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Abstract: The $N = 2$ spinning particle action describes the propagation of antisymmetric tensor fields, including vector fields as a special case. In this paper we study the path integral quantization on a one-dimensional torus of the $N = 2$ spinning particle coupled to spacetime gravity. The action has a local $N = 2$ worldline supersymmetry with a gauged U(1) symmetry that includes a Chern-Simons coupling. Its quantization on the torus produces the one-loop effective action for a single antisymmetric tensor. We use this worldline representation to calculate the first few Seeley-DeWitt coefficients for antisymmetric tensor fields of arbitrary rank in arbitrary dimensions. As side results we obtain the correct trace anomaly of a spin 1 particle in four dimensions as well as exact duality relations between differential form gauge fields. This approach yields a drastic simplification over standard heat-kernel methods. It contains on top of the usual proper time a new modular parameter implementing the reduction to a single tensor field. Worldline methods are generically simpler and more efficient in perturbative computations than standard QFT Feynman rules. This is particularly evident when the coupling to gravity is considered.

Keywords: Sigma Models, Duality in Gauge Field Theories, Anomalies in Field and String Theories.

Contents

1. Introduction

Worldline approaches often produce efficient tools for calculating Feynman diagrams of standard quantum field theories, see e.g. [1] for a review. Recently, also gravitational interactions have been discussed in this framework by considering the path integral quantization of worldlines of particles of spin 0 and 1/2 embedded in a curved spacetime [2−4]. This method has then been applied to study new processes [5]. In this paper we wish to study the propagation of particles of spin 1 and, more generally, of antisymmetric tensor fields coupled to gravity. The corresponding mechanical model that must be quantized is the $N = 2$ spinning particle discussed in [6], and further analyzed and extended in [7−11]. This model with a suitable Chern-Simons coupling is known to describe antisymmetric tensor fields of arbitrary rank [10]. Our aim is to describe the one-loop effective action of a spin 1 particle and, more generally, of antisymmetric tensor fields in terms of worldline path integrals. Indeed this is possible: proceeding with the quantization of the spinning particle one obtains a quite interesting representation of the one-loop effective action. This effective action is written in terms of an integration over two moduli: the standard proper time and a new parameter related to the gauge fixing of the U(1) symmetry. The effect of this new modular parameter is to restrict the propagation to the sector which corresponds to a single tensor field. The worldline representation produces a drastic simplification over standard heat-kernel methods, and gives us the chance of computing the first few Seeley-DeWitt coefficients for antisymmetric tensor fields of *arbitrary rank* in arbitrary dimensions. Previous studies for a worldline description of particles of spin 1 were presented in [12, 13], where the projection to the physical states of the spin 1 particle was achieved by using a certain limiting procedure. It differs from the construction presented here. However, we do not consider the couplings to background electromagnetism or Yang-Mills fields which, on the other hand, were investigated in [12, 13].

The paper is organized as follows. In section 2 we review the $N = 2$ spinning particle and remind that it describes the propagation of a gauge potential p-form A_p with standard gauge invariant Maxwell action. It turns out that the particle description is achieved directly in terms of the field strength, the $(p + 1)$-form $F_{p+1} = dA_p$. In section 3 we consider the path integral quantization of the $N = 2$ spinning particle action on the torus, and describe its gauge fixing. In particular, the gauge fixing of the U(1) symmetry produces a new modular parameter on top of the usual proper time. There is no need of summing over the spin structures of the fermions as the integration over the U(1) modulus effectively interpolates between all boundary conditions. The gauge fixed path integral thus obtained gives a novel representation of the one-loop effective action for a p-form. In section 4 we discuss the perturbative evaluation of this effective action using an expansion in the proper time. This way we are able to compute the first few Seeley-DeWitt coefficients for a p-form in arbitrary dimensions, namely the coefficients a_0, a_1 and a_2. As a side result we obtain the trace anomaly for a 1-form, i.e. a spin 1 particle, in 4 dimensions using worldline methods. In section 5 we derive exact duality relations between differential forms, and then present our conclusions. In the appendix A we collect some technical results on the worldline propagators and determinants, and on the dimensional regularization of the $N = 2$ nonlinear sigma model.

2. A brief review of the $N = 2$ spinning particle

The $N = 2$ spinning particle action is characterized by a $N = 2$ extended supergravity on the worldline. The gauge fields $(e, \chi, \bar{\chi}, a)$ of the $N = 2$ supergravity contain in particular the einbein e which gauges worldline translations, complex conjugate gravitinos χ and $\bar{\chi}$ which gauge the $N = 2$ worldline supersymmetry, and a standard gauge field a for the U(1) symmetry which rotates by a phase the worldline fermions and gravitinos. The einbein and the gravitinos correspond to constraints that eliminate negative norm states and make the particle model consistent with unitarity. The constraints arising from the gauge field a makes the model irreducible, eliminating some further degrees of freedom [8 − 10].

The action in flat target spacetime is most easily deduced by starting with a model with $N = 2$ extended rigid supersymmetry, and then gauging its symmetries. The rigid model is described in a graded phase space with real bosonic variables (x^μ, p_μ) and complex fermionic variables $(\psi^\mu, \bar{\psi}_\mu)$ ($\bar{\psi}_\mu$ is the complex conjugate of ψ_μ). It is given by the following real action

$$S = \int dt \left[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu - \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu \right] \qquad (2.1)$$

where the indices $\mu, \nu = 0, 1, \cdots, D-1$ are spacetime indices and $\eta_{\mu\nu} \sim (-, +, \cdots, +)$ is the Lorentz metric used to lower and raise spacetime indices. A dot denotes as usual the time derivative. The graded Poisson brackets are then given by $\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu$ and $\{\psi^\mu, \bar{\psi}_\nu\}_{PB} = -i\delta^\mu_\nu$. This action is manifestly Poincaré invariant in target space and thus describes a relativistic model which, however, is not unitary at this stage. The lack of unitarity is due to negative norm states which appear in the Hilbert space because of the timelike value the index μ can take on the bosonic and fermionic variables. As well-known in relativistic string and particle theory, unitarity can be recovered by imposing suitable constraints. This can be achieved as follows. The action (2.1) has on the worldline a rigid $N = 2$ supersymmetry generated by the charges

$$H = \frac{1}{2} p_\mu p^\mu \,, \qquad Q = p_\mu \psi^\mu \,, \qquad \bar{Q} = p_\mu \bar{\psi}^\mu \,, \qquad J = \bar{\psi}^\mu \psi_\mu \,. \tag{2.2}$$

The whole symmetry algebra can be gauged since the charges close under Poisson brackets and can be considered as a set of first class constraints

$$\{Q, \bar{Q}\}_{PB} = -2iH \,, \qquad \{J, Q\}_{PB} = iQ \,, \qquad \{J, \bar{Q}\}_{PB} = -i\bar{Q} \tag{2.3}$$

(other Poisson brackets vanish). These constraints are enough to recover unitarity, as it will be evident. Introducing the gauge fields $G = (e, \bar{\chi}, \chi, a)$ which correspond to the constraints $C = (H, Q, \bar{Q}, J)$ gives the action

$$\begin{aligned} S &= \int dt \left[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu - eH - i\bar{\chi} Q - i\chi \bar{Q} - aJ \right] \\ &= \int dt \left[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu - \frac{1}{2} e p_\mu p^\mu - i\bar{\chi} p_\mu \psi^\mu - i\chi p_\mu \bar{\psi}^\mu - a\bar{\psi}^\mu \psi_\mu \right] . \end{aligned} \tag{2.4}$$

The gauge transformations on the phase space variables are generated through Poisson brackets by the generator $G \equiv \xi H + i\bar{\epsilon} Q + i\epsilon \bar{Q} + \alpha J$, where $\xi, \bar{\epsilon}, \epsilon, \alpha$ are local parameters with appropriate Grassmann parity,

$$\begin{aligned} \delta x^\mu &= \{x^\mu, G\}_{PB} = \xi p^\mu + i\bar{\epsilon}\psi^\mu + i\epsilon\bar{\psi}^\mu \\ \delta p_\mu &= \{p_\mu, G\}_{PB} = 0 \\ \delta \psi^\mu &= \{\psi^\mu, G\}_{PB} = -\epsilon p^\mu - i\alpha\psi^\mu \\ \delta \bar{\psi}^\mu &= \{\bar{\psi}^\mu, G\}_{PB} = -\bar{\epsilon} p^\mu + i\alpha\bar{\psi}^\mu \end{aligned} \tag{2.5}$$

while on gauge fields the gauge transformations are easily obtained with the help of the constraint algebra (2.3)

$$\begin{aligned} \delta e &= \dot{\xi} + 2i\bar{\chi}\epsilon + 2i\chi\bar{\epsilon} \\ \delta \chi &= \dot{\epsilon} + ia\epsilon - i\alpha\chi \\ \delta \bar{\chi} &= \dot{\bar{\epsilon}} - ia\bar{\epsilon} + i\alpha\bar{\chi} \\ \delta a &= \dot{\alpha} \,. \end{aligned} \tag{2.6}$$

Eliminating algebraically the momenta p_μ by using their equations of motion

$$p^\mu = \frac{1}{e} \left(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu \right) \tag{2.7}$$

one obtains the action in configuration space

$$S = \int dt \left[\frac{1}{2} e^{-1} (\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)^2 + i\bar{\psi}_\mu \dot{\psi}^\mu - a\bar{\psi}^\mu \psi_\mu \right]. \tag{2.8}$$

The corresponding gauge invariances can be easily deduced from the phase space ones using (2.7).

In addition, one can add a Chern-Simons term for the gauge field a

$$S_{CS} = q \int dt\, a \tag{2.9}$$

which is obviously invariant under the gauge transformations (2.6). Absence of anomalies requires a quantization of the Chern-Simons coupling [14]

$$q = \frac{D}{2} - p - 1\,, \qquad p \text{ integer}\,. \tag{2.10}$$

With this precise coupling the $N = 2$ spinning particle describes an antisymmetric gauge field of rank p (and corresponding field strength of rank $p + 1$). In fact the gauge field a with this Chern-Simons coupling produces the constraint $J = \frac{D}{2} - p - 1$ instead of $J = 0$. A vector field (spin 1) has $p = 1$ and thus does not need a Chern-Simons coupling in $D = 4$, though such a term will be needed in different dimensions.

Let us derive some of these statements by briefly reviewing the canonical quantization of the model. The phase space variables are turned into operators satisfying the following (anti)commutation relations (we use $\hbar = 1$)

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu\,, \qquad \left\{ \hat{\psi}^\mu, \hat{\psi}^\dagger_\nu \right\} = \delta^\mu_\nu\,. \tag{2.11}$$

States of the full Hilbert space can be described by functions of the coordinates x^μ and ψ^μ. By x^μ we denote the eigenvalues of the operator $\hat{x}^\mu$, while for the fermionic variables we use bra coherent states defined by

$$\langle\psi|\hat{\psi}^\mu = \langle\psi|\psi^\mu \;= \psi^\mu\langle\psi|\,. \tag{2.12}$$

Any state $|\phi\rangle$ can then be described by the wave function

$$\phi(x, \psi) \equiv (\langle x| \otimes \langle\psi|)|\phi\rangle \tag{2.13}$$

and since the ψ's are Grassmann variables the wave function has the following general expansion

$$\phi(x, \psi) = F(x) + F_\mu(x)\psi^\mu + \frac{1}{2} F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + \cdots + \frac{1}{D!} F_{\mu_1\cdots\mu_D}(x)\psi^{\mu_1}\cdots\psi^{\mu_D}\,. \tag{2.14}$$

The classical constraints C now become operators $\hat{C}$ which are used to select the physical states through the requirement $\hat{C}|\phi_{phys}\rangle = 0$. In the above representation they take the form of differential operators

$$\hat{H} = -\frac{1}{2}\partial_\mu\partial^\mu\,, \qquad \hat{Q} = -i\psi^\mu\partial_\mu\,, \qquad \hat{Q}^\dagger = -i\partial_\mu \frac{\partial}{\partial\psi_\mu}\,, \qquad \hat{J} = -\frac{1}{2}\left[\psi^\mu, \frac{\partial}{\partial\psi^\mu}\right] - q \tag{2.15}$$

where we have redefined $\hat{J}$ to include the Chern-Simons coupling and antisymmetrized $\hat{\psi}^\mu$ and $\hat{\psi}^\dagger_\mu$ to resolve an ordering ambiguity. The constraint $\hat{J}|\phi_{phys}\rangle = 0$ selects states with only $p+1$ ψ's (recall that $q \equiv \frac{D}{2} - p - 1$), namely

$$\phi_{phys}(x,\psi) = \frac{1}{(p+1)!} F_{\mu_1\cdots\mu_{p+1}}(x)\psi^{\mu_1}\cdots\psi^{\mu_{p+1}}\,. \tag{2.16}$$

The constraints $\hat{Q}|\phi_{phys}\rangle = 0$ gives the Bianchi identities

$$\partial_{[\mu}F_{\mu_1\cdots\mu_{p+1}]}(x) = 0 \tag{2.17}$$

and the constraint $\hat{Q}^\dagger|\phi_{phys}\rangle = 0$ produces the Maxwell equations

$$\partial^{\mu_1}F_{\mu_1\cdots\mu_{p+1}}(x) = 0\,. \tag{2.18}$$

The constraint $\hat{H}|\phi_{phys}\rangle = 0$ is automatically satisfied as a consequence of the algebra $\{\hat{Q},\hat{Q}^\dagger\} = 2\hat{H}$.

Thus we see that the $N=2$ spinning particle describes the propagation of a standard p-form gauge potential $A_{\mu_1\cdots\mu_p}$ in a gauge invariant way, namely through its $F_{\mu_1\cdots\mu_{p+1}}$ field strength. The path integral approach of this model has also been used to obtain the free propagator for a tensor field, thus confirming the physical spectrum just discussed [15 – 18].

Finally, we review the coupling to spacetime gravity. This coupling can be achieved by suitably covariantizing the constraints $H, Q, \bar{Q}, J$. It is convenient, though not necessary, to use flat indices for the worldline fermions by introducing the vielbein $e_\mu{}^a$ and the corresponding spin connection $\omega_\mu{}^{ab}$. The action reads still as

$$S = \int dt \Big[p_\mu \dot{x}^\mu + i\bar{\psi}_a\dot{\psi}^a - eH - i\bar{\chi}Q - i\chi\bar{Q} - aJ\Big] \tag{2.19}$$

but with covariantized constraints (we now include the Chern-Simons term in J)

$$\begin{aligned} J &= \bar{\psi}^a\psi_a - q \\ Q &= \psi^a e_a{}^\mu \pi_\mu \\ \bar{Q} &= \bar{\psi}^a e_a{}^\mu \pi_\mu \\ H &= \frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu - \frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\,. \end{aligned} \tag{2.20}$$

Here we have defined the "covariant" momentum

$$\pi_\mu \equiv p_\mu - i\omega_{\mu ab}\bar{\psi}^a\psi^b \tag{2.21}$$

which becomes the Lorentz covariant derivative upon canonical quantization. The covariantizations of Q and $\bar{Q}$ are easy to guess. Then one may use the algebra to identify H. Of course one must also check that the full constraint algebra remains unchanged. For example, $\{Q,Q\}_{PB} = 0$ is verified using the cyclic identity satisfied by the Riemann tensor. Elimination of the momentum p_μ gives the configuration space action

$$\begin{aligned} S = \int dt \bigg[&\frac{1}{2}e^{-1}g_{\mu\nu}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)(\dot{x}^\nu - i\bar{\chi}\psi^\nu - i\chi\bar{\psi}^\nu) + \\ &+ i\bar{\psi}_a(\dot{\psi}^a + \dot{x}^\mu\omega_\mu{}^a{}_b\psi^b + ia\psi^a) + \frac{e}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d + qa\bigg]. \end{aligned} \tag{2.22}$$

This is the action we are going to quantize on the torus in the next sections. Actually, for simplicity, we prefer to use euclidean conventions. So we perform a Wick rotation to euclidean time ($t \to -i\tau$, and also $a \to ia$ to keep the gauge group U(1) compact) which produces the euclidean action ($S_E = -iS$)

$$S_E = \int_0^1 d\tau \bigg[\frac{1}{2} e^{-1} g_{\mu\nu} (\dot{x}^\mu - \bar{\chi}\psi^\mu - \chi\bar{\psi}^\mu)(\dot{x}^\nu - \bar{\chi}\psi^\nu - \chi\bar{\psi}^\nu) + \\ + \bar{\psi}_a(\dot{\psi}^a + \dot{x}^\mu \omega_\mu{}^a{}_b \psi^b + ia\psi^a) - \frac{e}{2} R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d - iqa \bigg] \tag{2.23}$$

where $\tau \in [0,1]$ parametrizes the torus. From now on we will drop the subscript on S_E as no confusion should arise. Before closing this section, we list the gauge transformations of the supergravity multiplet in euclidean time, as they will be needed to study the gauge fixing

$$\begin{aligned} \delta e &= \dot{\xi} + 2\bar{\chi}\epsilon + 2\chi\bar{\epsilon} \\ \delta\chi &= \dot{\epsilon} + ia\epsilon - i\alpha\chi \\ \delta\bar{\chi} &= \dot{\bar{\epsilon}} - ia\bar{\epsilon} + i\alpha\bar{\chi} \\ \delta a &= \dot{\alpha}\,. \end{aligned} \tag{2.24}$$

3. Quantization on a torus

We have seen that the action (2.23) describes the propagation of a p-form in a background metric $g_{\mu\nu}$, or vielbein $e_\mu{}^a$ (see figure 1). Its quantization on a torus is then expected to produce the one-loop effective action $\Gamma_p^{QFT}[g_{\mu\nu}]$ due to the virtual propagation of a p-form gauge field in a gravitational background (see figure 2)

$$\Gamma_p^{QFT}[g_{\mu\nu}] \sim Z[g_{\mu\nu}] = \int_{T^1} \frac{\mathcal{D}G\mathcal{D}X}{\text{Vol(Gauge)}}\, e^{-S[X,G;g_{\mu\nu}]} \tag{3.1}$$

where $G = (e, \chi, \bar{\chi}, a)$ and $X = (x^\mu, \psi^\mu, \bar{\psi}^\mu)$ indicate the dynamical fields that must be integrated over, and $S[X,G;g_{\mu\nu}]$ denotes the action in (2.23). Division by the volume of the gauge group reminds of the necessity of fixing the gauge symmetries.

The torus is described by taking the parameter $\tau \in [0,1]$ and imposing periodic boundary conditions on the bosonic fields x^μ and e (the gauge field a is instead treated as a

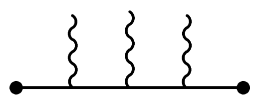

Figure 1: Propagation of a p-form, wavy lines represent external gravitons.

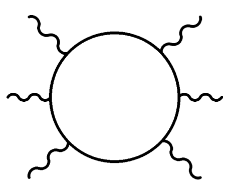

Figure 2: Loop of a p-form in a gravitational background.

connection). As for the fermions we take antiperiodic boundary conditions, and we shall soon understand why this is sufficient. The gauge symmetries can be used to fix the supergravity multiplet to $\hat{G} = (\beta, 0, 0, \phi)$, where β and ϕ are the leftover bosonic moduli that must be integrated over. The parameter β is the usual proper time [23 – 25], while the parameter ϕ is a phase that corresponds to the only modular parameter that the gauge field a can have on the torus. Note that the gravitinos χ and $\bar{\chi}$ are antiperiodic and can be completely gauged away using (2.24), leaving no moduli.

It is worthwhile to discuss more extensively how the modular parameter ϕ arises. The action for the fermions in (2.23) is of the standard form (the target space geometry is inessential for this particular gauge fixing, and one can take it flat)

$$S \sim \int_0^1 d\tau\, \bar{\psi}(\dot{\psi} + ia\psi)\,. \tag{3.2}$$

Finite gauge transformations are given by

$$\begin{array}{rcl} \psi(\tau) & \to & \psi'(\tau) = e^{-i\alpha(\tau)}\psi(\tau) \\ \bar{\psi}(\tau) & \to & \bar{\psi}'(\tau) = e^{i\alpha(\tau)}\bar{\psi}(\tau) \\ a(\tau) & \to & a'(\tau) = a(\tau) + \dot{\alpha}(\tau) \end{array} \tag{3.3}$$

where the gauge transformations $e^{-i\alpha(\tau)}$ are required to be periodic functions on $[0, 1]$. Note also that in one dimension the only gauge invariant quantity that can be constructed from the gauge field is the Wilson loop

$$w = e^{i\int_0^1 d\tau\, a(\tau)}\,. \tag{3.4}$$

Using "small" gauge transformations, i.e. those continuously connected to the identity, one can bring $a(\tau)$ to a constant value ϕ

$$\phi = \int_0^1 d\tau\, a(\tau)\,. \tag{3.5}$$

Then "large" gauge transformations with $\alpha(\tau) = 2\pi n\tau$ allow to identify

$$\phi \sim \phi + 2\pi n\,, \qquad n \text{ integer}. \tag{3.6}$$

Therefore one can take $\phi \in [0, 2\pi]$ as the fundamental region of the moduli space. The value of the Wilson loop is given by the phase $w = e^{i\phi}$, and once again one can recognize that ϕ is an angle.

Let us now comment on the choice of antiperiodic boundary conditions for the fermions. In the gauge $a(\tau) = \phi$ the action (3.2) becomes

$$S \sim \int_0^1 d\tau\, \bar{\psi}(\dot{\psi} + i\phi\psi)\,. \tag{3.7}$$

One may now redefine the fermion by $\psi' = e^{i\phi\tau}\psi$ to eliminate the gauge field from the action

$$S \sim \int_0^1 d\tau\, \bar{\psi}'\dot{\psi}'\,. \tag{3.8}$$

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However, the new field acquires twisted boundary conditions

$$\psi'(1) = -e^{i\phi}\psi'(0) \tag{3.9}$$

so that the modulus $\phi \in [0, 2\pi]$ interpolates between all possible boundary conditions specified by a phase. Therefore there was no loss of generality in the original assumption of antiperiodic boundary conditions: the sum over spin structures is automatically taken care of by the integration over the U(1) modulus. Note that a similar situation appears in the $N = 2$ string theory [21, 22]. For $\phi = \pi$ one obtains periodic boundary conditions. This is a delicate point, as the fermions acquire zero modes (and the gravitinos develop corresponding moduli) whose effects we will comment upon in the next section.

We are now ready to describe the gauge fixing of (3.1). We choose the gauge $\hat{G} = (\beta, 0, 0, \phi)$, insert the Faddeev-Popov determinants to eliminate the volume of the gauge group, and integrate over the moduli. This gives

$$\Gamma_p^{QFT}[g_{\mu\nu}] = \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(2\cos\frac{\phi}{2}\right)^{-2} \int_{T^1} \mathcal{D}X\, e^{-S[X,\hat{G};g_{\mu\nu}]} \tag{3.10}$$

where: $i)$ the measure over the proper time β takes into account the effect of the symmetry generated by the Killing vector on the torus (namely the constant vector); $ii)$ the Faddeev-Popov determinants from the commuting susy ghosts are obtained from (2.24) and are computed to give $\det^{-1}(\partial_\tau + i\phi)\det^{-1}(\partial_\tau - i\phi) = (2\cos\frac{\phi}{2})^{-2}$. These are the inverse of the fermionic determinant arising from (3.7) which is easily computed: antiperiodic boundary conditions produce a trace over the corresponding two-dimensional Hilbert space and thus $\det(\partial_\tau + i\phi) = e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}} = 2\cos\frac{\phi}{2}$. For more details see the appendix A. $iii)$ The other Faddeev-Popov determinants do not give rise to any moduli dependent term.

Thus, up to the final integration over the moduli, one is left with a standard path integral for a nonlinear $N = 2$ susy sigma model. This path integral cannot be evaluated exactly for arbitrary background metrics $g_{\mu\nu}$, but it is the starting point of various approximations schemes. In particular, we will consider here an expansion in terms of the proper time β which leads to the local heat-kernel expansion of the effective action [19, 20]. It is a derivative expansion depending on the so-called Seeley-DeWitt coefficients. Note that, strictly speaking, the effective action does not have a derivative expansion for massless fields, but the corresponding Seeley-DeWitt coefficients still characterize the field theoretical model.

4. Proper time expansion

In the previous section we have set up the worldline path integral representation for the one-loop effective action of a p-form gauge field coupled to gravity. We now wish to compute it in a proper time expansion. For this purpose we need to evaluate, perturbatively in β, the following path integral

$$\int_{T^1} \mathcal{D}X\, e^{-S[X,\hat{G};g_{\mu\nu}]+iq\phi} \tag{4.1}$$

where, for convenience, we have extracted the constant Chern-Simons term from the action, so that the nonlinear sigma model action reads as

$$S[X,\hat{G};g_{\mu\nu}] = \frac{1}{\beta}\int_0^1 d\tau \Bigg[\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \\ + \bar{\psi}_a(\dot{\psi}^a + i\phi\psi^a + \dot{x}^\mu {\omega_\mu}^a{}_b\psi^b) - \frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\Bigg]. \tag{4.2}$$

We have found it convenient to scale the fermion by $\psi^a \to \psi^a/\sqrt{\beta}$ to extract a global factor $1/\beta$. This shows that β can be used as a loop counting parameter and thus organizes the loop expansion. The perturbative calculation now is standard and mimics the worldline treatment of spin 0 and 1/2 particles described in [2–4]. One can extract the dependence on the zero modes x_0^μ of the coordinates and obtains

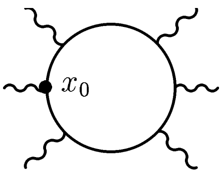

Figure 3: Loop with a marked point.

$$\Gamma_p^{QFT}[g_{\mu\nu}] = \int_0^\infty \frac{d\beta}{\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{iq\phi}\left(2\cos\frac{\phi}{2}\right)^{D-2} \times \\ \times \int \frac{d^Dx_0\sqrt{g(x_0)}}{(2\pi\beta)^{\frac{D}{2}}}\langle e^{-S_{int}}\rangle\,. \tag{4.3}$$

Let us comment on the various terms appearing in this formula.

i) The constant zero modes x_0^μ are factorized by setting $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ and imposing the Dirichlet boundary conditions $y^\mu(0) = y^\mu(1) = 0$ on the quantum fields $y^\mu(\tau)$. This describes a loop with a fixed point x_0 (see figure 3). The measure for integrating over x_0 is fixed by covariance and by checking the correct flat space limit, so that $\langle 1\rangle = 1$ as far as the path integral over the y's is concerned. Other options for factorizing the zero modes are available and have been carefully analyzed in [4].

ii) The extra factor $\left(2\cos\frac{\phi}{2}\right)^D$ comes form the normalization of the fermionic path integral and corresponds to $\det^D(\partial_\tau + i\phi)$.

iii) One can deduce the quantization law for the Chern-Simons coupling from (4.3). The point $\phi = 0$ is gauge equivalent to the point $\phi = 2\pi$. Thus in even dimensions periodicity in ϕ requires that q be an integer. On the other hand, in odd dimensions it must be a half integer to compensate the anomalous behavior of the fermionic determinants $\det^D(\partial_\tau + i\phi) = \left(2\cos\frac{\phi}{2}\right)^D$.

iv) The propagators are identified from the quadratic part of the action which is obtained by Taylor expanding the metric around the point x_0^μ

$$S_2 = \frac{1}{\beta}\int_0^1 d\tau\ \left[\frac{1}{2}g_{\mu\nu}(x_0)\dot{y}^\mu\dot{y}^\nu + \bar{\psi}_a(\dot{\psi}^a + i\phi\psi^a)\right]. \tag{4.4}$$

The interactions are then given by $S_{int} = S - S_2$. In particular, the fermion propagator is

$$\langle\psi^a(\tau)\bar{\psi}_b(\sigma)\rangle = \beta\,\delta^a_b\Delta_{AF}(\tau-\sigma,\phi) \tag{4.5}$$

where the function $\Delta_{AF}(x,\phi)$ is given for $x \in]-1,1[$ by

$$\Delta_{AF}(x,\phi) = \frac{e^{-i\phi x}}{2\cos\frac{\phi}{2}}\left[e^{i\frac{\phi}{2}}\theta(x) - e^{-i\frac{\phi}{2}}\theta(-x)\right] \tag{4.6}$$

with $\theta(x)$ the step function (taking the value 0 for $x < 0$ and 1 for $x > 0$). It satisfies

$$(\partial_x + i\phi)\Delta_{AF}(x,\phi) = \delta(x)\,. \tag{4.7}$$

For $\phi = 0$ it reduces to the propagator for antiperiodic fermions already used in [3], $\Delta_{AF}(x) = \frac{1}{2}[\theta(x) - \theta(-x)]$. For coinciding points ($\tau = \sigma$ i.e. $x = 0$) it takes the (regulated) value

$$\Delta_{AF}(0,\phi) = \frac{i}{2}\tan\frac{\phi}{2}\,. \tag{4.8}$$

In the appendix A we give some details on how this propagator is obtained.

v) It is convenient to use "measure" ghosts to exponentiate the nontrivial dependence of the path integral measure on $g_{\mu\nu}$. These ghosts can be added by replacing

$$\dot{x}^\mu \dot{x}^\nu \qquad \to \qquad \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \tag{4.9}$$

in the action. The a^μ ghosts are commuting while the b^μ, c^ν ghosts are anticommuting, and they keep track in all Feynman diagrams of the contributions coming from the path integral measure associated to nonlinear sigma models [26, 27]. They make the calculation finite, but a regularization is still needed to remove finite ambiguities. The relevant propagators are reported in the appendix A.

vi) To compute $\langle e^{-S_{int}}\rangle$ one must select a regularization scheme and add the corresponding counterterm. We employ dimensional regularization. We have checked that for our $N = 2$ model the counterterm from dimensional regularization vanish, just like in the $N = 1$ model [3]. Note that other regularization schemes may need a non-vanishing counterterm (as the time-slicing scheme [28]). We discuss this issue more extensively in the appendix A. Finally, we choose Riemann normal coordinates and a two-loop calculation on the worldline produces (we use conventions with $R > 0$ on the sphere and rewrite $\tan^2\frac{\phi}{2} = \cos^{-2}\frac{\phi}{2} - 1$ for convenience whenever it appears)

$$\begin{aligned}
\langle e^{-S_{int}}\rangle &= 1 + \beta R\left(\frac{1}{12} - \frac{1}{8}\cos^{-2}\frac{\phi}{2}\right) + \\
&\quad + \beta^2\Bigg[R^2_{abcd}\left(\frac{1}{720} - \frac{1}{192}\cos^{-2}\frac{\phi}{2} + \frac{1}{128}\cos^{-4}\frac{\phi}{2}\right) + \\
&\qquad + R^2_{ab}\left(-\frac{1}{720} + \frac{1}{32}\cos^{-2}\frac{\phi}{2} - \frac{1}{32}\cos^{-4}\frac{\phi}{2}\right) + \\
&\qquad + R^2\left(\frac{1}{288} - \frac{1}{96}\cos^{-2}\frac{\phi}{2} + \frac{1}{128}\cos^{-4}\frac{\phi}{2}\right) + \\
&\qquad + \nabla^2 R\left(\frac{1}{120} - \frac{1}{96}\cos^{-2}\frac{\phi}{2}\right)\Bigg] + O(\beta^3)\,.
\end{aligned} \tag{4.10}$$

We are now left to insert this perturbative result into eq. (4.3). As already mentioned, infrared divergences prevent a meaningful local expansion of the effective action for massless fields. This is signaled from the fact that when (4.10) is inserted into (4.3) the proper time integral does not converge at $\beta = \infty$ (the infrared region). In fact, the standard mass term $e^{-\frac{1}{2}m^2\beta}$ which insures convergence for massive theories is absent in this case. One may nevertheless write eq. (4.3) in the form

$$\begin{aligned} \Gamma_p^{QFT}[g_{\mu\nu}] &= \int_0^\infty \frac{d\beta}{\beta} Z_p(\beta) \\ Z_p(\beta) &= \int \frac{d^D x_0 \sqrt{g(x_0)}}{(2\pi\beta)^{\frac{D}{2}}} \left(a_0 + a_1\beta + a_2\beta^2 + \ldots\right) \end{aligned} \tag{4.11}$$

where the coefficients a_i are the Seeley-DeWitt coefficients (in the coincidence limit). Even if convergence of the proper time integral in the upper limit is not guaranteed, one can still compute these coefficients. They characterize the theory. For example they identify the counterterms needed to renormalize the full effective action. In addition, a_2 gives the trace anomaly for a spin 1 field in four dimensions, and a_1 the trace anomaly for a scalar field in two dimensions. To compute these coefficients and test the correctness of the previous set up we must integrate over the U(1) modulus. Before doing that let us parametrize the structure of the Seeley-DeWitt coefficients contained in the round bracket of $Z(\beta)$ as follows

$$\left(v_1 + v_2 R\beta + \left(v_3 R^2_{abcd} + v_4 R^2_{ab} + v_5 R^2 + v_6 \nabla^2 R\right)\beta^2 + O(\beta^3)\right). \tag{4.12}$$

Next we will compute the coefficients v_i.

4.1 Seeley-DeWitt coefficients

Let us now discuss the integration over the modular parameter ϕ. We see that at $\phi = \pi$ there is a potential divergence appearing in the integrand (4.10) (as $\cos^{-2}\frac{\pi}{2} = \infty$). In fact at this point of moduli space the fermions develop a zero mode and, as a consequence, their propagator develops a singularity. According to our previous discussion, the point $\phi = \pi$ corresponds to periodic boundary conditions and the kinetic operator in (3.8) can only be inverted in the space orthogonal to the space of constant fields. Note that at this particular point of moduli space the gravitinos cannot be completely gauged away and fermionic moduli appears (the constant gravitinos).

To take into account this singular point we are going to use an analytic regularization in moduli space. First we change coordinates and use the Wilson loop variable $w = e^{i\phi}$ instead of ϕ. The integration region of this new variable is the unit circle γ on the complex w-plane

$$\int_0^{2\pi} \frac{d\phi}{2\pi} = \oint_\gamma \frac{dw}{2\pi i w}. \tag{4.13}$$

The singular point $\phi = \pi$ is now mapped to $w = -1$. Our prescription is to use complex contour integration and deform the contour to exclude the point $w = -1$ (say by moving it outside to $w = -1 - \epsilon$ with $\epsilon > 0$, and then letting $\epsilon \to 0^+$).

This prescription allows us to recover all known results, though a deeper justification would be welcome. We may note that this prescription can be interpreted as giving the worldline fermions a small mass ϵ to lift the zero modes appearing at the point $\phi = \pi$ of moduli space, i.e. replacing $\phi = \pi$ by $\phi = \pi - i\epsilon$. However, it does not seem obvious to us why one should require $\epsilon > 0$.

The computation proceeds now as follows. From

$$Z_p(\beta) = \int \frac{d^D x \sqrt{g}}{(2\pi\beta)^{\frac{D}{2}}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{iq\phi} \left(2\cos\frac{\phi}{2}\right)^{D-2} \langle e^{-S_{int}} \rangle \tag{4.14}$$

with the expression (4.10) one can get all coefficients $v_1, ..., v_6$ for arbitrary (D, p). They are combinations of the following basic integrals (recall that $q = D/2 - p - 1$)

$$\begin{aligned} I_n(D,p) &\equiv 2^{D-2} \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{iq\phi} \left(\cos\frac{\phi}{2}\right)^{D-2n} \\ &= 2^{2n-2} \oint_\gamma \frac{dw}{2\pi i} \frac{(1+w)^{D-2n}}{w^{p+2-n}}\,. \end{aligned} \tag{4.15}$$

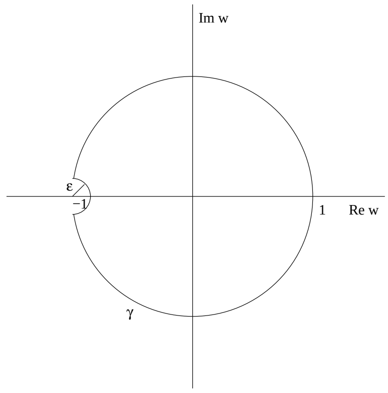

Figure 4: Regulated contour for the U(1) modular parameter.

As already mentioned, there is a possible pole at $w = -1$. With the prescription to push the pole out of the integration circle, the integrals are given by computing the residue at the pole $w = 0$, and the result is

$$\begin{aligned} I_n(D,p) &= \frac{2^{2n-2}}{(p+1-n)!} \frac{d^{p+1-n}}{dw^{p+1-n}} (1+w)^{D-2n}\Big|_{w=0} & \text{if} \quad p \geq n-1 \\ I_n(D,p) &= 0 & \text{if} \quad p < n-1\,. \end{aligned} \tag{4.16}$$

For the coefficients v_i one then gets

$$\begin{aligned} v_1(D,p) &= I_1(D,p) = \frac{(D-2)!}{p!(D-2-p)!} \\ v_2(D,p) &= \frac{1}{12} I_1(D,p) - \frac{1}{8} I_2(D,p) \\ v_3(D,p) &= \frac{1}{720} I_1(D,p) - \frac{1}{192} I_2(D,p) + \frac{1}{128} I_3(D,p) \\ v_4(D,p) &= -\frac{1}{720} I_1(D,p) + \frac{1}{32} I_2(D,p) - \frac{1}{32} I_3(D,p) \\ v_5(D,p) &= \frac{1}{288} I_1(D,p) - \frac{1}{96} I_2(D,p) + \frac{1}{128} I_3(D,p) \\ v_6(D,p) &= \frac{1}{120} I_1(D,p) - \frac{1}{96} I_2(D,p)\,. \end{aligned} \tag{4.17}$$

Using (4.16) one may check that all known values for $p = 0, 1, 2, 3$ are reproduced, see for example [19] and [20]. In particular, it is worth noticing the case of $p = 3$, which was derived after considerable algebra in volume II of [20] (see page 974). For completeness and future reference let us list these coefficients in the format

$$F_{p+1} \to (v_1; v_2; v_3; v_4; v_5; v_6)\,. \tag{4.18}$$

For $p = -1, 0, 1, 2, 3$ we have

$$
\begin{aligned}
F_0 &\to (0; 0; 0; 0; 0; 0) \\
F_1 &\to \left(1; \frac{1}{12}; \frac{1}{720}; -\frac{1}{720}; \frac{1}{288}; \frac{1}{120}\right) \\
F_2 &\to \left(D-2; \frac{D-8}{12}; \frac{D-17}{720}; \frac{92-D}{720}; \frac{D-14}{288}; \frac{D-7}{120}\right) \\
F_3 &\to \left(\frac{(D-2)(D-3)}{2}; \frac{D^2-17D+54}{24}; \frac{(D-17)(D-18)}{1440}; -\frac{D^2-185D+1446}{1440};\right. \\
&\qquad \left.\frac{D^2-29D+174}{576}; \frac{D^2-15D+46}{240}\right) \\
F_4 &\to \left(\frac{1}{6}(D-2)(D-3)(D-4); \frac{1}{72}(D-4)(D^2-23D+96);\right. \\
&\qquad \frac{1}{4320}(D^3-54D^2+971D-4164); -\frac{1}{4320}(D^3-279D^2+4616D-18384); \\
&\qquad \left.\frac{1}{1728}(D-12)(D^2-33D+170); \frac{1}{720}(D-4)(D^2-20D+81)\right). \qquad (4.19)
\end{aligned}
$$

The case F_1 corresponds to a massless scalar (without any improvement term), and the case F_2 to a massless spin 1 field. The latter contains, in particular, the correct trace anomaly of the photon in $D = 4$, already obtained with a worldline approach in [27], but with a different quantum mechanical model (which we shall consider in subsection 4.3). The coefficient v_1 counts the number of physical degrees of freedom and it is easily checked to be correct for any (D, p).

4.2 An application: F_5 and F_6

As an application of the previous general result we can make explicit the coefficients for the 4- and 5-form, which, to our knowledge, are not present in the literature. For the $p = 4$ case, we need the following values

$$
\begin{aligned}
I_1(D,4) &= \frac{(D-2)(D-3)(D-4)(D-5)}{24} && (4.20) \\
I_2(D,4) &= 4\frac{(D-4)(D-5)(D-6)}{6} && (4.21) \\
I_3(D,4) &= 16\frac{(D-6)(D-7)}{2}\,. && (4.22)
\end{aligned}
$$

Using the formulas given above we get for $F_5 = dA_4$

$$
\begin{aligned}
F_5 \to &\left(\frac{1}{24}(D-2)(D-3)(D-4)(D-5); \frac{1}{288}(D-4)(D-5)(D^2-29D+150);\right. \\
&\quad \frac{1}{17280}(D^4-74D^3+2051D^2-18634D+52680);
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{17280}(D^4-374D^3+9791D^2-82954D+224760);\\
&\frac{1}{6912}(D^4-62D^3+1223D^2-9322D+24024);\\
&\left.\frac{1}{2880}(D-4)(D-5)(D-7)(D-18)\right). \qquad (4.23)
\end{aligned}
$$

To test these coefficients, one may check that $F_5 \sim F_1$ in $D = 6$, as in such dimensions a 4-form gauge field gives a dual description of a scalar field. In $D = 6$ the topological Euler density appears at higher order in β so the duality holds exactly (in $D = 4$ a 2-form gives a dual description of a scalar field only up to the topological Euler density [29], the appearance of the Euler term is a general phenomenon which we discuss in section 5).

Moving on to the $p = 5$ case, we need the following integrals

$$
I_1(D,5) = \frac{(D-2)(D-3)(D-4)(D-5)(D-6)}{120} \qquad (4.24)
$$

$$
I_2(D,5) = 4\frac{(D-4)(D-5)(D-6)(D-7)}{24} \qquad (4.25)
$$

$$
I_3(D,5) = 16\frac{(D-6)(D-7)(D-8)}{6} \qquad (4.26)
$$

and the explicit coefficients for $F_6 = dA_5$ turn out to be

$$
\begin{aligned}
F_6 \to \Bigg(&\frac{1}{120}(D-2)(D-3)(D-4)(D-5)(D-6);\\
&\frac{1}{1440}(D-4)(D-5)(D-6)(D-8)(D-27);\\
&\frac{1}{86400}(D-6)(D^4-89D^3+3071D^2-33379D+111420);\\
&-\frac{1}{86400}(D-6)(D^4-464D^3+14471D^2-145504D+466320);\\
&\frac{1}{34560}(D-6)(D^4-74D^3+1751D^2-15934D+48840);\\
&\left.\frac{1}{14400}(D-4)(D-5)(D-6)(D^2-30D+181)\right). \qquad (4.27)
\end{aligned}
$$

These coefficients pass the expected duality tests. An immediate check to perform is to see that $F_6 \sim F_0 \sim 0$ in $D = 6$, as F_6 in six dimensions is dual to a scalar field strength which has no gauge potential. As a further check, one can for example verify that $F_6 \sim F_2$ in $D = 8$. Again, in $D = 6$ and $D = 8$ the expected dualities hold exactly since the topological Euler density appears at higher order in the proper time expansion.

4.3 Another application: susy ungauged, A_4 and A_5

The Seeley-DeWitt coefficients for the differential $(p+1)$-form A_{p+1}, not interpreted as a field strength, but as a differential form with kinetic operator given by the generalized laplacian $dd^\dagger + d^\dagger d$, can be obtained from the $N = 2$ model with ungauged susy (gauging susy enforces Maxwell equations and Bianchi identities). This corresponds to changing the factor $(2\cos\frac{\phi}{2})^{D-2} \to (2\cos\frac{\phi}{2})^{D}$ in (4.14). The coefficients will still be combinations of the basic integrals given in eq. (4.15) where we only need to replace $2^{D-2} \to 2^D$ as overall normalization factor. In this case I_0 will also enter the computation together with I_1 and I_2. In this way, one can reproduce for $p = -1, 0, 1, 2$ (remember that the Chern-Simons coupling $q = \frac{D}{2} - p - 1$ selects a $(p+1)$-form) the coefficients for A_0, A_1, A_2, A_3 already present in the literature, see again [19] and [20]. We list them here for completeness

$$\begin{aligned}
A_0 &\to \left(1; \frac{1}{12}; \frac{1}{720}; -\frac{1}{720}; \frac{1}{288}; \frac{1}{120}\right)\\
A_1 &\to \left(D; \frac{D-6}{12}; \frac{D-15}{720}; \frac{90-D}{720}; \frac{D-12}{288}; \frac{D-5}{120}\right)\\
A_2 &\to \left(\frac{D(D-1)}{2}; \frac{D^2-13D+24}{24}; \frac{D^2-31D+240}{1440}; -\frac{D^2-181D+1080}{1440};\right.\\
&\qquad \left.\frac{D^2-25D+120}{576}; \frac{D^2-11D+20}{240}\right)\\
A_3 &\to \left(\frac{1}{6}D(D-1)(D-2); \frac{1}{72}(D-2)(D^2-19D+54);\right.\\
&\qquad \frac{1}{4320}(D^3-48D^2+767D-2430); -\frac{1}{4320}(D^3-273D^2+3512D-10260);\\
&\qquad \left.\frac{1}{1728}(D-10)(D^2-29D+108); \frac{1}{720}(D-2)(D^2-16D+45)\right).
\end{aligned} \tag{4.28}$$

However, the $N = 2$ model has produced the result for all differential forms. As an example, we can make explicit the cases with $p = 3, 4$ to obtain the coefficients for A_4 and A_5 which are not known in the literature. We get

$$\begin{aligned}
A_4 \to &\left(\frac{1}{24}D(D-1)(D-2)(D-3); \frac{1}{288}(D-2)(D-3)(D^2-25D+96);\right.\\
&\frac{1}{17280}(D^4-66D^3+1631D^2-11286D+23040);\\
&-\frac{1}{17280}(D^4-366D^3+7571D^2-48246D+95040);\\
&\frac{1}{6912}(D^4-54D^3+875D^2-5142D+9792);\\
&\left.\frac{1}{2880}(D-2)(D-3)(D-5)(D-16)\right)
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
A_5 \to &\left(\frac{1}{120}D(D-1)(D-2)(D-3)(D-4);\right.\\
&\frac{1}{1440}(D-2)(D-3)(D-4)(D-6)(D-25);\\
&\frac{1}{86400}(D-4)(D^4-81D^3+2561D^2-22131D+56250);
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{86400}(D-4)(D^4 - 456D^3 + 11711D^2 - 93156D + 229500); \\
&\frac{1}{34560}(D-4)(D^4 - 66D^3 + 1331D^2 - 9786D + 23400); \\
&\left.\frac{1}{14400}(D-2)(D-3)(D-4)(D^2 - 26D + 125)\right). \qquad (4.30)
\end{aligned}
$$

As a test, one may check that Poincaré duality holds, for example $A_4 \sim A_2$ and $A_5 \sim A_1$ in $D = 6$.

As a final test of our results, one may note that these differential forms with generalized laplacian as kinetic operator appear in the covariantly gauge fixed action for an antisymmetric tensor gauge field. Denoting by W_p the one-loop effective action of one such a p-form and recalling the "triangular" structure of the gauge fixed action [30, 31] one can identify

$$
\begin{aligned}
\Gamma_p^{QFT} &= W_p - 2W_{p-1} + 3W_{p-2} + \cdots + (-1)^p (p+1) W_0 \\
&= \sum_{k=0}^{p} (-1)^k (k+1) W_{p-k} \, . \qquad (4.31)
\end{aligned}
$$

To check this relation, let us remember that the Seeley-DeWitt coefficients which characterize W_p can be obtained by taking (4.17), which gives the coefficients for Γ_p^{QFT}, and replacing everywhere $I_n(D,p) \to 4I_{n-1}(D,p-1)$. Having noticed this, eq. (4.31) can be seen as a consequence of the following identity involving the integrals in (4.15)

$$
\begin{aligned}
I_n(D,p) &= \frac{2^{2n-2}}{(p+1-n)!} \partial_w^{p+1-n} (1+w)^{D-2n} \Big|_{w=0} \\
&= \frac{2^{2n-2}}{(p+1-n)!} \partial_w^{p+1-n} \left[(1+w)^{D-2n+2} (1+w)^{-2} \right] \Big|_{w=0} \\
&= 2^{2n-2} \sum_{k=0}^{p+1-n} \frac{\left[\partial_w^{p+1-n-k} (1+w)^{D-2n+2}\right] \left[\partial_w^k (1+w)^{-2}\right]}{(p+1-n-k)! k!} \Bigg|_{w=0} \\
&= \sum_{k=0}^{p+1-n} \left[4 I_{n-1}(D, p-k-1)\right] (-1)^k (k+1) \, . \qquad (4.32)
\end{aligned}
$$

Using that $I_{n-1}(D, p-k-1)$ is zero if $k > p+1-n$, see (4.16), we may equivalently write this identity as

$$
I_n(D,p) = \sum_{k=0}^{p} (-1)^k (k+1) 4 I_{n-1}(D, p-k-1) \qquad (4.33)
$$

from which eq. (4.31) follows.

5. Discussion and conclusions

We have used the $N = 2$ spinning particle to compute one-loop effects due to the propagation of differential forms coupled to gravity, including as a particular case a spin 1 field. One of the most interesting points of the construction is the appearance of the U(1) modulus ϕ, a parameter which does not emerge in the standard derivation of Feynman rules. This is also a delicate point which deserves further discussions. For this purpose it is convenient to switch to an operatorial picture and cast the effective action (3.10) in the form

$$\Gamma_p^{QFT}[g_{\mu\nu}] = \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(2\cos\frac{\phi}{2}\right)^{-2} \mathrm{Tr}\left[e^{i\phi(\hat{N}-\frac{D}{2}+q)} e^{-\beta\hat{H}}\right] \tag{5.1}$$

where the path integral on the torus has been represented by the trace in the matter sector of the Hilbert space of the spinning particle (i.e. excluding the ghost sector). Here $\hat{N} = \hat{\psi}^\mu \hat{\psi}_\mu^\dagger$ is the (anti) fermion number operator (it counts the degree of the field strength form, and up to the ordering and to the Chern-Simons coupling coincides with the current $-\hat{J}$) and $\hat{H}$ is the quantum hamiltonian without the coupling to the gauge field, which has been explicitly factorized. Computing the trace and using the Wilson loop variable $w = e^{i\phi}$ gives an answer of the form

$$\Gamma_p^{QFT}[g_{\mu\nu}] = \int_0^\infty \frac{d\beta}{\beta} \oint_\gamma \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \sum_{n=0}^{D} w^{n-\frac{D}{2}+q}\, t_n(\beta) \tag{5.2}$$

where the coefficients $t_n(\beta)$ arise from the trace restricted to the sector of the Hilbert space with occupation number n. From the answer written in this form one can make various comments.

If susy is not gauged then the ghost term $\frac{w}{(1+w)^2}$ is absent, and one recovers the model described in section 4.3. Then, there is no pole at $w = -1$, at least for finite D, and the w integral projects onto the sector of the Hilbert space with occupation number $n - \frac{D}{2} + q = 0$, i.e. $n = p+1$. This describes a $(p+1)$-form with generalized laplacian as kinetic operator. The absence of the pole at $w = -1$ means that excluding or including this point in the regularized contour γ must produce the same answer. This is related to Poincaré duality, which tells that the result for a $(p+1)$-form must be equivalent to that of a $(D-p-1)$-form. The change from $(p+1)$ to $(D-p-1)$ in (5.2) is described by $q \to -q$, which can be undone by a change of integration variable $w \to w' = \frac{1}{w}$ (or $\phi \to -\phi$). This proves Poincaré equivalence.

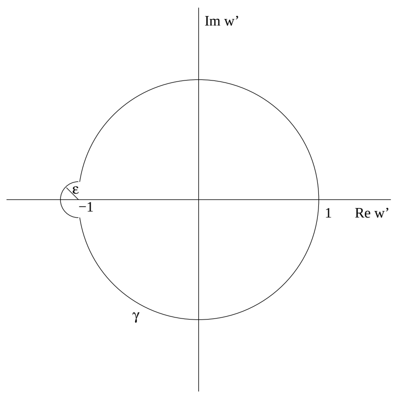

Figure 5: Regulated contour in the variable w'.

Next consider the case of gauged susy. Now one must include the contribution of the ghosts' determinants $\frac{w}{(1+w)^2}$ and face the appearance of a possible pole at $w = -1$. The duality between a gauge p-form and a gauge $(D-p-2)$-form is again described by $q \to -q$ and compensated by the change of integration variable $w \to w' = \frac{1}{w}$. However, the original contour which was regulated by excluding the pole at $w = -1$ gets mapped into a contour which now includes that pole. Therefore strict equivalence is not guaranteed. The mismatch corresponds to the residue at the pole $w=-1$, and can be computed as follows

$$
\begin{aligned}
\mathrm{Res}\left(\frac{1}{(1+w)^2}\,\mathrm{Tr}\left[w^{\hat{N}-\frac{D}{2}+q}e^{-\beta\hat{H}}\right], w=-1\right) &= \frac{d}{dw}\,\mathrm{Tr}\left[w^{\hat{N}-p-1}e^{-\beta\hat{H}}\right]\Big|_{w=-1} \\
&= \mathrm{Tr}\left[\left(\hat{N}-p-1\right)(-1)^{\hat{N}-p-2}\,e^{-\beta\hat{H}}\right] \\
&= (-1)^p\,\mathrm{Tr}\left[\hat{N}(-1)^{\hat{N}}e^{-\beta\hat{H}}\right]+ \\
&\quad +(-1)^{p+1}(p+1)\,\mathrm{Tr}\,[(-1)^{\hat{N}}e^{-\beta\hat{H}}] \\
&= (-1)^{p+1}Z_{D-1}(\beta)+(-1)^{p+1}(p+1)\chi\,. \qquad (5.3)
\end{aligned}
$$

In fact, the term in the last-but-one line is proportional to the Witten index [32], which is β independent and computes the topological Euler number χ of target space [33]. The previous term is instead similar to an index introduced in [34] for two dimensional field theories, and which in our case computes the partition function $Z_{D-1}(\beta)$ of a $(D-1)$-form A_{D-1} (with the top form F_D as field strength)

$$
\begin{aligned}
\mathrm{Tr}\left[\hat{N}(-1)^{\hat{N}}e^{-\beta\hat{H}}\right] &= \sum_{n=1}^{D}(-1)^n\,n\,t_n(\beta) = \sum_{n=1}^{D}(-1)^n\,n\,t_{D-n}(\beta) \\
&= -\sum_{n=0}^{D-1}(-1)^n(n+1)\,t_{(D-1)-n}(\beta) = -Z_{D-1}(\beta) \qquad (5.4)
\end{aligned}
$$

where we have used Poincaré duality ($t_n = t_{D-n}$) and recognized the "triangular" structure (4.31) for the gauge field A_{D-1}. The notation for the partition function $Z_{D-1}(\beta)$ is as in (4.11). Thus we arrive at the following equivalence between propagating p-forms and $(D-p-2)$-forms

$$
Z_p(\beta) = Z_{D-p-2}(\beta) + (-1)^p Z_{D-1}(\beta) + (-1)^p(p+1)\chi\,. \qquad (5.5)
$$

As the gauge field A_{D-1} does not have propagating degrees of freedom, the first Seeley-DeWitt coefficient a_0 is not spoiled by this duality.

For odd D the Euler number vanishes, and one has (at the level of field strengths) a duality of the form

$$
F_{p+1} \sim F_{D-(p+1)} + (-1)^p F_D\,. \qquad (5.6)
$$

One can easily check this relation in few examples, like $F_1 \sim F_4 + F_5$ and $F_2 \sim F_3 - F_5$ in $D=5$. Another example is $F_1 \sim F_2 + F_3$ in $D=3$, where one should remember to use the $D=3$ identity relating the Riemann to the Ricci tensor which makes the four dimensional Gauss-Bonnet combination vanish identically, $R^2_{abcd} - 4R^2_{ab} + R^2 = 0$.

For even D the partition function $Z_{D-1}(\beta)$ is proportional to the Euler number, namely $Z_{D-1}(\beta) = -\frac{D}{2}\chi$, as can be checked by using Poincaré duality, so that equation (5.5) simplifies to

$$
Z_p(\beta) = Z_{D-p-2}(\beta) + (-1)^p\left(p+1-\frac{D}{2}\right)\chi \qquad (5.7)
$$

or, equivalently,

$$
F_{p+1} \sim F_{D-(p+1)} + (-1)^p\left(1-\frac{2(p+1)}{D}\right)F_D\,. \qquad (5.8)
$$

Thus we see that the mismatch is purely topological, as already noticed in [29] for the duality between a scalar and 2-form gauge field in $D = 4$. The β independence of χ shows, for example, that a mismatch between a scalar and a 4-form in $D = 6$ will be visible in the coefficient a_3, and so on. The coefficient a_3 is laborious to compute, nevertheless the bosonic ($N = 0$) worldline coefficient has been already calculated in [35], and could with some effort be dressed up to the $N = 2$ model to obtain the a_3 coefficient for *all* differential forms in *all* dimensions. Finally, we may note that (5.8) is consistent with the selfduality of the $F_{D/2}$ form.

To conclude, we have seen that a worldline perspective on particles of spin 1 and on antisymmetric tensor gauge fields of arbitrary rank coupled to gravity has produced a quite interesting and useful representation of the one-loop effective action. We have tested that such a representation is correct by checking that the model correctly reproduces known Seeley-DeWitt coefficients, and in fact produces many more previously unknown ones. Other applications and extensions of this worldline approach will be left for future work.

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A. Appendix

A.1 Propagators and determinants

The relative coordinates $y^\mu(\tau) = x^\mu(\tau) - x_0^\mu$ and the measure ghosts a^μ, b^μ, c^μ are taken to vanish at the boundary of the space $[0, 1]$. Then their propagators are obtained from (4.4), with measure ghosts inserted by the shift (4.9), and read

$$\begin{aligned}
\langle y^\mu(\tau) y^\nu(\sigma)\rangle &= -\beta g^{\mu\nu}(x_0)\Delta(\tau,\sigma) \\
\langle a^\mu(\tau) a^\nu(\sigma)\rangle &= \beta g^{\mu\nu}(x_0)\Delta_{gh}(\tau,\sigma) \\
\langle b^\mu(\tau) c^\nu(\sigma)\rangle &= -2\beta g^{\mu\nu}(x_0)\Delta_{gh}(\tau,\sigma)
\end{aligned} \tag{A.1}$$

with the functions Δ and Δ_{gh} given by

$$\begin{aligned}
\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty}\left[\frac{-2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] = (\tau-1)\sigma\,\theta(\tau-\sigma) + (\sigma-1)\tau\,\theta(\sigma-\tau) \\
\Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{\infty} 2\,\sin(\pi m\tau)\sin(\pi m\sigma) = \partial_\tau^2\Delta(\tau,\sigma) = \delta(\tau,\sigma)
\end{aligned} \tag{A.2}$$

where $\theta(\tau-\sigma)$ is the standard step function and $\delta(\tau,\sigma)$ is the delta function which vanishes at the boundaries $\tau,\sigma = 0, 1$. These functions are not translationally invariant. One could

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also use translational invariant Green functions (the so-called "string inspired" propagators), which are quite efficient in computations, but one has to remember to include an extra Faddeev-Popov determinant due to the slightly more complicated factorization of the zero modes x_0^μ [36, 37, 4].

The fermionic fields with antiperiodic boundary conditions can be expanded in half-integer modes

$$\psi^a(\tau) = \sum_{r\in Z+\frac{1}{2}} \psi_r^a\, \mathrm{e}^{2\pi i r\tau}\,, \qquad \bar{\psi}^a(\tau) = \sum_{r\in Z+\frac{1}{2}} \bar{\psi}_r^a\, \mathrm{e}^{-2\pi i r\tau}\,. \tag{A.3}$$

Then from the action

$$S = \frac{1}{\beta}\int_0^1 d\tau\, \bar{\psi}_a(\partial_\tau + i\phi)\psi^a \tag{A.4}$$

one finds the propagator (AF stands for antiperiodic fermions)

$$\langle \psi^a(\tau)\bar{\psi}_b(\sigma)\rangle = \beta\delta_b^a \Delta_{AF}(\tau-\sigma,\phi)\,, \qquad \Delta_{AF}(x,\phi) = \sum_{r\in Z+\frac{1}{2}} \frac{-i}{2\pi r+\phi}\, e^{2\pi i r x} \tag{A.5}$$

which satisfies

$$(\partial_x + i\phi)\Delta_{AF}(x,\phi) = \sum_{r\in Z+\frac{1}{2}} e^{2\pi i r x} = \delta_{AF}(x) \tag{A.6}$$

where δ_{AF} is the delta function on the space of antiperiodic functions. For $x \in]-1,1[$ the propagator can be summed up to yield

$$\Delta_{AF}(x,\phi) = \frac{e^{-i\phi x}}{2\cos\frac{\phi}{2}}\left[e^{i\frac{\phi}{2}}\theta(x) - e^{-i\frac{\phi}{2}}\theta(-x)\right]. \tag{A.7}$$

For coinciding points ($\tau=\sigma$ i.e. $x=0$) it takes the regulated value

$$\Delta_{AF}(0,\phi) = \frac{i}{2}\tan\frac{\phi}{2} \tag{A.8}$$

which can be computed by "symmetric integration", i.e. symmetrically combining the modes $+r$ and $-r$ and then summing up the series. Note also that for $x \neq 0$

$$\Delta_{AF}(x,\phi)\Delta_{AF}(-x,\phi) = -\frac{1}{4}\cos^{-2}\frac{\phi}{2} \tag{A.9}$$

which, when combined with (A.8), shows that this function has a discontinuity at $x=0$, so that when multiplied by a distribution it necessitates a regularization. An example will be discussed at the end of next section.

Let us now review the calculation of the fermionic determinant

$$\int_{\mathrm{ABC}} D\bar{\psi}D\psi\; e^{-S} = \det{}^D(\partial_\tau + i\phi) \tag{A.10}$$

where ABC stands for antiperiodic boundary conditions and the action is the one in (A.4). The easiest way to obtain the determinant is to use the operator formalism

$$\int_{\mathrm{ABC}} D\bar{\psi}D\psi\; e^{-S} = \mathrm{Tr}\, e^{-\hat{H}_\phi} \tag{A.11}$$

where $\hat{H}_\phi$ is the hamiltonian operator of the system and equals

$$\hat{H}_\phi = i\phi\frac{1}{2}\left(\hat{\psi}_a^\dagger\hat{\psi}^a - \hat{\psi}^a\hat{\psi}_a^\dagger\right) = i\phi\left(\hat{\psi}_a^\dagger\hat{\psi}^a - \frac{D}{2}\right). \tag{A.12}$$

This is the hamiltonian for a D dimensional fermionic oscillator, and the trace is easily computed. In one dimension the eigenvalues of the fermionic number operator $\hat{\psi}^\dagger\hat{\psi}$ are either 0 or 1, thus one gets

$$\begin{aligned}\det{}^D(\partial_\tau + i\phi) &= \mathrm{Tr}\, e^{-i\phi\left(\hat{\psi}_a^\dagger\hat{\psi}^a - \frac{D}{2}\right)} \\ &= e^{i\phi\frac{D}{2}}(1+e^{-i\phi})^D = \left(2\cos\frac{\phi}{2}\right)^D.\end{aligned} \tag{A.13}$$

Alternatively, one can compute the determinant directly from the path integral by expanding the fermions in antiperiodic modes and taking the infinite product of eigenvalues

$$\begin{aligned}\det{}^D(\partial_\tau + i\phi) &= \det{}^D(\partial_\tau)\left[\frac{\det(\partial_\tau + i\phi)}{\det(\partial_\tau)}\right]^D \\ &= 2^D \prod_{n=-\infty}^{+\infty}\left(1 + \frac{\phi}{2\pi(n+\frac{1}{2})}\right)^D \\ &= \left(2\cos\frac{\phi}{2}\right)^D\end{aligned} \tag{A.14}$$

where we have used a standard representation of the cosine as an infinite product (after combining positive and negative frequencies together as part of our regularization prescription).

A.2 Dimensional regularization

We discuss here the dimensional regularization (DR) of the $N = 2$ supersymmetric nonlinear sigma models on a compact one-dimensional base space, and with target space of arbitrary dimensions, extending the treatment presented in [38] for bosonic and in [3] for $N = 1$ supersymmetric sigma models. In particular, we wish to fix the corresponding counterterm. Dimensional regularization had been previously applied to bosonic nonlinear sigma models on an infinite one-dimensional base space and one-dimensional target space in [39], where it was noticed that noncovariant counterterms did not arise, and in [40], where the correct counterterm for higher dimensional target space was found. A general discussion on the regularization issues of one dimensional nonlinear sigma models can be found in the forthcoming book [41].

The quantum hamiltonian for bosonic systems can be required to be proportional to the covariant scalar laplacian, $H = -\frac{1}{2}\nabla^2$, without any coupling to the scalar curvature R. Then the rules of dimensional regularization developed in [38] demand the use of a counterterm $V_{DR} = -\frac{1}{8}R$ to be added to the classical euclidean action, normalized as $\Delta S_{DR} = \frac{1}{\beta}\int_0^1 d\tau\,\beta^2 V_{DR}$ (the powers of β signal that this is due to a two-loop effect). On the other hand, the quantum hamiltonian of the $N = 1$ model acts on a spinor space and it is fixed by susy to be the square of the Dirac operator (the susy charge) $H = -\frac{1}{2}\nabla\!\!\!/\nabla\!\!\!/ = -\frac{1}{2}(\nabla^2 - \frac{1}{4}R)$. In such a case the total counterterm in dimensional regularization vanish, showing that DR respects $N = 1$ susy [3].

For the $N = 2$ model one might conjecture that the counterterm would vanish as well. This is correct, and can be proved in various ways. One way is to use an arbitrary counterterm proportional to R, and then fix the proportionality constant so to reproduce some known result, e.g. the a_1 coefficient for a 0-form. Then DR is seen to require a vanishing total counterterm. However, to obtain this result, see (4.19), one has to integrate over the $U(1)$ modulus ϕ, and one might feel uneasy as the modular integration acts effectively as a projection. A different test which does not require the modular integration is to compare DR with the unambiguous operatorial treatment [42] or with the time slicing regularization scheme worked out in [28]. In those papers one finds the transition amplitude for the $N = 2$ model. We have used that result to compute the partition function (the trace of the transition amplitude) which is directly related to our path integral on the torus with $\phi = 0$ and antiperiodic boundary conditions for the fermions. The comparison then requires the vanishing of the total counterterm for the $N = 2$ model. Counterterms are local effects arising from ultraviolet ambiguities and should not depend on the boundary conditions imposed on the quantum fields. The conclusion is that the counterterm vanishes for any value of the modular parameter ϕ.

Dimensional regularization requires the extension of the space $I = [0, 1]$ to $I \times \mathbb{R}^d$. Then also the action can be extended to $d + 1$ dimensions, so that one can recognize the structure of the vertices and propagators to dimensionally continue the various Feynman diagrams. The extended action is

$$S = \frac{1}{\beta}\int_{I\times\mathbb{R}^d} d^{d+1}t \left[\frac{1}{2} g_{\mu\nu}\partial^\alpha x^\mu \partial_\alpha x^\nu + \bar\psi_a\gamma^\alpha(\partial_\alpha\psi^a + \partial_\alpha x^\mu {\omega_\mu}^{ab}\psi_b) + i\phi\bar\psi_a\psi^a - \right.$$
$$\left. - \frac{1}{2}R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d\right] \tag{A.15}$$

where $t^\alpha \equiv (\tau, \mathbf{t})$ are coordinates in the extended space (bold face indicates vectors in the extra d dimensions) and γ^α are the corresponding Dirac matrices. The propagators in $d+1$ dimensions read

$$\Delta(t, s) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\sum_{m=1}^{\infty}\frac{-2}{(\pi m)^2 + \mathbf{k}^2}\sin(\pi m\tau)\sin(\pi m\sigma)\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{A.16}$$

$$\Delta_{gh}(t, s) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\sigma)\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{A.17}$$
$$= \delta(\tau, \sigma)\delta^d(\mathbf{t} - \mathbf{s})$$

$$\Delta_{AF}(t - s) = -i\int \frac{d^d\mathbf{k}}{(2\pi)^d}\sum_{r\in Z+\frac{1}{2}}\frac{2\pi r\gamma^0 + \mathbf{k}\cdot\vec\gamma - \phi}{(2\pi r)^2 + \mathbf{k}^2 - \phi^2}\mathrm{e}^{2\pi i r(\tau-\sigma)}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{A.18}$$

and satisfy

$$\partial^\alpha\partial_\alpha\Delta(t, s) = \Delta_{gh}(t, s) = \delta(\tau, \sigma)\,\delta^d(\mathbf{t} - \mathbf{s}) \tag{A.19}$$

$$\left(\gamma^\alpha \frac{\partial}{\partial t^\alpha} + i\phi\right) \Delta_{AF}(t-s) = \Delta_{AF}(t-s) \left(-\gamma^\beta \frac{\overleftarrow{\partial}}{\partial s^\beta} + i\phi\right) = \delta_{AF}(\tau - \sigma)\delta^d(\mathbf{t} - \mathbf{s})\,.$$

The index contractions in $d+1$ dimensions serve mostly as a bookkeeping device to keep track of which derivative can be contracted to which vertex to produce the $(d+1)$-dimensional delta function. The delta functions in (A.19) are only to be used in $d+1$ dimensions, as we assume that only in such a situation the regularization due to the extra dimensions is taking place. Then, by using partial integration one casts the various loop integrals in a form which can be computed by sending $d \to 0$ first. At this stage one can use the propagators in one dimensions, and $\gamma^0 = 1$ (with no extra factors arising from the Dirac algebra in $d+1$ dimensions).

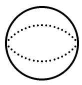
This procedure will be now exemplified in one of the most relevant graphs coming from the vertex

$$\dot{x}^\mu \omega_{\mu ab}(x) \bar{\psi}^a \psi^b \qquad \rightsquigarrow \qquad \partial_\alpha x^\mu \omega_{\mu ab}(x) \bar{\psi}^a \gamma^\alpha \psi^b \tag{A.20}$$

(we indicate also the extension to $d+1$ dimensions). Expanding the vertex around $x_0^\mu = x^\mu(\tau) - y^\mu(\tau)$, and using Riemann normal coordinates and a Lorentz gauge such that $\omega_{\mu ab}(x_0) = 0$ and $\partial_{(\nu}\omega_{\mu)ab}(x_0) = 0$, one obtains a quartic vertex of the form

$$\begin{aligned} \Delta S &= \frac{1}{\beta}\int_0^1 d\tau \frac{1}{2}\, \dot{y}^\mu y^\nu R_{\nu\mu ab}(x_0) \bar{\psi}^a \psi^b \\ &\rightsquigarrow \frac{1}{\beta}\int d^{d+1}t\, \frac{1}{2}\partial_\alpha y^\mu y^\nu R_{\nu\mu ab}(x_0) \bar{\psi}^a \gamma^\alpha \psi^b\,. \end{aligned} \tag{A.21}$$

This vertex can be used to construct the graph

$$\frac{1}{2}\langle (\Delta S_3)^2 \rangle \; = $$  $$ = \frac{\beta^2}{8} R^2_{\mu\nu ab}(x_0) I$$

$$\begin{aligned} I &= \int_0^1 d\tau \int_0^1 d\sigma [{}^\bullet\Delta^\bullet(\tau,\sigma)\Delta(\tau,\sigma) - {}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)]\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau) \\ &\rightsquigarrow \int d^{d+1}t \int d^{d+1}s \Big\{ [{}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \times \\ &\qquad\qquad \times \mathrm{tr}[\gamma^\alpha \Delta_{AF}(t-s)\gamma^\beta \Delta_{AF}(s-t)] \Big\} \end{aligned} \tag{A.22}$$

where dots and indices on the left/right of the propagators indicate derivatives with respect to the first/second variable. Regularization is needed as ${}^\bullet\Delta^\bullet$ contains a delta function multiplying the step functions contained in Δ_{AF}, and these products of distributions are ambiguous and must be carefully regularized. Thus we have extended the integrals in (A.22) to $d+1$ dimensions. In DR we can use partial integrations to obtain the relations (A.19) and we can enforce the delta functions at the regulated level. In the present case we proceed as follows. We integrate by part the ∂_α from ${}_\alpha\Delta_\beta$. This produces a boundary term which vanish, a term which doubles the other term in (A.22), and the following extra term

$$\int d^{d+1}t \int d^{d+1}s \Big\{ \Delta_\beta(t,s)\Delta(t,s)\,\mathrm{tr}\left[\left(\gamma^\alpha \frac{\partial}{\partial t^\alpha}\Delta_{AF}(t,s)\right)\gamma^\beta \Delta_{AF}(s,t) + \right. \\ \left. + \Delta_{AF}(t,s)\gamma^\beta \left(\Delta_{AF}(s,t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\right)\right]\Big\}. \tag{A.23}$$

The "mass" term $i\phi$ can be added for free to obtain the Dirac equations, then using the second line in (A.19) and enforcing the delta function shows that this extra contribution vanish

$$2\int d^{d+1}t\Delta_\beta(t,t)\Delta(t,t)\,\mathrm{tr}[\gamma^\beta \Delta_{AF}(0)] \rightsquigarrow 2\Delta_{AF}(0)\int_0^1 d\tau \Delta^\bullet(\tau,\tau)\Delta(\tau,\tau) = 0\,. \tag{A.24}$$

We have used $\Delta^\bullet(\tau,\tau) = \tau - \frac{1}{2}$ and $\Delta(\tau,\tau) = \tau^2 - \tau$. Note that we have removed the regularization only when it was obvious that the integral did not contain any dangerous product of distributions at $d = 0$. Thus, we are left with

$$\begin{aligned} I &= -2\int d^{d+1}t \int d^{d+1}s_\alpha \Delta(t,s)\Delta_\beta(t,s)\,\mathrm{tr}[\gamma^\alpha \Delta_{AF}(t-s)\gamma^\beta \Delta_{AF}(s-t)] \\ &\rightsquigarrow -2\int_0^1 d\tau \int_0^1 d\sigma\, {}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau) \\ &= -\frac{1}{24}\cos^{-2}\frac{\phi}{2}\,. \end{aligned} \tag{A.25}$$

Here we have used that ${}^\bullet\Delta(\tau,\sigma) = \sigma - \theta(\sigma-\tau)$, $\Delta^\bullet(\tau,\sigma) = \tau - \theta(\tau-\sigma)$, and $\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau) = -\frac{1}{4}\cos^{-2}\frac{\phi}{2}$ (which are the correct limits of the Fourier sums up to a set of points of zero measure). Thus we have obtained

$$\frac{1}{2}\langle(\Delta S_3)^2\rangle = -\frac{\beta^2}{192}R^2_{\mu\nu ab}\cos^{-2}\frac{\phi}{2} \tag{A.26}$$

which is one of the contributions appearing in (4.10), and in fact the only one containing fermions and needing a regularization. However DR is still needed in other purely bosonic graphs.

References

[1] C. Schubert, *Perturbative quantum field theory in the string-inspired formalism*, *Phys. Rept.* **355** (2001) 73 [hep-th/0101036].

[2] F. Bastianelli and A. Zirotti, *Worldline formalism in a gravitational background*, *Nucl. Phys.* **B 642** (2002) 372 [hep-th/0205182].

[3] F. Bastianelli, O. Corradini and A. Zirotti, *Dimensional regularization for $N = 1$ SUSY sigma models and the worldline formalism*, *Phys. Rev.* **D 67** (2003) 104009 [hep-th/0211134].

[4] F. Bastianelli, O. Corradini and A. Zirotti, *BRST treatment of zero modes for the worldline formalism in curved space*, *JHEP* **01** (2004) 023 [hep-th/0312064].

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[5] F. Bastianelli and C. Schubert, *One loop photon graviton mixing in an electromagnetic field. I*, *JHEP* **02** (2005) 069 [gr-qc/0412095].

[6] L. Brink, P. Di Vecchia and P.S. Howe, *A lagrangian formulation of the classical and quantum dynamics of spinning particles*, *Nucl. Phys.* **B 118** (1977) 76.

[7] F.A. Berezin and M.S. Marinov, *Particle spin dynamics as the grassmann variant of classical mechanics*, *Ann. Phys. (NY)* **104** (1977) 336.

[8] V.D. Gershun and V.I. Tkach, *Classical and quantum dynamics of particles with arbitrary spin*, *Zh. Eksp. Teor. Fiz.* **29** (1979) 320, *Sov. Phys. JETP* **29** (1979) 288.

[9] P.S. Howe, S. Penati, M. Pernici and P.K. Townsend, *Wave equations for arbitrary spin from quantization of the extended supersymmetric spinning particle*, *Phys. Lett.* **B 215** (1988) 555.

[10] P.S. Howe, S. Penati, M. Pernici and P.K. Townsend, *A particle mechanics description of antisymmetric tensor fields*, *Class. and Quant. Grav.* **6** (1989) 1125.

[11] N. Marcus, *Kahler spinning particles*, *Nucl. Phys.* **B 439** (1995) 583 [hep-th/9409175].

[12] M.J. Strassler, *Field theory without Feynman diagrams: one loop effective actions*, *Nucl. Phys.* **B 385** (1992) 145 [hep-ph/9205205].

[13] M. Reuter, M.G. Schmidt and C. Schubert, *Constant external fields in gauge theory and the spin* $0, 1/2, 1$ *path integrals*, *Ann. Phys. (NY)* **259** (1997) 313 [hep-th/9610191].

[14] S. Elitzur, Y. Frishman, E. Rabinovici and A. Schwimmer, *Origins of global anomalies in quantum mechanics*, *Nucl. Phys.* **B 273** (1986) 93.

[15] G. Papadopoulos, *The path integral quantization of the spinning particles with extended supersymmetries*, *Class. and Quant. Grav.* **6** (1989) 1745.

[16] M. Pierri and V.O. Rivelles, *BRST quantization of spinning relativistic particles with extended supersymmetries*, *Phys. Lett.* **B 251** (1990) 421.

[17] V.O. Rivelles and J. Sandoval, L., *BRST quantization of relativistic spinning particles with a Chern-Simons term*, *Class. and Quant. Grav.* **8** (1991) 1605.

[18] R. Marnelius, *Proper BRST quantization of relativistic particles*, *Nucl. Phys.* **B 418** (1994) 353 [hep-th/9309002].

[19] B.S. DeWitt, *Dynamical theory of groups and fields*, in *Relativity, groups and topology* Les Houches 1963, B. and C. DeWitt eds., Gordon Breach, New York 1964; *The spacetime approach to quantum field theory*, in *Relativity, groups and topology, II*, Les Houches 1983, B. DeWitt and R. Stora eds., North Holland, Amsterdam 1984

[20] B.S. DeWitt, *The global approach to quantum field theory*, vol. I and II, published in Int. Ser. Monogr.Phys. **114** Oxford University Press, Oxford 2003.

[21] S.D. Mathur and S. Mukhi, *The* $N = 2$ *fermionic string: path integral, spin structures and supermoduli on the torus*, *Nucl. Phys.* **B 302** (1988) 130.

[22] H. Ooguri and C. Vafa, *Geometry of* $N = 2$ *strings*, *Nucl. Phys.* **B 361** (1991) 469.

[23] M. Henneaux and C. Teitelboim, *Relativistic quantum mechanics of supersymmetric particles*, *Ann. Phys. (NY)* **143** (1982) 127.

[24] A.G. Cohen, G.W. Moore, P. Nelson and J. Polchinski, *An off-shell propagator for string theory*, *Nucl. Phys.* **B 267** (1986) 143.

[25] A.M. Polyakov, *Gauge fields and strings*, Harwood, Chur 1987.

[26] F. Bastianelli, *The path integral for a particle in curved spaces and weyl anomalies*, *Nucl. Phys.* **B 376** (1992) 113 [hep-th/9112035].

[27] F. Bastianelli and P. van Nieuwenhuizen, *Trace anomalies from quantum mechanics*, *Nucl. Phys.* **B 389** (1993) 53 [hep-th/9208059].

[28] J.de Boer, B. Peeters, K. Skenderis and P. van Nieuwenhuizen, *Loop calculations in quantum-mechanical non-linear sigma models sigma models with fermions and applications to anomalies*, *Nucl. Phys.* **B 459** (1996) 631 [hep-th/9509158].

[29] M.J. Duff and P. van Nieuwenhuizen, *Quantum inequivalence of different field representations*, *Phys. Lett.* **B 94** (1980) 179.

[30] W. Siegel, *Hidden ghosts*, *Phys. Lett.* **B 93** (1980) 170.

[31] J. Thierry-Mieg, *BRS structure of the antisymmetric tensor gauge theories*, *Nucl. Phys.* **B 335** (1990) 334.

[32] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.

[33] L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, *Commun. Math. Phys.* **90** (1983) 161.

[34] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, *A new supersymmetric index*, *Nucl. Phys.* **B 386** (1992) 405 [hep-th/9204102].

[35] F. Bastianelli and O. Corradini, 6*d trace anomalies from quantum mechanical path integrals*, *Phys. Rev.* **D 63** (2001) 065005 [hep-th/0010118].

[36] D.H. Friedan, *Nonlinear models in* $2+\epsilon$ *dimensions*, *Ann. Phys. (NY)* **163** (1985) 318.

[37] H. Kleinert and A. Chervyakov, *Perturbatively defined effective classical potential in curved space*, *Int. J. Mod. Phys.* **A 18** (2003) 5521 [quant-ph/0301081].

[38] F. Bastianelli, O. Corradini and P. van Nieuwenhuizen, *Dimensional regularization of nonlinear sigma models on a finite time interval*, *Phys. Lett.* **B 494** (2000) 161 [hep-th/0008045].

[39] H. Kleinert and A. Chervyakov, *Reparametrization invariance of path integrals*, *Phys. Lett.* **B 464** (1999) 257 [hep-th/9906156].

[40] F. Bastianelli, O. Corradini and P. van Nieuwenhuizen, *Dimensional regularization of the path integral in curved space on an infinite time interval*, *Phys. Lett.* **B 490** (2000) 154 [hep-th/0007105].

[41] F. Bastianelli and P. van Nieuwenhuizen, *Path integrals and anomalies in curved space*, Cambridge University Press, Cambridge, to appear.

[42] B. Peeters and P. van Nieuwenhuizen, *The hamiltonian approach and phase space path integration for nonlinear sigma models with and without fermions*, hep-th/9312147.