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Abstract: The spectral density of euclidean Dirac operator is investigated in partially quenched QCD with arbitrary quark masses. A representation of scalar and pseudoscalar correlators in terms of the spectral density is discussed. The spectral density, obtained from the partially quenched chiral perturbation theory is shown to be compatible with low energy sum rules, obtained earlier.

Keywords: Chiral Lagrangians, QCD, Sum Rules.

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Contents

1. Introduction

One of most important characteristics in QCD is the spectral density of Dirac operator [1]:

$$\rho(\lambda) = \frac{1}{V}\left\langle \sum_n \delta\left(\lambda - \lambda_n[A]\right)\right\rangle_A , \tag{1.1}$$

where $\lambda_n[A]$ are eingenvalues of the Dirac operator D with gluon field A in 4-dimensional euclidean volume V : $iDu_n = \lambda_n u_n$. The average $\langle\cdots\rangle_A$ denotes integration over gluon fields with Yang-Mills action and the product of N fermionic determinants $\det(D + m_i)$. Due to this averaging the spectral density depends on the quark masses m_i.

A useful tool for the investigation of the spectral density is partially quenched QCD [2]. In addition to N usual quark with masses m_i it includes one unphysical (valence) quark with mass m_{v1} and one unphysical quark of opposite statistics with mass m_{v2}. The generation functional of this SU$(N+1|1)$ theory is written as follows:

$$Z^{pq}(m_1,\ldots,m_N;m_{v1},m_{v2}) = \int [dA]\,\frac{\det(D+m_{v1})}{\det(D+m_{v2})}\prod_{i=1}^{N}\det(D+m_i)\,e^{S_{\rm YM}[A]} . \tag{1.2}$$

Then one finds the condensate of unphysical fermionic quark for equal masses m_{v1} and m_{v2}:

$$\begin{aligned}\Sigma(m_v) &= \frac{1}{V}\left\langle \sum_n \frac{1}{m_v - i\,\lambda_n[A]}\right\rangle_A \\ &= \frac{1}{V}\,\frac{\partial}{\partial m_{v1}}\ln Z^{pq}(m_1,\ldots,m_N;m_{v1},m_{v2})|_{m_{v1}=m_{v2}=m_v} .\end{aligned} \tag{1.3}$$

It depends on the masses of all quarks, but we shall write explicitly only the dependence on the unphysical quark mass m_v. If we put it equal to the mass of some physical quark, we get its chiral condensate:

$$\Sigma(m_i) = -\langle \bar{q}_i\, q_i \rangle\,. \tag{1.4}$$

The spectral density is the discontinuity of $\Sigma(m_v)$ across the imaginary axis:

$$2\pi\,\rho(\lambda) = \Sigma(i\,\lambda + 0) - \Sigma(i\,\lambda - 0)\,. \tag{1.5}$$

This equation with condensate $\Sigma(m)$ given by (1.2) and (1.3) is just trivial rewritting of the definition (1.1) of the spectral density. Nevertheless it allows us to compute the spectral density using well known rules and methods of evaluating the Feynman graphs.

Indeed, the condensate of unphysical fermionic quark $\Sigma(m_v)$ is the fermion Green function at equal coordinates. It can be obtained from the quark propagator $S(p)$ in momentum space in d dimensions as the expansion over QCD coupling constant α_s:

$$\begin{aligned}
\Sigma(m) &= \mathrm{tr} \int \frac{d^d p}{(2\pi)^d}\, S(p) \\
&= \int \frac{d^d p}{(2\pi)^d}\, \frac{12\, m}{p^2 + m^2} - \\
&\quad - 64\,\pi\,\alpha_s\, m \int \frac{d^d\, p\, d^d k}{(2\,\pi)^{2d}}\, \frac{2(d-2)(p\,k) + (d-4)p^2 + dm^2}{(p^2+m^2)^2 k^2[(p+k)^2+m^2]} + O\left(\alpha_s^2\right) \\
&= 12\, m^{d-1}\, \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} - 2^7 \pi \alpha_s\, m^{2d-5} \frac{(d-1)}{(d-3)}\, \frac{\Gamma^2(2-d/2)}{(4\pi)^d} + O\left(\alpha_s^2\right),
\end{aligned} \tag{1.6}$$

"tr" here is the trace over color and spinor indices, $\mathrm{tr}\, 1 = 3 \cdot 4$; euclidean space is assumed. The two-loop integral in (1.6) can be computed with help of technique described, for instance, in [3]. The result (1.6) is divergent in four dimensions. If we put $d = 4 - 2\,\epsilon$ and expand over ϵ, we get the following result:

$$\Sigma(m) = \frac{m^3}{\tilde{\mu}^{2\epsilon}} \left\{ \mathrm{const} + \frac{3}{4\,\pi^2}\, \ln \frac{m^2}{\mu^2} + \frac{\alpha_s}{\tilde{\mu}^{2\epsilon}\,\pi^3} \left[\left(\frac{3}{\epsilon} - 2 \right) \ln \frac{m^2}{\mu^2} - 3\, \ln^2 \frac{m^2}{\mu^2} \right] \right\}, \tag{1.7}$$

where $\mu^2 = 4\,\pi\, e^{\Gamma'(1)+1} \tilde{\mu}^2$ according to $\overline{\mathrm{MS}}$ convention. The terms $\sim m^3$ are not important for our purposes, since they are regular for $m^2 < 0$, while logarithmic terms give us desired discontinuity. The term with $1/\epsilon$ is removed by renormalization of the mass and condensate itself. Now we may apply the eq. (1.5) in order to get the spectral density with α_s correction:

$$\rho(\lambda) = \frac{3\,|\lambda|^3}{4\,\pi^2} \left[1 - \frac{8\,\alpha_s}{\pi} \left(\ln \frac{\lambda^2}{\mu^2} + \frac{1}{3} \right) \right]. \tag{1.8}$$

Notice, that the spectral density in QCD depends on the quark masses only starting from α_s^2 level, where the loops with usual quarks appear.

The eq. (1.8) works in the QCD range of validity, i.e. for $|\lambda| \gg \Lambda_{\rm QCD}$. In the region $|\lambda| \sim \Lambda_{\rm QCD}$ the formula (1.2) is useless, since non-perturbative effects become important. Some attempts to extract the behavior of the spectral density in the region $|\lambda| \ll \Lambda_{\rm QCD}$ from low energy sum rules were undertaken in [4, 5, 6]. Two-point scalar correlators, known from the chiral perturbation theory (ChPT), can be presented as certain integrals over the spectral density. This analysis demonstrates linear behavior $\rho(\lambda) \sim (N^2-4)\,|\lambda|$ for $|\lambda| \gg m$.

The spectral density has been obtained in [7] within 1-loop partially quenched chiral perturbation theory [2] with N quarks of equal mass. This result correctly reproduces the 3-point scalar correlator, obtained in [4].

In section 2 of this paper the result of [7] is extended to the case of arbitrary quark masses. We take into account complete 1-loop generation functional of partially quenched ChPT with $O(p^4)$ terms, which absorb divergencies. In section 3 the scalar and pseudoscalar correlators are analyzed with help of spectral density representation. Many of them can be reduced to the condensate $\Sigma(m_v)$ (and its derivatives) and to the correlators of topological charge densities. Our result is shown to be in agreement with the sum rules, obtained in [5, 6].

2. Spectral density in partially quenched ChPT

At low energies the chiral perturbation theory [8, 9] is a good replacement of QCD. Its degrees of freedom are N^2-1 mesons instead of N quarks. In analogy with this construction one supposes that the low energy approximation of the partially quenched QCD (1.2) is partially quenched ChPT with chiral symmetry SU($N+1|1$) [2]. The effective degrees of freedom are mesons ϕ_{ij} of the same statistics as the product of relevant quarks $\bar{q}_i\, q_j$.

The generalization of the SU(N) chiral lagrangian to the partially quenched case is obvious: everywhere the flavor trace should be replaced with supertrace. Nevertheless, it is simpler to consider the U($N+1|1$) theory with super-η' instead of pure SU($N+1|1$). In this case there is no need to impose constraint $\mathrm{Str}\,\phi = 0$ and one can consider diagonal fields ϕ_{ii} as independent variables. In the final results one can take the limit of heavy super-η' which corresponds to the SU($N+1|1$) case.

The following terms of the chiral lagrangian are responsible for the chiral condensate:

$$L = -V_0(\phi_0) + V_1(\phi_0)\,\mathrm{Str}(\partial_\mu U^\dagger\, \partial^\mu U) + 2\;\mathrm{Re}\;\mathrm{Str}\,[V_2(\phi_0) m\, U] + \\ + V_3(\phi_0)\,\partial_\mu\,\phi_0\,\partial^\mu \phi_0\,, \qquad (2.1)$$

where U is unitary matrix of meson fields:

$$U = \exp\left(\frac{i\sqrt{2}\,\phi}{F}\right), \qquad \phi_0 = \mathrm{Str}\,\phi = \sum_{i=1}^{N+2} \epsilon_i\,\phi_{ii}\,,$$

$$\epsilon_i = \begin{cases} +1 & i = 1,\ldots,N+1\,, \\ -1 & i = N+2\,, \end{cases}$$

$m = \mathrm{diag}(m_1,\ldots,m_N,m_{v1};m_{v2})$ is the quark mass matrix, F is pion decay constant. The freedom of field redefinition $\phi_{ij} \to \phi_{ij} + \delta_{ij} f(\phi_0)$ can be used either to vanish V_3 or to make V_2 real. We shall use the last choice, since in this case the calculation is simpler. Expanding the lagrangian (2.1) up to quadratic terms, one needs the following constants:

$$V_1(0) = \frac{F^2}{4}\,, \qquad V_2(0) = \frac{BF^2}{2}\,, \qquad V_0''(0) = \mu^2\,, \qquad V_3(0) = \frac{\alpha}{2}\,.$$

There is also term $V_2''(0)\,\mathrm{Str}\,m\,\phi_0^2$. It belongs to the same class, as the terms of order $O(p^4)$, so the constant $V_2''(0)$ is supposed to be small. Since it contributes to the condensate only at the loop level, we shall ignore it.

The derivation of the one-loop chiral condensate (1.3) is standard. The propagator of the diagonal mesons ϕ_{ii} was found in [2]:

$$G_{ij} = \frac{\delta_{ij}\,\epsilon_i}{p^2 - M_i^2} - \frac{1}{(p^2 - M_i^2)(p^2 - M_j^2)}\left(\frac{1}{\alpha\,p^2 - \mu^2} + \sum_k \frac{\epsilon_k}{p^2 - M_k^2}\right)^{-1}, \qquad (2.2)$$

where $M_i^2 = 2\,B\,m_i$. Later we consider only the limit $\mu^2 \to \infty$. In this case the super-η' decouples leaving us with pure SU$(N+1|1)$ theory. In euclidean space p^2 changes the sign and the chiral condensate (1.3) can be written as

$$\Sigma(m_v) = BF^2 - B\sum_p \Bigg[\sum_{i=1}^{N} \frac{1}{p^2 + B(m_i + m_v)} - \frac{1}{(p^2 + 2\,B\,m_v)^2}\left(\sum_{k=1}^{N} \frac{1}{p^2 + 2\,B\,m_k}\right)^{-1}\Bigg]. \qquad (2.3)$$

When the volume goes to infinity the sum over momenta in (2.3) should be replaced with the integral over $d^4p/(2\pi)^4$. The integral is quadratically divergent, so we shall use dimensional regularization. We separate divergent terms according to usual conventions of ChPT [9]:

$$\Sigma(m_v) = \Sigma^r(m_v) - 2\,B^2 c \sum_{i=1}^{N}\left[\left(1 + \frac{2}{N^2}\right)m_i + \left(1 - \frac{4}{N^2}\right)m_v\right], \qquad (2.4)$$

where

$$c = \frac{1}{16\,\pi^2}\,\mu^{d-4}\left[\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln 4\,\pi + 1\right)\right],$$

$\Sigma^r(m_v)$ is regular in $d = 4$ part. The divergencies are absorbed by the coupling constants of the $O(p^4)$ lagrangian. The terms without derivatives of this lagrangian:

$$
\begin{aligned}
L^{(4)} &= L_6 \,\mathrm{Str}\,(\chi^+ U + U^+ \chi)^2 \,+\, L_7 \,\mathrm{Str}\,(\chi^+ U - U^+ \chi)^2 + \\
&+ L_8 \,\mathrm{Str}\,(\chi^+ U \chi^+ U + U^+ \chi U^+ \chi) + H_2 \,\mathrm{Str}\,(\chi^+ \chi)\,,
\end{aligned} \tag{2.5}
$$

where $\chi = 2Bm$. Adding the contribution of these terms to (2.4), we get the final result for the one-loop chiral condensate:

$$
\begin{aligned}
\Sigma(m_v) = BF^2 + \frac{B^2}{16\,\pi^2} \Bigg\{ 2\int_0^\infty \frac{x\,dx}{(x+m_v)^2} \left[\left(\sum_{i=1}^N \frac{1}{x+m_i} \right)^{-1} - \frac{1}{N^2} \sum_{i=1}^N (x+m_i) \right] - \\
- \sum_{i=1}^N \Bigg[(m_i + m_v) \ln \frac{B(m_i+m_v)}{\mu^2} + \\
+ \frac{2}{N^2}(m_i - 2\,m_v) \ln \frac{2\,Bm_v}{\mu^2} + \frac{2}{N^2}(m_i - m_v) \Bigg] \Bigg\} \\
+ 8\,B^2 \left[4\,L_6^r(\mu) \sum_{i=1}^N m_i \,+ (2\,L_8^r(\mu) + H_2^r(\mu))\, m_v \right],
\end{aligned} \tag{2.6}
$$

where $L^r(\mu)$, $H^r(\mu)$ are renormalized constants which depend on the normalization scale μ [9]:

$$
\begin{aligned}
L_6^r(\mu) = L_6 - \frac{N^2+2}{16\,N^2}\,c\,, \qquad L_8^r(\mu) = L_8 - \frac{N^2-4}{16\,N}\,c\,, \\
H_2^r(\mu) = H_2 - \frac{N^2-4}{8N}\,c\,.
\end{aligned} \tag{2.7}
$$

In the integral we subtracted two terms, divergent at $x = p^2/2B \to \infty$, so the total integral is finite. Due to boson-fermion loop cancellation the couplings of $\mathrm{SU}(N{+}1|1)$ partially quenched ChPT are renormalized exactly in the same way as the ones of usual $\mathrm{SU}(N)$ theory. Obviously, the total result (2.6) does not depend on μ.

Now one can compute the discontinuity of the condensate $\Sigma(m_v)$ across the imaginary axis according to (1.5) to find the spectral density. Nevertheless, the r.h.s. of the eq. (2.6), as it stands, does not have appropriate analytical properties for complex m_v. Indeed, according to Banks-Casher relation the condensate must have the form $\Sigma(m) = mf(m^2)$, where the function $f(m^2)$ has a cut in m^2 plane along half-axis $m^2 < 0$. The r.h.s. of (2.6) does not satisfy this requirement. Consequently, in order to get the spectral density from (2.6), the formula (1.5) should be applied with the following modification [7]:

$$
2\pi\,\rho(\lambda) = \mathrm{Disc}\,\frac{m_v}{\sqrt{m_v^2}}\,\Sigma\left(\sqrt{m_v^2}\right)|_{m_v = i\lambda} = \Sigma(i\,\lambda) + \Sigma(-i\,\lambda)\,. \tag{2.8}
$$

Notice, that in QCD the equations (1.5) and (2.8) give the same result (1.8).

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Final result for the Dirac operator spectral density:

$$\rho(\lambda) = \frac{BF^2}{\pi} + \frac{B^2}{32\,\pi^3}\left\{4\int_0^{\infty}\frac{x(x^2-\lambda^2)}{(x^2+\lambda^2)^2}\left[\left(\sum_{i=1}^{N}\frac{1}{x+m_i}\right)^{-1} - \frac{1}{N^2}\sum_{i=1}^{N}(x+m_i)\right]dx - \right.$$
$$-\sum_{i=1}^{N}\left[m_i\ln\frac{B^2(\lambda^2+m_i^2)}{\mu^4} - 2\,\lambda\,\mathrm{arctg}\frac{\lambda}{m_i} + \right.$$
$$+\frac{4}{N^2}\,m_i\ln\frac{2B|\lambda|}{\mu^2} + \frac{4\pi}{N^2}\,|\lambda| +$$
$$\left.\left. + \frac{4}{N^2}\,m_i - (32\,\pi)^2 L_6^r(\mu)\,m_i\right]\right\}. \qquad (2.9)$$

In particular case of equal quark masses m_i, the integral in (2.9) disappears and our result coincides with eq. (83) of ref. [7] up to the terms, linear in mass. In our case there is no ambiguity among these terms, since we have included all the $O(p^4)$ contributions.

In deriving the eq. (2.9) we neglected the contribution of higher loops. Each diagram in ChPT brings extra factor F^{-2} together with loop factor $(4\pi)^{-2}$. Consequently the expression (2.9) for the spectral density is valid for $|\lambda|$, $m_i \ll 4\pi F$, which is typical expansion parameter in ChPT.

Two flavor case deserves special attention. For $N = 2$ there appears extra term of order $O(p^4)$ in the chiral lagrangian:

$$L^{(2)} = \frac{l_3}{16}\,\mathrm{Str}\left(\chi^+U + U^+\chi\right)^2 - \frac{l_7}{16}\,\mathrm{Str}\left(\chi^+U - U^+\chi\right)^2 +$$
$$+\frac{1}{4}\,(h_1+h_3)\,\mathrm{Str}\,(\chi^+\chi) + \frac{1}{4}\,(h_1-h_3)\,\mathrm{Re\,Sdet}\,\chi\,,$$

the couplings l_i, h_i are the constants of the two flavor chiral lagrangian [8]. In this case the last line of eq. (2.6) should be replaced with:

$$\Sigma(m_v) = \cdots + B^2\left[2\,l_3(\mu)\,(m_1+m_2) + 2(h_1+h_3)\,m_v + (h_1-h_3)\,\frac{m_1 m_2}{m_v}\right]. \qquad (2.10)$$

The spectral density for $N = 2$ is given by (2.9) with obvious replacement $L_6 \Rightarrow l_3/16$. Again, the high energy constants h_i do not contribute to the spectral density.

Few steps further can be done to evaluate the integral in (2.9). One may notice, that the expression inside this integral can be written in the following form:

$$\left(\sum_{i=1}^{N}\frac{1}{x+m_i}\right)^{-1} = \frac{1}{N^2}\sum_{i=1}^{N}(\,x+m_i) + \sum_{j=1}^{N-1}\frac{a_j}{x+\tilde{m}_j}\,. \qquad (2.11)$$

The masses $\tilde{m}_j$ are the roots (with opposite sign) of the polinomial, which stands in denominator of the l.h.s. in eq. (2.11). The constants a_j have dimension $(\mathrm{mass})^2$;

they vanish when all m_i are equal. In particular case of two quarks:

$$N=2: \ \tilde{m}_1 = \frac{1}{2}\,(m_1+m_2)\,, \qquad a_1 = -\frac{1}{8}\,(m_1-m_2)^2\,.$$

For $N=3$ the constants $\tilde{m}_j$ and a_j also can be found in explicit form, but we shall not write them here. Knowing these constants, the integral in (2.9) can be easily computed:

$$\int_0^\infty \frac{x(x^2-\lambda^2)}{(x^2+\lambda^2)^2}\left[\left(\sum_{i=1}^N \frac{1}{x+m_i}\right)^{-1} - \frac{1}{N^2}\sum_{i=1}^N (\,x+m_i)\right] dx \;=$$
$$= \sum_{j=1}^{N-1} \frac{a_j\,\tilde{m}_j}{\left(\lambda^2+\tilde{m}_j^2\right)^2}\left[\left(\lambda^2-\tilde{m}_j^2\right)\ln\frac{|\lambda|}{\tilde{m}_j} - \lambda^2 - \tilde{m}_j^2 + \pi\,|\lambda|\,\tilde{m}_j\right]. \quad (2.12)$$

3. Low energy sum rules

The low energy sum rules are formulated for the correlators of charge densities which can be written as certain integrals over the spectral density. In particular, the two point correlators of scalar and pseudoscalar charge densities

$$S^{ij} = \bar{q}^i q^j\,, \qquad P^{ij} = i\,\bar{q}^i\,\gamma^5 q^j$$

were expressed in terms of the spectral density in [5]. This technique can be extended to the case of different quark masses:

$$i\int dx\,\left\langle S^{ij}(x)S^{kl}(0)^\dagger\right\rangle = 2\,\delta^{ik}\delta^{jl}\int_0^\infty \frac{(\lambda^2-m_i\,m_j)\,\rho(\lambda)}{(\lambda^2+m_i^2)(\lambda^2+m_j^2)}\,d\lambda\,, \quad i\neq j,\ k\neq l\,; \quad (3.1)$$

$$i\int dx\,\left\langle P^{ij}(x)P^{kl}(0)^\dagger\right\rangle = 2\,\delta^{ik}\delta^{jl}\int_0^\infty \frac{(\lambda^2+m_i\,m_j)\,\rho(\lambda)}{(\lambda^2+m_i^2)(\lambda^2+m_j^2)}\,d\lambda + \delta^{ij}\delta^{kl}\langle\nu^2\rangle\,, \quad (3.2)$$

where $\langle\nu^2\rangle$ is the topological susceptibility:

$$\langle\nu^2\rangle = i\int dx\langle\omega(x)\,\omega(0)\rangle\,, \quad \omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\left(G_{\mu\nu}\,\tilde{G}^{\mu\nu}\right)\,. \quad (3.3)$$

Correlators of diagonal scalar charges include additional contributions of disconnected graphs. The insertion of the operator $-i\int dx\,S^{ii}(x)$ corresponds to the differentiation over the mass m_i. In particular

$$i\int dx\,\left\langle S^{ii}(x)\,S^{jj}(0)^\dagger\right\rangle \;=\; \frac{\partial\Sigma(m_j)}{\partial m_i}\,. \quad (3.4)$$

Since the condensate is itself the derivative, l.h.s. is symmetric over ij.

There is no need to compute the integrals in (3.1), (3.2), since they can be expressed in terms of the chiral condensate:

$$i\int dx\,\langle S^{ij}(x)\,S^{kl}(0)^\dagger\rangle = \delta^{ik}\,\delta^{jl}\,\frac{\Sigma(m_i)-\Sigma(m_j)}{m_i-m_j}\,,\quad i\neq j,\ k\neq l\,; \tag{3.5}$$

$$i\int dx\,\langle P^{ij}(x)P^{kl}(0)^\dagger\rangle = \delta^{ik}\delta^{jl}\,\frac{\Sigma(m_i)+\Sigma(m_j)}{m_i+m_j} + \delta^{ij}\delta^{kl}\langle\nu^2\rangle\,. \tag{3.6}$$

These equations directly follow from the limit $q^\mu \to 0$ in the appropriate vector/axial correlators $q^\mu q^\nu \int e^{iqx} dx\langle V_\mu(x)V_\nu(0)\rangle$. Nevertheless, the eq. (3.5) within the framework of the partially quenched theory can be also applied if some quarks have equal masses: the ratio should be replaced with derivative. The result of this limit $m_i \to m_j$ is however not the same as the l.h.s. of (3.4) due to different order of operations: one should compute the derivative $\partial\Sigma(m_v)/\partial m_v$ at first and substitute $m_v = m_i$ after then.

In the same way other n-point off-diagonal correlators can be expressed in terms of the chiral condensate. In particular

$$i^{n-1}\int dx_1\cdots dx_{n-1}\,\langle S^{i_1j_1}(x_1)\cdots S^{i_{n-1}j_{n-1}}(x_{n-1})S^{i_nj_n}(0)\rangle = \tag{3.7}$$

$$= \left(\delta^{i_1j_2}\cdots\delta^{i_{n-1}j_n}\delta^{i_nj_1} + \mathrm{perm}(j)\right)\sum_{a=1}^{n}\frac{\Sigma(m_{i_a})}{\prod\limits_{b=1,\,b\neq a}^{n}(m_{i_b}-m_{i_a})}\,,\quad i_c\neq j_c \text{ for any } c\,.$$

where perm(j) denotes all permutations of indices j.

A large set of sum rules can be formulated using spectral density representation. Below we consider the simplest realistic model with $N = 2$ flavors of equal mass $m_u = m_d = m$. The charges are combined in triplet $S^a = \bar q\sigma^a q$ and singlet $S^0 = \bar q\, q$. The following sum rules have been formulated earlier [5, 6]:

$$i\int dx\,\langle S^a(x)\,S^b(0) - \delta^{ab}P^0(x)P^0(0)\rangle = 2\,\delta^{ab}\left(\left.\frac{\partial\Sigma(m_v)}{\partial m_v}\right|_{m_v=m} - \frac{\Sigma(m)}{m} - 2\langle\nu^2\rangle\right)$$
$$= -\delta^{ab}8\,B^2 l_7\,; \tag{3.8}$$

$$i\int dx\,\langle\delta^{ab}\,S^0(x)S^0(0) - P^a(x)P^b(0)\rangle = 2\,\delta^{ab}\left(\frac{\partial\Sigma(m)}{\partial m} - \frac{\Sigma(m)}{m}\right)$$
$$= \delta^{ab}\left(-\frac{2BF^2}{m} - \frac{3B^2}{8\pi^2}\right). \tag{3.9}$$

We have expressed the integrals over the spectral density in terms of the condensate according to (3.4)–(3.6). Equation (3.9) is satisfied for $\Sigma(m_v)$ given by (2.6). The sum rule (3.8) gives the following result for the topological susceptibility:

$$\langle\nu^2\rangle = \frac{1}{2}\,BF^2m + B^2m^2\left(2\,l_3^r(\mu) - 2\,l_7 - \frac{3}{32\pi^2}\ln\frac{M_\pi^2}{\mu^2}\right). \tag{3.10}$$

This result can be obtained directly from the two flavor chiral lagrangian [8], which verifies the partially quenched approach used here.

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