. More importantly though, it is dual to an operator in a non-relativistic theory, and in such a theory there is a sort of polarization: a simple example of this occurs in free field theories, in which the elementary field creates a particle (and not anti-particle) state.

Now, in this paper we consider correlators of various types. In this regard, as developed by Skenderis and van Rees [6, 7], we regard the metric (2.1) as defined formally for complex  $t$ , and a given correlator is constructed from a particular contour in the complex  $t$  plane. Here, we consider two such cases, in which the contour is constructed from horizontal (Lorentzian time) and vertical (Euclidean time) contour segments (see figure 1).

In the next two subsections, we consider scalar fields in Lorentzian time and in Euclidean time, respectively.



**Figure 1.** Contours corresponding to the time-ordered correlator and the Wightman function, respectively.

## 2.1 Lorentzian signature

Given the metric (2.1) for real time, the scalar equation of motion takes the form

$$z^2 \partial_z^2 \phi - (d+1)z \partial_z \phi + z^2 (2\partial_t \partial_\xi + \partial_i^2) \phi + b^2 \partial_\xi^2 \phi - m_0^2 \phi = 0. \quad (2.9)$$

We look for solutions of the form

$$\phi_{(n)} = e^{in\xi} e^{-i\omega t + i\vec{k} \cdot \vec{x}} f_{\omega, n, \vec{k}}(z), \quad \bar{\phi}_{(n)} = e^{-in\xi} e^{i\omega t - i\vec{k} \cdot \vec{x}} \bar{f}_{\omega, n, \vec{k}}(z) \quad (2.10)$$

in which case  $f$  satisfies

$$z^2 \partial_z^2 f - (d+1)z \partial_z f + z^2 (2\omega n - \vec{k}^2) f - m^2 f = 0, \quad (2.11)$$

where  $m^2 = m_0^2 + n^2 b^2$ . The general solution of (2.11) can be written in terms of modified Bessel functions as

$$f_{n, \omega, \vec{k}}(z) = A(\omega, \vec{k}) z^{\frac{d}{2}+1} K_\nu(qz) + B(\omega, \vec{k}) z^{\frac{d}{2}+1} I_\nu(qz) \quad (2.12)$$

with  $\nu = \sqrt{(\frac{d}{2}+1)^2 + m^2}$  and  $q = \sqrt{q^2} = \sqrt{\vec{k}^2 - 2\omega n}$ .  $K_\nu$  and  $I_\nu$  correspond to non-normalizable and normalizable modes, respectively. Their asymptotic behavior is as follows

$$z^{\frac{d}{2}+1} K_\nu(qz \rightarrow 0) = \Gamma(\nu) \frac{z^{\frac{d}{2}+1-\nu}}{2^{-\nu+1} q^\nu} + \dots \quad (2.13)$$

$$z^{\frac{d}{2}+1} I_\nu(qz \rightarrow 0) = \frac{1}{\Gamma(\nu+1)} \frac{z^{\frac{d}{2}+1+\nu}}{2^\nu q^{-\nu}} + \dots \quad (2.14)$$

$$z^{\frac{d}{2}+1} K_\nu(|qz| \rightarrow \infty) = \sqrt{\frac{\pi z^{d+1}}{2q}} e^{-qz} + \dots \quad (2.15)$$

$$z^{\frac{d}{2}+1} I_\nu(|qz| \rightarrow \infty) = \sqrt{\frac{z^{d+1}}{2\pi q}} \left[ e^{qz} (1 + \dots) + e^{-qz - i\pi(\nu+1/2)} (1 + \dots) \right] \dots \quad (2.16)$$

For  $q^2 < 0$ , both  $K_\nu$  and  $I_\nu$  are regular everywhere, while for  $q^2 > 0$ ,  $I_\nu$  diverges for large  $z$  and should be discarded. This situation is very similar to that of a scalar field

propagating on  $AdS_{d+3}$ , where the solution can also be written in terms of modified Bessel functions. In fact this similarity is very useful and was employed in ref. [11] to compute the non-relativistic bulk-to-boundary propagator. We note though that there is a small but important difference due to the non-relativistic nature of the boundary theory, that we will explain presently.

Without loss of generality, we take  $n > 0$ . To construct the most general solution (with fixed  $n$ ), we must integrate over all values of  $\omega, \vec{k}$ . However,  $q$  has a branch point at  $\omega = \vec{k}^2/2n$ , and we must then say how to integrate over  $\omega$ . Following [6], we do so by moving the branch point off of the real  $\omega$  axis by defining  $q_\epsilon = \sqrt{-2\omega n + \vec{k}^2 - i\epsilon}$ ,  $\bar{q}_\epsilon = \sqrt{-2\omega n + \vec{k}^2 + i\epsilon}$ . The branch cut is taken along the negative real axis. Clearly, we have made a choice here, but we will see later that this is the correct choice, for physical reasons. Notice that since  $Re(q_\epsilon), Re(\bar{q}_\epsilon) > 0$ ,  $K_\nu$  always decays exponentially as  $|qz| \rightarrow \infty$ . In contrast, the large  $z$  behavior of  $I_\nu$  tells us that  $q, \bar{q}$  cannot have a real part. As a result, the  $i\epsilon$  insertion should not be applied for the normalizable mode.<sup>1</sup>

With these comments, we arrive at the general solution to (2.11) in Lorentzian signature

$$\phi_{(n)}(t, \vec{x}) = e^{in\xi} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega t + i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}+1} \left( A(\omega, \vec{k}) K_\nu(q_\epsilon z) + \theta(-q^2) B(\omega, \vec{k}) J_\nu(|q|z) \right) \quad (2.17)$$

where we have used  $I_\nu(\sqrt{q^2}z) = I_\nu(-i|q|z) \sim J_\nu(|q|z)$ .

## 2.2 Euclidean signature

Next, we consider a similar analysis in Euclidean signature. To do so, we Wick rotate the metric (2.1) to [15]

$$ds^2 = L^2 \left( b^2 \frac{d\tau^2}{z^4} + \frac{-2id\tau d\xi + d\vec{x}^2 + dz^2}{z^2} \right) \quad (2.18)$$

Although this metric is complex and thus not physical, it is possible to trace carefully through the analysis, and this is what we need to do in any case for Euclidean signature.

The general solution is

$$\phi_{(n)}(\tau, \vec{x}) = e^{in\xi} \int \frac{d\omega_E}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega_E \tau + i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}+1} A(\omega_E, \vec{k}) K_\nu(q_E z) \quad (2.19)$$

$$\bar{\phi}_{(n)}(\tau, \vec{x}) = e^{-in\xi} \int \frac{d\omega_E}{2\pi} \frac{d^d k}{(2\pi)^d} e^{i\omega_E \tau - i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}+1} \bar{A}(\omega_E, \vec{k}) K_\nu(\bar{q}_E z) \quad (2.20)$$

where now  $q_E = \sqrt{q_E^2} = \sqrt{\vec{k}^2 - i2\omega_E n}$ . Note that in this case, the branch point is at imaginary  $\omega_E$ , and so no  $i\epsilon$  insertion is necessary.

In contrast to the Lorentzian case, the Euclidean scalar does not have a normalizable mode. This is because  $q_E$  and  $\bar{q}_E$  cannot be pure imaginary, so  $I_\nu(q_E z)$  is never regular in the interior. It is important to note, however, that this statement applies to the case  $\tau \in$

---

<sup>1</sup>This fact was not clearly spelled out in ref. [6] in the relativistic analogue, but we will see later that it is an important point.

$(-\infty, \infty)$ . If  $\tau$  is restricted, a normalizable mode can emerge. For example, if  $\tau \in [0, \infty)$ , we write  $\omega_E = -i\omega$  for  $\phi$  and  $\omega_E = i\omega$  for  $\bar{\phi}$  and the following mode is allowable

$$\phi \sim e^{in\xi} e^{-\omega\tau + i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}+1} I_\nu(qz) \tag{2.21}$$

$$\bar{\phi} \sim e^{-in\xi} e^{-\omega\tau - i\vec{k}\cdot\vec{x}} z^{\frac{d}{2}+1} I_\nu(\bar{q}z) \tag{2.22}$$

as long as  $\omega > 0$  and  $-2\omega n + \vec{k}^2 < 0$ , or equivalently  $\omega > \vec{k}^2/2n$ .

A similar result pertains in the finite temperature case where  $\tau \in [0, \beta]$ . Observe however that in contrast to the relativistic real-time formalism, there is no normalizable mode for the Euclidean segment if we restrict  $\tau \in (-\infty, 0)$ . This is because we would need both  $\omega < 0$  and  $-2\omega n + \vec{k}^2 < 0$ , and these contradict each other. This will have important consequences. In particular we note that there is no normalizable mode in the segment  $M_0$  of either contour in figure 1.

### 3 Non-relativistic holography and correlators

#### 3.1 Matching conditions

To construct correlation functions, we must match solutions at the interfaces between contour segments. We will label field values on a contour segment  $M_n$  by a subscript,  $\phi_n$ . Let us begin by considering the Lorentzian( $M_1$ )-Lorentzian( $M_2$ ) interface in figure 1b, where  $t_1 \in [0, T]$  and  $t_2 \in [T, 2T]$  (where  $T \rightarrow \infty$  is a large time). The total action (for these two segments) is

$$\begin{aligned} S &= S_{M_1} + S_{M_2} \tag{3.1} \\ &= \int_0^T dt_1 \left( g_{M_1}^{\mu\nu} \partial_\mu \bar{\phi}_1 \partial_\nu \phi_1 + m_0^2/L^2 \bar{\phi}_1 \phi_1 \right) - \int_T^{2T} dt_2 \left( g_{M_1}^{\mu\nu} \partial_\mu \bar{\phi}_2 \partial_\nu \phi_2 + m_0^2/L^2 \bar{\phi}_2 \phi_2 \right) \end{aligned}$$

The relative minus sign arises because  $M_1$  and  $M_2$  have opposite orientation. For the same reason, the metric in  $M_2$  is

$$ds_{M_2}^2 = L^2 \left( -\frac{dt_2^2}{z^4} + \frac{-2dt_2 d\xi + d\vec{x}^2 + dz^2}{z^2} \right), \tag{3.2}$$

which has an extra minus sign in the off-diagonal component.

Requiring continuity of the momentum conjugate to  $\bar{\phi}$  at the intersection  $t_1 = t_2 = T$ , we get

$$\partial_\xi \phi_1 = \partial_\xi \phi_2. \tag{3.3}$$

Along with the continuity of  $\phi$ , we conclude that the matching conditions at  $t_1 = t_2 = T$  are

$$\phi_1(T) = \phi_2(T) \tag{3.4}$$

$$n_1 = n_2 \tag{3.5}$$

Thus, we do not need to impose first-order time derivative continuity of fields along the contour as in the relativistic case — it is just replaced by particle number conservation. It turns out that (3.4), (3.5) are also the matching conditions for Euclidean-Lorentzian interfaces.

### 3.2 Convergence and the choice of vacuum

The non-relativistic holographic correspondence is in general the same as its relativistic counterpart, where the path integral with specified boundary conditions in the bulk is identified with the partition function with sources inserted in the boundary theory. In the case of a complex bulk scalar, we must temporarily treat the sources  $\phi_{(0)}$  and  $\bar{\phi}_{(0)}$  as independent. The near boundary expansion of the fields are qualitatively the same as scalars on  $AdS_{d+3}$

$$\phi_{(n)} = e^{in\xi} \left\{ z^{\Delta_-} (\phi_{(0)} + z^2 \phi_{(2)} + o(z^4)) + z^{\Delta_+} (v_{(0)} + z^2 v_{(2)} + o(z^4)) \right\} \quad (3.6)$$

$$\bar{\phi}_{(n)} = e^{in\xi} \left\{ z^{\Delta_-} (\bar{\phi}_{(0)} + z^2 \bar{\phi}_{(2)} + o(z^4)) + z^{\Delta_+} (\bar{v}_{(0)} + z^2 \bar{v}_{(2)} + o(z^4)) \right\}, \quad (3.7)$$

with  $\Delta_{\pm} = 1 + d/2 \pm \nu$  and

$$\phi_{(2m)} = \frac{1}{2m(2\Delta_+ - (d+2) - 2m)} \square_0 \phi_{(2m-2)}, \quad (3.8)$$

where here  $\square_0 = 2in\partial_t + \partial_i^2$  is the non-relativistic Laplacian. As usual the holographic correspondence implies

$$e^{iS_C^{\text{bulk}}[\bar{\phi}_{(0)}, \phi_{(0)}]} = \langle e^{i \int_C (\hat{O}^\dagger \phi_{(0)} + \bar{\phi}_{(0)} \hat{O})} \rangle, \quad (3.9)$$

where  $C$  denotes the contour. Although we have a very different geometry, it's easily seen that in each patch of the contour the bulk (either Euclidean or Lorentzian) on-shell action

$$S_{\text{os}} = \frac{1}{2} \int_{\epsilon} d^{d+1} x d\xi \sqrt{|g|} \bar{\phi} g^{zz} \partial_z \phi \quad (3.10)$$

is essentially the same as scalars on  $AdS_{d+3}$ . As a result, the renormalization procedure proceeds in the same way as  $AdS_{d+3}/CFT_{d+2}$ , which was carried out in much details in [8]. In specific, for Lorentzian signature the counter terms take the form,

$$S_{\text{ct}} = \int_{\epsilon} d^{d+1} x d\xi \sqrt{-\gamma} \left( \frac{d+2-\Delta_+}{2} \bar{\phi} \phi + \frac{1}{2(\Delta_+ - d - 4)} \bar{\phi} \square_{\gamma} \phi + \dots \right), \quad (3.11)$$

where  $\sqrt{-\gamma} = z^{-(d+2)}$  is the  $(d+2)$ -dimensional induced metric determinant and  $\square_{\gamma} = z^2(2in\partial_t + \partial_i^2)$  (we will set  $L = 1$  from now on). The dots represent higher derivative terms. For special cases where  $\nu$  is an integer, logarithmic counter terms  $\sim \log \epsilon$  may appear [8]. It's important to note that  $S_{\text{ct}}$  preserves the Galilean subalgebra, since  $[\square_{\gamma}, K_i] = 0$ . This is in parallel with relativistic holography where the Poincare subalgebra is preserved by the counter terms. In any case,  $v_{(0)}$  will determine the v.e.v of the dual operator and its derivative with respect to the source  $\phi_{(0)}$  gives us the 2-point functions.

There is, however, a subtlety of which we must be cognizant. Unlike relativistic field theories, in non-relativistic field theories an elementary field  $\Psi$  and its Hermitian conjugate  $\Psi^\dagger$  play the role of creation and annihilation operators. There is a freedom to choose which is an annihilator, or equivalently a freedom to pick the vacuum. Once a convention is chosen,  $\Psi$  and  $\Psi^\dagger$  are no longer on the same footing. This is also true for any operator  $\hat{O}$ ,

$\hat{\mathcal{O}}^\dagger$ , in which  $\hat{\mathcal{O}}$  is constructed only from annihilators. This corresponds to the fact that there is only a single pole in the complex  $\omega$ -plane in the non-relativistic case. Consequently, the time-ordered propagator will in fact have only a single temporal  $\theta$ -function present. We expect to see this coming about in the analysis, but to see this properly, one has to be careful with the convergence of various integrals.

## 4 Correlation functions

In both cases shown in figure 1, we have an initial vertical contour  $M_0$ . The correlation functions of interest are computed by including source(s) on horizontal component(s) of the contour. We first show that given such a contour component  $M_0$ , there is no normalizable mode (such a mode would be everywhere subleading in the  $z \rightarrow 0$  expansion). This implies that any solution with a specific boundary condition is *unique*. Indeed, we argued in section 2.2 that there is no non-trivial normalizable solution in  $M_0$ . So in the cases of interest (no sources on  $M_0$ ),  $\phi_0 = 0$  identically. The matching condition between  $\phi_0$  and  $\phi_1$  then requires that  $\phi_1(t_1 = 0, \vec{x}, z) = 0$ . The most general normalizable solution on  $M_1$  is

$$\phi_1^{\text{norm}}(t_1, \vec{x}, z) = e^{in\xi} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega t_1 + i\vec{k} \cdot \vec{x}} z^{\frac{d}{2}+1} \theta(-q^2) B(\omega, \vec{k}) J_\nu(|q|z). \quad (4.1)$$

Multiply by  $z^{-\frac{d}{2}} e^{-in\xi - i\vec{k}' \cdot \vec{x}} J_\nu(|q'|z)$  with  $q'^2 = -2\omega'n + \vec{k}'^2 < 0$  and integrate over  $\vec{x}$  and  $z$ . We then find

$$0 = \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} d^d x e^{i\vec{x} \cdot (\vec{k} - \vec{k}')} B(\omega, \vec{k}) \theta(-q^2) \left( \int_0^\infty dz z J_\nu(|q|z) J_\nu(|q'|z) \right) \quad (4.2)$$

The  $z$ -integral is elementary (see appendix, eq. (A.1)) and this becomes

$$0 = \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} d^d x e^{i\vec{x} \cdot (\vec{k} - \vec{k}')} B(\omega, \vec{k}) \theta(-q^2) \frac{1}{|q'|} \delta(|q| - |q'|) \quad (4.3)$$

$$= \frac{1}{n} \int \frac{d\omega}{2\pi} B(\omega, \vec{k}') \theta(2\omega n - \vec{k}'^2) \delta(\omega - \omega') \quad (4.4)$$

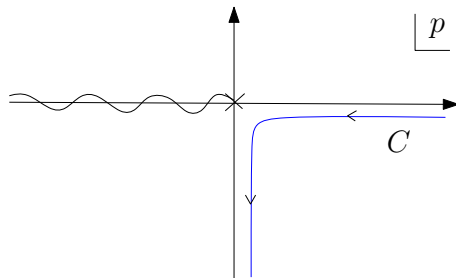
$$= \frac{1}{2\pi n} B(\omega', \vec{k}') \theta(-q'^2). \quad (4.5)$$

Thus, if  $\phi_1(t, \vec{x}, z) = 0$  at some time, there is no non-trivial normalizable mode. This reasoning in fact applies for all segments of both contours in figure 1.

### 4.1 Bulk-Boundary Propagator and Time-ordered Correlator

Given the absence of a normalizable mode, any solution with sources that we find for the two contours in figure 1 is unique. In this subsection, we consider contour figure 1a, with segments  $M_0$  ( $\tau_0 \in (-\infty, 0]$ ),  $M_1$  ( $t_1 \in [0, T]$ ),  $M_2$  ( $\tau_2 \in [0, \infty)$ ). We place a single  $\delta$ -function source at  $\vec{x} = 0, t_1 = \hat{t}_1$  on  $M_1$ . From our discussions above,  $\phi_1$  must be of the form

$$\phi_{1,(n)}(t_1, \vec{x}, z) = \frac{2}{\Gamma(\nu)} e^{in\xi} z^{1+d/2} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t_1 - \hat{t}_1) + i\vec{k} \cdot \vec{x}} \left(\frac{q_\epsilon}{2}\right)^\nu K_\nu(q_\epsilon z). \quad (4.6)$$



**Figure 2.** Contour of integration in the complex  $p$ -plane for the Lorentzian bulk-boundary propagator.

as this satisfies  $z^{-\Delta_-} \phi_{1,(n)}(t_1, \vec{x}, z)|_{z \rightarrow 0} = e^{in\xi} \delta(t_1 - \hat{t}_1) \delta(\vec{x})$ , and any ambiguity corresponds to normalizable modes, which we have argued are zero. Since there are no sources on  $M_2$ ,  $\phi_2$  takes the form

$$\phi_{2,(n)} = \frac{2\pi i}{\Gamma(\nu)} e^{in\xi} z^{1+d/2} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-\omega(\tau+iT-i\hat{t}_1)+i\vec{k}\cdot\vec{x}} \theta(-q^2) \left(\frac{|q|}{2}\right)^\nu J_\nu(|q|z). \quad (4.7)$$

which has been deduced from the matching condition  $\phi_1(t_1 = T) = \phi_2(\tau = 0)$  as follows. For any time  $t_1 > \hat{t}_1$ , we can re-expand  $\phi_1$  in terms of  $J_\nu$ 's. In particular, at  $t_1 = T$ , we should have

$$\int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(T-\hat{t}_1)+i\vec{k}\cdot\vec{x}} q_\epsilon^\nu z K_\nu(q_\epsilon z) = \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(T-\hat{t}_1)+i\vec{k}\cdot\vec{x}} C(\omega, \vec{k}) \theta(-q^2) z J_\nu(|q|z) \quad (4.8)$$

for some  $C(\omega, \vec{k})$ . To find this coefficient we use the same trick as in the last subsection: multiply both sides by  $e^{i\omega'(T-\hat{t}_1)-i\vec{k}'\cdot\vec{x}} J_\nu(|q'|z)$  with  $q'^2 = -2\omega'n + \vec{k}'^2 < 0$  and integrate over  $\vec{x}, z$ . The right-hand side gives  $\frac{1}{2\pi n} \theta(-q^2) C(\omega', \vec{k}')$ , while the left-hand side can be computed using (A.2) to give  $\frac{i}{2n} |q'|^\nu$ .

The bulk-boundary propagator is essentially identified with  $\phi_1$  itself: if we simply strip off the  $e^{in\xi}$  factor, we can write

$$K_{n,n'}(t, \vec{x}, z) = \delta_{n,n'} K_{(n)}(t, \vec{x}, z) \quad (4.9)$$

$$K_{(n)}(t, \vec{x}, z; \hat{t}) = \frac{2z^{1+d/2}}{\Gamma(\nu)} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t-\hat{t})+i\vec{k}\cdot\vec{x}} \left(\frac{q_\epsilon}{2}\right)^\nu K_\nu(q_\epsilon z). \quad (4.10)$$

As shown in ref. [11] for example, this is closely related to the bulk-boundary propagator in  $AdS_{d+3}$ . Alternatively, we may perform the integration directly, following the analogous treatment in ref. [6]. To do so, it is convenient to convert the  $\omega$ -integral to an integration over  $p = q_\epsilon$ , and the contour in the  $p$ -plane is as shown in figure 2.

Here though there is just one branch point (at  $\omega = \vec{k}^2/2n - i\epsilon$ ) and the  $i\epsilon$  tells us in which sense to traverse the cut. One arrives at

$$K_{(n)}(t, \vec{x}, z; \hat{t}) = \theta(t_1 - \hat{t}_1) \frac{1}{\pi^{d/2} \Gamma(\nu)} \left(\frac{n}{2i}\right)^{\Delta_+ - 1} \left(\frac{z}{t_1 - \hat{t}_1}\right)^{\Delta_+} e^{in \frac{z^2 + \vec{x}^2 + i\epsilon}{2(t_1 - \hat{t}_1)}} \quad (4.11)$$

where  $\Delta_\pm = 1 + d/2 \pm \nu$ .

The correlator is then identified with the  $z^{\Delta_+}$  coefficient in the near boundary expansion of  $\phi_1$  (without the  $e^{in\xi}$  factor)

$$\langle T(\hat{\mathcal{O}}_{(n)}(\vec{x}, t_1)\hat{\mathcal{O}}_{(n)}^\dagger(\vec{x}', t'_1)) \rangle = \frac{1}{\pi^{d/2}\Gamma(\nu)} \left(\frac{n}{2i}\right)^{\Delta_+-1} \frac{\theta(t_1 - t'_1)}{(t_1 - t'_1)^{\Delta_+}} e^{in\frac{(\vec{x}-\vec{x}')^2+i\epsilon}{2(t_1-t'_1)}}. \quad (4.12)$$

## 4.2 Wightman function

The time-ordered correlator, as we have explained, contains a single temporal  $\theta$ -function. It does not tell us about  $\langle \hat{\mathcal{O}}(\vec{x}, t_1)\hat{\mathcal{O}}^\dagger(\vec{x}', t'_1) \rangle$  for  $t'_1 > t_1$ . To find this 2-point function we work with the contour of figure 1b. Denote the segments by  $M_0$  ( $\tau_0 \in (-\infty, 0]$ ),  $M_1$  ( $t_1 \in [0, T]$ ),  $M_2$  ( $t_2 \in [T, 2T]$ ) and  $M_3$  ( $\tau_3 \in [0, \infty)$ ) as sketched in the figure. We place a  $\delta$ -function source at  $\vec{x} = 0, t_1 = \hat{t}_1$  on  $M_1$  and nowhere else. The Wightman function is obtained then from  $\phi_2$ , the field on  $M_2$ . Here  $\phi_0 = 0$  and  $\phi_1$  remain the same as (4.6). Given experience from the last subsection, we can see immediately that  $\phi_2$  should be

$$\phi_{2,(n)} = \frac{2\pi i}{\Gamma(\nu)} e^{in\xi} z^{1+d/2} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(2T-t_2-\hat{t}_1)+i\vec{k}\cdot\vec{x}} \left(\frac{|q|}{2}\right)^\nu \theta(-q^2) J_\nu(|q|z). \quad (4.13)$$

This has been determined by requiring the matching condition  $\phi_1(t_1 = T) = \phi_2(t_2 = T)$ . Notice the unusual  $e^{+i\omega t_2+i\vec{k}\cdot\vec{x}}$  wave factor. It is related to the fact mentioned before that along this part of the contour, the metric has an extra minus sign in the  $g_{t_2\xi}$  component.

It is now necessary to compute  $\phi_2$  in coordinate space. We make a change of variable  $p = |q| = \sqrt{2\omega n - \vec{k}^2}$

$$\phi_2 = \frac{i}{n\Gamma(\nu)2^\nu} e^{in\xi} z^{1+d/2} \int_0^\infty dp e^{-ip^2(2T-t_2-\hat{t}_1)/2n} p^{\nu+1} J_\nu(pz) \int \frac{d^d k}{(2\pi)^d} e^{-ik^2(2T-t_2-\hat{t}_1)/2n} e^{i\vec{k}\cdot\vec{x}}. \quad (4.14)$$

We note that both integrals converge if  $2T - t_2 - \hat{t}_1 \rightarrow 2T - t_2 - \hat{t}_1 - i\epsilon$ . The first integral can be computed using (A.3), while the second one is just a Gaussian integral. The final result is

$$\phi_2 = e^{in\xi} \frac{1}{\pi^{d/2}\Gamma(\nu)} \left(\frac{n}{2i}\right)^{\Delta_+-1} \left(\frac{z}{\tilde{t}_2 - \hat{t}_1 - i\epsilon}\right)^{\Delta_+} e^{in\frac{z^2+\vec{x}^2}{2(\tilde{t}_2-\hat{t}_1-i\epsilon)}}. \quad (4.15)$$

where  $\tilde{t}_2 = 2T - t_2$ . Observe that  $\phi_2$  is closely related to the bulk-boundary propagator (4.11) except for the absence of the step function and a different  $i\epsilon$  insertion, as expected.

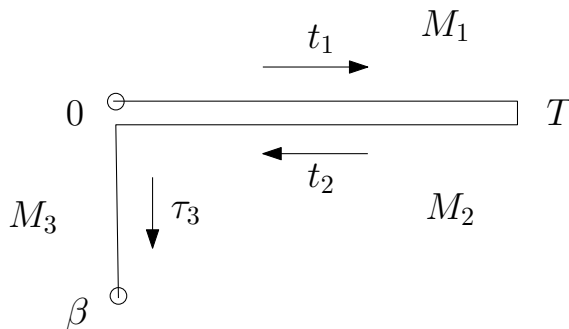
The vacuum expectation value of  $\hat{\mathcal{O}}(\tilde{t}_2, \vec{x})$  is

$$\langle \hat{\mathcal{O}}(\tilde{t}_2, \vec{x}) e^{i(\phi_{1(0)}\hat{\mathcal{O}}^\dagger + \bar{\phi}_{1(0)}\hat{\mathcal{O}})} \rangle = \frac{1}{\pi^{d/2}\Gamma(\nu)} \left(\frac{n}{2i}\right)^{\Delta_+-1} \int dt_1 d^d x' \frac{e^{in\frac{(\vec{x}-\vec{x}')^2}{2(\tilde{t}_2-t_1-i\epsilon)}}}{(\tilde{t}_2 - t_1 - i\epsilon)^{\Delta_+}} \phi_{1(0)}(t_1, \vec{x}'). \quad (4.16)$$

Taking a derivative with respect to  $\phi_{1(0)}$  and setting the source to zero, we get the Wightman function

$$\langle \hat{\mathcal{O}}(\tilde{t}_2, \vec{x}) \hat{\mathcal{O}}^\dagger(t_1, \vec{x}') \rangle = \frac{1}{\pi^{d/2}\Gamma(\nu)} \left(\frac{n}{2i}\right)^{\Delta_+-1} \frac{e^{in\frac{(\vec{x}-\vec{x}')^2}{2(\tilde{t}_2-t_1-i\epsilon)}}}{(\tilde{t}_2 - t_1 - i\epsilon)^{\Delta_+}} \quad (4.17)$$

Notice that  $\hat{\mathcal{O}}^\dagger$  is always in the front of  $\hat{\mathcal{O}}$  because  $t_1$  is always the earlier contour time.



**Figure 3.** Thermal contour. Points with a circle are identified.

### 4.3 Thermal correlator

Finally, we compute a thermal correlator by taking the time direction to be compact of period  $\beta$ .

To compute the thermal time-ordered correlator and Wightman function, we consider the thermal contour shown in figure 3, where  $t = 0$  and  $t = -i\beta$  are identified. We place a  $\delta$ -function source at  $t_1 = \hat{t}_1, \vec{x} = 0$ . Note that in contrast to the previous discussions, here there is no  $M_0$  component of the contour. It is convenient in this context to write the general solution along  $M_1$  in the form

$$\phi_1 = \frac{2e^{in\xi} z^{1+d/2}}{\Gamma(\nu)} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t_1 - \hat{t}_1) + i\vec{k} \cdot \vec{x}} \times \quad (4.18)$$

$$\times \left( A(\omega, \vec{k}) \left(\frac{q_\epsilon}{2}\right)^\nu K_\nu(q_\epsilon z) + B(\omega, \vec{k}) \left(\frac{q_{-\epsilon}}{2}\right)^\nu K_\nu(q_{-\epsilon} z) \right).$$

where  $q_{-\epsilon} = \bar{q}_\epsilon = \sqrt{-2\omega n + \vec{k}^2 + i\epsilon}$ . In order that this correspond to a  $\delta$ -function source for  $z \rightarrow 0$ , we must have  $A + B = 1$ . (Furthermore, the case  $B = -A$  corresponds to a normalizable mode.) Note that because of the condition on  $A, B$ , although  $A$  and  $B$  are not necessarily analytic functions, their sum is analytic. Thus for example, for any pole in  $A$ , there will be a corresponding pole in  $B$  with opposite residue. All of their poles will contribute opposite residues and cancel out each other in the limit  $\epsilon \rightarrow 0$ . In (4.18), the first term has support for  $t_1 > \hat{t}_1$ , while the second has support for  $t_1 < \hat{t}_1$ .

The matching condition at  $(M_1, M_2)$  and  $(M_2, M_3)$  intersections imply that

$$\phi_2 = \frac{2\pi i e^{in\xi} z^{1+d/2}}{\Gamma(\nu)} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(2T - t_2 - \hat{t}_1) + i\vec{k} \cdot \vec{x}} A(\omega, \vec{k}) \left(\frac{|q|}{2}\right)^\nu J_\nu(|q|z) \theta(-q^2) \quad (4.19)$$

$$\phi_3 = \frac{2\pi i e^{in\xi} z^{1+d/2}}{\Gamma(\nu)} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-\omega(\tau_3 - i\hat{t}_1) + i\vec{k} \cdot \vec{x}} A(\omega, \vec{k}) \left(\frac{|q|}{2}\right)^\nu J_\nu(|q|z) \theta(-q^2) \quad (4.20)$$

The thermal condition  $\phi_1(t_1 = 0) = \phi_3(\tau_3 = \beta)$  along with  $A + B = 1$  then gives

$$A = \frac{1}{1 - e^{-\beta\omega}}, \quad B = \frac{1}{1 - e^{+\beta\omega}}. \quad (4.21)$$

As usual, the time-ordered propagator is the coefficient of  $z^{\Delta+}$  in the small  $z$  expansion of  $\phi_1$  (without the  $e^{in\xi}$  factor). Hence we get<sup>2</sup>

$$\langle T(\hat{O}(x)\hat{O}^\dagger(x')) \rangle \sim \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \times \left( \frac{(-2\omega n + \vec{k}^2 - i\epsilon)^\nu}{1 - e^{-\beta\omega}} + \frac{(-2\omega n + \vec{k}^2 + i\epsilon)^\nu}{1 - e^{\beta\omega}} \right). \quad (4.22)$$

Note that this has the expected form for a thermal correlator [6]

$$\langle T(\hat{O}(x)\hat{O}^\dagger(x')) \rangle = -N(\omega)\Delta_A(\omega, \vec{k}) + (1 + N(\omega))\Delta_R(\omega, \vec{k}) \quad (4.23)$$

In the present notation,  $N = -B$ . We can also write this as the zero temperature result plus a finite temperature piece:

$$\langle T(\hat{O}(x)\hat{O}^\dagger(x')) \rangle \sim \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \left[ q_\epsilon^{2\nu} - \frac{1}{1 - e^{\beta\omega}} (q_\epsilon^{2\nu} - q_{-\epsilon}^{2\nu}) \right] \quad (4.24)$$

The Wightman function can also be read off from  $\phi_2$

$$\langle \hat{O}(x)\hat{O}^\dagger(x') \rangle \sim i\pi \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t-t'-i\epsilon) + i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{(2\omega n - \vec{k}^2)^\nu}{1 - e^{-\beta\omega}} \theta(2\omega n - \vec{k}^2) \quad (4.25)$$

## A Useful formulas

We record integrals that have been useful in the above analysis.

$$\int_0^\infty t J_\nu(qt) J_\nu(q't) dt = \frac{1}{q} \delta(q - q'), \quad q, q' \text{ real}, \nu > -\frac{1}{2} \quad (A.1)$$

$$\int_0^\infty K_\mu(at) J_\nu(bt) t^{\mu+\nu+1} dt = \frac{(2a)^\mu (2b)^\nu \Gamma(\mu + \nu + 1)}{(a^2 + b^2)^{\mu+\nu+1}}, \quad Re(\nu + 1) > Re(\mu), Re(a) > |Im(b)| \quad (A.2)$$

$$\int_0^\infty e^{-a^2 t^2} t^{\nu+1} J_\nu(bt) dt = \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-\frac{b^2}{4a^2}}, \quad Re(\nu) > -1, Re(a^2) > 0 \quad (A.3)$$

## References

- [1] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[SPIRES](#)].
- [2] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d+1) correspondence*, *Nucl. Phys. B* **546** (1999) 96 [[hep-th/9804058](#)] [[SPIRES](#)].

---

<sup>2</sup>For integer  $\nu$ , there is an extra logarithmic factor, namely  $q_{\pm\epsilon}^{2\nu}$  is replaced by  $q_{\pm\epsilon}^{2\nu} \ln q_{\pm\epsilon}^2$ .

- [3] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [[hep-th/0205051](#)] [[SPIRES](#)].
- [4] C.P. Herzog and D.T. Son, *Schwinger-Keldysh propagators from AdS/CFT correspondence*, *JHEP* **03** (2003) 046 [[hep-th/0212072](#)] [[SPIRES](#)].
- [5] N. Iqbal and H. Liu, *Real-time response in AdS/CFT with application to spinors*, *Fortsch. Phys.* **57** (2009) 367 [[arXiv:0903.2596](#)] [[SPIRES](#)].
- [6] K. Skenderis and B.C. van Rees, *Real-time gauge/gravity duality: prescription, renormalization and examples*, *JHEP* **05** (2009) 085 [[arXiv:0812.2909](#)] [[SPIRES](#)].
- [7] K. Skenderis and B.C. van Rees, *Real-time gauge/gravity duality*, *Phys. Rev. Lett.* **101** (2008) 081601 [[arXiv:0805.0150](#)] [[SPIRES](#)].
- [8] K. Skenderis *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)] [[SPIRES](#)].
- [9] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev. D* **78** (2008) 046003 [[arXiv:0804.3972](#)] [[SPIRES](#)].
- [10] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [[arXiv:0804.4053](#)] [[SPIRES](#)].
- [11] A. Volovich and C. Wen, *Correlation functions in non-relativistic holography*, *JHEP* **05** (2009) 087 [[arXiv:0903.2455](#)] [[SPIRES](#)].
- [12] A. Akhavan, M. Alishahiha, A. Davody and A. Vahedi, *Fermions in non-relativistic AdS/CFT correspondence*, [arXiv:0902.0276](#) [[SPIRES](#)].
- [13] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up galilean holography*, *JHEP* **11** (2008) 080 [[arXiv:0807.1099](#)] [[SPIRES](#)].
- [14] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [[arXiv:0807.1111](#)] [[SPIRES](#)].
- [15] C.A. Fuertes and S. Moroz, *Correlation functions in the non-relativistic AdS/CFT correspondence*, *Phys. Rev. D* **79** (2009) 106004 [[arXiv:0903.1844](#)] [[SPIRES](#)].