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Abstract: We extend earlier ideas about the appearance of noncommutative geometry in string theory with a nonzero B-field. We identify a limit in which the entire string dynamics is described by a minimally coupled (supersymmetric) gauge theory on a noncommutative space, and discuss the corrections away from this limit. Our analysis leads us to an equivalence between ordinary gauge fields and noncommutative gauge fields, which is realized by a change of variables that can be described explicitly. This change of variables is checked by comparing the ordinary Dirac-Born-Infeld theory with its noncommutative counterpart. We obtain a new perspective on noncommutative gauge theory on a torus, its T-duality, and Morita equivalence. We also discuss the $D0/D4$ system, the relation to M-theory in DLCQ, and a possible noncommutative version of the six-dimensional $(2,0)$ theory.

Keywords: Bosonic Strings, D-branes, Space-Time Symmetries, Gauge Symmetry.

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Contents

1. Introduction

The idea that the spacetime coordinates do not commute is quite old [1]. It has been studied by many authors both from a mathematical and a physical perspective.

The theory of operator algebras has been suggested as a framework for physics in noncommutative spacetime — see [2] for an exposition of the philosophy — and Yang-Mills theory on a noncommutative torus has been proposed as an example [3]. Though this example at first sight appears to be neither covariant nor causal, it has proved to arise in string theory in a definite limit [4], with the noncovariance arising from the expectation value of a background field. This analysis involved toroidal compactification, in the limit of small volume, with fixed and generic values of the worldsheet theta angles. This limit is fairly natural in the context of the matrix model of M-theory [5, 6], and the original discussion was made in this context. Indeed, early work relating membranes to large matrices [7], has motivated in [8, 9] constructions somewhat similar to [3]. For other thoughts about applications of noncommutative geometry in physics, see e.g. [10]. Noncommutative geometry has also been used as a framework for open string field theory [11].

Part of the beauty of the analysis in [4] was that T-duality acts within the noncommutative Yang-Mills framework, rather than, as one might expect, mixing the modes of noncommutative Yang-Mills theory with string winding states and other stringy excitations. This makes the framework of noncommutative Yang-Mills theory seem very powerful.

Subsequent work has gone in several directions. Additional arguments have been presented extracting noncommutative Yang-Mills theory more directly from open strings without recourse to matrix theory [12]–[16]. The role of Morita equivalence in establishing T-duality has been understood more fully [17, 18]. The modules and their T-dualities have been reconsidered in a more elementary language [19]–[21], and the relation to the Dirac-Born-Infeld lagrangean has been explored [20, 21]. The BPS spectrum has been more fully understood [19, 20, 22]. Various related aspects of noncommutative gauge theories have been discussed in [23] -[32]. Finally, the authors of [33] suggested interesting relations between noncommutative gauge theory and the little string theory [34].

Large instantons and the α' expansion. Our work has been particularly influenced by certain further developments, including the analysis of instantons on a noncommutative $\mathbb{R}^4$ [35]. It was shown that instantons on a noncommutative $\mathbb{R}^4$ can be described by adding a constant (a Fayet-Iliopoulos term) to the ADHM equations. This constant had been argued, following [36], to arise in the description of instantons on D-branes upon turning on a constant B-field [37],[1] so putting the two facts together it was proposed that instantons on branes with a B-field should be described by noncommutative Yang-Mills theory [35, 38].

Another very cogent argument for this is as follows. Consider N parallel three-branes of type IIB. They can support supersymmetric configurations in the form of

[1]One must recall that in the presence of a D-brane, a constant B-field cannot be gauged away and can in fact be reinterpreted as a magnetic field on the brane.

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U(N) instantons. If the instantons are large, they can be described by the classical self-dual Yang-Mills equations. If the instantons are small, the classical description of the instantons is no longer good. However, it can be shown that, at $B = 0$, the instanton moduli space $\mathcal{M}$ in string theory coincides precisely with the classical instanton moduli space. The argument for this is presented in section 2.3. In particular, $\mathcal{M}$ has the small instanton singularities that are familiar from classical Yang-Mills theory. The significance of these singularities in string theory is well known: they arise because an instanton can shrink to a point and escape as a -1-brane [39, 40]. Now if one turns on a B-field, the argument that the stringy instanton moduli space coincides with the classical instanton moduli space fails, as we will also see in section 2.3. Indeed, the instanton moduli space must be corrected for nonzero B. The reason is that, at nonzero B (unless B is anti-self-dual) a configuration of a threebrane and a separated -1-brane is not BPS,[2] so an instanton on the threebrane cannot shrink to a point and escape. The instanton moduli space must therefore be modified, for non-zero B, to eliminate the small instanton singularity. Adding a constant to the ADHM equations resolves the small instanton singularity [41], and since going to noncommutative $\mathbb{R}^4$ does add this constant [35], this strongly encourages us to believe that instantons with the B-field should be described as instantons on a noncommutative space.

This line of thought leads to an apparent paradox, however. Instantons come in all sizes, and however else they can be described, big instantons can surely be described by conventional Yang-Mills theory, with the familiar stringy α' corrections that are of higher dimension, but possess the standard Yang-Mills gauge invariance. The proposal in [35] implies, however, that the large instantons would be described by classical Yang-Mills equations with corrections coming from the noncommutativity of spacetime. For these two viewpoints to agree means that noncommutative Yang-Mills theory must be equivalent to ordinary Yang-Mills theory perturbed by higher dimension, gauge-invariant operators. To put it differently, it must be possible (at least to all orders in a systematic asymptotic expansion) to map noncommutative Yang-Mills fields to ordinary Yang-Mills fields, by a transformation that maps one kind of gauge invariance to the other and adds higher dimension terms to the equations of motion. This at first sight seems implausible, but we will see in section 3 that it is true.

Applying noncommutative Yang-Mills theory to instantons on $\mathbb{R}^4$ leads to another puzzle. The original application of noncommutative Yang-Mills to string theory [4] involved toroidal compactification in a small volume limit. The physics of noncompact $\mathbb{R}^4$ is the opposite of a small volume limit! The small volume limit is also puzzling even in the case of a torus; if the volume of the torus the strings prop-

[2] This is shown in a footnote in section 4.2; the configurations in question are further studied in section 5.

agate on is taken to zero, how can we end up with a noncommutative torus of finite size, as has been proposed? Therefore, a reappraisal of the range of usefulness of noncommutative Yang-Mills theory seems called for. For this, it is desireable to have new ways of understanding the description of D-brane phenomena in terms of physics on noncommuting spacetime. A suggestion in this direction is given by recent analyses arguing for noncommutativity of string coordinates in the presence of a B-field, in a hamiltonian treatment [14] and also in a worldsheet treatment that makes the computations particularly simple [15]. In the latter paper, it was suggested that rather classical features of the propagation of strings in a constant magnetic field [42, 43] can be reinterpreted in terms of noncommutativity of spacetime.

In the present paper, we will build upon these suggestions and reexamine the quantization of open strings ending on D-branes in the presence of a B-field. We will show that noncommutative Yang-Mills theory is valid for some purposes in the presence of any nonzero constant B-field, and that there is a systematic and efficient description of the physics in terms of noncommutative Yang-Mills theory when B is large. The limit of a torus of small volume with fixed theta angle (that is, fixed periods of B) [4, 12] is an example with large B, but it is also possible to have large B on $\mathbb{R}^n$ and thereby make contact with the application of noncommutative Yang-Mills to instantons on $\mathbb{R}^4$. An important element in our analysis is a distinction between two different metrics in the problem. Distances measured with respect to one metric are scaled to zero as in [4, 12]. However, the noncommutative theory is on a space with a different metric with respect to which all distances are nonzero. This guarantees that both on $\mathbb{R}^n$ and on $\mathbf{T}^n$ we end up with a theory with finite metric.

Organization of the paper. This paper is organized as follows. In section 2, we reexamine the behavior of open strings in the presence of a constant B-field. We show that, if one introduces the right variables, the B dependence of the effective action is completely described by making spacetime noncommutative. In this description, however, there is still an α' expansion with all of its usual complexity. We further show that by taking B large or equivalently by taking $\alpha' \to 0$ holding the effective open string parameters fixed, one can get an effective description of the physics in terms of noncommutative Yang-Mills theory. This analysis makes it clear that two different descriptions, one by ordinary Yang-Mills fields and one by noncommutative Yang-Mills fields, differ by the choice of regularization for the world-sheet theory. This means that (as we argued in another way above) there must be a change of variables from ordinary to noncommutative Yang-Mills fields. Once one is convinced that it exists, it is not too hard to find this transformation explicitly: it is presented in section 3. In section 4, we make a detailed exploration of the two descriptions by ordinary and noncommutative Yang-Mills fields, in the case of almost constant fields where one can use the Born-Infeld action for the ordinary Yang-Mills fields. In section 5, we explore the behavior of instantons at nonzero B by quantization of the D0-D4 system. Other aspects of instantons are studied in sections 2.3 and 4.2. In section 6, we consider the behavior of noncommutative Yang-Mills theory on a torus and analyze the action of T-duality, showing how the standard action of T-duality on the underlying closed string parameters induces the action of T-duality on the noncommutative Yang-Mills theory that has been described in the literature [17]–[21]. We also show that many mathematical statements about modules over a noncommutative torus and their Morita equivalences — used in analyzing T-duality mathematically — can be systematically derived by quantization of open strings. In the remainder of the paper, we reexamine the relation of noncommutative Yang-Mills theory to DLCQ quantization of M-theory, and we explore the possible noncommutative version of the $(2,0)$ theory in six dimensions.

Conventions. We conclude this introduction with a statement of our main conventions about noncommutative gauge theory.

For $\mathbb{R}^n$ with coordinates x^i whose commutators are c-numbers, we write

$$[x^i, x^j] = i\theta^{ij}\,, \tag{1.1}$$

with real θ. Given such a Lie algebra, one seeks to deform the algebra of functions on $\mathbb{R}^n$ to a noncommutative, associative algebra $\mathcal{A}$ such that $f * g = fg + \frac{1}{2}i\theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^2)$, with the coefficient of each power of θ being a local differential expression bilinear in f and g. The essentially unique solution of this problem (modulo redefinitions of f and g that are local order by order in θ) is given by the explicit formula

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)g(x+\zeta)\big|_{\xi=\zeta=0} = fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^2)\,. \tag{1.2}$$

This formula defines what is often called the Moyal bracket of functions; it has appeared in the physics literature in many contexts, including applications to old and new matrix theories [8, 9, 44]–[46]. We also consider the case of $N\times N$ matrix-valued functions f, g. In this case, we define the $*$ product to be the tensor product of matrix multiplication with the $*$ product of functions as just defined. The extended $*$ product is still associative.

The $*$ product is compatible with integration in the sense that for functions f, g that vanish rapidly enough at infinity, so that one can integrate by parts in evaluating the following integrals, one has

$$\int \mathrm{Tr}\, f * g = \int \mathrm{Tr}\, g * f\,. \tag{1.3}$$

Here Tr is the ordinary trace of the $N\times N$ matrices, and $\int$ is the ordinary integration of functions.

For ordinary Yang-Mills theory, we write the gauge transformations and field strength as

$$\begin{aligned} \delta_\lambda A_i &= \partial_i \lambda + i[\lambda, A_i]\,, \\ F_{ij} &= \partial_i A_j - \partial_j A_i - i[A_i, A_j]\,, \\ \delta_\lambda F_{ij} &= i[\lambda, F_{ij}]\,, \end{aligned} \tag{1.4}$$

where A and λ are $N \times N$ hermitian matrices. The Wilson line is

$$W(a,b) = Pe^{i\int_b^a A}\,, \tag{1.5}$$

where in the path ordering $A(b)$ is to the right. Under the gauge transformation (1.4)

$$\delta W(a,b) = i\lambda(a)W(a,b) - iW(a,b)\lambda(b)\,. \tag{1.6}$$

For noncommutative gauge theory, one uses the same formulas for the gauge transformation law and the field strength, except that matrix multiplication is replaced by the $*$ product. Thus, the gauge parameter $\widehat{\lambda}$ takes values in $\mathcal{A}$ tensored with $N \times N$ hermitian matrices, for some N, and the same is true for the components $\widehat{A}_i$ of the gauge field $\widehat{A}$. The gauge transformations and field strength of noncommutative Yang-Mills theory are thus

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_i &= \partial_i \widehat{\lambda} + i\widehat{\lambda} * \widehat{A}_i - i\widehat{A}_i * \widehat{\lambda} \\ \widehat{F}_{ij} &= \partial_i \widehat{A}_j - \partial_j \widehat{A}_i - i\widehat{A}_i * \widehat{A}_j + i\widehat{A}_j * \widehat{A}_i \\ \widehat{\delta}_{\widehat{\lambda}} F_{ij} &= i\widehat{\lambda} * \widehat{F}_{ij} - i\widehat{F}_{ij} * \widehat{\lambda}\,. \end{aligned} \tag{1.7}$$

The theory obtained this way reduces to conventional $\mathrm{U}(N)$ Yang-Mills theory for $\theta \to 0$. Because of the way that the theory is constructed from associative algebras, there seems to be no convenient way to get other gauge groups. The commutator of two infinitesimal gauge transformations with generators $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ is, rather as in ordinary Yang-Mills theory, a gauge transformation generated by $i(\widehat{\lambda}_1 * \widehat{\lambda}_2 - \widehat{\lambda}_2 * \widehat{\lambda}_1)$. Such commutators are nontrivial even for the rank 1 case, that is $N = 1$, though for $\theta = 0$ the rank 1 case is the abelian $\mathrm{U}(1)$ gauge theory. For rank 1, to first order in θ, the above formulas for the gauge transformations and field strength read

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_i &= \partial_i \widehat{\lambda} - \theta^{kl} \partial_k \widehat{\lambda} \partial_l \widehat{A}_i + \mathcal{O}(\theta^2) \\ \widehat{F}_{ij} &= \partial_i \widehat{A}_j - \partial_j \widehat{A}_i + \theta^{kl} \partial_k \widehat{A}_i \partial_l \widehat{A}_j + \mathcal{O}(\theta^2) \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{F}_{ij} &= -\theta^{kl} \partial_k \widehat{\lambda} \partial_l \widehat{F}_{ij} + \mathcal{O}(\theta^2)\,. \end{aligned} \tag{1.8}$$

Finally, a matter of terminology: we will consider the opposite of a "noncommutative" Yang-Mills field to be an "ordinary" Yang-Mills field, rather than a "commutative" one. To speak of ordinary Yang-Mills fields, which can have a nonabelian gauge group, as being "commutative" would be a likely cause of confusion.

2. Open strings in the presence of constant B-field

2.1 Bosonic strings

In this section, we will study strings in flat space, with metric g_{ij}, in the presence of a constant Neveu-Schwarz B-field and with Dp-branes. The B-field is equivalent to a constant magnetic field on the brane; the subject has a long history and the basic formulas with which we will begin were obtained in the mid-80's [42, 43].

We will denote the rank of the matrix B_{ij} as r; r is of course even. Since the components of B not along the brane can be gauged away, we can assume that $r \leq p+1$. When our target space has lorentzian signature, we will assume that $B_{0i} = 0$, with "0" the time direction. With a euclidean target space we will not impose such a restriction. Our discussion applies equally well if space is $\mathbb{R}^{10}$ or if some directions are toroidally compactified with $x^i \sim x^i + 2\pi r^i$. (One could pick a coordinate system with $g_{ij} = \delta_{ij}$, in which case the identification of the compactified coordinates may not be simply $x^i \sim x^i + 2\pi r^i$, but we will not do that.) If our space is $\mathbb{R}^{10}$, we can pick coordinates so that B_{ij} is nonzero only for $i, j = 1, \ldots, r$ and that g_{ij} vanishes for $i = 1, \ldots, r$, $j \neq 1, \ldots, r$. If some of the coordinates are on a torus, we cannot pick such coordinates without affecting the identification $x^i \sim x^i + 2\pi r^i$. For simplicity, we will still consider the case $B_{ij} \neq 0$ only for $i, j = 1, \ldots, r$ and $g_{ij} = 0$ for $i = 1, \ldots, r$, $j \neq 1, \ldots, r$.

The worldsheet action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{ij} \partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j \right) \\ &= \frac{1}{4\pi\alpha'} \int_\Sigma g_{ij} \partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j \,, \end{aligned} \tag{2.1}$$

where Σ is the string worldsheet, which we take to be with euclidean signature. (With Lorentz signature, one would omit the "i" multiplying B.) ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$. The equations of motion determine the boundary conditions. For i along the Dp-branes they are

$$g_{ij} \partial_n x^j + 2\pi i \alpha' B_{ij} \partial_t x^j |_{\partial\Sigma} = 0 \,, \tag{2.2}$$

where ∂_n is a normal derivative to $\partial\Sigma$. (These boundary conditions are not compatible with real x, though with a lorentzian worldsheet the analogous boundary conditions would be real. Nonetheless, the open string theory can be analyzed by determining the propagator and computing the correlation functions with these boundary conditions. In fact, another approach to the open string problem is to omit or not specify the boundary term with B in the action (2.1) and simply impose the boundary conditions (2.2).)

For $B = 0$, the boundary conditions in (2.2) are Neumann boundary conditions. When B has rank $r = p$ and $B \to \infty$, or equivalently $g_{ij} \to 0$ along the spatial directions of the brane, the boundary conditions become Dirichlet; indeed, in this limit, the second term in (2.2) dominates, and, with B being invertible, (2.2) reduces to $\partial_t x^j = 0$. This interpolation from Neumann to Dirichlet boundary conditions will be important, since we will eventually take $B \to \infty$ or $g_{ij} \to 0$. For B very large or g very small, each boundary of the string worldsheet is attached to a single point in the Dp-brane, as if the string is attached to a zero-brane in the Dp-brane. Intuitively, these zero-branes are roughly the constituent zero-branes of the Dp-brane as in the matrix model of M-theory [5, 6], an interpretation that is supported by the fact that in the matrix model the construction of Dp-branes requires a nonzero B-field.

Our main focus in most of this paper will be the case that Σ is a disc, corresponding to the classical approximation to open string theory. The disc can be conformally mapped to the upper half plane; in this description, the boundary conditions (2.2) are

$$g_{ij}(\partial - \bar\partial)x^j + 2\pi\alpha' B_{ij}(\partial + \bar\partial)x^j|_{z=\bar z} = 0\,, \tag{2.3}$$

where $\partial = \partial/\partial z$, $\bar\partial = \partial/\partial \bar z$, and $\operatorname{Im} z \geq 0$. The propagator with these boundary conditions is [42, 43]

$$\begin{aligned}\langle x^i(z)x^j(z')\rangle = -\alpha'\Bigg[&g^{ij}\log|z-z'| - g^{ij}\log|z-\bar z'| + \\ &+ G^{ij}\log|z-\bar z'|^2 + \frac{1}{2\pi\alpha'}\theta^{ij}\log\frac{z-\bar z'}{\bar z - z'} + D^{ij}\Bigg]\,.\end{aligned} \tag{2.4}$$

Here

$$\begin{aligned}G^{ij} &= \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_S = \left(\frac{1}{g+2\pi\alpha' B}\,g\,\frac{1}{g-2\pi\alpha' B}\right)^{ij},\\ G_{ij} &= g_{ij} - (2\pi\alpha')^2(Bg^{-1}B)_{ij}\,,\\ \theta^{ij} &= 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}\,B\,\frac{1}{g-2\pi\alpha' B}\right)^{ij},\end{aligned} \tag{2.5}$$

where $(\ )_S$ and $(\ )_A$ denote the symmetric and antisymmetric part of the matrix. The constants D^{ij} in (2.4) can depend on B but are independent of z and z'; they play no essential role and can be set to a convenient value. The first three terms in (2.4) are manifestly single-valued. The fourth term is single-valued, if the branch cut of the logarithm is in the lower half plane.

In this paper, our focus will be almost entirely on the open string vertex operators and interactions. Open string vertex operators are of course inserted on the boundary of Σ. So to get the relevant propagator, we restrict (2.4) to real z and z', which we denote τ and τ'. Evaluated at boundary points, the propagator is

$$\langle x^i(\tau)x^j(\tau')\rangle = -\alpha' G^{ij}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau')\,, \tag{2.6}$$

where we have set D^{ij} to a convenient value. $\epsilon(\tau)$ is the function that is 1 or -1 for positive or negative τ.

The object G_{ij} has a very simple intuitive interpretation: it is the effective metric seen by the open strings. The short distance behavior of the propagator between interior points on Σ is $\langle x^i(z)x^j(z')\rangle = -\alpha' g^{ij}\log|z-z'|$. The coefficient of the logarithm determines the anomalous dimensions of closed string vertex operators, so that it appears in the mass shell condition for closed string states. Thus, we will refer to g_{ij} as the closed string metric. G_{ij} plays exactly the analogous role for open strings, since anomalous dimensions of open string vertex operators are determined by the coefficient of $\log(\tau-\tau')^2$ in (2.6), and in this coefficient G^{ij} enters in exactly the way that g^{ij} would enter at $\theta = 0$. We will refer to G_{ij} as the open string metric.

The coefficient θ^{ij} in the propagator also has a simple intuitive interpretation, suggested in [15]. In conformal field theory, one can compute commutators of operators from the short distance behavior of operator products by interpreting time ordering as operator ordering. Interpreting τ as time, we see that

$$[x^i(\tau), x^j(\tau)] = T\left(x^i(\tau)x^j(\tau^-) - x^i(\tau)x^j(\tau^+)\right) = i\theta^{ij}\,. \tag{2.7}$$

That is, x^i are coordinates on a noncommutative space with noncommutativity parameter θ.

Consider the product of tachyon vertex operators $e^{ip\cdot x}(\tau)$ and $e^{iq\cdot x}(\tau')$. With $\tau > \tau'$, we get for the leading short distance singularity

$$e^{ip\cdot x}(\tau)\cdot e^{iq\cdot x}(\tau') \sim (\tau-\tau')^{2\alpha' G^{ij}p_i q_j} e^{-\frac{1}{2}i\theta^{ij}p_i q_j} e^{i(p+q)\cdot x}(\tau') + \cdots\,. \tag{2.8}$$

If we could ignore the term $(\tau-\tau')^{2\alpha' p\cdot q}$, then the formula for the operator product would reduce to a $*$ product; we would get

$$e^{ip\cdot x}(\tau)e^{iq\cdot x}(\tau') \sim e^{ip\cdot x} * e^{iq\cdot x}(\tau')\,. \tag{2.9}$$

This is no coincidence. If the dimensions of all operators were zero, the leading terms of operator products $\mathcal{O}(\tau)\mathcal{O}'(\tau')$ would be independent of $\tau-\tau'$ for $\tau\to\tau'$, and would give an ordinary associative product of multiplication of operators. This would have to be the $*$ product, since that product is determined by associativity, translation invariance, and (2.7) (in the form $x^i * x^j - x^j * x^i = i\theta^{ij}$).

Of course, it is completely wrong in general to ignore the anomalous dimensions; they determine the mass shell condition in string theory, and are completely essential to the way that string theory works. Only in the limit of $\alpha'\to 0$ or equivalently small momenta can one ignore the anomalous dimensions. When the dimensions are nontrivial, the leading singularities of operator products $\mathcal{O}(\tau)\mathcal{O}'(\tau')$ depend on $\tau-\tau'$ and do not give an associative algebra in the standard sense. For precisely this reason, in formulating open string field theory in the framework of noncommutative geometry [39], instead of using the operator product expansion directly, it was necessary to define the associative $*$ product by a somewhat messy procedure of gluing strings. For the same reason, most of the present paper will be written in a limit with $\alpha' \to 0$ that enables us to see the $*$ product directly as a product of vertex operators.

B **dependence of the effective action.** However, there are some important general features of the theory that do not depend on taking a zero slope limit. We will describe these first.

Consider an operator on the boundary of the disc that is of the general form $P(\partial x, \partial^2 x, \ldots)e^{ip\cdot x}$, where P is a polynomial in derivatives of x, and x are coordinates along the Dp-brane (the transverse coordinates satisfy Dirichlet boundary conditions). Since the second term in the propagator (2.6) is proportional to $\epsilon(\tau - \tau')$, it does not contribute to contractions of derivatives of x. Therefore, the expectation value of a product of k such operators, of momenta $p^1, \ldots, p^k$, satisfies

$$\left\langle \prod_{n=1}^{k} P_n(\partial x(\tau_n), \partial^2 x(\tau_n), \ldots) e^{ip^n\cdot x(\tau_n)} \right\rangle_{G,\theta} = \tag{2.10}$$
$$= e^{-\frac{i}{2}\sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)} \left\langle \prod_{n=1}^{k} P_n(\partial x(\tau_n), \partial^2 x(\tau_n), \ldots) e^{ip^n\cdot x(\tau_n)} \right\rangle_{G,\theta=0},$$

where $\langle \cdots \rangle_{G,\theta}$ is the expectation value with the propagator (2.6) parametrized by G and θ. We see that when the theory is described in terms of the open string parameters G and θ, rather than in terms of g and B, the θ dependence of correlation functions is very simple. Note that because of momentum conservation ($\sum_m p^m = 0$), the crucial factor

$$\exp\left(-\frac{i}{2}\sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)\right) \tag{2.11}$$

depends only on the cyclic ordering of the points $\tau_1, \ldots, \tau_k$ around the circle.

The string theory S-matrix can be obtained from the conformal field theory correlators by putting external fields on shell and integrating over the τ's. Therefore, it has a structure inherited from (2.10). To be very precise, in a theory with $N \times N$ Chan-Paton factors, consider a k point function of particles with Chan-Paton wave functions W_i, $i = 1, \ldots, k$, momenta p_i, and additional labels such as polarizations or spins that we will generically call ϵ_i. The contribution to the scattering amplitude in which the particles are cyclically ordered around the disc in the order from 1 to k depends on the Chan-Paton wave functions by a factor Tr $W_1 W_2 \ldots W_k$. We suppose, for simplicity, that N is large enough so that there are no identities between this factor and similar factors with other orderings. (It is trivial to relax this assumption.) By studying the behavior of the S-matrix of massless particles of small momenta, one can extract order by order in α' a low energy effective action for the theory. If Φ_i is an $N \times N$ matrix-valued function in spacetime representing a wavefunction for the i^{th} field, then at $B = 0$ a general term in the effective action is a sum of expressions of the form

$$\int d^{p+1}x \sqrt{\det G}\, \mathrm{Tr}\, \partial^{n_1}\Phi_1 \partial^{n_2}\Phi_2 \cdots \partial^{n_k}\Phi_k \,. \tag{2.12}$$

Here ∂^{n_i} is, for each i, the product of n_i partial derivatives with respect to some of the spacetime coordinates; which coordinates it is has not been specified in the notation. The indices on fields and derivatives are contracted with the metric G, though this is not shown explicitly in the formula.

Now to incorporate the B-field, at fixed G, is very simple: if the effective action is written in momentum space, we need only incorporate the factor (2.11). Including this factor is equivalent to replacing the ordinary product of fields in (2.12) by a $*$ product. (In this formulation, one can work in coordinate space rather than momentum space.) So the term corresponding to (2.12) in the effective action is given by the same expression but with the wave functions multiplied using the $*$ product:

$$\int d^{p+1}x \sqrt{\det G}\, \mathrm{Tr}\, \partial^{n_1}\Phi_1 * \partial^{n_2}\Phi_2 * \cdots * \partial^{n_k}\Phi_k \,. \tag{2.13}$$

It follows, then, that the B dependence of the effective action for fixed G and constant B can be obtained in the following very simple fashion: replace ordinary multiplication by the $*$ product. We will make presently an explicit calculation of an S-matrix element to illustrate this statement, and we will make a detailed check of a different kind in section 4 using almost constant fields and the Dirac-Born-Infeld theory.

Though we have obtained a simple description of the B-dependence of the effective action, the discussion also makes clear that going to the noncommutative description does not in general enable us to describe the effective action in closed form: it has an α' expansion that is just as complicated as the usual α' expansion at $B = 0$. To get a simpler description, and increase the power of the description by noncommutative Yang-Mills theory, we should take the $\alpha' \to 0$ limit.

The $\alpha' \to 0$ limit. For reasons just stated, and to focus on the low energy behavior while decoupling the string behavior, we would like to consider the zero slope limit ($\alpha' \to 0$) of our open string system. Clearly, since open strings are sensitive to G and θ, we should take the limit $\alpha' \to 0$ keeping fixed these parameters rather than the closed string parameters g and B.

So we consider the limit

$$\begin{aligned} &\alpha' \sim \epsilon^{1/2} \to 0 \\ &g_{ij} \sim \epsilon \to 0\,, \qquad \text{for } i,j = 1,\ldots,r \end{aligned} \tag{2.14}$$

with everything else, including the two-form B, held fixed. Then (2.5) become

$$G^{ij} = \begin{cases} -\frac{1}{(2\pi\alpha')^2}\left(\frac{1}{B}g\frac{1}{B}\right)^{ij} & \text{for } i,j=1,\ldots,r \\ g^{ij} & \text{otherwise} \end{cases}$$

$$G_{ij} = \begin{cases} -(2\pi\alpha')^2(Bg^{-1}B)_{ij} & \text{for } i,j=1,\ldots,r \\ g_{ij} & \text{otherwise} \end{cases}$$

$$\theta^{ij} = \begin{cases} \left(\frac{1}{B}\right)^{ij} & \text{for } i,j=1,\ldots,r \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.15}$$

Clearly, G and θ are finite in the limit. In this limit the boundary propagator (2.6) becomes

$$\langle x^i(\tau)x^j(0)\rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau)\,. \tag{2.16}$$

In this $\alpha' \to 0$ limit, the bulk kinetic term for the x^i with $i = 1,\ldots,r$ (the first term in (2.1)) vanishes. Hence, their bulk theory is topological. The boundary degrees of freedom are governed by the following action:

$$-\frac{i}{2}\int_{\partial\Sigma} B_{ij}x^i\partial_t x^j\,. \tag{2.17}$$

(A sigma model with only such a boundary interaction, plus gauge fixing terms, is a special case of the theory used by Kontsevich in studying deformation quantization [47], as has been subsequently elucidated [48].) If one regards (2.17) as a one-dimensional action (ignoring the fact that $x^i(\tau)$ is the boundary value of a string), then it describes the motion of electrons in the presence of a large magnetic field, such that all the electrons are in the first Landau level. In this theory the spatial coordinates are canonically conjugate to each other, and $[x^i, x^j] \neq 0$. As we will discuss in section 6.3, when we construct the representations or modules for a noncommutative torus, the fact that $x^i(\tau)$ is the boundary value of a string changes the story in a subtle way, but the general picture that the $x^i(\tau)$ are noncommuting operators remains valid.

With the propagator (2.16), normal ordered operators satisfy

$$:e^{ip_ix^i(\tau)}:\,:e^{iq_ix^i(0)}: = e^{-\frac{i}{2}\theta^{ij}p_iq_j\epsilon(\tau)}\;:e^{ipx(\tau)+iqx(0)}:\,, \tag{2.18}$$

or more generally

$$:f(x(\tau)):\,:g(x(0)): = :e^{\frac{i}{2}\epsilon(\tau)\theta^{ij}\frac{\partial}{\partial x^i(\tau)}\frac{\partial}{\partial x^j(0)}}f(x(\tau))g(x(0)):\,, \tag{2.19}$$

and

$$\lim_{\tau\to 0^+}\;:f(x(\tau)):\,:g(x(0)): = :f(x(0)) * g(x(0)):\,, \tag{2.20}$$

where

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0} \tag{2.21}$$

is the product of functions on a noncommutative space.

As always in the zero slope limit, the propagator (2.16) is not singular as $\tau \to 0$. This lack of singularity ensures that the product of operators can be defined without a subtraction and hence must be associative. It is similar to a product of functions, but on a noncommutative space.

The correlation functions of exponential operators on the boundary of a disc are

$$\left\langle \prod_n e^{ip_i^n x^i(\tau_n)} \right\rangle = e^{-\frac{i}{2}\sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)} \delta\left(\sum p^n\right). \tag{2.22}$$

Because of the δ function and the antisymmetry of θ^{ij}, the correlation functions are unchanged under cyclic permutation of τ_n. This means that the correlation functions are well defined on the boundary of the disc. More generally,

$$\left\langle \prod_n f_n(x(\tau_n)) \right\rangle = \int dx f_1(x) * f_2(x) * \cdots * f_n\,, \tag{2.23}$$

which is invariant under cyclic permutations of the f_n's. As always in the zero slope limit, the correlation functions (2.22) and (2.23) do not exhibit singularities in τ, and therefore there are no poles associated with massive string states.

Adding gauge fields. Background gauge fields couple to the string worldsheet by adding

$$-i \int d\tau A_i(x)\partial_\tau x^i \tag{2.24}$$

to the action (2.1). We assume for simplicity that there is only a rank one gauge field; the extension to higher rank is straightforward. Comparing (2.1) and (2.24), we see that a constant B-field can be replaced by the gauge field $A_i = -\frac{1}{2}B_{ij}x^j$, whose field strength is $F = B$. When we are working on $\mathbb{R}^n$, we are usually interested in situations where B and F are constant at infinity, and we fix the ambiguity be requiring that F is zero at infinity.

Naively, (2.24) is invariant under ordinary gauge transformations

$$\delta A_i = \partial_i \lambda \tag{2.25}$$

because (2.24) transforms by a total derivative

$$\delta \int d\tau A_i(x)\partial_\tau x^i = \int d\tau \partial_i\lambda \partial_\tau x^i = \int d\tau \partial_\tau \lambda\,. \tag{2.26}$$

However, because of the infinities in quantum field theory, the theory has to be regularized and we need to be more careful. We will examine a point splitting regularization, where different operators are never at the same point.

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Then expanding the exponential of the action in powers of A and using the transformation law (2.25), we find that the functional integral transforms by

$$-\int d\tau A_i(x)\partial_\tau x^i \cdot \int d\tau' \partial_{\tau'}\lambda \tag{2.27}$$

plus terms of higher order in A. The product of operators in (2.27) can be regularized in a variety of ways. We will make a point-splitting regularization in which we cut out the region $|\tau - \tau'| < \delta$ and take the limit $\delta \to 0$. Though the integrand is a total derivative, the τ' integral contributes surface terms at $\tau - \tau' = \pm\delta$. In the limit $\delta \to 0$, the surface terms contribute

$$\begin{aligned} -\int d\tau \ :A_i(x(\tau))\partial_\tau x^i(\tau)::\big(\lambda(x(\tau^-)) - \lambda(x(\tau^+))\big): = \\ = -\int d\tau \ :\left(A_i(x) * \lambda - \lambda * A_i(x)\right)\partial_\tau x^i: \end{aligned} \tag{2.28}$$

Here we have used the relation of the operator product to the $*$ product, and the fact that with the propagator (2.16) there is no contraction between $\partial_\tau x$ and x. To cancel this term, we must add another term to the variation of the gauge field; the theory is invariant not under (2.25), but under

$$\widehat{\delta}\widehat{A}_i = \partial_i\lambda + i\lambda * \widehat{A}_i - i\widehat{A}_i * \lambda\,. \tag{2.29}$$

This is the gauge invariance of noncommutative Yang-Mills theory, and in recognition of that fact we henceforth denote the gauge field in the theory defined with point splitting regularization as $\widehat{A}$. A sigma model expansion with Pauli-Villars regularization would have preserved the standard gauge invariance of open string gauge field, so whether we get ordinary or noncommutative gauge fields depends on the choice of regulator.

We have made this derivation to lowest order in $\widehat{A}$, but it is straightforward to go to higher orders. At the n-th order in $\widehat{A}$, the variation is

$$\begin{aligned} &\frac{i^{n+1}}{n!}\int \widehat{A}(x(t_1))\ldots\widehat{A}(x(t_n))\partial_t\lambda(x(t)) + \\ &\quad + \frac{i^{n+1}}{(n-1)!}\int \widehat{A}(x(t_1))\ldots\widehat{A}(x(t_{n-1}))\left(\lambda * \widehat{A}(x(t_n)) - \widehat{A} * \lambda(x(t_n))\right), \end{aligned} \tag{2.30}$$

where the integration region excludes points where some t's coincide. The first term in (2.30) arises by using the naive gauge transformation (2.25), and expanding the action to n-th order in $\widehat{A}$ and to first order in λ. The second term arises from using the correction to the gauge transformation in (2.29) and expanding the action to the same order in $\widehat{A}$ and λ. The first term can be written as

$$\begin{aligned} &\frac{i^{n+1}}{n!}\sum_j\int \widehat{A}(x(t_1))\cdots\widehat{A}(x(t_{j-1}))\widehat{A}(x(t_{j+1}))\cdots\widehat{A}(x(t_n))\left(\widehat{A}*\lambda(x(t_j)) - \lambda*\widehat{A}(x(t_j))\right) = \\ &\quad = \frac{i^{n+1}}{(n-1)!}\int \widehat{A}(x(t_1))\cdots\widehat{A}(x(t_{n-1}))\left(\widehat{A} * \lambda(x(t_n)) - \lambda * \widehat{A}(x(t_n))\right), \end{aligned} \tag{2.31}$$

making it clear that (2.30) vanishes. Therefore, there is no need to modify the gauge transformation law (2.29) at higher orders in $\widehat{A}$.

Let us return to the original theory before taking the zero slope limit (2.14), and examine the correlation functions of the physical vertex operators of gauge fields

$$V = \int \xi \cdot \partial x e^{ip\cdot x} \, . \tag{2.32}$$

These operators are physical when

$$\xi \cdot p = p \cdot p = 0 \, , \tag{2.33}$$

where the dot product is with the open string metric G (2.5). We will do an explicit calculation to illustrate the statement that the B dependence of the S-matrix, for fixed G, consists of replacing ordinary products with $*$ products. Using the conditions (2.33) and momentum conservation, the three point function is

$$\begin{aligned} \left\langle \xi^1 \cdot \partial x e^{ip^1\cdot x(\tau_1)} \, \xi^2 \cdot \partial x e^{ip^2\cdot x(\tau_2)} \, \xi^3 \cdot \partial x e^{ip^3\cdot x(\tau_3)} \right\rangle &\sim \frac{1}{(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)} \times \\ \times \left(\xi^1\cdot\xi^2 p^2\cdot\xi^3 + \xi^1\cdot\xi^3 p^1\cdot\xi^2 + \xi^2\cdot\xi^3 p^3\cdot\xi^1 + 2\alpha' p^3\cdot\xi^1 p^1\cdot\xi^2 p^2\cdot\xi^3 \right) &\times \\ \times e^{-\frac{i}{2}\left(p_i^1\theta^{ij}p_j^2\epsilon(\tau_1-\tau_2)+p_i^2\theta^{ij}p_j^3\epsilon(\tau_2-\tau_3)+p_i^3\theta^{ij}p_j^1\epsilon(\tau_3-\tau_1)\right)} \, . \end{aligned} \tag{2.34}$$

This expression should be multiplied by the Chan-Paton matrices. The order of these matrices is correlated with the order of τ_n. Therefore, for a given order of these matrices we should not sum over different orders of τ_n. Generically, the vertex operators (2.32) should be integrated over τ_n, but in the case of the three point function on the disc, the gauge fixing of the $\mathrm{SL}(2;\mathbb{R})$ conformal group cancels the integral over the τ's. All we need to do is to remove the denominator $(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)$. This leads to the amplitude

$$\left(\xi^1\cdot\xi^2 p^2\cdot\xi^3 + \xi^1\cdot\xi^3 p^1\cdot\xi^2 + \xi^2\cdot\xi^3 p^3\cdot\xi^1 + 2\alpha' p^3\cdot\xi^1 p^1\cdot\xi^2 p^2\cdot\xi^3 \right) \cdot e^{-\frac{i}{2}p_i^1\theta^{ij}p_j^2} \, . \tag{2.35}$$

The first three terms are the same as the three point function evaluated with the action

$$\frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s} \int \sqrt{\det G} G^{ii'} G^{jj'} \,\mathrm{Tr}\, \widehat{F}_{ij} * \widehat{F}_{i'j'}, \tag{2.36}$$

where G_s is the string coupling and

$$\widehat{F}_{ij} = \partial_i \widehat{A}_j - \partial_j \widehat{A}_i - i\widehat{A}_i * \widehat{A}_j + i\widehat{A}_j * \widehat{A}_i \tag{2.37}$$

is the noncommutative field strength. The normalization is the standard normalization in open string theory. The effective open string coupling constant G_s in (2.36) can differ from the closed string coupling constant g_s. We will determine the relation between them shortly. The last term in (2.35) arises from the $(\partial\widehat{A})^3$ part of a term

$\alpha'\widehat{F}^3$ in the effective action. This term vanishes for $\alpha' \to 0$ (and in any event is absent for superstrings).

Gauge invariance of (2.36) is slightly more subtle than in ordinary Yang-Mills theory. Since under gauge transformations $\widehat{\delta}\widehat{F} = i\lambda * \widehat{F} - i\widehat{F} * \lambda$, the gauge variation of $\widehat{F} * \widehat{F}$ is not zero. But this gauge variation is $\lambda * (i\widehat{F} * \widehat{F}) - (i\widehat{F} * \widehat{F}) * \lambda$, and the integral of this vanishes by virtue of (1.3). Notice that, because the scaling in (2.14) keeps all components of G fixed as $\epsilon \to 0$, (2.36) is uniformly valid whether the rank of B is $p+1$ or smaller.

The three point function (2.34) can easily be generalized to any number of gauge fields. Using (2.10)

$$\left\langle \prod_n \xi^n \cdot \partial x e^{ip^n \cdot x(\tau_n)} \right\rangle_{G,\theta} = e^{-\frac{i}{2}\sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)} \left\langle \prod_n \xi^n \cdot \partial x e^{ip^n \cdot x(\tau_n)} \right\rangle_{G,\theta=0} . \tag{2.38}$$

This illustrates the claim that when the effective action is expressed in terms of the open string variables G, θ and G_s (as opposed to g, B and g_s), θ appears only in the $*$ product.

The construction of the effective lagrangean from the S-matrix elements is always subject to a well-known ambiguity. The S-matrix is unchanged under field redefinitions in the effective lagrangean. Therefore, there is no canonical choice of fields. The vertex operators determine the linearized gauge symmetry, but field redefinitions $A_i \to A_i + f_i(A_j)$ can modify the nonlinear terms. It is conventional in string theory to define an effective action for ordinary gauge fields with ordinary gauge invariances that generates the S-matrix. In this formulation, the B-dependence of the effective action is very simple: it is described by everywhere replacing F by $F + B$. (This is manifest in the sigma model approach that we mention presently.)

We now see that it is also natural to generate the S-matrix from an effective action written for noncommutative Yang-Mills fields. In this description, the B-dependence is again simple, though different. For fixed G and G_s, B affects only θ, which determines the $*$ product. Being able to describe the same S-matrix with the two kinds of fields means that there must be a field redefinition of the form $A_i \to A_i + f_i(A_j)$, which relates them.

This freedom to write the effective action in terms of different fields has a counterpart in the sigma model description of string theory. Here we can use different regularization schemes. With Pauli-Villars regularization (such as the regularization we use in section 2.3), the theory has ordinary gauge symmetry, as the total derivative in (2.26) integrates to zero. Additionally, with such a regularization, the effective action can depend on B and F only in the combination $F + B$, since there is a symmetry $A \to A + \Lambda$, $B \to B - d\Lambda$, for any one-form Λ. With point-splitting regularization, we have found noncommutative gauge symmetry, and a different description of the B-dependence.

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The difference between different regularizations is always in a choice of contact terms; theories defined with different regularizations are related by coupling constant redefinition. Since the coupling constants in the worldsheet lagrangean are the spacetime fields, the two descriptions must be related by a field redefinition. The transformation from ordinary to noncommutative Yang-Mills fields that we will describe in section 3 is thus an example of a transformation of coupling parameters that is required to compare two different regularizations of the same quantum field theory.

In the $\alpha' \to 0$ limit (2.14), the amplitudes and the effective action are simplified. For example, the $\alpha' \widehat{F}^3$ term coming from the last term in the amplitude (2.35) is negligible in this limit. More generally, using dimensional analysis and the fact that the θ dependence is only in the definition of the $*$ product, it is clear that all higher dimension operators involve more powers of α'. Therefore they can be neglected, and the $\widehat{F}^2$ action (2.36) becomes exact for $\alpha' \to 0$.

The lack of higher order corrections to (2.36) can also be understood as follows. In the limit (2.14), there are no on-shell vertex operators with more derivatives of x, which would correspond to massive string modes. Since there are no massive string modes, there cannot be corrections to (2.36). As a consistency check, note that there are no poles associated with such operators in (2.22) or in (2.38) in our limit.

All this is standard in the zero slope limit, and the fact that the action for $\alpha' \to 0$ reduces to $\widehat{F}^2$ is quite analogous to the standard reduction of open string theory to ordinary Yang-Mills theory for $\alpha' \to 0$. The only novelty in our discussion is the fact that for $B \neq 0$, we have to take $\alpha' \to 0$ keeping fixed G rather than g. Even before taking the $\alpha' \to 0$ limit, the effective action, as we have seen, can be written in terms of the noncommutative variables. The role of the zero slope limit is just to remove the higher order corrections to $\widehat{F}^2$ from the effective action.

It remains to determine the relation between the effective open string coupling G_s which appears in (2.36) and the closed string variables g, B and g_s. For this, we examine the constant term in the effective lagrangean. For slowly varying fields, the effective lagrangean is the Dirac-Born-Infeld lagrangean (for a recent review of the DBI theory see [49] and references therein)

$$\mathcal{L}_{DBI} = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B+F))} \,. \tag{2.39}$$

The coefficient is determined by the Dp-brane tension which for $B = 0$ is

$$T_p(B=0) = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \,. \tag{2.40}$$

Therefore

$$\mathcal{L}(F=0) = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha' B)} \,. \tag{2.41}$$

Above we argued that when the effective action is expressed in terms of noncommutative gauge fields and the open string variables G, θ and G_s, the θ dependence is entirely in the $*$ product. In this description, the analog of (2.39) is

$$\mathcal{L}(\widehat{F}) = \frac{1}{G_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{\det G + 2\pi\alpha'\widehat{F}}\,, \tag{2.42}$$

and the constant term in the effective lagrangean is

$$\mathcal{L}(\widehat{F}=0) = \frac{1}{G_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{\det G}\,. \tag{2.43}$$

Therefore,

$$G_s = g_s\left(\frac{\det G}{\det\left(g+2\pi\alpha' B\right)}\right)^{/2} = g_s\left(\frac{\det G}{\det g}\right)^{1/4} = g_s\left(\frac{\det(g+2\pi\alpha' B)}{\det g}\right)^{1/2}, \tag{2.44}$$

where the definition (2.5) of G has been used. As a (rather trivial) consistency check, note that when $B=0$ we have $G_s = g_s$. In the zero slope limit (2.14) it becomes

$$G_s = g_s \det{}'(2\pi\alpha' B g^{-1})^{1/2}\,, \tag{2.45}$$

where $\det'$ denotes a determinant in the $r\times r$ block with nonzero B.

The effective Yang-Mills coupling is determined from the $\widehat{F}^2$ term in (2.42) and is

$$\frac{1}{g_{YM}^2} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}G_s} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}g_s}\left(\frac{\det(g+2\pi\alpha' B)}{\det G}\right)^{1/2}. \tag{2.46}$$

Using (2.45) we see that in order to keep it finite in our limit such that we end up with a quantum theory, we should scale

$$\begin{aligned} G_s &\sim \epsilon^{\frac{3-p}{4}} \\ g_s &\sim \epsilon^{\frac{3-p+r}{4}}\,. \end{aligned} \tag{2.47}$$

Note that the scaling of g_s depends on the rank r of the B field, while the scaling of G_s is independent of B. The scaling of G_s just compensates for the dimension of the Yang-Mills coupling, which is proportional to $p-3$ as the Yang-Mills theory on a brane is scale-invariant precisely for threebranes.

If several D-branes are present, we should scale g_s such that all gauge couplings of all branes are finite. For example, if there are some $D0$-branes, we should scale $g_s \sim \epsilon^{3/4}$ ($p=r=0$ in (2.47)). In this case, all branes for which $p>r$ can be treated classically, and branes with $p=r$ are quantum.

If we are on a torus, then the limit (2.14) with $g_{ij}\to 0$ and B_{ij} fixed is essentially the limit used in [4]. This limit takes the volume to zero while keeping fixed the periods of B. On the other hand, if we are on $\mathbb{R}^n$, then by rescaling the coordinates, instead of taking $g_{ij} \to 0$ with B_{ij} fixed, one could equivalently keep g_{ij} fixed and take $B_{ij} \to \infty$. (Scaling the coordinates on $\mathbf{T}^n$ changes the periodicity, and therefore it is more natural to scale the metric in this case.) In this sense, the $\alpha' \to 0$ limit can, on $\mathbb{R}^n$, be interpreted as a large B limit.

It is crucial that g_{ij} is taken to zero with fixed G_{ij}. The latter is the metric appearing in the effective lagrangean. Therefore, either on $\mathbb{R}^n$ or on a torus, all distances measured with the metric g scale to zero, but the noncommutative theory is sensitive to the metric G, and with respect to this metric the distances are fixed. This is the reason that we end up with finite distances even though the closed string metric g is taken to zero.

2.2 Worldsheet supersymmetry

We now add fermions to the theory and consider worldsheet supersymmetry. Without background gauge fields we have to add to the action (2.1)

$$\frac{i}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\psi^i\bar\partial\psi^j + g_{ij}\bar\psi^i\partial\bar\psi^j\right) \tag{2.48}$$

and the boundary conditions are

$$g_{ij}(\psi^j - \bar\psi^j) + 2\pi\alpha' B_{ij}(\psi^j + \bar\psi^j)|_{z=\bar z} = 0 \tag{2.49}$$

($\bar\psi$ is not the complex conjugate of ψ). The action and the boundary conditions respect the supersymmetry transformations

$$\begin{aligned} \delta x^i &= -i\eta(\psi^i + \bar\psi^i)\,, \\ \delta\psi^i &= \eta\partial x^i\,, \\ \delta\bar\psi^i &= \eta\bar\partial x^i\,. \end{aligned} \tag{2.50}$$

In studying sigma models, the boundary interaction (2.24) is typically extended to

$$L_A = -i\int d\tau \left(A_i(x)\partial_\tau x^i - iF_{ij}\Psi^i\Psi^j\right)\,, \tag{2.51}$$

with $F_{ij} = \partial_i A_j - \partial_j A_i$ and

$$\Psi^i = \frac{1}{2}(\psi^i + \bar\psi^i) = \left(\frac{1}{g - 2\pi\alpha' B}g\right)^i_{\ j}\psi^j\,. \tag{2.52}$$

The expression (2.51) seems to be invariant under (2.50) because its variation is a total derivative

$$\delta\int d\tau\left(A_i(x)\partial_\tau x^i - iF_{ij}\Psi^i\Psi^j\right) = -2i\eta\int d\tau\partial_\tau(A_i\Psi^i)\,. \tag{2.53}$$

However, as in the derivation of (2.28), with point splitting regularization, a total derivative such as the one in (2.53) can contribute a surface term. In this case, the surface term is obtained by expanding the $\exp(-L_A)$ term in the path integral in powers of A. The variation of the path integral coming from (2.53) reads, to first order in L_A,

$$i \int d\tau \int d\tau' \left(A_i \partial_\tau x^i(\tau) - iF_{ij}\Psi^i\Psi^j(\tau)\right)\left(-2i\eta\partial_{\tau'} A_k \Psi^k(\tau')\right). \tag{2.54}$$

With point splitting regularization, one picks up surface terms as $\tau' \to \tau^+$ and $\tau' \to \tau^-$, similar to those in (2.28). The surface terms can be canceled by the supersymmetric variation of an additional interaction term $\int d\tau A_i * A_j \Psi^i\Psi^j(\tau)$, and the conclusion is that with point-splitting regularization, (2.51) should be corrected to

$$-i \int d\tau \left(A_i(x)\partial_\tau x^i - i\widehat{F}_{ij}\Psi^i\Psi^j\right), \tag{2.55}$$

with $\widehat{F}$ the noncommutative field strength (2.37).

Once again, if supersymmetric Pauli-Villars regularization were used (an example of an explicit regularization procedure will be given presently in discussing instantons), the more naive boundary coupling (2.51) would be supersymmetric. Whether "ordinary" or "noncommutative" gauge fields and symmetries appear in the formalism depends on the regularization used, so there must be a transformation between them.

2.3 Instantons on noncommutative $\mathbb{R}^4$

As we mentioned in the introduction, one of the most fascinating applications of noncommutative Yang-Mills theory has been to instantons on $\mathbb{R}^4$. Given a system of N parallel D-branes with worldvolume $\mathbb{R}^4$, one can study supersymmetric configurations in the $\mathrm{U}(N)$ gauge theory. (Actually, most of the following discussion applies just as well if $\mathbb{R}^4$ is replaced by $\mathbf{T}^n \times \mathbb{R}^{4-n}$ for some n.) In classical Yang-Mills theory, such a configuration is an instanton, that is a solution of $F^+ = 0$. (For any two-form on $\mathbb{R}^4$ such as the Yang-Mills curvature F, we write F^+ and F^- for the self-dual and anti-self-dual projections.) So the objects we want are a stringy generalization of instantons. A priori one would expect that classical instantons would be a good approximation to stringy instantons only when the instanton scale size is very large compared to $\sqrt{\alpha'}$. However, we will now argue that with a suitable regularization of the worldsheet theory, the classical or field theory instanton equation is exact if $B = 0$. This implies that with any regularization, the stringy and field theory instanton moduli spaces are the same. The argument, which is similar to an argument about sigma models with K3 target [50], also suggests that for $B \neq 0$, the classical instanton equations and moduli space are *not* exact. We have given some arguments for this assertion in the introduction, and will give more arguments below and in the rest of the paper.

At $B = 0$, the free worldsheet theory in bulk

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{ij}\partial_a x^i \partial^a x^i + i g_{ij}\psi^i \bar{\partial}\psi^j + i g_{ij}\bar{\psi}^i \partial \bar{\psi}^j \right) \tag{2.56}$$

actually has a $(4, 4)$ worldsheet supersymmetry. This is a consequence of the $\mathcal{N} = 1$ worldsheet supersymmetry described in (2.50) plus an R symmetry group. In fact, we have a symmetry group $\mathrm{SO}(4)_L$ acting on the ψ^i and another $\mathrm{SO}(4)_R$ acting on $\bar{\psi}^i$. We can decompose $\mathrm{SO}(4)_L = \mathrm{SU}(2)_{L,+} \times \mathrm{SU}(2)_{L,-}$, and likewise $\mathrm{SO}(4)_R = \mathrm{SU}(2)_{R,+} \times \times \mathrm{SU}(2)_{R,-}$. $\mathrm{SU}(2)_{R,+}$, together with the $\mathcal{N} = 1$ supersymmetry in (2.50), generates an $\mathcal{N} = 4$ supersymmetry of the right-movers, and $\mathrm{SU}(2)_{L,+}$, together with (2.50), likewise generates an $\mathcal{N} = 4$ supersymmetry of left-movers. So altogether in bulk we get an $\mathcal{N} = (4, 4)$ free superconformal model. Of course, we could replace $\mathrm{SU}(2)_{R,+}$ by $\mathrm{SU}(2)_{R,-}$ or $\mathrm{SU}(2)_{L,+}$ by $\mathrm{SU}(2)_{L,-}$, so altogether the free theory has (at least) four $\mathcal{N} = (4, 4)$ superconformal symmetries. But for the instanton problem, we will want to focus on just one of these extended superconformal algebras.

Now consider the case that Σ has a boundary, but with $B = 0$ and no gauge fields coupled to the boundary. The boundary conditions on the fermions are, from (2.49), $\psi^j = \bar{\psi}^j$. This breaks $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$ down to a diagonal subgroup $\mathrm{SO}(4)_D = \mathrm{SU}(2)_{D,+} \times \mathrm{SU}(2)_{D,-}$ (here $\mathrm{SU}(2)_{D,+}$ is a diagonal subgroup of $\mathrm{SU}(2)_{L,+} \times \mathrm{SU}(2)_{R,+}$, and likewise for $\mathrm{SU}(2)_{D,-}$). We can define an $\mathcal{N} = 4$ superconformal algebra in which the R-symmetry is $\mathrm{SU}(2)_{D,+}$ (and another one with R-symmetry $\mathrm{SU}(2)_{D,-}$). As is usual for open superstrings, the currents of this $\mathcal{N} = 4$ algebra are mixtures of left and right currents from the underlying $\mathcal{N} = (4, 4)$ symmetry in bulk.

Now let us include a boundary interaction as in (2.51):

$$L_A = -i \int d\tau \left(A_i(x)\partial_\tau x^i - i F_{ij}\Psi^i\Psi^j \right). \tag{2.57}$$

The condition that the boundary interaction preserves some spacetime supersymmetry is that the theory with this interaction is still an $\mathcal{N} = 4$ theory. This condition is easy to implement, at the classical level. The Ψ^i transform as $(1/2, 1/2)$ under $\mathrm{SU}(2)_{D,+} \times \mathrm{SU}(2)_{D,-}$. The $F_{ij}\Psi^i\Psi^j$ coupling in L_A transforms as the antisymmetric tensor product of this representation with itself, or $(1, 0) \oplus (0, 1)$, where the two pieces multiply, respectively, F^+ and F^-, the self-dual and anti-self-dual parts of F. Hence, the condition that L_A be invariant under $\mathrm{SU}(2)_{D,+}$ is that $F^+ = 0$, in other words that the gauge field should be an instanton. For invariance under $\mathrm{SU}(2)_{D,-}$ we need $F^- = 0$, an anti-instanton. Thus, at the classical level, an instanton or anti-instanton gives an $\mathcal{N} = 4$ superconformal theory,[3] and hence a supersymmetric or BPS configuration.

[3]Our notation is not well adapted to nonabelian gauge theory. In this case, the factor e^{-L_A} in the path integral must be reinterpreted as a trace $\mathrm{Tr}\, P \exp \oint_{\partial\Sigma} \left(iA_i\partial_\tau x^i + F_{ij}\Psi^i\Psi^j \right)$ where the exponent is Lie algebra valued. This preserves $\mathrm{SU}(2)_{D,\pm}$ if $F^\pm = 0$.

To show that this conclusion is valid quantum mechanically, we need a regularization that preserves (global) $\mathcal{N}=1$ supersymmetry and also the $\mathrm{SO}(4)_D$ symmetry. This can readily be provided by Pauli-Villars regularization. First of all, the fields $x^i, \psi^i, \bar{\psi}^i$, together with auxiliary fields F^i, can be interpreted in the standard way as components of $\mathcal{N}=1$ superfields Φ^i, $i=1,\ldots,4$.

To carry out Pauli-Villars regularization, we introduce two sets of superfields C^i and E^i, where E^i are real-valued and C^i takes values in the same space ($\mathbb{R}^4$ or more generally $\mathbf{T}^n \times \mathbb{R}^{4-n}$) that Φ^i does, and we write $\Phi^i = C^i - E^i$. For C^i and E^i, we consider the following lagrangean:

$$L = \int d^2x d^2\theta \left(\epsilon^{\alpha\beta} D_\alpha C^i D_\beta C^i\right) - \int d^2x d^2\theta \left(\epsilon^{\alpha\beta} D_\alpha E^i D_\beta E^i + M^2 (E^i)^2\right). \quad (2.58)$$

This regularization of the bulk theory is manifestly invariant under global $\mathcal{N}=1$ supersymmetry. But since it preserves an $\mathrm{SO}(4)_D$ (which under which all left and right fermions in C or E transform as $(1/2,1/2)$), it actually preserves a global $\mathcal{N}=4$ supersymmetry.

This symmetry can be preserved in the presence of boundaries. We simply consider free boundary conditions for both C^i and E^i. The usual short distance singularity is absent in the Φ^i propagator (as it cancels between C^i and E^i). Now, include a boundary coupling to gauge fields by the obvious superspace version of (2.51):

$$L_A = -i \int d\tau d\theta \, A_i(\Phi) D\Phi^i = -i \int d\tau \left(A_i(x)\partial_\tau x^i - iF_{ij}\Psi^i\Psi^j\right). \quad (2.59)$$

Classically (as is clear from the second form, which arises upon doing the θ integral), this coupling preserves $\mathrm{SU}(2)_{D,+}$ if $F^+ = 0$, or $\mathrm{SU}(2)_{D,-}$ if $F^- = 0$. Because of the absence of a short distance singularity in the Φ propagator, all Feynman diagrams are regularized.[4] Hence, for every classical instanton, we get a two-dimensional quantum field theory with global $\mathcal{N}=4$ supersymmetry.

If this theory flows in the infrared to a conformal field theory, this theory is $\mathcal{N}=4$ superconformal and hence describes a configuration with spacetime supersymmetry. On the other hand, the global $\mathcal{N}=4$ supersymmetry, which holds precisely if $F^+=0$, means that any renormalization group flow that occurs as $M \to \infty$ would be a flow on classical instanton moduli space. Such a flow would mean that stringy corrections generate a potential on instanton moduli space. But there is too much supersymmetry for this, and therefore there is no flow on the space; i.e. different classical instantons lead to distinct conformal field theories. We conclude that, with this regularization, every classical instanton corresponds in a natural way to a supersymmetric configuration in string theory or in other words to a stringy instanton.

[4]In most applications, Pauli-Villars regularization fails to regularize the one-loop diagrams, because it makes the vertices worse while making the propagators better. The present problem has the unusual feature that Pauli-Villars regularization eliminates the short distance problems even from the one-loop diagrams.

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Thus, with this regularization, the stringy instanton equation is just $F^+ = 0$. Since the moduli space of conformal field theories is independent of the regularization, it also follows that with any regularization, the stringy instanton moduli space coincides with the classical one.

Turning on B. Now, let us reexamine this issue in the presence of a constant B field. The boundary condition required by supersymmetry was given in (2.49):

$$(g_{ij} + 2\pi\alpha' B_{ij})\psi^j = (g_{ij} - 2\pi\alpha B_{ij})\bar{\psi}^j \,. \tag{2.60}$$

To preserve (2.60), if one rotates ψ^i by an SO(4) matrix h, one must rotate $\bar{\psi}^i$ with a *different* SO(4) matrix $\bar{h}$. The details of the relation between h and $\bar{h}$ will be explored below, in the context of point-splitting regularization. At any rate, (2.60) does preserve a diagonal subgroup $\mathrm{SO}(4)_{D,B}$ of $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$, but as the notation suggests, which diagonal subgroup it is depends on B.

The Pauli-Villars regularization introduced above preserves $\mathrm{SO}(4)_D$, which for $B \neq 0$ does not coincide with $\mathrm{SO}(4)_{D,B}$. The problem arises because the left and right chiral fermions in the regulator superfields E^i are coupled by the mass term in a way that breaks $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$ down to $\mathrm{SO}(4)_D$, but they are coupled by the boundary condition in a way that breaks $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$ down to $\mathrm{SO}(4)_{D,B}$. Thus, the argument that showed that classical instanton moduli space is exact for $B = 0$ fails for $B \neq 0$.

This discussion raises the question of whether a different regularization would enable us to prove the exactness of classical instantons for $B \neq 0$. However, a very simple argument mentioned in the introduction shows that one must expect stringy corrections to instanton moduli space when $B \neq 0$. In fact, if $B^+ \neq 0$, a configuration containing a threebrane and a separated -1-brane is not BPS (we will explore it in section 5), so the small instanton singularity that is familiar from classical Yang-Mills theory should be absent when $B^+ \neq 0$.

It has been proposed [35, 38] that the stringy instantons at $B^+ \neq 0$ are the instantons of noncommutative Yang-Mills theory, that is the solutions of $\widehat{F}^+ = 0$ with a suitable $*$ product. We can now make this precise in the $\alpha' \to 0$ limit. In this limit, the effective action is, as we have seen, $\widehat{F}^2$, with the indices in $\widehat{F}$ contracted by the open string metric G. In this theory, the condition for a gauge field to leave unbroken half of the linearly realized supersymmetry on the branes is $\widehat{F}^+ = 0$, where the projection of $\widehat{F}$ to selfdual and antiselfdual parts is made with respect to the open string metric G, rather than the closed string metric g. Hence, at least in the $\alpha' = 0$ limit, BPS configurations are described by noncommutative instantons, as has been suggested in [35, 38]. If we are on $\mathbb{R}^4$, then, as shown in [35], deforming the classical instanton equation $\widehat{F}^+ = 0$ to the noncommutative instanton equation $\widehat{F}^+ = 0$ has the effect of adding a Fayet-Iliopoulos (FI) constant term to the ADHM equations, removing the small instanton singularity.[5] The ADHM equations with the FI term have a natural interpretation in terms of the DLCQ description of the six-dimensional $(2,0)$ theory [37], and have been studied mathematically in [41].

What happens if $B \neq 0$ but we do not take the $\alpha' \to 0$ limit? In this case, the stringy instanton moduli space must be a hyper-Kähler deformation of the classical instanton moduli space, with the small instanton singularities eliminated if $B^+ \neq 0$, and reducing to the classical instanton moduli space for instantons of large scale size if we are on $\mathbb{R}^4$. We expect that the most general hyper-Kähler manifold meeting these conditions is the moduli space of noncommutative instantons, with some θ parameter and with some effective metric on spacetime G.[6]

Details for instanton number one. Though we do not know how to prove this in general, one can readily prove it by hand for the case of instantons of instanton number one on $\mathbb{R}^4$. The ADHM construction for such instantons, with gauge group $\mathrm{U}(N)$, expresses the moduli space as the moduli space of vacua of a $\mathrm{U}(1)$ gauge theory with N hypermultiplets H^a of unit charge (times a copy of $\mathbb{R}^4$ for the instanton position). In the $\alpha' \to 0$ limit with non-zero B, there is a FI term. If we write the hypermultiplets H^a, in a notation that makes manifest only half the supersymmetry, as a pair of chiral superfields A^a, B_a, with respective charges $1, -1$, then the ADHM equations read

$$\sum_a A^a B_a = \zeta_c \, . \qquad \sum_a |A^a|^2 - \sum_a |B_a|^2 = \zeta \, . \tag{2.61}$$

One must divide by $A^a \to e^{i\alpha} A^a$, $B_a \to e^{-i\alpha} B_a$. Here ζ_c is a complex constant, and ζ a real constant. ζ_c and ζ are the FI parameters. The real and imaginary part of ζ_c, together with ζ, transform as a triplet of an $\mathrm{SU}(2)$ R-symmetry group, which is broken to $\mathrm{U}(1)$ (rotations of ζ_c) by our choice of writing the equations in terms of chiral superfields. To determine the topology of the moduli space $\mathcal{M}$, we make an $\mathrm{SU}(2)_R$ transformation (or a judicious choice of A^a and B_a) to set $\zeta_c = 0$ and $\zeta > 0$. Then, if we set $B_a = 0$, the A^a, modulo the action of $\mathrm{U}(1)$, determine a point in $\mathbb{CP}^N$; the equation $\sum_a A^a B_a = 0$ means that the B_a determine a cotangent vector of $\mathbb{CP}^N$, so $\mathcal{M}$ is the cotangent bundle $T^*\mathbb{CP}^N$.

The second homology group of $\mathcal{M}$ is of rank one, being generated by a two-cycle in $\mathbb{CP}^N$. Moduli space of hyper-Kähler metrics is parametrized by the periods of the three covariantly constant two-forms I, J, K. As there is only one period, there are precisely three real moduli, namely ζ, $\mathrm{Re}\,\zeta_c$, and $\mathrm{Im}\,\zeta_c$.

[5] Actually, it was assumed in [35] that θ is self-dual. The general situation, as we will show at the end of section 5, is that the small instanton singularity is removed precisely if $B^+ \neq 0$, or equivalently $\theta^+ \neq 0$.

[6] The effective metric on spacetime must be hyper-Kähler for supersymmetry, so it is a flat metric if we are on $\mathbb{R}^4$ or $\mathbf{T}^n \times \mathbb{R}^{4-n}$, or a hyper-Kähler metric if we are bold enough to extrapolate the discussion to a K3 manifold or a Taub-NUT or ALE space.

Hence, at least for instanton number one, the stringy instanton moduli space on $\mathbb{R}^4$, for any B, must be given by the solutions of $\widehat{F}^+ = 0$, with some effective metric on spacetime and some effective theta parameter. It is tempting to believe that these may be the metric and theta parameter found in (2.5) from the open string propagator.

Noncommutative instantons and $\mathcal{N} = 4$ supersymmetry. We now return to the question of what symmetries are preserved by the boundary condition (2.60). We work in the $\alpha' \to 0$ limit, so that we know the boundary couplings and the gauge invariances precisely. The goal is to show, by analogy with what happened for $B = 0$, that noncommutative gauge fields that are self-dual with respect to the open string metric lead to $\mathcal{N} = 4$ worldsheet superconformal symmetry.

It is convenient to introduce a vierbein e_a^i for the closed string metric Thus $g^{-1} = ee^t$ (e^t is the transpose of e) or $g^{ij} = \sum_a e_a^i e_a^j$. Then, we express the fermions in terms of the local Lorentz frame in spacetime

$$\psi^i = e_a^i \chi^a \,, \qquad \bar{\psi}^i = e_a^i \bar{\chi}^a \,. \tag{2.62}$$

The $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$ automorphism group of the supersymmetry algebra rotates these four fermions by $\chi \to h\chi$ and $\bar{\chi} \to \bar{h}\bar{\chi}$. The boundary conditions (2.60) breaks $\mathrm{SO}(4)_L \times \mathrm{SO}(4)_R$ to a diagonal subgroup $\mathrm{SO}(4)_{D,B}$ defined by

$$\bar{h} = e^{-1} \frac{1}{g - 2\pi\alpha' B} (g + 2\pi\alpha' B) e h e^{-1} \frac{1}{g + 2\pi\alpha' B} (g - 2\pi\alpha' B) e \,. \tag{2.63}$$

In terms of χ and $\bar{\chi}$, the boundary coupling of the original fermions (2.55) becomes

$$\chi^t e^t g \frac{1}{g + 2\pi\alpha' B} \widehat{F} \frac{1}{g - 2\pi\alpha' B} g e \chi \,. \tag{2.64}$$

We have used (2.52) to express Ψ in terms of ψ, and (2.62) to express ψ in terms of χ. Under $\mathrm{SO}(4)_{D,B}$, this coupling transforms as

$$\chi^t e^t g \frac{1}{g + 2\pi\alpha' B} \widehat{F} \frac{1}{g - 2\pi\alpha' B} g e \chi \to \chi^t h^t e^t g \frac{1}{g + 2\pi\alpha' B} \widehat{F} \frac{1}{g - 2\pi\alpha' B} g e h \chi \,, \tag{2.65}$$

and the theory is invariant under the subgroup of $\mathrm{SO}(4)_{D,B}$ for which

$$e^t g \frac{1}{g + 2\pi\alpha' B} \widehat{F} \frac{1}{g - 2\pi\alpha' B} g e = h^t e^t g \frac{1}{g + 2\pi\alpha' B} \widehat{F} \frac{1}{g - 2\pi\alpha' B} g e h \,. \tag{2.66}$$

In order to analyze the consequences of this equation, we define a vierbein for the open string metric by the following very convenient formula:

$$E = \frac{1}{g - 2\pi\alpha' B} g e \,. \tag{2.67}$$

To verify that this is a vierbein, we compute

$$EE^t = \frac{1}{g-2\pi\alpha' B} g \frac{1}{g+2\pi\alpha' B} = \frac{1}{g+2\pi\alpha' B} g \frac{1}{g-2\pi\alpha' B} = G^{-1}\,. \tag{2.68}$$

In terms of E, (2.66) reads

$$E^t \widehat{F} E = h^t E^t \widehat{F} E h\,. \tag{2.69}$$

For this equation to hold for h in an SU(2) subgroup of $\mathrm{SO}(4)_{D,B}$, $E^t \widehat{F} E^t$ must be selfdual, or anti-selfdual, with respect to the trivial metric of the local Lorentz frame. This is equivalent to $\widehat{F}$ being selfdual or anti-selfdual with respect to the open string metric G. Thus, we have shown that the boundary interaction preserves an SU(2) R symmetry, and hence an $\mathcal{N}=4$ superconformal symmetry, if $\widehat{F}^+=0$ or $\widehat{F}^-=0$ with respect to the open string metric.

3. Noncommutative gauge symmetry vs ordinary gauge symmetry

We have by now seen that ordinary and noncommutative Yang-Mills fields arise from the same two-dimensional field theory regularized in different ways. Consequently, there must be a transformation from ordinary to noncommutative Yang-Mills fields that maps the standard Yang-Mills gauge invariance to the gauge invariance of noncommutative Yang-Mills theory. Moreover, this transformation must be local in the sense that to any finite order in perturbation theory (in θ) the noncommutative gauge fields and gauge parameters are given by local differential expressions in the ordinary fields and parameters.

At first sight, it seems we want a local field redefinition $\widehat{A} = \widehat{A}(A, \partial A, \partial^2 A, \ldots; \theta)$ of the gauge fields, and a simultaneous reparametrization $\widehat{\lambda} = \widehat{\lambda}(\lambda, \partial\lambda, \partial^2\lambda, \ldots; \theta)$ of the gauge parameters that maps one gauge invariance to the other. However, this must be relaxed. If there were such a map intertwining with the gauge invariances, it would follow that the gauge group of ordinary Yang-Mills theory is isomorphic to the gauge group of noncommutative Yang-Mills theory. This is not the case. For example, for rank one, the ordinary gauge group, which acts by

$$\delta A_i = \partial_i \lambda\,, \tag{3.1}$$

is abelian, while the noncommutative gauge invariance, which acts by

$$\delta A_i = \partial_i \lambda + i\lambda * A_i - iA_i * \lambda, \tag{3.2}$$

is nonabelian. An abelian group cannot be isomorphic to a nonabelian group, so no redefinition of the gauge parameter can map the ordinary gauge parameter to the noncommutative one while intertwining with the gauge symmetries.

What we actually need is less than an identification between the two gauge groups. To do physics with gauge fields, we only need to know when two gauge fields A and A' should be considered gauge-equivalent. We do not need to select a particular set of generators of the gauge equivalence relation — a gauge group that generates the equivalence relation.[7] In the problem at hand, it turns out that we can map A to $\widehat{A}$ in a way that preserves the gauge equivalence relation, even though the two gauge groups are different.

What this means in practice is as follows. We will find a mapping from ordinary gauge fields A to noncommutative gauge fields $\widehat{A}$ which is local to any finite order in θ and has the following further property. Suppose that two ordinary gauge fields A and A' are equivalent by an ordinary gauge transformation by $U = \exp(i\lambda)$. Then, the corresponding noncommutative gauge fields $\widehat{A}$ and $\widehat{A}'$ will also be gauge-equivalent, by a noncommutative gauge transformation by $\widehat{U} = \exp(i\widehat{\lambda})$. However, $\widehat{\lambda}$ will depend on both λ and A. If $\widehat{\lambda}$ were a function of λ only, the ordinary and noncommutative gauge groups would be the same; since $\widehat{\lambda}$ is a function of A as well as λ, we do not get any well-defined mapping between the gauge groups, and we get an identification only of the gauge equivalence relations.

Note that the situation that we are considering here is the opposite of a gauge theory in which the gauge group has field-dependent structure constants or only closes on shell. This means (see [51] for a fuller explanation) that one has a well-defined gauge equivalence relation, but the equivalence classes are not the orbits of any useful group, or are such orbits only on shell. In the situation that we are considering, there is more than one group that generates the gauge equivalence relation; one can use either the ordinary gauge group or (with one's favorite choice of θ) the gauge group of noncommutative Yang-Mills theory.

Finally, we point out in advance a limitation of the discussion. The arguments in section 2 (which involved, for example, comparing two different ways of constructing an α' expansion of the string theory effective action) show only that ordinary and noncommutative Yang-Mills theory must be equivalent to all finite orders in a long wavelength expansion. By dimensional analysis, this means that they must be equivalent to all finite orders in θ. However, it is not clear that the transformation between A and $\widehat{A}$ should always work nonperturbatively. Indeed, the small instanton problem discussed in section 2.3 seems to give a situation in which the transformation between $\widehat{A}$ and A breaks down, presumably because the perturbative series that we will construct does not converge.

3.1 The change of variables

Once one is convinced that a transformation of the type described above exists, it is

[7] Fadde'ev-Popov quantization of gauge theories is formulated in terms of the gauge group, but in the more general Batalin-Vilkovisky approach to quantization, the emphasis is on the equivalence relation generated by the gauge transformations. For a review of this approach, see [51].

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not too hard to find it. We take the gauge fields to be of arbitrary rank N, so that all fields and gauge parameters are $N \times N$ matrices (with entries in the ordinary ring of functions or the noncommutative algebra defined by the $*$ product of functions, as the case may be). We look for a mapping $\widehat{A}(A)$ and $\widehat{\lambda}(\lambda, A)$ such that

$$\widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}}\widehat{A}(A) = \widehat{A}(A + \delta_\lambda A)\,, \tag{3.3}$$

with infinitesimal λ and $\widehat{\lambda}$. This will ensure that an ordinary gauge transformation of A by λ is equivalent to a noncommutative gauge transformation of $\widehat{A}$ by $\widehat{\lambda}$, so that ordinary gauge fields that are gauge-equivalent are mapped to noncommutative gauge fields that are likewise gauge-equivalent. The gauge transformation laws δ_λ and $\widehat{\delta}_{\widehat{\lambda}}$ were defined at the end of the introduction. We first work to first order in θ. We write $\widehat{A} = A + A'(A)$ and $\widehat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$, with A' and λ' local function of λ and A of order θ. Expanding (3.3) in powers of θ, we find that we need

$$A'_i(A+\delta_\lambda A) - A'_i(A) - \partial_i\lambda' - i[\lambda', A_i] - i[\lambda, A'_i] = -\frac{1}{2}\theta^{kl}(\partial_k\lambda\partial_l A_i + \partial_l A_i \partial_k\lambda) + \mathcal{O}(\theta^2)\,. \tag{3.4}$$

In arriving at this formula, we have used the expansion $f*g = fg + \frac{1}{2}i\theta^{ij}\partial_i f\partial_j g + \mathcal{O}(\theta^2)$, and have written the $\mathcal{O}(\theta)$ part of the $*$ product explicitly on the right hand side. All products in (3.4) are therefore ordinary matrix products, for example $[\lambda', A_i] = \lambda' A_i - A_i\lambda'$, where (as λ' is of order θ), the multiplication on the right hand side should be interpreted as ordinary matrix multiplication at $\theta = 0$.

Equation (3.4) is solved by

$$\begin{aligned}
\widehat{A}_i(A) &= A_i + A'_i(A) = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2) \\
\widehat{\lambda}(\lambda, A) &= \lambda + \lambda'(\lambda, A) = \lambda + \frac{1}{4}\theta^{ij}\{\partial_i\lambda, A_j\} + \mathcal{O}(\theta^2)
\end{aligned} \tag{3.5}$$

where again the products on the right hand side, such as $\{A_k, \partial_l A_i\} = A_k \cdot \partial_l A_i + \partial_l A_i \cdot A_k$ are ordinary matrix products. From the formula for $\widehat{A}$, it follows that

$$\widehat{F}_{ij} = F_{ij} + \frac{1}{4}\theta^{kl}\left(2\{F_{ik}, F_{jl}\} - \{A_k, D_l F_{ij} + \partial_l F_{ij}\}\right) + \mathcal{O}(\theta^2)\,. \tag{3.6}$$

These formulas exhibit the desired change of variables to first nontrivial order in θ.

By reinterpreting the above formulas, it is a rather short step to write down a differential equation that generates the desired change of variables to all finite orders in θ. Consider the problem of mapping noncommutative gauge fields $\widehat{A}(\theta)$ defined with respect to the $*$ product with one choice of θ, to noncommutative gauge fields $\widehat{A}(\theta + \delta\theta)$, defined for a nearby choice of θ. To first order in $\delta\theta$, the problem of converting from $\widehat{A}(\theta)$ to $\widehat{A}(\theta + \delta\theta)$ is equivalent to what we have just solved. Indeed, apart from associativity, the only property of the $*$ product that one needs to verify that (3.5) obeys (3.3) to first order in θ is that for any variation $\delta\theta^{ij}$ of θ,

$$\delta\theta^{ij}\frac{\partial}{\partial\theta^{ij}}(f*g) = \delta\theta^{ij}\frac{\partial f}{\partial x^i} * \frac{\partial g}{\partial x^j} \tag{3.7}$$

at $\theta = 0$. But this is true for any value of θ, as one can verify with a short perusal of the explicit formula for the $*$ product in (1.2). Hence, adapting the above formulas, we can write down a differential equation that describes how $\widehat{A}(\theta)$ and $\widehat{\lambda}(\theta)$ should change when θ is varied, to describe equivalent physics:

$$\begin{aligned}
\delta\widehat{A}_i(\theta) &= \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\widehat{A}_i(\theta) = -\frac{1}{4}\delta\theta^{kl}\left[\widehat{A}_k * (\partial_l\widehat{A}_i + \widehat{F}_{li}) + (\partial_l\widehat{A}_i + \widehat{F}_{li}) * \widehat{A}_k\right] \\
\delta\widehat{\lambda}(\theta) &= \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\widehat{\lambda}(\theta) = \frac{1}{4}\delta\theta^{kl}\left(\partial_k\lambda * A_l + A_l * \partial_k\lambda\right) \\
\delta\widehat{F}_{ij}(\theta) &= \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\widehat{F}_{ij}(\theta) = \frac{1}{4}\delta\theta^{kl}\Big[2\widehat{F}_{ik} * \widehat{F}_{jl} + 2\widehat{F}_{jl} * \widehat{F}_{ik} - \widehat{A}_k * \left(\widehat{D}_l\widehat{F}_{ij} + \partial_l\widehat{F}_{ij}\right) - \\
&\qquad - \left(\widehat{D}_l\widehat{F}_{ij} + \partial_l\widehat{F}_{ij}\right) * \widehat{A}_k\Big].
\end{aligned} \tag{3.8}$$

On the right hand side, the $*$ product is meant in the generalized sense explained in the introduction: the tensor product of matrix multiplication with the $*$ product of functions. This differential equation generates the promised change of variables to all finite orders in θ. To what extent the series in θ generates by this equation converges is a more delicate question, beyond the scope of the present paper. The equation is invariant under a scaling operation in which θ has degree -2 and A and $\partial/\partial x$ have degree one, so one can view the expansion it generates as an expansion in powers of θ for any A, which is how we have derived it, or as an expansion in powers of A and $\partial/\partial x$ for any θ.

The differential equation (3.8) can be solved explicitly for the important case of a rank one gauge field with constant $\widehat{F}$. In this case, the equation can be written

$$\delta\widehat{F} = -\widehat{F}\delta\theta\widehat{F} \tag{3.9}$$

(the Lorentz indices are contracted as in matrix multiplication). Its solution with the boundary condition $\widehat{F}(\theta = 0) = F$ is

$$\widehat{F} = \frac{1}{1 + F\theta}F\,. \tag{3.10}$$

From (3.10) we find F in terms of $\widehat{F}$

$$F = \widehat{F}\frac{1}{1 - \theta\widehat{F}}\,. \tag{3.11}$$

We can also write these relations as

$$\widehat{F} - \frac{1}{\theta} = -\frac{1}{\theta(\frac{1}{\theta} + F)\theta}\,. \tag{3.12}$$

We see that when $F = -\theta^{-1}$ we cannot use the noncommutative description because $\widehat{F}$ has a pole. Conversely, F is singular when $\widehat{F} = \theta^{-1}$, so in that case, the commutative description does not exist. Using our identification in the zero slope limit (or in a natural regularization scheme which will be discussed below) $\theta = \frac{1}{B}$, equations (3.10), (3.11) and (3.12) become

$$\begin{aligned} \widehat{F} &= B\frac{1}{B+F}F\,, \\ F &= \widehat{F}\frac{1}{B-\widehat{F}}B \end{aligned} \tag{3.13}$$

and

$$\widehat{F} - B = -B\frac{1}{B+F}B\,. \tag{3.14}$$

So an ordinary abelian gauge field with constant curvature F and Neveu-Schwarz two-form field B is equivalent to a noncommutative gauge field with $\theta = 1/B$ and the value of $\widehat{F}$ as in (3.13). When $B + F = 0$ we cannot use the noncommutative description. It is natural that this criterion depends only on $B + F$, since in the description by ordinary abelian gauge theory, B and F are mixed by a gauge symmetry, with only the combination $B + F$ being gauge-invariant.

Application to instantons. Another interesting application is to instantons in four dimensions. We have argued in section 2.3 (following [35, 38]) that a stringy instanton is a solution of the noncommutative instanton equation

$$\widehat{F}^+_{ij} = 0\,. \tag{3.15}$$

We can evaluate this equation to first nontrivial order in θ using (3.6). Since $\theta^{kl}\{A_k, D_l F_{ij} + \partial_l F_{ij}\}^+ = 0$ if $F^+_{ij} = 0$, to evaluate the $\mathcal{O}(\theta)$ deviation of (3.15) from the classical instanton equation $F^+_{ij} = 0$, we can drop those non-gauge-invariant terms in (3.6). We find that to first order in θ, the noncommutative instanton equation can be written in any of the following equivalent forms:

$$\begin{aligned} 0 &= F^+_{ij} + \frac{1}{2}\left(\theta^{kl}\{F_{ik}, F_{jl}\}\right)^+ + \mathcal{O}(\theta^2) \\ &= F^+_{ij} - \frac{1}{8}\frac{1}{\sqrt{\det G}}\epsilon^{rstu}F_{rs}F_{tu}G_{ik}G_{jl}(\theta^+)^{kl} + \mathcal{O}(\theta^2) \\ &= F^+_{ij} - \frac{1}{4}(F\widetilde{F})\theta^+_{ij} + \mathcal{O}(\theta^2). \end{aligned} \tag{3.16}$$

Here G is the open string metric, which is used to determine the self-dual parts of F and θ. In (3.16), we used the facts that $F^- = \mathcal{O}(1)$ and $F^+ = \mathcal{O}(\theta)$, along with various identities of SO(4) group theory. For example, in evaluating $(\theta^{kl}\{F_{ik}, F_{jl}\})^+$ to order θ, one can replace F by F^-. According to SO(4) group theory, a product of any number of anti-selfdual tensors can never make a selfdual tensor, so we can likewise replace θ by θ^+. SO(4) group theory also implies that there is only one self-dual tensor linear in θ^+ and quadratic in F^-, namely $\theta^+(F^-)^2$, so the $\mathcal{O}(\theta)$ term in the equation is a multiple of this. To first order in θ, we can replace $(F^-)^2$ by $(F^-)^2 - (F^+)^2$, which is a multiple of $F\widetilde{F} = \frac{1}{2\sqrt{\det G}}\epsilon^{rstu}F_{rs}F_{tu}$; this accounts for the other ways of writing the equation given in (3.16).

In (3.16), we see that to first order, the corrections to the instanton equation depend only on θ^+ and not θ^-; in section 5, we explore the extent to which this is true to all orders.

More freedom in the description. What we have learned is considerably more than was needed to account for the results of section 2. In section 2, we found that, using a point-splitting regularization, string theory with given closed string parameters g and B can be described, in the open string sector, by a noncommutative Yang-Mills theory with θ given in eqn. (2.5). There must, therefore, exist a transformation from commutative Yang-Mills to noncommutative Yang-Mills with that value of θ.

In our present discussion, however, we have obtained a mapping from ordinary Yang-Mills to non-commutative Yang-Mills that is completely independent of g and B and hence allows us to express the open string sector in terms of a noncommutative Yang-Mills theory with an arbitrary value of θ. It is plausible that this type of description would arise if one uses a suitable regularization that somehow interpolates between Pauli-Villars and point-splitting.

What would the resulting description look like? In the description by ordinary Yang-Mills fields, the effective action is a function of $F+B$, and is written using ordinary multiplication of functions. In the description obtained with point-splitting regularization, the effective action is a function of $\widehat{F}$, but the multiplication is the $*$ product with θ in (2.5). If one wishes a description with an arbitrary θ, the variable in the action will have to somehow interpolate from $F+B$ in the description by ordinary Yang-Mills fields to $\widehat{F}$ in the description with the canonical value of θ in (2.5). The most optimistic hypothesis is that there is some two-form Φ, which depends on B, g, and θ, such that the θ dependence of the effective action is completely captured by replacing $\widehat{F}$ by

$$\widehat{F} + \Phi\,, \tag{3.17}$$

using the appropriate θ-dependent $*$ product, and using an appropriate effective metric G and string coupling G_s.

We propose that this is so, with G, G_s, and Φ determined in terms of g, B, and θ by the following formulas, whose main justification will be given in section 4.1:

$$\begin{aligned}\frac{1}{G+2\pi\alpha'\Phi} &= -\frac{\theta}{2\pi\alpha'} + \frac{1}{g+2\pi\alpha' B} \\ G_s &= g_s\left(\frac{\det(G+2\pi\alpha'\Phi)}{\det(g+2\pi\alpha' B)}\right)^{1/2} = g_s\frac{1}{\det\left[(\frac{1}{g+2\pi\alpha' B}-\frac{\theta}{2\pi\alpha'})(g+2\pi\alpha' B)\right]^{1/2}}.\end{aligned} \tag{3.18}$$

In the first equation, G and Φ are determined because they are symmetric and antisymmetric respectively. The second equation is motivated, as in (2.44), by demanding that for $F = \widehat{F} = 0$ the constant terms in the lagrangeans using the two set of variables are the same.

We will show in section 4 that for slowly varying fields — governed by the Dirac-Born-Infeld action — such a general description, depending on an arbitrary θ, does exist. The first equation in (3.19) has been determined because it is the unique formula compatible with the analysis in section 4.1. We will also see in section 3.2 that a special case of the transformation in (3.19) has a natural microscopic explanation in noncommutative Yang-Mills theory. We do not have a general proof of the existence of a description with the properties proposed in (3.17) and (3.19). Such a proof might be obtained by finding a regularization that suitably generalizes point-splitting and Pauli-Villars and leads to these formulas.

A few special cases of (3.19) are particularly interesting:

1. $\theta = 0$. Here we recover the commutative description, where $G = g$, $G_s = g_s$ and $\Phi = B$.

2. $\Phi = 0$. This is the description we studied in section 2.

3. In the zero slope limit with fixed G, B and Φ, we take $g = \epsilon g^{(0)} + \mathcal{O}(\epsilon^2)$, $B = B^{(0)} + \epsilon B^{(1)} + \mathcal{O}(\epsilon^2)$ and $\alpha' = \mathcal{O}(\epsilon^{1/2})$ (we assume for simplicity that the rank of B is maximal, i.e. $r = p+1$). Expanding the first expression in (3.19) in powers of ϵ we find

$$\begin{aligned}\frac{1}{G} - 2\pi\alpha'\frac{1}{G}\Phi\frac{1}{G} + \mathcal{O}(\epsilon) = -\frac{\theta}{2\pi\alpha'} &+ \frac{1}{2\pi\alpha' B^{(0)}} - \frac{\epsilon}{(2\pi\alpha')^2}\frac{1}{B^{(0)}}g^{(0)}\frac{1}{B^{(0)}} + \\ &+ \frac{\epsilon^2}{(2\pi\alpha')^3}\frac{1}{B^{(0)}}g^{(0)}\frac{1}{B^{(0)}}g^{(0)}\frac{1}{B^{(0)}} - \\ &- \frac{\epsilon}{2\pi\alpha'}\frac{1}{B^{(0)}}B^{(1)}\frac{1}{B^{(0)}} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.19}$$

Equating the different orders in ϵ we have

$$\begin{aligned}\theta &= \frac{1}{B^{(0)}} + \mathcal{O}(\epsilon) \\ G &= -\frac{(2\pi\alpha')^2}{\epsilon}B^{(0)}\frac{1}{g^{(0)}}B^{(0)} + \mathcal{O}(\epsilon) \\ \Phi &= -B^{(0)} + \frac{(2\pi\alpha')^2}{\epsilon}B^{(0)}\frac{1}{g^{(0)}}B^{(1)}\frac{1}{g^{(0)}}B^{(0)} + \mathcal{O}(\epsilon) \\ G_s &= g_s \det\left(\frac{2\pi\alpha'}{\epsilon}B^{(0)}\frac{1}{g^{(0)}}\right)^{1/2}(1 + \mathcal{O}(\epsilon))\,, \end{aligned} \tag{3.20}$$

which agree with our zero slope values (2.15), (2.45) (except the new value of Φ). The freedom in our description is the freedom in the way we take the zero slope limit; i.e. in the value of $B^{(1)}$. It affects only the value of Φ. For example, for $B^{(1)} = 0$ we have $\Phi = -B^{(0)}$, and for $B^{(1)} = \frac{\epsilon}{(2\pi\alpha')^2} g^{(0)} \frac{1}{B^{(0)}} g^{(0)}$ we have $\Phi = 0$, as in the discussion in section 2. The fact that there is freedom in the value of Φ in the zero slope limit has a simple explanation. In this limit the effective lagrangean is proportional to $\mathrm{Tr}(\widehat{F} + \Phi)^2 = \mathrm{Tr}(\widehat{F}^2 + 2\Phi\widehat{F} + \Phi^2)$. The Φ dependence affects only a total derivative term and a constant shift of the lagrangean. Such terms are neglected when the effective lagrangean is derived, as in perturbative string theory, from the equations of motion or the S-matrix elements.

4. We can extend the leading order expressions in the zero slope limit (3.20) (again in the case of maximal rank $r = p + 1$) with $B^{(1)} = 0$ to arbitrary value of ϵ, away from the zero slope limit, and find

$$\begin{aligned} \theta &= \frac{1}{B} \\ G &= -(2\pi\alpha')^2 B \frac{1}{g} B \\ \Phi &= -B, \end{aligned} \tag{3.21}$$

which satisfy (3.19). With this choice of θ the string coupling (3.19) keeps its zero slope limit value (2.45)

$$G_s = g_s \det(2\pi\alpha' B g^{-1})^{1/2} \,. \tag{3.22}$$

In the next subsection, we will see that the existence of a description with these values of the parameters is closely related to background independence of noncommutative Yang-Mills theory. These are also the values for which the pole in $\widehat{F}(F)$, given in (3.10), occurs at $F + B = 0$ as in (3.13).

3.2 Background independence of noncommutative Yang-Mills on $\mathbb{R}^n$

In the language of ordinary Yang-Mills theory, the gauge-invariant combination of B and F is $M = 2\pi\alpha'(B + F)$. (The $2\pi\alpha'$ is for later convenience.) The same gauge-invariant field M can be split in different ways as $2\pi\alpha'(B + F)$ or $2\pi\alpha'(B' + F')$ where B and B' are constant two-forms. Given such a splitting, we incorporate the background B or B' as a boundary condition in an exactly soluble conformal field theory, as described in section 2. Then we treat the rest of M by a boundary interaction. As we have seen in section 2 and above, the boundary interaction can be regularized either by Pauli-Villars, leading to ordinary Yang-Mills theory, or by point splitting, leading to noncommutative Yang-Mills.

In the present discussion, we will focus on noncommutative Yang-Mills, and look at the background dependence. Thus, by taking the background to be B or B', we should get a noncommutative description with appropriate θ or θ', and different

$\widehat{F}$'s. Note the contrast with the discussion in sections 2 and 3.1: here we are sticking with point-splitting regularization, and changing the background from B to B', while in our previous analysis, we kept the background fixed at B, but changed the regularization.

We make the following remarks:

1. If we are on a torus, a shift in background from B to B' must be such that the difference $B - B'$ obeys Dirac quantization (the periods of $B - B'$ are integer multiples of 2π) because the ordinary gauge fields with curvatures F and F' each obey Dirac quantization, so their difference $F - F'$ does also. Such quantized shifts in B are elements of the T-duality group.

2. Even if we are on $\mathbb{R}^n$, there can be at most one value of B for which the noncommutative curvature vanishes at infinity. Thus, if we are going to investigate background independence in the form proposed above, we have to be willing to consider noncommutative gauge fields whose curvature measured at infinity is constant.

3. This has a further consequence. Since the condition for $\widehat{F}$ to vanish at infinity will not be background independent, there is no hope for the noncommutative action as we have written it so far, namely,

$$\frac{1}{g_{YM}^2}\int d^n x\sqrt{G}G^{ik}G^{jl}\,\mathrm{Tr}\,\widehat{F}_{ij} * \widehat{F}_{kl} \tag{3.23}$$

to be background independent. Even the condition that this action converges will not be background independent. We will find it necessary to extend the action to

$$\widehat{L} = \frac{1}{g_{YM}^2}\int d^n x\sqrt{G}G^{ik}G^{jl}(\widehat{F}_{ij} - \theta^{-1}_{ij})(\widehat{F}_{kl} - \theta^{-1}_{kl})\,, \tag{3.24}$$

which will be background independent. The constant we added corresponds to $\Phi = -\theta^{-1}$ in (3.17). It is easy to see that with this value of Φ equation (3.19) determines $\theta = B^{-1}$, $G = -(2\pi\alpha')^2 Bg^{-1}B$ as in (3.21). It is important that even though the expressions for θ and G are as in the zero slope limit (2.15), in fact they are exact even away from this limit as they satisfy (3.19).

Note that these remarks apply to background independence, and not to behavior under change in regularization. (Change in regularization is not particularly restricted by being on a torus, since for instance (3.8) makes perfect sense on a torus; leaves fixed the condition that the curvature vanishes at infinity; and does not leave fixed any particular lagrangean.) Note also that in the description of open strings by ordinary gauge theory, the symmetry of shift in B (keeping fixed $B + F$) is made at fixed closed string metric g, so we want to understand background independence of noncommutative Yang-Mills at fixed g.

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Remark 1 above makes it clear that Morita equivalence must be an adequate tool for proving background independence, since more generally [4, 17]–[21], Morita equivalence is an effective tool for analyzing T-duality of noncommutative Yang-Mills theory. We will here consider only background independence for noncommutative Yang-Mills on $\mathbb{R}^n$, and not surprisingly in this case the discussion reduces to something very concrete that we can write down naively without introducing the full machinery of Morita equivalence.

We will consider first the case that the rank r of θ equals the dimension n of the space, so that θ is invertible. (The generalization is straightforward and is briefly indicated below.) The gauge fields are described by covariant derivatives

$$D_i = \frac{\partial}{\partial x^i} - iA_i\,, \tag{3.25}$$

where the A_i are elements of the algebra $\mathcal{A}$ generated by the x_i (tensored with $N \times N$ matrices if the gauge field has rank N). We recall that $\mathcal{A}$ is defined by the relations $[x^i, x^j] = i\theta^{ij}$.

The $\partial/\partial x^i$ do not commute with the x's that appear in A_i, and this is responsible for the usual complexities of gauge theory. The surprising simplification of noncommutative Yang-Mills theory is that this complexity can be eliminated by a simple change of variables. We write

$$D_i = \partial_i' - iC_i\,, \tag{3.26}$$

where

$$\partial_i' = \frac{\partial}{\partial x^i} + iB_{ij}x^j \tag{3.27}$$

and

$$C_i = A_i + B_{ij}x^j\,. \tag{3.28}$$

The point of this is that the ∂_i' commute with the x^i. Hence, the curvature $\widehat{F}_{ij} = i[D_i, D_j]$ is simply

$$\widehat{F}_{ij} = i[\partial_i', \partial_j'] - i[C_i, C_j] \tag{3.29}$$

or more explicitly

$$\widehat{F}_{ij} = B_{ij} - i[C_i, C_j]\,. \tag{3.30}$$

Now we are almost ready to explain what background independence means. The C_i are given as functions of the x^i, and as such they are elements of an algebra $\mathcal{A}$ that depends on θ. However, as an abstract algebra, $\mathcal{A}$ only depends on the rank of θ, and because no derivatives of C appear in the formula for the curvature, we can treat the C_i as elements of an abstract algebra. For example, we can take any fixed algebra $[y^a, y^b] = it^{ab}$, $a, b = 1, \ldots, n$ with t^{ab} being any invertible antisymmetric tensor. Then, picking a "vierbein" f_a^k, such that $\theta^{jk} = f_a^j f_b^k t^{ab}$, we write the x^k that appear in the argument of C_i as $x^k = f_a^k y^a$, and regard the C_i as functions of y^a. We make no such transformation of the x^i that appear in the definition of ∂'_i. Thus, the covariant derivatives are

$$D_i = \partial'_i - iC_i(y^a) = \frac{\partial}{\partial x^i} + iB_{ij}x^j - iC_i(y_a)\,. \tag{3.31}$$

Because $[\partial'_i, x^j] = 0$, we can make this change of variables for the "internal" x's that appear as arguments of C_i, without touching the x's that appear explicitly on the right hand side of (3.31), and without changing the formula for the curvature. There is no analog of this manipulation in ordinary Yang-Mills theory.

One is now tempted to define background independence by varying θ^{ij}, and its inverse B_{ij}, while keeping fixed $C_i(y_a)$. Then, writing (3.30) in the form $\widehat{F}_{ij} = \theta^{-1}_{ij} - i[C_i, C_j]$, we see that under this operation

$$N_{ij} = \widehat{F}_{ij} - \theta^{-1}_{ij} = -i[C_i, C_j] \tag{3.32}$$

is invariant. However, this operation, taken with fixed open string metric G, does not leave fixed the action (3.24), since G depends on θ: $G = -(2\pi\alpha')^2 Bg^{-1}B = -(2\pi\alpha')^2\theta^{-1}g^{-1}\theta^{-1}$. Instead, we want to vary θ while keeping fixed the components of C_i in a fixed local Lorentz frame. If e^a_i is a vierbein for the closed string metric g (so $g_{ij} = \sum_a e^a_i e^a_j$), then a vierbein for G is $E^a_i = 2\pi\alpha' B_{ij} {e^j}_a$. We write $C_i = E^a_i C_a$. Now we can formulate background independence: it is an operation in which one varies θ, keeping fixed g and C_a.

It is easy to see now that

$$G^{ik}G^{jl}\,\mathrm{Tr}\,(\widehat{F}_{ij} - \theta^{-1}_{ij}) * (\widehat{F}_{kl} - \theta^{-1}_{kl}) \tag{3.33}$$

is invariant under this operation. With $G^{il} = -(2\pi\alpha')^{-2}\theta^{ij}g_{jk}\theta^{kl}$, this follows from (3.32) and the fact that $\theta^{ij}C_j$ is background independent. Background independence of (3.33) is equivalent to the form of G^{ij} and the fact that the quantity defined by $Q^{il} = -i\theta^{ij}[C_j, C_k]\theta^{kl}$, or more simply

$$Q = \theta\widehat{F}\theta - \theta\,, \tag{3.34}$$

is background independent. In the rank one case, for constant $\widehat{F}$, we can via (3.14) express Q in terms of the equivalent ordinary abelian gauge field that could be used in an alternative description of the same physics. We find simply

$$Q = -\frac{1}{B+F}\,. \tag{3.35}$$

This is a satisfying result; it says that in this case the background independent object Q defined with point-splitting regularization and noncommutative Yang-Mills theory is a function of the background independent object $B + F$ found with Pauli-Villars regularization and ordinary Yang-Mills theory. It also shows that we should not try to use the noncommutative description if $B+F=0$ where Q is infinite, and we should not try to expand around $Q=0$, where $B+F$ is infinite.

For background independence of the action (3.24), background independence of (3.33) is not quite enough. We need, in addition, that the measure

$$\frac{d^n x\sqrt{G}}{g_{YM}^2} \tag{3.36}$$

should be background independent. This will tell how g_{YM}^2 must transform under the change of background. Since the action density is most naturally written as a function of the y's, we should convert the integration measure to an integral over y. From $x^k = f_a^k y^a$ and $f_a^k f_b^l t^{ab} = \theta^{kl}$, we get $d^n x = d^n y\ \det(f) = d^n y\sqrt{\det\theta}/\sqrt{\det\, t}$. We also have $G = -(2\pi\alpha')^2\theta^{-1}g\theta^{-1}$ so $\det G = (2\pi\alpha')^{2n}(\det\theta)^{-2}\det g$. So the measure is

$$\frac{d^n y(2\pi\alpha')^n\sqrt{\det g}}{g_{YM}^2\sqrt{\det\theta}\sqrt{\det\, t}}. \tag{3.37}$$

So g_{YM} must transform under a change in θ in such a way that $g_{YM}^2\sqrt{\det\theta}$ is invariant. Since $g_{YM}^2 \sim G_s$ this means that $G_s/\sqrt{\det B}$ is invariant. This is clearly the case for the value of $G_s = g_s\det(2\pi\alpha' Bg^{-1})^{1/2}$ as determined in (3.22). This means that under a shift in the B-field,

$$B \to B' = B + b \tag{3.38}$$

(with b a constant antisymmetric tensor), which induces

$$\theta \to \theta' = \theta\frac{1}{1+\theta b}\,, \tag{3.39}$$

we require

$$g_{YM} \to g'_{YM} = g_{YM}\left(\det(1+\theta b)\right)^{1/4}\ . \tag{3.40}$$

If we are on a torus, we require b to have periods that are integer multiples of 2π, and then (3.38) is a special case of a T-duality transformation. The transformation (3.40) is in this situation a special case of the T-duality transformations of noncommutative Yang-Mills, as analyzed in [4, 17]–[21]; in section 6 (see equation (6.14)), we derive this formula from the standard T-duality transformations of closed strings and the mapping from closed string to open string parameters.

Finally, let us consider the more general case that θ might have rank $r < n$. The algebra then has a center generated by $n-r$ coordinates, which we can call $x^1, \ldots, x^{n-r}$. θ is only invertible in the space of $x^{n-r+1}, \ldots, x^n$. We let b_{ij} be a partial inverse of θ^{ij}, with b_{ij} zero unless $i,j > n-r$, and $b_{ij}\theta^{jk} = \delta_i{}^k$ for $i,r > n-r$. A construction just as above, defining ∂_i' and C_i by the same formulas, gives now invariance under change of background, as long as one preserves the center of the algebra and the rank of θ. But otherwise, one can change θ as one pleases.

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4. Slowly varying fields

The purpose of the present section is to do some explicit calculations verifying and illustrating our theoretical claims. We have argued that the same open string theory effective action can be expressed in terms of either ordinary Yang-Mills theory or noncommutative Yang-Mills theory. In the description by ordinary Yang-Mills, the B-dependence is described by replacing everywhere F by $B+F$. In the description by noncommutative Yang-Mills, the B-dependence is entirely contained in the dependence on B of the open string metric G, the open string effective coupling G_s, and the θ dependence of the $*$ product. We have also argued in section 3.1 that there must exist a continuous interpolation between these two descriptions with arbitrary θ.

To compare these different descriptions, we need a situation in which we can compute in all of them. For this we will take the limit of slowly varying, but not necessarily small, gauge fields of rank one. The effective action is [42, 43] the Dirac-Born-Infeld (DBI) action. We will compare the ordinary DBI action as a function of the closed string metric and coupling and $B+F$, to its noncommutative counterpart, as a function of the open string metric and coupling and $\widehat{F}+\Phi$.[8] After proving the equivalence between them and exploring the zero slope limit in section 4.1, we specialize in section 4.2 to the case of four dimensions, and compare the respective BPS conditions — the stringy generalization of the instanton equation.

4.1 Dirac-Born-Infeld action

For slowly varying fields on a single Dp-brane, the effective lagrangean is the Dirac-Born-Infeld lagrangean

$$\mathcal{L}_{DBI} = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B+F))} \,. \tag{4.1}$$

We discussed in section 2 the normalization and the fact that $B+F$ is the gauge invariant combination.

We have argued in section 2 that the effective lagrangean of the noncommutative gauge fields $\widehat{A}$ must be such that when expressed in terms of the open string variables G, θ, and G_s given in (2.5) and (2.45), the dependence on θ is only in the $*$ product. Therefore, for slowly varying $\widehat{F}$ it is

$$\widehat{\mathcal{L}}_{DBI} = \frac{1}{G_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(G + 2\pi\alpha'\widehat{F})} \,. \tag{4.2}$$

[8] The comparison cannot be made just using the formula (3.13) for constant F, since as we will see, terms in (3.8) of the general form $A\partial F$ contribute in the analysis. It is necessary to integrate by parts in comparing the DBI actions, and one cannot naively treat F as a constant.

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We also proposed in section 3.1 that there is a more general description with an arbitrary θ and parameters G, Φ, G_s determined in (3.19)

$$\begin{aligned}\frac{1}{G+2\pi\alpha'\Phi} &= -\frac{\theta}{2\pi\alpha'} + \frac{1}{g+2\pi\alpha' B} \\ G_s = g_s\left(\frac{\det(G+2\pi\alpha'\Phi)}{\det(g+2\pi\alpha' B)}\right)^{1/2} &= g_s \frac{1}{\det\left[\left(\frac{1}{g+2\pi\alpha' B} - \frac{\theta}{2\pi\alpha'}\right)(g+2\pi\alpha' B)\right]^{1/2}}. \end{aligned} \tag{4.3}$$

Under the assumption of (3.17), the effective action in these variables should be

$$\widehat{\mathcal{L}}_{DBI} = \frac{1}{G_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{\det(G+2\pi\alpha'(\widehat{F}+\Phi))} \tag{4.4}$$

We will here demonstrate that in the limit of slowly varying fields, (4.1), (4.2), and (4.4) are all equivalent. In particular, this will verify the equivalence of (4.1) and (4.2), for which we gave an *a priori* explanation in section 2, it will give (along with the special case considered in section 3.2) our main evidence that there exists a description with an arbitrary θ and $\widehat{F}$ shifted as in (3.17), and it will show that in such a description G, Φ, and G_s must be given by the formulas in (4.3).

In all of the above formulas, we can expand $\widehat{\mathcal{L}}_{DBI}$ in powers of $\widehat{F}$ and all the resulting products can be regarded as $*$ products. If instead we treat them as ordinary products, our answer will differ by terms including derivatives of $\widehat{F}$. Since the DBI lagrangean is obtained in string theory after dropping such terms, there is no reason to keep some of them but not others. Therefore, we will ignore all such derivatives of $\widehat{F}$ and regard the products in the expansion of (4.4) as ordinary products, i.e. the θ dependence will be only in the definition of $\widehat{F}$.

We want to show that using the change of variables (4.3) and the transformations of the fields $\widehat{A}(A)$ given in (3.6), the two lagrangeans are related as

$$\mathcal{L}_{DBI} = \widehat{\mathcal{L}}_{DBI} + \text{total derivative} + \mathcal{O}(\partial F)\,. \tag{4.5}$$

The difference in total derivative arises from the fact that the action is derived in string theory by using the equations of motion or the S-matrix elements, which are not sensitive to such total derivatives. Furthermore, the effective lagrangean in terms of $\widehat{A}$ is gauge invariant only up to total derivatives, and we will permit ourselves to integrate by parts and discard total derivatives. The $\mathcal{O}(\partial F)$ term in (4.5) is possible because these two lagrangeans are derived in string theory in the approximation of neglecting derivatives of F, and therefore they can differ by such terms.

For $\theta = 0$, the change of variables (4.3) is trivial and so is (4.5). Therefore, in order to prove (4.5) it is enough to prove its derivative with respect to θ holding fixed the closed string parameters g, B, g_s and the commutative gauge field A. In other words, we will show that this variation of the right hand side of (4.5) vanishes.

In order to keep the equations simple we will set $2\pi\alpha' = 1$; the α' dependence can be easily restored on dimensional grounds. In preparation for the calculation we differentiate (4.3) holding g, B and g_s fixed, and express the variation of G, Φ and G_s in terms of the variation of θ:

$$\begin{aligned}\delta G + \delta\Phi &= (G+\Phi)\delta\theta(G+\Phi)\\ \delta G_s &= \frac{1}{2}G_s \operatorname{Tr}\frac{1}{G+\Phi}(\delta G+\delta\Phi) = \frac{1}{2}G_s \operatorname{Tr}(G+\Phi)\delta\theta = \frac{1}{2}G_s \operatorname{Tr}\Phi\delta\theta\,,\end{aligned}\tag{4.6}$$

where δG and $\delta\Phi$ are symmetric and antisymmetric respectively. We also need the variation of $\widehat{F}$ with respect to θ. Adapting (3.8) to the rank one case we have

$$\delta\widehat{F}_{ij}(\theta) = \frac{1}{2}\delta\theta^{kl}\Big[\widehat{F}_{ik}*\widehat{F}_{jl}+\widehat{F}_{jl}*\widehat{F}_{ik}-\widehat{A}_k*\partial_l\widehat{F}_{ij}-\partial_l\widehat{F}_{ij}*\widehat{A}_k\Big].\tag{4.7}$$

Since we are going to ignore derivatives of $\widehat{F}$, we can replace the $*$ product by ordinary products. However, we cannot neglect the terms with explicit derivatives of $\widehat{F}$ since they multiply $\widehat{A}$ and can become terms without derivatives of $\widehat{F}$ after integration by parts. Therefore, we write (4.7) as

$$\delta\widehat{F}_{ij} = \delta\theta^{kl}\left(\widehat{F}_{ik}\widehat{F}_{jl}-\widehat{A}_k\partial_l\widehat{F}_{ij}\right)+\mathcal{O}(\partial\widehat{F})\,.\tag{4.8}$$

We are now ready to vary $\widehat{\mathcal{L}}_{DBI}$ (4.4):

$$\begin{aligned}\delta&\left[\frac{1}{G_s}\det(G+\widehat{F}+\Phi)^{1/2}\right] =\\ &= \frac{\det(G+\widehat{F}+\Phi)^{1/2}}{G_s}\left[-\frac{\delta G_s}{G_s}+\frac{1}{2}\operatorname{Tr}\frac{1}{G+\widehat{F}+\Phi}(\delta G+\delta\widehat{F}+\delta\Phi)\right]\\ &= \frac{1}{2}\frac{\det(G+\widehat{F}+\Phi)^{1/2}}{G_s}\Bigg[-\operatorname{Tr}\delta\theta(G+\Phi)+\operatorname{Tr}\frac{1}{G+\widehat{F}+\Phi}(G+\Phi)\delta\theta(G+\Phi)+\\ &\qquad +\left(\frac{1}{G+\widehat{F}+\Phi}\right)_{ji}\delta\theta^{kl}\left(\widehat{F}_{ik}\widehat{F}_{jl}-\widehat{A}_k\partial_l\widehat{F}_{ij}\right)\Bigg]+\mathcal{O}(\partial\widehat{F})\\ &= \frac{1}{2}\frac{\det(G+\widehat{F}+\Phi)^{1/2}}{G_s}\Bigg[-\operatorname{Tr}\frac{1}{G+\widehat{F}+\Phi}\widehat{F}\delta\theta(G+\Phi)-\\ &\qquad -\operatorname{Tr}\frac{1}{G+\widehat{F}+\Phi}\widehat{F}\delta\theta\widehat{F}+\operatorname{Tr}\delta\theta\widehat{F}\Bigg]+\mathcal{O}(\partial\widehat{F})+\text{tot. deriv.}\\ &= \mathcal{O}(\partial\widehat{F})+\text{total derivative}\,,\end{aligned}\tag{4.9}$$

where in the third step we used

$$\partial_l\det(G+\widehat{F}+\Phi)^{1/2} = \frac{1}{2}\det(G+\widehat{F}+\Phi)^{1/2}\left(\frac{1}{G+\widehat{F}+\Phi}\right)_{ji}\partial_l F_{ij}\,,\tag{4.10}$$

and then integrated by parts. This completes the proof of (4.5) and of our change of variables (3.19).

In fact, we first found the change of variables (3.19) by demanding that (4.5) is satisfied. It is, however, quite nontrivial that there exists a change of variables like (3.19) for which (4.5) is satisfied. This depends on our relation between the commutative and the noncommutative gauge fields (4.8) and on the particular form of the DBI lagrangean. One could actually use this computation to motivate the DBI lagrangean as the only lagrangean for which (4.5) is true.

Even though we proved (4.5) for every value of the parameters, it is instructive to examine it in various limits, comparing the commutative and the noncommutative sides of (4.5) explicitly. Some of the technical aspects of these comparisons are similar to the calculation in (4.9), but they are conceptually different. Rather than varying the description by changing θ for a fixed background (fixed g, B and g_s), here we will use a description with $\Phi = 0$ and will vary the background. We will perform two computations. The first will be for small $\alpha' B$ and the second will be in the zero slope limit. The reader who is not interested in the details could jump to section 4.2.

Comparison for small B. First, we consider the comparison for small B. In this regime, the open string variables are

$$
\begin{aligned}
G &= g - (2\pi\alpha')^2 B g^{-1} B\,, \qquad \theta = -(2\pi\alpha')^2 g^{-1} B g^{-1} + \mathcal{O}(B^3) \\
G_s &= g_s\left(1 - (\pi\alpha')^2\,\mathrm{Tr}(g^{-1}B)^2 + \mathcal{O}(B^4)\right).
\end{aligned}
\tag{4.11}
$$

Since θ is small for small B, we begin by expanding (4.2) in powers of θ. The change of variables in the rank one case

$$
\widehat{F}_{ij} = F_{ij} + \theta^{kl}\left(F_{ik}F_{jl} - A_k\partial_l F_{ij}\right) + \mathcal{O}(\theta^2)
\tag{4.12}
$$

leads to

$$
\begin{aligned}
&\det(G + 2\pi\alpha'\widehat{F})^{1/2} = \\
&\quad = \det\left[G_{ij} + 2\pi\alpha'\left(F_{ij} + \theta^{kl}\left(F_{ik}F_{jl} - A_k\partial_l F_{ij}\right) + \mathcal{O}(\theta^2)\right)\right]^{1/2} \\
&\quad = \det(G + 2\pi\alpha' F)^{1/2}\left[1 + \pi\alpha'\left(\frac{1}{G + 2\pi\alpha' F}\right)_{ji}\theta^{kl}\left(F_{ik}F_{jl} - A_k\partial_l F_{ij}\right) + \mathcal{O}(\theta^2)\right] \\
&\quad = \det(G+2\pi\alpha' F)^{1/2}\left[1 - \pi\alpha'\,\mathrm{Tr}\,\frac{1}{G+2\pi\alpha' F}F\theta F + \frac{1}{2}\,\mathrm{Tr}\,\theta F + \mathcal{O}(\theta^2)\right] + \text{tot. deriv.} \\
&\quad = \det(G + 2\pi\alpha' F)^{1/2}\left[1 - \frac{1}{4\pi\alpha'}\,\mathrm{Tr}\,\frac{1}{G + 2\pi\alpha' F}G\theta G + \mathcal{O}(\theta^2)\right] + \text{tot. deriv.}\,,
\end{aligned}
\tag{4.13}
$$

where in the second step we used (4.10) and then integrated by parts, and in the third step we used $\mathrm{Tr}\ G\theta = 0$. Using (4.11), to first order in B, we find as expected

$$
\begin{aligned}
\widehat{\mathcal{L}}_{DBI} &= \frac{1}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\det(g + 2\pi\alpha' F)^{1/2}\left(1 + \pi\alpha'\,\mathrm{Tr}\,\frac{1}{g + 2\pi\alpha' F}B\right) + \\
&\quad + \mathcal{O}(B^2) + \text{total derivative} \\
&= \mathcal{L}_{DBI} + \mathcal{O}(B^2) + \text{total derivative}\,.
\end{aligned}
\tag{4.14}
$$

To extend this comparison to order B^2, we would need to use (3.8) to determine the order θ^2 terms in (4.12). However, since the order θ^2 terms in (4.12) involve three factors of F or terms which become three factors of F after integration by parts, we can compare the B^2F^2 terms in the two lagrangeans without needing the corrections to (4.12).

We use the identity for antisymmetric M

$$\det(1+M)^{1/2} = 1 - \frac{1}{4}\operatorname{Tr} M^2 - \frac{1}{8}\operatorname{Tr} M^4 + \frac{1}{32}\left(\operatorname{Tr} M^2\right)^2 + \mathcal{O}(M^6) \tag{4.15}$$

to write the DBI lagrangean density as

$$\begin{aligned}
\mathcal{L}_{DBI} &= \frac{\sqrt{\det g}}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\Big[1 - (\pi\alpha')^2\operatorname{Tr}(g^{-1}(B+F))^2 - 2(\pi\alpha')^4\operatorname{Tr}(g^{-1}(B+F))^4 + \\
&\qquad + \frac{(\pi\alpha')^4}{2}\left(\operatorname{Tr}(g^{-1}(B+F))^2\right)^2 + \mathcal{O}((B+F)^6)\Big] \\
&= \frac{(\pi\alpha')^2\sqrt{\det g}}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\Big[-\operatorname{Tr}(g^{-1}F)^2 - 2(\pi\alpha')^2\operatorname{Tr}(g^{-1}F)^4 + \\
&\qquad + \frac{(\pi\alpha')^2}{2}\left(\operatorname{Tr}(g^{-1}F)^2\right)^2 - 8(\pi\alpha')^2\operatorname{Tr} g^{-1}B(g^{-1}F)^3 + \\
&\qquad + 2(\pi\alpha')^2\operatorname{Tr}(g^{-1}F)^2\operatorname{Tr} g^{-1}Fg^{-1}B - \\
&\qquad - 8(\pi\alpha')^2\operatorname{Tr}(g^{-1}B)^2(g^{-1}F)^2 + \\
&\qquad + (\pi\alpha')^2\operatorname{Tr}(g^{-1}B)^2\operatorname{Tr}(g^{-1}F)^2 + \\
&\qquad + \text{constant} + \text{total derivative} + \mathcal{O}(B^3, BF^5, B^2F^4)\Big],
\end{aligned} \tag{4.16}$$

where in the last step we used the fact that

$$2\operatorname{Tr}(g^{-1}Bg^{-1}F)^2 - \left(\operatorname{Tr} g^{-1}Bg^{-1}F\right)^2 = \text{total derivative}\,. \tag{4.17}$$

Similarly,

$$\begin{aligned}
\widehat{\mathcal{L}}_{DBI} &= \frac{\sqrt{\det G}}{G_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\Big[1 - (\pi\alpha')^2\operatorname{Tr}(G^{-1}\widehat{F})^2 - 2(\pi\alpha')^4\operatorname{Tr}(G^{-1}\widehat{F})^4 + \\
&\qquad + \frac{(\pi\alpha')^4}{2}\left(\operatorname{Tr}(G^{-1}\widehat{F})^2\right)^2 + \mathcal{O}(\widehat{F}^6)\Big] \\
&= \frac{(\pi\alpha')^2\sqrt{\det g}}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\Big[-\operatorname{Tr}(g^{-1}F)^2 - 2(\pi\alpha')^2\operatorname{Tr}(g^{-1}F)^4 + \\
&\qquad + \frac{(\pi\alpha')^2}{2}\left(\operatorname{Tr}(g^{-1}F)^2\right)^2 - 8(\pi\alpha')^2\operatorname{Tr} g^{-1}B(g^{-1}F)^3 + \\
&\qquad + 2(\pi\alpha')^2\operatorname{Tr}(g^{-1}F)^2\operatorname{Tr} g^{-1}Fg^{-1}B -
\end{aligned}$$

$$
\begin{aligned}
&\qquad - 8(\pi\alpha')^2 \operatorname{Tr}(g^{-1}B)^2(g^{-1}F)^2 + \\
&\qquad + (\pi\alpha')^2 \operatorname{Tr}(g^{-1}B)^2 \operatorname{Tr}(g^{-1}F)^2 + \\
&\qquad + \text{constant} + \text{total derivative} + \mathcal{O}(B^3, BF^5, B^2F^4)\Big] \\
&= \mathcal{L}_{DBI} + \text{total derivative} + \mathcal{O}(B^3, BF^5, B^2F^4)\,, \qquad (4.18)
\end{aligned}
$$

where we have used (4.12) and (4.11).

This demonstrates explicitly that

$$
\widehat{\mathcal{L}}_{DBI} = \mathcal{L}_{DBI} + \text{total derivative} + \mathcal{O}(B^3F^3, B^2F^4)\,. \qquad (4.19)
$$

Comparison in the zero slope limit. We now turn to another interesting limit — our zero slope limit. In this limit the entire string effective lagrangean becomes quadratic in $\widehat{F}$. The same is true for the DBI lagrangean

$$
\begin{aligned}
\widehat{\mathcal{L}}_{DBI} &= \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s}\sqrt{\det G}\,G^{im}G^{jn}\widehat{F}_{ij} * \widehat{F}_{mn} + \\
&\qquad + \text{total derivative} \ + \ \text{constant} \ + \ \text{higher powers of } \alpha'. \qquad (4.20)
\end{aligned}
$$

Ignoring the total derivative and the constant we can set $\Phi = 0$.

If we take the zero slope limit, and F is slowly varying, then we can get a description either using ordinary gauge fields and the DBI lagrangean $\mathcal{L}_{DBI}$, or using noncommutative gauge fields using the $\widehat{F}^2$ lagrangean (4.20).

We work on a single euclidean p-brane with B of rank $r = p+1$, and for simplicity we consider the metric $g_{ij} = \epsilon\delta_{ij}$. We are interested in the zero slope limit, i.e. $\alpha' \sim \epsilon^{1/2} \to 0$. Expanding the DBI action density in powers of ϵ, we find

$$
\begin{aligned}
\mathcal{L}_{DBI} &= \frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}}g_s}\left(|\mathrm{Pf}(M)| + \frac{\epsilon}{2}|\mathrm{Pf}(M)| \operatorname{Tr}\frac{1}{M} + \frac{\epsilon^2}{8}|\mathrm{Pf}(M)|\left(\operatorname{Tr}\frac{1}{M}\right)^2 - \right.\\
&\qquad \left. - \frac{\epsilon^2}{4}|\mathrm{Pf}(M)| \operatorname{Tr}\frac{1}{M^2} + \mathcal{O}(\epsilon^3)\right), \qquad (4.21)
\end{aligned}
$$

where

$$
M = 2\pi\alpha'(B+F)\,. \qquad (4.22)
$$

The absolute value sign arises from the branch of the square root in (4.1). Since M is antisymmetric, the second and third terms vanish. The first term is a constant plus a total derivative in spacetime, which we ignore in this discussion. In the limit $\epsilon \to 0$, the leading term is

$$
\begin{aligned}
\mathcal{L}_{DBI} &= -\frac{\epsilon^2}{4(2\pi)^{\frac{p+3}{2}}(\alpha')^2 g_s}|\mathrm{Pf}(B+F)| \operatorname{Tr}\frac{1}{(B+F)^2} + \\
&\qquad + \text{total derivative} \ + \ \text{constant} \ + \ \text{higher powers of } \alpha'\,. \qquad (4.23)
\end{aligned}
$$

Our general discussion above shows that the lagrangean (4.23) must be the same as the $\widehat{F}^2$ lagrangean (4.20). We now verify this explicitly in a power series in F.

We define three auxiliary functions

$$\begin{aligned} f(B,F) &= |\mathrm{Pf}(B+F)|\,\mathrm{Tr}\,\frac{1}{(B+F)^2} - |\mathrm{Pf}B|\,\mathrm{Tr}\,\frac{1}{B^2}\,, \\ q(B,F,\eta) &= \left|\mathrm{Pf}\left(B+F+\eta\frac{1}{B}\right)\right|, \\ h(B,F) &= 4\frac{\partial q}{\partial \eta}(\eta=0) - q(\eta=0)\,\mathrm{Tr}\,\frac{1}{B^2}\,. \end{aligned} \tag{4.24}$$

q and therefore also all its derivatives with respect to η are total derivatives in spacetime. In particular, $h(B,F)$ is a total derivatives in spacetime. Expanding $f(B,F)$ in powers of F we find

$$\begin{aligned} f(B,F) = h(B,F) + |\mathrm{Pf}B|\,\mathrm{Tr}\left(\frac{1}{B^2}F\frac{1}{B^2}F\right) + \frac{1}{2}|\mathrm{Pf}B|\,\mathrm{Tr}\left(\frac{1}{B}F\right)\mathrm{Tr}\left(\frac{1}{B^2}F\frac{1}{B^2}F\right) - \\ - 2|\mathrm{Pf}B|\,\mathrm{Tr}\left(\frac{1}{B}F\frac{1}{B^2}F\frac{1}{B^2}F\right) + \mathcal{O}(F^4). \end{aligned} \tag{4.25}$$

If $\widehat{F}$ is small, the $\widehat{F}^2$ action can be expressed in terms of ordinary gauge fields. In the rank one case (3.6) becomes

$$\widehat{F}_{ij} = F_{ij} + \theta^{kl}\left(F_{ik}F_{jl} - A_k\partial_l F_{ij}\right) + \mathcal{O}(F^3)\,. \tag{4.26}$$

Here in asserting that the corrections are $\mathcal{O}(F^3)$, we consider two derivatives or two powers of A (or one of each) to be equivalent to one power of F. Substituting (4.26) in (4.20) we find

$$\begin{aligned} \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s}\sqrt{\det G}G^{im}G^{jn}\widehat{F}_{ij} * \widehat{F}_{mn} &= \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s}\sqrt{\det G}\Big[G^{im}G^{jn}F_{ij}F_{mn} + \\ &+2G^{im}G^{jn}\theta^{kl}\left(F_{ik}F_{jl} - A_k\partial_l F_{ij}\right)F_{mn}\Big] + \text{total derivatives} + \mathcal{O}(F^4) \\ &= \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s}\sqrt{\det G}\Big[-\mathrm{Tr}\left(G^{-1}FG^{-1}F\right) + 2\,\mathrm{Tr}\left(F\theta FG^{-1}FG^{-1}\right) - \\ &- \frac{1}{2}\,\mathrm{Tr}\,(F\theta)\left(FG^{-1}FG^{-1}\right)\Big] + \text{total derivatives} + \mathcal{O}(F^4)\,. \end{aligned} \tag{4.27}$$

For $g_{ij} = \epsilon\delta_{ij}$ with B of rank $r = p+1$, we find from (2.15) that $G = -\epsilon^{-1}(2\pi\alpha')^2B^2$ (which is finite as $\epsilon \to 0$), $\theta = 1/B$ and $G_s = g_s(2\pi\alpha'/\epsilon)^{\frac{p+1}{2}}|\mathrm{Pf}B|$. Using these formulas, with (4.25) and (4.27), we get

$$\mathcal{L}_{DBI} = \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_s}\sqrt{\det G}G^{im}G^{jn}\widehat{F}_{ij} * \widehat{F}_{mn} + \text{total derivatives} + \text{constant} + \mathcal{O}(F^4)\,. \tag{4.28}$$

We conclude that the zero slope limit of $\mathcal{L}_{DBI}$ is the nonpolynomial action (4.23). It conicides with the zero slope limit of $\widehat{\mathcal{L}}_{DBI}$, which is simply $\widehat{F}^2$ (we have checked it explicitly only up to terms of order F^4).

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4.2 Supersymmetric configurations

Now we will specialize to four dimensions (though some of the introductory remarks are more general) and analyze supersymmetric configurations, the stringy instantons.

We recall first that, in general, a Dp-brane preserves only half of the supersymmetry of type II superstring theory. In an interpretation [52] that actually predates the D-brane era, this means the theory along the brane has spontaneously broken (or "nonlinearly realized") supersymmetry along with its unbroken (or "linearly realized") supersymmetry.

For example, in the extreme low energy limit, the theory along a threebrane is the F^2 theory. Its minimal supersymmetric extension in four dimensions is obtained by adding a positive chirality "photino" field λ_α,[9] with its CPT conjugate $\bar{\lambda}_{\dot{\alpha}}$ of opposite chirality. The linearly realized or unbroken supersymmetry acts by the standard formulas

$$\delta\lambda_\alpha = \frac{1}{2\pi\alpha'} M^+_{ij}\sigma^{ij\beta}_\alpha \eta_\beta \tag{4.29}$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = \frac{1}{2\pi\alpha'} M^-_{ij}\bar{\sigma}^{ij\dot{\beta}}_{\dot{\alpha}} \bar{\eta}_{\dot{\beta}}, \tag{4.30}$$

where we have included a B-field, and written the standard formula in terms of M (defined in (4.22)) rather than F. As usual, $\sigma^{ij} = \frac{1}{2}[\Gamma^i, \Gamma^j]$, while α, β and $\dot{\alpha}$, $\dot{\beta}$ are spinor indices of respectively positive and negative chiralities. The nonlinearly realized or spontaneously broken supersymmetry of the F^2 theory acts simply by

$$\delta^*\lambda_\alpha = \frac{\epsilon^2}{4\pi\alpha'}\eta^*_\alpha\,, \qquad \delta^*\bar{\lambda}_{\dot{\alpha}} = \frac{\epsilon^2}{4\pi\alpha'}\bar{\eta}^*_{\dot{\alpha}}\,. \tag{4.31}$$

Here η and $\bar{\eta}$ are constants, and we have chosen a convenient normalization.

One of the many special properties of the DBI theory is that [53] it has in four dimensions a supersymmetric extension that preserves not only the linearly realized supersymmetry — many bosonic theories have such a supersymmetric version — but also, what is much more special, the nonlinearly realized supersymmetry. The transformation law of the photino under the linearly realized supersymmetry is unchanged from (4.29) in going to the DBI theory. The nonlinearly realized supersymmetries, however, become much more complicated. The generalization of (4.31) is

$$\delta^*\lambda_\alpha = \frac{1}{4\pi\alpha'}\left[\epsilon^2 - \mathrm{Pf}M + \sqrt{\det_{ij}(\epsilon\delta_{ij} + M_{ij})}\right]\eta^*_\alpha =$$

[9] We recall that SO(4) decomposes as $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. A positive chirality spinor transforms as $(1/2, 0)$, while a negative chirality spinor transforms as $(0, 1/2)$. Our conventions are such that a selfdual antisymmetric tensor transforms as $(1, 0)$ and an anti-selfdual one as $(0, 1)$. The full low energy $D3$-brane action has additional fields and supersymmetries beyond those discussed here, but we do not expect them to affect the particular issues we will address.

$$
\begin{aligned}
&= \frac{1}{4\pi\alpha'}\left[\epsilon^2 - \mathrm{Pf}M + \sqrt{\epsilon^4 - \frac{\epsilon^2}{2}\,\mathrm{Tr}\,M^2 + (\mathrm{Pf}M)^2}\right]\eta^*_\alpha \\
&= \frac{1}{4\pi\alpha'}\left[-\mathrm{Pf}M + |\mathrm{Pf}M| + \epsilon^2\frac{4|\mathrm{Pf}M| - \mathrm{Tr}\,M^2}{4|\mathrm{Pf}M|} + \mathcal{O}(\epsilon^4)\right]\eta^*_\alpha, \\
\delta^*\bar\lambda_{\dot\alpha} &= \frac{1}{4\pi\alpha'}\left[\epsilon^2 + \mathrm{Pf}M + \sqrt{\det_{ij}(\epsilon\delta_{ij} + M_{ij})}\right]\bar\eta^*_{\dot\alpha} \\
&= \frac{1}{4\pi\alpha'}\left[\epsilon^2 + \mathrm{Pf}M + \sqrt{\epsilon^4 - \frac{\epsilon^2}{2}\,\mathrm{Tr}\,M^2 + (\mathrm{Pf}M)^2}\right]\bar\eta^*_{\dot\alpha} \\
&= \frac{1}{4\pi\alpha'}\left[\mathrm{Pf}M + |\mathrm{Pf}M| + \epsilon^2\frac{4|\mathrm{Pf}M| - \mathrm{Tr}\,M^2}{4|\mathrm{Pf}M|} + \mathcal{O}(\epsilon^4)\right]\bar\eta^*_{\dot\alpha}\,. \qquad (4.32)
\end{aligned}
$$

In both (4.32) and (4.32), the first two formulas for δ^* are taken directly from [53], and the last is an expansion in small ϵ aimed at taking the by now familiar $\alpha' \to 0$ limit.

When $B = 0$ and we expand around $F = 0$, the supersymmetry (4.29)and (4.30) is realized linearly, while the supersymmetry of (4.32) and (4.32) is spontaneously broken and is realized nonlinearly. In expanding around any constant B, there is always a linear combination of the δ and δ^* supersymmetries that is unbroken. However, this combination depends on B. To see why, consider an open string ending on the threebrane, and let $\bar\psi$ and ψ be the left and right-moving worldsheet fermions. In reflection from the end of the string, we get $\bar\psi = R(B)\psi$ where $R(B)$ is a rotation matrix that can be found from (2.49) to be

$$
R(B) = \left(1 - 2\pi\alpha' g^{-1}B\right)^{-1}\left(1 + 2\pi\alpha' g^{-1}B\right). \qquad (4.33)
$$

Let Q_L and Q_R be the spacetime supersymmetries carried by left and right-moving worldsheet degrees of freedom in type II superstring theory. (Because we are mainly focusing now on threebranes, we are in type IIB, and Q_L and Q_R both have the same chirality.) A general supersymmetry of the closed string theory is generated by $\epsilon_R^\alpha Q_{R,\alpha} + \epsilon_L^\beta Q_{L,\beta}$ with constants ϵ_L, ϵ_R. Reflection at the end of the open string breaks this down to a subgroup with

$$
\epsilon_L = R(B)\epsilon_R\,, \qquad (4.34)
$$

where now of course the rotation matrix $R(B)$ must be taken in the spinor representation. Here we see explicitly that which supersymmetries are unbroken depends on B, though the number of unbroken supersymmetries is independent of B.[10]

[10] We can also see explicitly that a system of a three-brane with a B-field and a separated -1-brane is only supersymmetric if $B^+ = 0$, as asserted in the introduction. The supersymmetry left unbroken by the -1-brane obeys $\epsilon_L = \Gamma_{0123}\epsilon_R$, where Γ_{0123} is the four-dimensional chirality operator. At $B = 0$, this is compatible with (4.34) for spinors of $\Gamma_{0123} = 1$, the expected result that the -1-brane (like an instanton) preserves the supersymmetry of positive chirality. If we now turn on $B \neq 0$, compatibility with (4.34) fails unless $R(B) = 1$ for states of $\Gamma_{0123} = 1$, that is, unless $B^+ = 0$.

We can easily match the stringy parameters ϵ_L, ϵ_R with the parameters η, η^* of the supersymmetrized DBI action. We have (up to possible inessential constants)

$$\eta = \epsilon_L + \epsilon_R\,, \qquad \eta^* = \epsilon_L - \epsilon_R\,. \tag{4.35}$$

In fact, η can be identified with $\epsilon_L + \epsilon_R$ as the generator of a supersymmetry that is unbroken at $B = 0$. And η^* can be identified with $\epsilon_L - \epsilon_R$ as being odd under a $\mathbb{Z}_2$ symmetry that acts by $\lambda \to -\lambda$, $F \to -F$ in the field theory, and by reversal of worldsheet orientation in the string theory.

Now, specializing again to four dimensions, we want to identify the unbroken supersymmetry in the $\alpha' \to 0$ limit, which we temporarily think of as the limit with g fixed and $B \to \infty$. Here we meet the interesting fact that there are *two* inequivalent zero slope limits in four dimensions. For nondegenerate B with all eigenvalues becoming large, we get from (4.33) that $R(B) \to -1$ in the vector representation for $B \to \infty$. But the element -1 of the vector representation can be lifted to spinors in two different ways: as a group element that is -1 on positive chirality spinors and 1 on negative chirality spinors, or vice-versa. Starting from $R(B) = 1$ (on both vectors and spinors) at $B = 0$, what limit we get for $B \to \infty$ depends on the sign of $\mathrm{Pf}(B)$. In fact, the limit of $R(B)$ is, in acting on spinors,

$$R(B) \to -\Gamma_0\Gamma_1\Gamma_2\Gamma_3 \cdot \mathrm{sign}(\mathrm{Pf}(B))\,. \tag{4.36}$$

To prove this, it is enough to consider the special case that B is selfdual or anti-selfdual. If B is selfdual, then from (4.33), $R(B) \in \mathrm{SU}(2)_+$ and is identically 1 on negative chirality spinors, and hence approaches -1 for $B \to \infty$ on positive chirality spinors. For B anti-selfdual, we get the opposite result, leading to (4.36).

There is no loss of essential generality in assuming that we want to study instantons, or anti-selfdual gauge fields, for which the unbroken supersymmetry generators are of positive chirality. In this case, (4.36) reduces to $R(B) = -\mathrm{sign}(\mathrm{Pf}(B))$. Eqs. (4.35) tells us that the unbroken supersymmetry is η when $R(B) = 1$ and η^* when $R(B) = -1$. We conclude, then, that for instantons, when $B \to \infty$ with $\mathrm{Pf}(B) > 0$, the unbroken supersymmetry is the one that at $B = 0$ is nonlinearly realized, while for $B \to \infty$ with $\mathrm{Pf}(B) < 0$, the unbroken supersymmetry is the one that is linearly realized at $B = 0$. For anti-instantons, the two cases are reversed. We shall see this difference both in the discussion below, based on the supersymmetric DBI action of [53], and in section 5 in the analysis of the $D0/D4$ system, which will also lead to an alternative intuitive explanation of the difference.

BPS states of supersymmetric Born-Infeld. We assume that we are in four dimensions with B of rank four. Instead of taking $B \to \infty$, it is equivalent, of course, to take the metric to be $g_{ij} = \epsilon\delta_{ij}$ and take $\epsilon \to 0$ with the two-form B-fixed. Using $\mathrm{Tr}\, M^2 = \mathrm{Tr}(M^-)^2 + \mathrm{Tr}(M^+)^2$ and $4\mathrm{Pf}M = \mathrm{Tr}(M^-)^2 - \mathrm{Tr}(M^+)^2$, the equations (4.32) and (4.32) become in this limit

$$\delta^*\lambda_\alpha = -\frac{1}{2\pi\alpha'}\eta^*_\alpha \begin{cases} \mathrm{Pf}M + \mathcal{O}(\epsilon^2) & \text{for } \mathrm{Pf}M < 0 \\ \epsilon^2 \dfrac{\mathrm{Tr}(M^+)^2}{4\mathrm{Pf}M} + \mathcal{O}(\epsilon^4) & \text{for } \mathrm{Pf}M > 0 \end{cases} \tag{4.37}$$

$$\delta^*\bar{\lambda}_{\dot{\alpha}} = \frac{1}{2\pi\alpha'}\bar{\eta}^*_{\dot{\alpha}} \begin{cases} \mathrm{Pf}M + \mathcal{O}(\epsilon^2) & \text{for } \mathrm{Pf}M > 0 \\ \epsilon^2 \dfrac{\mathrm{Tr}(M^-)^2}{4\mathrm{Pf}M} + \mathcal{O}(\epsilon^4) & \text{for } \mathrm{Pf}M < 0\,. \end{cases} \tag{4.38}$$

For $F = 0$ and constant B, the unbroken supersymmetries are a linear combination of the original unbroken supersymmetries (4.29) and (4.30) with the supersymmetries described in the last paragraph. The parameters of the unbroken supersymmetries obey

$$\eta^*_\alpha = C^+_{ij}\sigma^{ij\beta}_\alpha \eta_\beta \tag{4.39}$$

$$\bar{\eta}^*_{\dot{\alpha}} = C^-_{ij}\bar{\sigma}^{ij\dot{\beta}}_{\dot{\alpha}} \bar{\eta}_{\dot{\beta}} \tag{4.40}$$

where

$$C^+ = \begin{cases} \dfrac{1}{2\pi\alpha'\mathrm{Pf}B} B^+ & \text{for } \mathrm{Pf}B < 0 \\ \dfrac{8\pi\alpha'\mathrm{Pf}B}{\epsilon^2\,\mathrm{Tr}(B^+)^2} B^+ & \text{for } \mathrm{Pf}B > 0 \end{cases} \tag{4.41}$$

$$C^- = \begin{cases} -\dfrac{1}{2\pi\alpha'\mathrm{Pf}B} B^- & \text{for } \mathrm{Pf}B > 0 \\ -\dfrac{8\pi\alpha'\mathrm{Pf}B}{\epsilon^2\,\mathrm{Tr}(B^-)^2} B^- & \text{for } \mathrm{Pf}B < 0\,. \end{cases} \tag{4.42}$$

The dichotomy between the two cases described above is visible here in the dependence of the $\epsilon \to 0$ limit on the sign of $\mathrm{Pf}(B)$.[11] Now we want to consider BPS configurations on $\mathbb{R}^4$, with constant B and F approaching zero at infinity. Since $F \to 0$ at infinity, an unbroken supersymmetry must be of the form (4.41) or (4.42). Without loss of generality, we examine the condition for instantons, configurations that leave invariant the positive chirality supersymmetry (4.41). The condition for this to be so is

$$M^+ = \begin{cases} C^+\mathrm{Pf}M & \text{for } \mathrm{Pf}B < 0 \\ C^+\epsilon^2 \dfrac{\mathrm{Tr}(M^+)^2}{4\mathrm{Pf}M} & \text{for } \mathrm{Pf}B > 0\,, \end{cases} \tag{4.43}$$

[11]The relation to the discussion surrounding (4.36) is somewhat obscured by the fact that we have done the scaling with $\epsilon \to 0$ rather than $B \to \infty$, as assumed in the discussion of (4.36). Taking B to infinity with g_{ij}, η, and η^* fixed is equivalent to $\epsilon \to 0$ with B-fixed and a nontrivial scaling of η and η^*. To compare most directly to the discussion surrounding (4.36), if one takes $B \to \infty$ with fixed ϵ, then according to (4.41) and (4.42), in the limit one has $C^\pm$ tending to zero or infinity depending on the sign of $\mathrm{Pf}(B)$. Here $C \to 0$ means that the unbroken supersymmetry is generated by η (the generator of the original linearly realized supersymmetry) and $C \to \infty$ means that the unbroken supersymmetry is generated instead by η^*.

with the appropriate constants C^+ in the two cases (4.41). This choice of C^+ guarantees that the equations are satisfied at infinity, where F vanishes. It turns out that these two equations are the same, and we get one condition regardless of the sign of $\mathrm{Pf}B$, namely

$$M^+ = \frac{B^+}{2\pi\alpha' \mathrm{Pf}B} \mathrm{Pf}M \tag{4.44}$$

or equivalently, since $M = 2\pi\alpha'(B + F)$,

$$F^+ = \frac{B^+}{8\mathrm{Pf}B} \epsilon^{ijkl}(2B_{ij}F_{kl} + F_{ij}F_{kl}) \,. \tag{4.45}$$

The reduction of (4.43) to (4.44) is clear enough, using (4.41), if $\mathrm{Pf}(B) < 0$. For $\mathrm{Pf}(B) > 0$, one must work a bit harder. By taking the trace squared of (4.43) in the case $\mathrm{Pf}(B) > 0$, one gets $\mathrm{Tr}(M^+)^2 = \mathrm{Tr}(C^+)^2\epsilon^4(\mathrm{Tr}(M^+)^2)^2/16\mathrm{Pf}(M)^2$, or $\mathrm{Tr}(M^+)^2 = 16\mathrm{Pf}(M)^2/\epsilon^4\,\mathrm{Tr}(C^+)^2$. Using this together with the formula in (4.41) for C^+ when $\mathrm{Pf}(B) > 0$, one finally arrives at (4.45) also for $\mathrm{Pf}(B) > 0$.

The fact that the BPS condition ends up being the same (even though the unbroken supersymmetry is completely different) for $\mathrm{Pf}(B) > 0$ or $\mathrm{Pf}(B) < 0$ looks rather miraculous from the point of view of the supersymmetric DBI theory. However, in noncommutative gauge theory, the BPS condition is that $\widehat{F}$ should be anti-selfdual in the open string metric; this condition, when mapped to commutative gauge theory by (4.12), is manifestly independent of the sign of the pfaffian.

To compare (4.45) to $\widehat{F}^+ = 0$ in greater detail, it is convenient to consider the vierbein E of (2.67). Since here we set $g_{ij} = \epsilon\delta_{ij}$, and we are only interested in the limit $\epsilon \to 0$, we have

$$E = -\frac{\sqrt{\epsilon}}{2\pi\alpha'}\frac{1}{B} \,, \tag{4.46}$$

which is finite in the limit. Given any antisymmetric tensor Λ, such as B or F, denote its selfdual projection relative to the open string metric $G = (EE^t)^{-1}$ by Λ^+_G (as before, we write simply Λ^+ and B^+ for the selfdual projections of Λ in the closed string metric g). We have, for example, $F^+_G = (E^t)^{-1}(E^tFE)^+E^{-1}$, the idea being that to compute F^+_G, we first map F to a local orthonormal frame by $F \to E^tFE$, then take the ordinary selfdual projection, and then map back to the original frame using the inverse vierbein. Antisymmetric bi-vectors like $\frac{1}{B}$ are treated similarly except that we use $(E^t)^{-1}$ instead of E, so $\left(\frac{1}{B}\right)^+_G = E\left(E^{-1}\frac{1}{B}(E^t)^{-1}\right)^+ E^t$. Since the vierbein is proportional to $\frac{1}{B}$, we first derive a useful identity. We do that in a frame where B has special form:

$$B = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix} . \tag{4.47}$$

Then, for any

$$F = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix} \tag{4.48}$$

we find

$$\begin{aligned}
&\left(\frac{1}{B^t}F\frac{1}{B}\right)^{+} = \\
&= \begin{pmatrix} 0 & \frac{1}{b_1^2}F_{12} & \frac{1}{b_1b_2}F_{24} & -\frac{1}{b_1b_2}F_{23} \\ -\frac{1}{b_1^2}F_{12} & 0 & -\frac{1}{b_1b_2}F_{14} & \frac{1}{b_1b_2}F_{13} \\ -\frac{1}{b_1b_2}F_{24} & \frac{1}{b_1b_2}F_{14} & 0 & \frac{1}{b_2^2}F_{34} \\ \frac{1}{b_1b_2}F_{23} & -\frac{1}{b_1b_2}F_{13} & -\frac{1}{b_2^2}F_{34}0 & \end{pmatrix}^{+} \\
&= \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{b_1^2}F_{12}+\frac{1}{b_2^2}F_{34} & \frac{1}{b_1b_2}(F_{24}-F_{13}) & -\frac{1}{b_1b_2}(F_{23}+F_{14}) \\ -\frac{1}{b_1^2}F_{12}-\frac{1}{b_2^2}F_{34} & 0 & -\frac{1}{b_1b_2}(F_{14}+F_{23}) & \frac{1}{b_1b_2}(F_{13}-F_{24}) \\ -\frac{1}{b_1b_2}(F_{24}-F_{13}) & \frac{1}{b_1b_2}(F_{14}+F_{23}) & 0 & \frac{1}{b_2^2}F_{34}+\frac{1}{b_1^2}F_{12} \\ \frac{1}{b_1b_2}(F_{23}+F_{14}) & -\frac{1}{b_1b_2}(F_{13}-F_{24}) & -\frac{1}{b_2^2}F_{34}-\frac{1}{b_1^2}F_{12} & 0 \end{pmatrix} \\
&= -\frac{1}{b_1b_2}F^{+} + \frac{b_2F_{12}+b_1F_{34}}{b_1^2b_2^2}\left(\frac{b_1+b_2}{2}\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
&= -\frac{1}{\mathrm{Pf}B}F^{+} + \frac{\epsilon^{ijkl}B_{ij}F_{kl}}{4(\mathrm{Pf}B)^2}B^{+}\,.
\end{aligned} \tag{4.49}$$

We conclude that

$$\left(\frac{1}{B^t}F\frac{1}{B}\right)^{+} = -\frac{1}{\mathrm{Pf}B}F^{+} + \frac{\epsilon^{ijkl}B_{ij}F_{kl}}{4(\mathrm{Pf}B)^2}B^{+} \tag{4.50}$$

for any antisymmetric B, not necessarily of the form (4.47). Using the explicit form of E in the present case (4.46) and equation (4.50), one computes

$$\begin{aligned}
F_G^{+} &= (E^t)^{-1}\left(E^tFE\right)^{+}E^{-1} = \mathrm{Pf}E(E^t)^{-1}\left(-F^{+} + \frac{B^{+}}{4\mathrm{Pf}B}\epsilon^{ijkl}B_{ij}F_{kl}\right)E^{-1}\,, \\
B_G^{+} &= (E^t)^{-1}\left(E^tBE\right)^{+}E^{-1} = \mathrm{Pf}E(E^t)^{-1}B^{+}E^{-1}\,, \\
G\left(\frac{1}{B}\right)_G^{+}G &= (E^t)^{-1}\left(E^{-1}\frac{1}{B}(E^t)^{-1}\right)^{+}E^{-1} = -\frac{\sqrt{\det G}}{\mathrm{Pf}B}B_G^{+}\,,
\end{aligned} \tag{4.51}$$

where the second equation is derived by substituting B for F in the first equation, and the third equation is derived by writing E and G in terms of B and using the second equation. Using (4.51) equation (4.45) becomes

$$F_G^{+} = -\frac{1}{8\mathrm{Pf}B}B_G^{+}\epsilon^{ijkl}F_{ij}F_{kl} = \frac{1}{8\sqrt{\det G}}G\left(\frac{1}{B}\right)_G^{+}G\epsilon^{ijkl}F_{ij}F_{kl} = \frac{1}{4}G\theta_G^{+}GF\widetilde{F}, \tag{4.52}$$

which is the same as (3.16).

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We note that the $\epsilon \to 0$ limit of the Dirac-Born-Infeld action (4.23) has an infinite power series expansion in F, but the condition for a BPS configuration (4.45) or (4.52) is polynomial in F and in θ. Since the action (4.1) and the supersymmetry transformation laws (4.29) and (4.32) are exact when derivatives of F are neglected, the same is true in the limit $\epsilon \to 0$. Therefore, the action (4.23) and the BPS condition (4.45) or (4.52) are exact when such derivatives are neglected. We matched the action (4.23) with the noncommutative action (2.36) and the BPS condition (4.52) with the noncommutative BPS condition (3.16) at leading order in F, but it is now clear that (4.52) should agree with (3.16) to all orders in F, if derivatives of F are neglected.

Classical solution of (4.52). We conclude this section with a computation that is really only offered for entertainment, as there is no solid basis for interpreting it. Nekrasov and Schwarz [35] found, using the noncommutative ADHM equations, an explicit rank one solution (with instanton number one) of the noncommutative instanton equation $\widehat{F}^+ = 0$ on $\mathbb{R}^4$. What makes this interesting is that the corresponding equation $F^+ = 0$ in ordinary rank one (abelian) gauge theory has no such smooth, finite action solution on $\mathbb{R}^4$.

It is amusing to ask whether one can find the noncommutative rank one solution as a classical solution of DBI theory. There is no reason to expect this to work, since the fields in the rank one instanton are not slowly varying, and on the contrary it cannot work, since a nonsingular abelian gauge field with $F = 0$ at infinity cannot have a nonzero value of the instanton number

$$\int_{\mathbb{R}^4} d^4x F\widetilde{F} \,. \tag{4.53}$$

It is nonetheless interesting to see how far we can get. We will see that the result is as good as possible: the solution we will find of (4.52) has the mildest possible singularity compatible with a nonzero value of (4.53), and in particular has a milder singularity than an analogous solution of the linear equation $F^+ = 0$.

The equation we wish to solve is

$$F^+_{ij} = \omega^+_{ij}(F\widetilde{F}) \tag{4.54}$$

for an abelian gauge field in four dimensions, where ω is given in (4.52) in terms of B. We take $F\widetilde{F} = 4(F_{12}F_{34} + F_{14}F_{23} + F_{13}F_{42})$. There is no loss of essential generality in assuming that the nonzero components of ω are $\omega_{12} = \omega_{34} = 1$.

The noncommutative solution obtained in [35] is invariant under the U(2) subgroup of SO(4) that leaves fixed ω (or B^+). So we look for a solution of (4.54) with the same symmetry. Up to a gauge transformation, we can take

$$A_i = \omega_{ij}x^j h(r) \,, \tag{4.55}$$

where $r = \sqrt{\sum_i (x^i)^2}$. Since this is the most general U(2)-invariant ansatz, it must be compatible with the equations.

We compute

$$F_{12} = -2h - (x_1^2 + x_2^2) \cdot \frac{h'}{r}\,, \qquad F_{34} = -2h - (x_3^2 + x_4^2) \cdot \frac{h'}{r}\,,$$
$$F_{13} = F_{24} = (x_1 x_4 - x_2 x_3) \cdot \frac{h'}{r}\,, \qquad F_{23} = -F_{14} = (x_2 x_4 + x_1 x_3) \cdot \frac{h'}{r}\,, \tag{4.56}$$

with $h' = dh/dr$. It follows that

$$F\widetilde{F} = 8(2h^2 + hh'r)\,. \tag{4.57}$$

The only nontrivial component of the equation is the 1-2 component, which becomes

$$-4h - rh' = 16(2h^2 + hh'r) \tag{4.58}$$

or

$$r\frac{d}{dr}(h + 8h^2) = -4(h + 8h^2)\,. \tag{4.59}$$

Hence the solution is

$$h + 8h^2 = Cr^{-4}\,, \tag{4.60}$$

for some constant C. Note that $h \sim r^{-4}$ for $r \to \infty$, which is the correct behavior for this partial wave ("dipole") in abelian gauge theory, while $h \sim r^{-2}$ for $r \to 0$, which is the singularity of the solution. Since

$$\begin{aligned} \int d^4x F\widetilde{F} &= 16\pi^2 \int_0^\infty r^3 dr \left(2h^2 + hh'r\right) \\ &= 8\pi^2 \int_0^\infty dr \frac{d}{dr}\left(r^4 h^2\right) \\ &= -8\pi^2 \lim_{r\to 0}(r^4 h^2) = -\pi^2 C\,, \end{aligned} \tag{4.61}$$

the behavior near $r = 0$ is exactly what is needed to give the solution a finite and nonzero instanton number. This contrasts with the solution of the linear equation $F^+ = 0$ with the same symmetry; in that case, $h \sim 1/r^4$ near $r = 0$, and the instanton number is divergent.

5. *D*-branes and small instantons in the presence of constant *B* field

As we mentioned in the introduction, one of the most interesting applications of noncommutative Yang-Mills theory has been to instantons, especially small instantons [35, 38]. Noncommutative Yang-Mills has been related to the possibility of adding a Fayet-Iliopoulos term to the ADHM equations, a possibility also seen [37] in the study of small instantons via D-branes. In this section, we will reexamine the D-brane approach to small instantons in the context of the $\alpha' \to 0$ limit (2.14) with fixed open string parameters.

For this, we have to study the $D(p-4)$-Dp system for some $p \geq 3$. The first case, $p = 3$, has the advantage that quantum noncommutative super Yang-Mills is apparently well-defined in four dimensions, since it seems that in general the deformation to nonzero θ does not change the renormalization properties of Yang-Mills theory [54]–[61]. If so, the structure we will find in the small instanton problem for $\mathrm{Pf}(B) < 0$ must be already contained in the $\widehat{F}^2$ theory in the small instanton regime. However, the $D0$-$D4$ system is richer, because one has the chance to study the quantum mechanics on instanton moduli space, so we will focus on this. (It takes in any case only relatively minor modifications of the formulas to convert to $D(p-4)$-Dp for other p.)

For simplicity, we set $g_{ij} = \epsilon\delta_{ij}$. We consider a $D0$-brane embedded in the $D4$-brane. The effective hamiltonian for this case governs possible deformations to a $D0$-brane outside the $D4$-brane or a non-small instanton in the $D4$-brane. The boundary conditions on the 0-4 open strings are

$$\partial_\sigma x^0|_{\sigma=0,\pi} = \partial_\tau x^{1\ldots9}|_{\sigma=0} = \partial_\tau x^{5\ldots9}|_{\sigma=\pi} = g_{ij}\partial_\sigma x^j + 2\pi i\alpha' B_{ij}\partial_\tau x^j|_{\sigma=\pi} = 0 \qquad (5.1)$$

for $i = 1, \ldots, 4$. We bring B to a canonical form

$$B = \frac{\epsilon}{2\pi\alpha'}\begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix}. \qquad (5.2)$$

We will eventually be interested in the limit (2.14) with finite B and hence with $|b_I| \sim \epsilon^{-1/2}$. Because of the mass shell condition $\alpha' p^2/2 = N$, where p is the momentum in the "time" direction common to the $D0$ and $D4$, and N is the worldsheet hamiltonian for the string oscillators, we see that states of finite (spacetime) energy must have

$$N \sim \alpha' \sim \frac{1}{|b|} \qquad (5.3)$$

for $\epsilon \to 0$. Excitations with higher energy than this are not part of the limiting theory obtained in the zero slope limit.

As in section 4.2, the qualitative properties of the $\epsilon \to 0$ limit will depend on the sign of $\mathrm{Pf}(B)$. For large b, the boundary conditions (5.1) become Dirichlet boundary conditions at both ends. Those are the boundary conditions of both a supersymmetric $D0$-$D0$ system, and a nonsupersymmetric $D0$-$\overline{D0}$ system. We will see that for $\mathrm{Pf}(B) > 0$, the $D0$-$D4$ system behaves like $D0$-$\overline{D0}$, and for $\mathrm{Pf}(B) < 0$, it behaves like $D0$-$D0$. Intuitively, this is because the induced instanton number on a $D4$-brane with a B-field is proportional to $-\frac{1}{2}\int \mathrm{Pf}(B)$, so the $D4$-brane carries $D0$ charge if $\mathrm{Pf}(B) < 0$, and $\overline{D0}$ charge if $\mathrm{Pf}(B) > 0$.

We define $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ and express the boundary conditions on x^i as

$$\begin{aligned} \partial_\tau z_I\big|_{\sigma=0} &= \partial_\sigma z_I + b_I \partial_\tau z_I\big|_{\sigma=\pi} = 0 \\ \partial_\tau \bar{z}_I\big|_{\sigma=0} &= \partial_\sigma \bar{z}_I - b_I \partial_\tau \bar{z}_I\big|_{\sigma=\pi} = 0\,. \end{aligned} \tag{5.4}$$

A boson z with boundary conditions (5.4) has a mode expansion

$$\begin{aligned} z &= i \sum_n \left(e^{(n+\nu)(\tau+i\sigma)} - e^{(n+\nu)(\tau-i\sigma)}\right) \frac{\alpha_{n+\nu}}{n+\nu} \\ \bar{z} &= i \sum_n \left(e^{(n-\nu)(\tau+i\sigma)} - e^{(n-\nu)(\tau-i\sigma)}\right) \frac{\bar{\alpha}_{-n+\nu}}{n-\nu} \end{aligned} \tag{5.5}$$

with

$$e^{2\pi i\nu} = -\frac{1+ib}{1-ib}\,, \qquad 0 \leq \nu < 1\,. \tag{5.6}$$

With a euclidean worldsheet, $\bar{z}$ is not the complex conjugate of z because the boundary conditions (2.2) are not compatible with real $x^{1,2,3,4}$. With lorentzian signature, $\tau = it$ and $\bar{\alpha}$ is the complex conjugate of α in (5.5).

We find

$$\nu \approx \begin{cases} -\dfrac{1}{\pi b} & \text{for } b \to -\infty \\ \dfrac{1}{2} + \dfrac{b}{\pi} & \text{for } b \approx 0 \\ 1 - \dfrac{1}{\pi b} & \text{for } b \to +\infty \end{cases} \tag{5.7}$$

so ν changes from 0 to 1 as b changes from minus infinity to infinity. The complex boson ground state energy is

$$E(\nu) = \frac{1}{24} - \frac{1}{2}\left(\nu - \frac{1}{2}\right)^2. \tag{5.8}$$

It is invariant under $\nu \to 1-\nu$, as it should be because this exchanges the oscillators of z with those of $\bar{z}$ in (5.5).

We are interested in the spectrum of the 0-4 strings. In the R sector the ground state energy is zero, and we find massless fermions. In the NS sector the ground state energy is

$$\begin{aligned} E_t = 3E(0) + E(\nu_1) + E(\nu_2) - E(0) - 3E\left(\frac{1}{2}\right) - \\ - E\left(\left|\nu_1 - \frac{1}{2}\right|\right) - E\left(\left|\nu_2 - \frac{1}{2}\right|\right) + E\left(\frac{1}{2}\right), \end{aligned} \tag{5.9}$$

where the first term is from $x^{0,5,\ldots,9}$, the second and third from $z_{1,2}$, and the fourth from the bosonic ghosts. The other terms arise from the corresponding fermions, whose energies differ by 1/2 from the corresponding bosons — they are $n \pm |\nu - 1/2|$. From (5.9) we find

$$E_t = -\frac{1}{2}\left(\left|\nu_1 - \frac{1}{2}\right| + \left|\nu_2 - \frac{1}{2}\right|\right). \tag{5.10}$$

The four lowest energy states are the ground state and three states obtained by acting with the fermion creation operators with energies $|\nu_I - 1/2|$ on the ground state. Their energies are

$$\pm\frac{1}{2}\left(\nu_1 - \frac{1}{2}\right) \pm \frac{1}{2}\left(\nu_2 - \frac{1}{2}\right). \tag{5.11}$$

Of these the states with energies

$$E_{\pm}^{+} = \pm\frac{1}{2}\left(\nu_1 + \nu_2 - 1\right) \tag{5.12}$$

have one sign of $(-1)^F$, while those with energies

$$E_{\pm}^{-} = \pm\frac{1}{2}\left(\nu_1 - \nu_2\right), \tag{5.13}$$

have the opposite sign. The GSO projection projects out one of these pairs. Which pair is being projected out depends on whether we study D0-branes or anti-D0-branes. (This is a general feature of D-brane physics, exploited for instance in [62]: strings ending on $D0$-branes or $\overline{D0}$-branes have the same boundary condition but opposite GSO projection.) We use the conventions that for D0-branes we keep the states with energies $E_{\pm}^{+}$ and for anti-D0-branes the states with energies $E_{\pm}^{-}$. With this choice, a D0-brane has instanton number +1.

For D0-branes

$$E_{\pm}^{+} = \pm\frac{1}{2}(\nu_1 + \nu_2 - 1) \approx \pm\frac{1}{2\pi}\begin{cases} b_1 + b_2 & \text{for } b_1, b_2 \approx 0 \\ \pi - \left|\dfrac{1}{b_1} + \dfrac{1}{b_2}\right| & \text{for } |b_I| \to \infty \text{ with } \mathrm{Pf}(B) > 0 \\ \dfrac{1}{b_1} + \dfrac{1}{b_2} & \text{for } |b_I| \to \infty \text{ with } \mathrm{Pf}(B) < 0. \end{cases} \tag{5.14}$$

For small b_I, a tachyon appears, meaning that turning on a small $B \neq 0$ perturbs the small instanton problem, adding a Fayet-Iliopoulos term to the ADHM equations. This is in keeping, at least qualitatively, with the proposal in [37] that the B-field should add this term to the low energy physics. The tachyon mass squared is small for small b_I, so the $D0$-$D4$ system is almost on shell and gives a reliable description of small instanton behavior in this range. For generic b_I, there is still a tachyon in the $D0$-$D4$ system, suggesting that the instanton moduli space has no small instanton singularity, but since the tachyon mass squared is not small, the $D0$-$D4$ system is significantly off shell, and it is hard to use it for a quantitative study of instantons.

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Now let us consider the $\alpha' \to 0$ limit, or more precisely the two cases of $|b_I| \to \infty$, with $\mathrm{Pf}(B) > 0$ or $\mathrm{Pf}(B) < 0$. Here we may hope for a more precise description.

For $\mathrm{Pf}(B) > 0$, the $D0$-$D4$ system is tachyonic. The tachyon mass squared, in units of $1/\alpha'$, is of order 1, since $E^+_\pm$ is of order 1, so the tachyon "mass" is of order $1/\sqrt{\alpha'}$. The tachyon mass squared is actually that of the standard $D0$-$\overline{D0}$ tachyon, and we interpret the tachyon to mean that the $D0$ can annihilate one of the induced $\overline{D0}$-branes in the $D4$, along the lines of [62]. Thus, the $D0$-$D4$ system is an excitation of the $D4$ system. Its excitation energy is much too big to obey (5.3). In fact, the tachyon mass squared gives an estimate of the excitation energy required to deform a $D4$-brane to a $D0$-$D4$ system (with an extra induced $\overline{D0}$-brane in the $D4$), so this excitation energy corresponds to $N = |E^+_\pm| \sim 1$. Thus, although the $D0$-$D4$ system is a possible excitation of string theory, it is not one of the excitations of the $D4$-brane that survives in our favorite $\alpha' \to 0$ limit (2.14) of the open string theory. Hence, in particular, the $D0$-$D4$ system is not part of the physics that will be described by the $\widehat{F}^2$ theory if $\mathrm{Pf}(B) > 0$. And instanton moduli space of the $\widehat{F}^2$ theory should have no small instanton singularity (which would be governed presumably by a $D0$-$D4$ system) if $\mathrm{Pf}(B) > 0$. This last statement is in agreement with the analysis in [35], where θ (and hence B) was assumed to be self-dual, and the instanton moduli space was found, using a noncommutative ADHM transform, to have no small instanton singularity.

For $\mathrm{Pf}(B) < 0$, the situation is completely different. There is still a tachyon for generic B, but $N = |E^+_\pm| \sim 1/|b| \sim \alpha'$, which means that (5.3) is obeyed. Hence, for $\mathrm{Pf}(B) < 0$, the $D0$-$D4$ still represents (for generic B) an excitation of the $D4$-brane with a positive excitation energy, but the energy of this excitation scales correctly so that it survives as part of the limiting theory for $\alpha' \to 0$. Hence, the $D0$-$D4$ system is part of the physics that the limiting $\widehat{F}^2$ theory should describe. Moreover, if $\mathrm{Pf}(B) < 0$, it is possible to have $B^+ = 0$, in which case, since $b_1 = -b_2$, the tachyon mass squared vanishes. In this particular case, the $D0$-$D4$ system is supersymmetric and BPS and should represent a point on noncommutative instanton moduli space. Thus, the moduli space of noncommutative instantons should have a small instanton singularity precisely if $B^+ = 0$. The small instantons at or very near $B^+ = 0$ should be described by the $D0$-$D4$ system and the associated ADHM equations; in this description, the Fayet-Iliopoulos term vanishes, and the small instanton singularity appears, if $B^+ = 0$.

Let us now examine the excitation spectrum of the $D0$-$D4$ system for negative $\mathrm{Pf}(B)$ as $|b_I| \to \infty$. As we will see, this system has, in addition to the ground state, excited states that are also part of the limiting theory for $\alpha' \to 0$. We assume, without loss of generality, that $b_1 > 0$ and $b_2 < 0$ (their product is negative because $\mathrm{Pf}(B)$ is negative). Therefore $\nu_1 \approx 1 - 1/\pi b_1$ and $\nu_2 \approx -1/\pi b_2$.

Before the GSO projection, the lowest energy state $|0\rangle$ has energy:

$$E_-^- = -\frac{1}{2}(\nu_1 - \nu_2) \approx -\frac{1}{2} + \frac{1}{2\pi b_1} - \frac{1}{2\pi b_2}\,. \tag{5.15}$$

Four low energy states that survive the GSO projection are obtained by acting on $|0\rangle$ with the lowest oscillators of the fermionic partners of z_I and $\bar{z}_I$, whose energies are $\nu_1 - \frac{1}{2} \approx \frac{1}{2} - \frac{1}{\pi b_1}$, $\frac{3}{2} - \nu_1 \approx \frac{1}{2} + \frac{1}{\pi b_1}$, $\nu_2 + \frac{1}{2} \approx \frac{1}{2} - \frac{1}{\pi b_2}$, $\frac{1}{2} - \nu_2 \approx \frac{1}{2} + \frac{1}{\pi b_2}$. The resulting four states have energies

$$\begin{aligned} E_\pm^+ &\approx \pm\left(\frac{1}{2\pi b_1} + \frac{1}{2\pi b_2}\right) \\ E_1^+ &\approx \frac{3}{2\pi b_1} - \frac{1}{2\pi b_2} \\ E_2^+ &\approx \frac{1}{2\pi b_1} - \frac{3}{2\pi b_2}\,. \end{aligned} \tag{5.16}$$

Six more states are obtained by acting on $|0\rangle$ with the lowest oscillators of the fermionic partners of $x^{0,5,\ldots,9}$, whose energy is $1/2$. These states have energy

$$E_i^+ \approx \frac{1}{2\pi b_1} - \frac{1}{2\pi b_2} \qquad \text{for } i = 0, 5, \ldots, 9. \tag{5.17}$$

Of these ten states the lowest are the first two, which have already been mentioned above. The other eight states have larger energies, but since they have $N = E \sim 1/|b| \sim \alpha'$, they obey (5.3) and survive as part of the limiting quantum mechanics for $\alpha' \to 0$.

We can also act on any one of these states with an arbitrary polynomial in the lowest bosonic creation operator in z_1 and in $\bar{z}_2$, whose energies are $1-\nu_1 \approx 1/\pi b_1$ and $\nu_2 \approx -1/\pi b_2$. These states again have low enough energy to survive in the $\alpha' \to 0$ limit. Of the states just described, some will obey the physical state conditions, and some will be projected out of the spectrum.

The existence of this large number of light states can be simply understood as follows. For infinite $|b_I|$ with negative $\mathrm{Pf}(B)$, our string is effectively stretched between two D0-branes. One $D0$-brane has a fixed position at the origin, but the second $D0$-brane can be anywhere in the $D4$ (since a $D4$ with a strong B field of $\mathrm{Pf}(B) < 0$ contains a continuous distribution of $D0$'s). The fluctuation in position of the second $D0$-brane are given by the $n = 0$ bosonic oscillators in (5.5). Note that for $n = 0$, as $|b| \to \infty$ or $\nu \to 0$, the exponential factor $(\exp((n+\nu)(\tau+i\sigma)) - \exp((n+\nu)(\tau-i\sigma)))\,/(n+\nu)$ in (5.5) becomes a simple linear function of σ, describing a straight string connecting the $D0$-brane at the origin with an arbitrary point in the $D4$. The fluctuation in the free endpoint of the straight string is governed by an effective hamiltonian which is that of a charged particle in a magnetic field in its lowest Landau level. In section 6, we revisit these low-lying excitations (and their analogs in other cases) and use them to construct modules for the ring of functions on a noncommutative space.

We would like to interpret the low lying $D0$-$D4$ excitation spectrum that we have found as corresponding to small fluctuations around a point-like instanton. In ordinary Yang-Mills theory, the fluctuations about an instanton solution are given by eigenfunctions of a small fluctuation operator that one might describe as a generalized laplacean. The eigenfunctions depend on all four coordinates of $\mathbb{R}^4$. In noncommutative Yang-Mills theory, the fluctuations about an instanton should be described by a noncommutative analog of a Laplace operator. We do not know how to explicitly describe the appropriate operator directly, especially in the small instanton limit. But we believe that near $B^+ = 0$, for perturbing around a small instanton, its spectrum is given by the states we have just described in the $D0$-$D4$ system. These states are naturally regarded as functions of just two of the four spacetime coordinates, something which at least intuitively is compatible with noncommutativity of the spatial coordinates. For instance, charged particles in a constant magnetic field in the first Landau level have wave functions that are functions of just half of the coordinates. In fact, their wave functions are the functions of two bosons we found above.

Quantitative analysis of FI coupling. Let us consider the more general case of k D4-branes and N D0-branes. The effective theory of $D0$-branes has eight supersymmetries. From the 0-0 strings, we get a $\mathrm{U}(N)$ gauge group and (in a language making manifest half of the supersymmetry) two chiral superfields X and Y in the adjoint representation of $\mathrm{U}(N)$. Quantization of the 0-4 strings gives two chiral superfields q and p in the fundamental of $\mathrm{U}(N)$. Their lagrangean includes a potential proportional to

$$\mathrm{Tr}\left\{\left([X, X^\dagger] + [Y, Y^\dagger] + qq^\dagger - p^\dagger p - \zeta\right)^2 + |[X, Y] + qp|^2\right\}, \tag{5.18}$$

where ζ is an FI term, determined by the existence of a tachyon mass in quantizing the 0-4 strings. In fact, for nonzero ζ the spectrum of the theory at the origin includes a tachyon, which should match the tachyon found in the 0-4 spectrum (at least when ζ is small and the $D0$-$D4$ system is almost stable). The space of zeros of the potential (5.18) does not include the origin in this case. This means that the pointlike instanton, corresponding to $X = Y = q = p = 0$, does not exist when $\zeta \neq 0$.

Comparing (5.18) at $X = Y = q = p = 0$ with the analysis above of the energy of 0-4 strings, we see that for $b_I \approx 0$ the spectrum of the theory is consistent with

$$\zeta^2 \sim B^+_{ij} B^{+ij}\,. \tag{5.19}$$

We are more interested in the limit $|b_I| \sim \epsilon^{-1/2} \to \infty$. For positive $\mathrm{Pf}(B)$, the tachyon mass squared $m^2 = -E^+/\alpha' = -1/2\alpha' \sim \epsilon^{-1/2}$ diverges, signaling a strong instability and leading us to propose that in the $\alpha' \to 0$ limit, for $\mathrm{Pf}(B) = 0$, the small instanton is not part of the physics described by the $\widehat{F}^2$ theory. For negative $\mathrm{Pf}(B)$ the tachyon mass squared

$$m^2 = -\frac{E^+}{\alpha'} = -\frac{1}{2\alpha'}|\nu_1+\nu_2-1| \approx -\frac{1}{2\pi\alpha'}\left|\frac{1}{b_1}+\frac{1}{b_2}\right| = -\frac{(\det g)^{1/4}}{(2\pi\alpha')^2(\det G)^{1/4}}|\theta_G^+| \tag{5.20}$$

scales correctly so that this tachyon, and the $D0$-$D4$ system whose instability it describes, can be part of the physics described by the $\widehat{F}^2$ theory. In (5.20), we used (4.51) to express m^2 in terms of $|\theta_G^+|$, where

$$|\theta_G^+|^2 = -\operatorname{Tr} G\theta_G^+ G\theta_G^+ \,. \tag{5.21}$$

This relation leads us to identify the FI term

$$\zeta \sim |\theta_G^+| \,. \tag{5.22}$$

The space of zeros of (5.18) is, according to [35], the moduli space of instantons on noncommutative $\mathbb{R}^4$. Our identification of ζ in terms of the B-field and hence as the noncommutativity parameter θ gives an independent derivation of this fact.

The moduli space of instantons depends only on ζ, which in our limit is proportional to $|\theta_G^+|$. This means that if $\theta_G^+ = 0$, the moduli space of noncommutative instantons is not deformed from its value at $\theta = 0$, and includes the singularity of point-like instantons. When $\theta_G^- \neq 0$ but $\theta_G^+ = 0$, the underlying space is noncommutative, and the instanton solutions depends on θ, but the moduli space of instantons is independent of θ.

The removal of the small instanton singularity for generic θ seems, intuitively, to be in accord with the idea that the spacetime coordinates do not commute for $\theta \neq 0$, and hence an instanton cannot be completely localized. However, this line of thought cannot be pushed too far, since the small instanton singularity is present for $\theta^+ = 0$.

Since B^+ induces negative $D0$-brane charge and instantons are $D0$-branes, we found an instability when the $D0$-branes are point-like that prevents them from separating from the $D4$-branes. This suggests a relation between the problem of instantons on a noncommutative space and K-theory.

Instanton moduli space depends only on θ^+. By now we have seen several indications that the moduli space of noncommutative instantons depends only of θ^+, not θ^-. One indication is that the explicit equation (3.16) describing the first noncommutative correction to the instanton equation (for small θ or F) depends only on θ^+. Also, by determining the FI term from the $D0$-$D4$ system, we have just obtained a more general argument that the moduli space depends only on θ_G^+. We now want to show that this conclusion can actually be obtained using the methods in [35] for direct analysis of noncommutative instantons via the ADHM construction.

In effect, this gives a direct mathematical argument for identifying the FI term with θ^+_G. (In our terminology, the analysis in [35] is done in the open string metric G — the right metric for the $\widehat{F}^2$ action that they work with — and in extending their reasoning below, we work entirely in this metric. We will work in coordinates in which the metric G is δ_{ij}.)

Given any θ^{ij} on $\mathbb{R}^4$, we can always pick a complex structure on $\mathbb{R}^4$, with complex coordinates z_0 and z_1, such that the nonzero commutators are

$$\begin{aligned} [z_0, \bar{z}_0] &= -\zeta_0 \\ [z_1, \bar{z}_1] &= -\zeta_1. \end{aligned} \tag{5.23}$$

This can be done in such a way that the instanton equation says that $\widehat{F}^{2,0} = \widehat{F}^{0,2} = 0$ (that is, the $(2,0)$ and $(0,2)$ parts of $\widehat{F}$ vanish) and $\widehat{F}_{0\bar{0}} + \widehat{F}_{1\bar{1}} = 0$. (The complex structure with these properties is the one for which the metric G is Kähler, and θ is of type $(1,1)$.)

In the study of noncommutative instantons in [35], only the case $\zeta_0 = \zeta_1 = \zeta/2$ (or in other words $\theta^- = 0$) was considered. But it is straightforward to repeat the computation in greater generality and show that the moduli space only depends on the sum $\zeta_0 + \zeta_1$, or in other words it only depends on θ^+. The key step in [35] is the verification of the second part of eqn. (3.6) of that paper (namely $\tau_z\tau_z^\dagger = \sigma_z^\dagger\sigma_z$), where the τ's and σ's were defined earlier in their eqn. (2.2). The classical ADHM construction is formulated in eqn (2.1) of [35], with $\mu_r = \mu_c = 0$, and assumes that the z's commute. If one wants to turn on nonzero μ's, one can (by a rotation) assume that $\mu_c = 0$ and turn on only μ_r. The basic idea in [35] is that if $\mu_r \neq 0$, one can compensate for this by letting the z's and $\bar{z}$'s no longer commute. It is shown in this paper that in verifying the key equation $\tau_z\tau_z^\dagger = \sigma_z^\dagger\sigma_z$, a c-number term coming from the commutator of a z and a $\bar{z}$ can cancel the contribution of μ_r. Indeed, the term coming from the $[z, \bar{z}]$ commutators is in general $[z_0, \bar{z}_0] + [z_1, \bar{z}_1]$, so the value of μ_r that one needs depends only on the sum $\zeta_0 + \zeta_1$. In particular, the moduli space of noncommutative instantons, as determined from the ADHM construction, only depends on the sum $\zeta = \zeta_0 + \zeta_1$, as we wished to show.

Although the moduli space only depends on ζ, the instanton solutions themselves (whose construction is explained in [35]) depend on both ζ_0 and ζ_1. Indeed, the components of the instanton connection take values in an algebra that depends on both ζ_0 and ζ_1, so it is hard to compare gauge fields for different ζ's. (There is a notion of background independence for noncommutative gauge fields, which we discuss in section 3.2, but it involves a transformation that adds a constant to the curvature measured at infinity, so it is not directly relevant to comparing instantons on $\mathbb{R}^4$ with different values of the ζ's.)

6. Noncommutative gauge theory on a torus

In this section we consider D-branes compactified on a p-torus $\mathbf{T}^p$ in our usual limit of $\alpha' \to 0$ with the open string parameters G, θ fixed. Surprisingly, the effective field theory based on the $\widehat{F}^2$ action inherits the T-duality symmetry of the underlying string theory. This is surprising because without the B-field, the effective theory based on the F^2 action is not invariant under T-duality.

This T-duality appeared in the mathematical literature as Morita equivalence of different modules. In the physics literature it was explored in [4, 17]–[21] using the DLCQ description of M-theory. We will devote section 6.1 to deriving this duality using our point of view. It will not involve the DLCQ description of M-theory, but instead, will use the zero slope limit. (In section 7 we will explain how these two approaches are related.) A crucial element of our discussion will be the difference between the closed string parameters g, B and g_s and the open string parameters G, θ and G_s.

It is important that unlike T-duality of string theory, which is the reason for T-duality of these theories, here we do not relate a theory on a torus to a theory on the dual torus. In particular, we do not have the standard exchange of momentum modes and winding modes. Since we are studying open strings there are no winding modes. Instead, in open string theory T-duality has the effect of changing the dimensionality of the D-branes. Therefore, this T-duality acts on the D-brane charges. It changes the rank of the gauge fields and their topological charges.

In subsections 6.2–6.4, we will interpret many mathematical results about construction of modules over a noncommutative torus and Morita equivalences between them in terms of standard techniques of quantizing open strings.

6.1 T-duality

We start by deriving the T-duality of the theories using our point of view. We consider Dp-branes on T^p parametrized by $x^i \sim x^i + 2\pi r$ with the closed string metric g. The periods of the B field are $(2\pi r)^2 B$ and this motivates us to express the noncommutativity in terms of the dimensionless matrix $\Theta = 1/2\pi r^2 \theta$.

The $\mathrm{SO}(p,p,\mathbb{Z})$ T-duality group is represented by the matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.1}$$

where a, b, c and d are $p \times p$ matrices with integer entries. T satisfies

$$T^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.2}$$

and therefore

$$c^t a + a^t c = 0\,, \qquad d^t b + b^t d = 0\,, \qquad c^t b + a^t d = 1\,. \tag{6.3}$$

T acts on

$$E = \frac{r^2}{\alpha'}(g + 2\pi\alpha' B) \tag{6.4}$$

and the string coupling g_s as

$$E' = (aE + b)\frac{1}{cE + d} \tag{6.5}$$

$$g_s' = g_s \left(\frac{\det g'}{\det g}\right)^{1/4} . \tag{6.6}$$

Using (6.3), (6.4) and (6.5)

$$g' = \frac{\alpha'}{2r^2}(E' + (E')^t) = \frac{1}{(cE+d)^t}\, g\, \frac{1}{cE+d}\,, \tag{6.7}$$

and (6.6) becomes

$$g_s' = \frac{g_s}{\det(cE + d)^{1/2}}\,. \tag{6.8}$$

We are interested in the action of T on the open string parameters G, Θ and G_s. For simplicity we ignore the more general variables (3.19) including Φ. The role of Φ in T-duality was elucidated in [21]. Using

$$\frac{1}{E} = \frac{\alpha'}{r^2}G^{-1} + \Theta \tag{6.9}$$

and (6.3) we find

$$G' = \frac{2\alpha'}{r^2}\left(\frac{1}{E'} + \frac{1}{(E')^t}\right)^{-1} = (a + bE^{-1})G(a + bE^{-1})^t\,. \tag{6.10}$$

Similarly

$$\Theta' = \frac{1}{2}\left(\frac{1}{E'} - \frac{1}{(E')^t}\right) = \left[(c + dE^{-1})\frac{1}{a + bE^{-1}}\right]_A\,, \tag{6.11}$$

where $[\;]_A$ denotes the antisymmetric part. Finally,

$$G_s = g_s\left(\frac{\det\left(\frac{\alpha'}{r^2}E\right)}{\det g}\right)^{1/2} \tag{6.12}$$

transforms to

$$G_s' = G_s\left[\det(a + bE^{-1})\right]^{1/2}, \tag{6.13}$$

and therefore $\sqrt{\det G}/G_s^2$ is T-duality invariant. From (6.13) we find the transformation law of the Yang-Mills gauge coupling

$$g_{YM}' = g_{YM}\left[\det(a + bE^{-1})\right]^{1/4}, \tag{6.14}$$

In the zero slope limit with finite r we have $E^{-1} \approx \Theta$ and (6.10), (6.11), (6.14) become

$$\begin{aligned} G' &= (a+b\Theta)G(a+b\Theta)^t \\ \Theta' &= (c+d\Theta)\frac{1}{a+b\Theta} \\ g'_{YM} &= g_{YM}\left[\det(a+b\Theta)\right]^{1/4} \end{aligned} \tag{6.15}$$

(it is easy to check using the antisymmetry of Θ and the relations (6.3) that Θ' is antisymmetric). These transformation rules have already appeared in the mathematical literature. In the physics literature they appeared in [4, 17]–[21]. Our expressions (6.15) are similar to those in these papers but differ by $a \leftrightarrow d$ and $b \leftrightarrow c$; i.e. by conjugation by $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In section 7 we will explain the origin for this difference.

We note that the transformation of the metric G is unlike the typical T-duality transformation of a metric (like (6.7)). Since it is linear in G, for every T-duality transformation the transformed metric G' scales like the original metric G. If one of them scales to zero, so does the other one; for example, there is no transformation which maps G to G^{-1}. Although for closed strings there are transformations which map momentum modes to winding modes, this is not true for the open strings we consider; yet the theory is T-duality invariant!

If some of the components of Θ are rational, they can be transformed to zero. For example, if $\Theta = -d^{-1}c$, where c and d have integer entries, there is a T dual description with $\Theta' = 0$. It is given by $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with appropriate a and b. Θ' vanishes because (using (6.3)) $a + b\Theta = a - bd^{-1}c = (d^t)^{-1}$ is invertible. This also guarantees that the transformed metric $G' = d^{-1}G(d^{-1})^t$ and Yang-Mills coupling constant $g'_{YM} = g'_{YM}(\det d)^{-1/4}$ are finite. In this case the noncommutative theory with nonzero Θ is T dual to a commutative theory. The volume of the torus of this dual commutative theory is smaller by a factor of $\det d$ relative to the original torus. More generally, if only some of the components of Θ are rational some of the coordinates could be transformed to commuting coordinates.

There is another point we should mention about the case with rational Θ. Then, there exist T-duality transformations for which (6.15) is singular. In particular, for $a + b\Theta = 0$ the transformed G vanishes and the transformed Θ diverges. One way to understand it is to first use a T-duality transformation, as above, to transform to $\Theta = 0$. Then, all the transformations with $a = 0$ are singular. They include the transformation with $a = d = 0$, $b = c = 1$ which can be referred to as "T-duality on all sides of the torus." More generally, if only some of the components of Θ are rational, there exist T-duality transformations to Θ with infinite entries and to G with vanishing eigenvalues. We conclude that when Θ is rational not all the $\mathrm{SO}(p, p; \mathbb{Z})$ duality group acts.

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This discussion becomes more clear for the simplest case of the two torus. Then, the T-duality group is $\mathrm{SO}(4,4;\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. One $\mathrm{SL}(2,\mathbb{Z})$ factor acts geometrically on G leaving Θ, g_{YM} and the volume V of the torus with the metric G unchanged. The other $\mathrm{SL}(2,\mathbb{Z})$ acts as

$$\begin{aligned} V' &= V(a+b\Theta)^2\,, \\ \Theta' &= \frac{c+d\Theta}{a+b\Theta}\,, \\ g'_{YM} &= g_{YM}(a+b\Theta)^{1/2}\,, \end{aligned} \tag{6.16}$$

where now Θ, a, b, c and d are numbers, rather than matrices as above, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an $\mathrm{SL}(2,\mathbb{Z})$ matrix. We again see that when $\Theta = -c/d$ is rational it can be transformed to $\Theta' = 0$ by an $\mathrm{SL}(2,\mathbb{Z})$ transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with appropriate a and b. For such a transformation $a + b\Theta = 1/d$, and therefore $V' = V/d^2$ and $g'_{YM} = g_{YM}/d^{1/2}$ are finite. However, the transformation $\left(\begin{smallmatrix} c & d \\ -a & -b \end{smallmatrix}\right)$ does not act regularly.

The fact that for rational $\Theta = -c/d$ the theory is equivalent to another theory on a commutative torus whose volume is smaller by a factor of d^2 can be understood as follows. Before the T-duality transformation the torus is parametrized by $x^i \sim x^i + 2\pi r$. The algebra of functions on the torus is generated by $U_i = e^{ix^i/r}$, which satisfy $U_1U_2 = e^{-2\pi i\Theta}U_2U_1$. The T-duality transformation does not act on the complex structure of the torus but affects its volume. This can be achieved by rescaling the two coordinates x^i such that the identification is $x^i \sim x^i + 2\pi r/d$, thus reducing the volume by a factor of d^2. Now the algebra of functions on the torus is generated by $\widetilde{U}_i = e^{ix^i d/r} = U_i^d$. They satisfy $\widetilde{U}_1\widetilde{U}_2 = e^{-2\pi i d^2\Theta}\widetilde{U}_2\widetilde{U}_1 = \widetilde{U}_2\widetilde{U}_1$; i.e. the new torus is commutative.

We stress that this transformation to an ordinary theory on another torus is unrelated to the transformation to ordinary gauge fields we exhibited in section 3, which exists also on $\mathbb{R}^n$ and on a torus for any Θ, not necessarily rational. The transformation in section 3 does not act on the space the theory is formulated on — if the theory is formulated on a torus, this transformation maps us to another theory formulated on the same torus. This transformation also does not change the rank of the gauge group, but it converts the simple $\widehat{F}^2$ action to a complicated non-polynomial action. The transformation we discuss here for rational Θ acts on the torus, changes the rank of the gauge group and maps the simple $\widehat{F}^2$ to a simple F^2 action.

One might be concerned that because of the special properties of the zero slope limit with rational Θ the noncommutative theory behaves in a discontinuous fashion as a function of Θ. Furthermore, for generic g and B their map into the fundamental domain varies ergodically in the zero slope limit.[12] However, these discontinuities

[12]This fact was mentioned in [63], where the role of noncommutative geometry was anticipated.

affect only the closed strings, which decouple in the limit. The open string parameters G, Θ and G_s transform smoothly under T duality. Correspondingly, the open string dynamics varies smoothly with Θ for fixed G in the zero slope limit.

6.2 Modules over a noncommutative torus

Our goal in the rest of this section is to construct and analyze directly from quantization of open strings a natural class of representations for the algebra $\mathcal{A}$ of functions on a noncommutative torus. Mathematically, representations of a ring are usually called modules. We will aim to understand in physical terms the usual mathematical constructions [64, 65, 4, 17] of projective modules over a noncommutative torus, and the "Morita equivalences" between them, which [4], [17]–[20] are intimately related to T-duality.

For simplicity, we will discuss bosonic strings. (Incorporating supersymmetry does not affect the ring $\mathcal{A}$ and so does not alter the modules.) Consider a p-brane wrapped on the torus. The ground states of the p-p open strings are tachyons. For every function f on $\mathbf{T}^p$, there is a corresponding tachyon vertex operator $\mathcal{O}_f$. In the limit $\alpha' \to 0$, the dimensions of the $\mathcal{O}_f$'s vanish. The operator product algebra of the $\mathcal{O}_f$'s reduces, as we have seen in section 2, to the $*$-product of functions on $\mathbf{T}^p$, essentially

$$\mathcal{O}_f(t) \cdot \mathcal{O}_g(t') \to \mathcal{O}_{f*g}(t')\,, \qquad \text{for } t > t'\,. \tag{6.17}$$

A p-p open string has a world-sheet with topology $\Sigma = I \times \mathbb{R}$, where $\mathbb{R}$ is a copy of the real line parametrizing the proper time, and $I = [0, \pi]$ is an interval that parametrizes the string at fixed time. The algebra $\mathcal{A}$ of tachyon vertex operators can be taken to act at either end of the open string. Operators acting on the left of the string obviously commute with those acting on the right, and the open string states form a bimodule for $\mathcal{A} \times \mathcal{A}$. By a bimodule for a product of rings $\mathcal{A} \times \mathcal{A}'$, we mean a space that is a left module for the first factor $\mathcal{A}$, and a right module for the second factor $\mathcal{A}'$, with the two actions commuting.[13] That the strings are a left module for the first factor and a right module for the second can be seen as follows. The interaction of open strings with the B-field comes from a term

$$-\frac{i}{2}\int_\Sigma \epsilon^{ab} B_{ij}\partial_a x^i \partial_b x^j = -\frac{i}{2}\int_{\{0\}\times\mathbb{R}} B_{ij}x^i \frac{dx^j}{dt} + \frac{i}{2}\int_{\{\pi\}\times\mathbb{R}} B_{ij}x^i\frac{dx^j}{dt}\,. \tag{6.18}$$

The relative minus sign between the two boundary terms in (6.18) means that if for vertex operators inserted on $\{0\} \times \mathbb{R}$, (6.17) holds with some given θ, then for

[13] A left module for a ring $\mathcal{A}$ is a set M on which $\mathcal{A}$ acts, obeying a condition that will be stated momentarily. For $a \in \mathcal{A}$ and $m \in M$, we write am for the product of a with m. The defining property of a left module is that for $a, b \in \mathcal{A}$ and $m \in M$, one has $(ab)m = a(bm)$. In the case that M is a right module, the action of $\mathcal{A}$ on M is usually written on the right: the product of $a \in \mathcal{A}$ with $m \in M$ is written ma. The defining property of a right module is that $m(ba) = (mb)a$.

operators inserted on $\{\pi\} \times \mathbb{R}$, the same OPE holds with θ replaced by $-\theta$. Changing the sign of θ is equivalent to reversing the order of multiplication of functions, so if the conventions are such that (6.17) holds as written for vertex operators inserted at the left end of the string, then for operators inserted at the right end we have

$$\mathcal{O}_f(t) \cdot \mathcal{O}_g(t') \rightarrow \mathcal{O}_{g*f}(t') \,, \qquad \text{if } t > t' \,. \tag{6.19}$$

Comparing to the definition in the footnote, we see that the open string states form a left module for $\mathcal{A}$ acting on the left end of the open string, and a right module for $\mathcal{A}$ acting on the right end of the open string.

In the limit $\alpha' \rightarrow 0$, the excited string states decouple, and we can get an $\mathcal{A} \times \mathcal{A}$ bimodule M by just taking the open string ground states. In fact, this module is simply a free module, that is $M = \mathcal{A}$, since the open string ground states are a copy of $\mathcal{A}$.

We can construct many other left modules for $\mathcal{A}$ as follows. Consider an arbitrary boundary condition γ for open strings on $\mathbf{T}^p$ with the given open string parameters G and θ. (Physically, γ is determined by a collection of Dq-branes for $q = p, p-2, \ldots$) Then consider the p-γ open strings, that is the strings whose left end is on a fixed p-brane and whose right end obeys the boundary condition γ. The p-p algebra $\mathcal{A}$ acts on the p-γ open string ground states for any given γ, giving a left module M_γ for $\mathcal{A}$. In section 6.3, we will construct the usual projective left modules for $\mathcal{A}$ in this way. In section 6.4, we examine theoretical issues connected with this construction.

6.3 Construction of modules

We turn now to detailed description of the modules. For brevity of exposition, we will concentrate on modules over a noncommutative $\mathbf{T}^2$. The generalization to $\mathbf{T}^p$ does not involve essential novelty for the type of modules we will construct, which are obtained from constant curvature connections over an ordinary torus. (For $p \geq 4$, there should be additional modules constructed from instantons, but we are not able to describe them very concretely.)

The constructions all start with ordinary actions for open strings on an ordinary $\mathbf{T}^2$, in the presence of a B-field, with twobrane boundary conditions on the left end of the string and varying boundary conditions on the right. We consider four cases for the boundary conditions on the right of the open string: twobrane boundary conditions; zerobrane boundary conditions; strings ending on a twobrane with m units of magnetic flux; and finally the general case of open strings ending on a system of n twobranes with m units of magnetic flux. Quantization of the open strings will give standard modules over a noncommutative $\mathbf{T}^2$. These modules have been described and used in section 3.2 of [4], and reconsidered in [19, 20, 21].

6.3.1 Twobrane boundary conditions

Naively, the limit $\alpha' \to 0$ can be taken by simply dropping the kinetic energy term from the open string action. This leaves only boundary terms, which are determined by the interaction with the B-field. We describe the torus by angular coordinates x^i, $i = 1, 2$, with $0 \le x^i \le 2\pi$, and we describe the open string worldsheet by functions $x^i(\sigma, \tau)$, where τ is the proper time and σ, which ranges from 0 to π, parametrizes the position along the open string. If we set $x^i = x^i(0, \tau)$, $\widetilde{x}^i = x^i(\pi, \tau)$, then the boundary terms in the action become

$$L_B = -\frac{i}{2}\int dt B_{ij} x^i \frac{dx^j}{dt} + \frac{i}{2}\int dt B_{ij}\widetilde{x}^i \frac{d\widetilde{x}^j}{dt}\,. \tag{6.20}$$

If these were the only variables and L_B the full action, then x^1 would be the canonical conjugate of x^2 — and similarly $\widetilde{x}^1$ and $\widetilde{x}^2$ would be canonically paired — so the classical phase space would be a copy of $\mathbf{T}^2 \times \mathbf{T}^2$. This cannot be the full answer since ($\mathbf{T}^2$ being compact) the quantization would be inconsistent unless $\int_\Sigma B$ is an integral multiple of 2π. In fact, as we will now see, the phase space is $\mathbf{T}^2 \times \mathbb{R}^2$.

We must remember the string connecting the two endpoints. The ordinary kinetic energy of the string is

$$L_K = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ij}\partial_a x^i \partial^a x^j\,. \tag{6.21}$$

Here we recall that in the $\alpha' \to 0$ limit with fixed open string metric, g_{ij} is of order $(\alpha')^2$, so L_K formally vanishes. For a given set of endpoints, the contribution to the energy coming from L_K is minimized by a string that is a geodesic from x^i to $\widetilde{x}^i$. In each homotopy class of paths from x^i to $\widetilde{x}^i$, there is a unique geodesic. The fluctuations around the minimum involve modes with mass squared of order $1/\alpha'$. Since we are not interested in such high energy excitations, we can ignore the fluctuations and identify the phase space as consisting of a pair of points x^i and $\widetilde{x}^i$ together with a straight line (or geodesic) connecting them. (We can also reach this conclusion by just formally setting $\alpha' = 0$ and dropping L_K completely. Then there is a gauge invariance under arbitrary variations of $x^i(\sigma, \tau)$ keeping the endpoints fixed. We can use this gauge invariance to fix a gauge in which $x^i(\sigma, \tau)$ is a geodesic.)

Now we write

$$\begin{aligned} x^i &= y^i + \frac{1}{2}s^i\,, \\ \widetilde{x}^i &= y^i - \frac{1}{2}s^i\,. \end{aligned} \tag{6.22}$$

Here y^i is $\mathbf{T}^2$-valued, but s^i is real-valued. In fact, y^i is the midpoint of the geodesic from x^i to $\widetilde{x}^i$, while the real-valuedness of s^i enables us to keep track of how many times this geodesic wraps around $\mathbf{T}^2$.

The symplectic structure derived from L_B is

$$\omega = Bdx^1 \wedge dx^2 - Bd\widetilde{x}^1 \wedge d\widetilde{x}^2 = B\left(ds^1 \wedge dy^2 - ds^2 \wedge dy^1\right), \tag{6.23}$$

where we set $B_{12} = B$. In canonical quantization, we can take the y^i to be multiplication operators, and identify the s^i as the canonical momenta:

$$\begin{aligned} s^1 &= -\frac{i}{B}\frac{\partial}{\partial y^2} = 2\pi i\Theta\frac{\partial}{\partial y^2}\,, \\ s^2 &= \frac{i}{B}\frac{\partial}{\partial y^1} = -2\pi i\Theta\frac{\partial}{\partial y^1}\,. \end{aligned} \tag{6.24}$$

Here we have set as in the discussion of T-duality $\Theta = -1/2\pi B$. (Since we are studying a two dimensional situation, Θ like B is a number.) The physical Hilbert space thus consists of functions on an ordinary $\mathbf{T}^2$, with coordinates y^i. The algebra $\mathcal{A}$ of functions on the noncommutative $\mathbf{T}^2$ is generated by

$$\begin{aligned} U_1 &= \exp(ix^1) = \exp\left(iy^1 - \pi\Theta\left(\frac{\partial}{\partial y^2}\right)\right), \\ U_2 &= \exp(ix^2) = \exp\left(iy^2 + \pi\Theta\left(\frac{\partial}{\partial y^1}\right)\right). \end{aligned} \tag{6.25}$$

They obey

$$U_1U_2 = e^{-2\pi i\Theta}U_2U_1\,. \tag{6.26}$$

The commutant of $\mathcal{A}$ is generated by

$$\begin{aligned} \widetilde{U}_1 &= \exp(i\widetilde{x}^1) = \exp\left(iy^1 + \pi\Theta\left(\frac{\partial}{\partial y^2}\right)\right) \\ \widetilde{U}_2 &= \exp(i\widetilde{x}^2) = \exp\left(iy^2 - \pi\Theta\left(\frac{\partial}{\partial y^1}\right)\right). \end{aligned} \tag{6.27}$$

These operators obey

$$\widetilde{U}_1\widetilde{U}_2 = e^{-2\pi i\widetilde{\Theta}}\widetilde{U}_2\widetilde{U}_1\,, \tag{6.28}$$

with

$$\widetilde{\Theta} = -\Theta\,. \tag{6.29}$$

The relative minus sign between Θ and $\widetilde{\Theta}$ means, as we have explained in section 6.2, that the open strings are an $\mathcal{A}\times\mathcal{A}$ bimodule (left module for the first factor, acting at $\sigma = 0$, and right module for the second factor, acting at $\sigma = \pi$). The formulas that we have arrived at are standard formulas for a free $\mathcal{A}\times\mathcal{A}$ bimodule.

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6.3.2 Zerobrane boundary conditions

Now, without changing the boundary conditions at $\sigma = 0$, we replace the boundary conditions at $\sigma = \pi$ by zerobrane boundary conditions. For example, we can place the zerobrane at the origin and take the boundary condition at $\sigma = \pi$ to be $\widetilde{x}^i = 0$. The phase space therefore consists now of a point x^i on $\mathbf{T}^2$ together with a geodesic from that point to $x^i = 0$. A shift $x^i \to x^i + 2\pi n^i$ (with integers n^i) acts freely on the phase space, changing the winding number of the geodesic. We can forget about the geodesic if we consider the x^i to be real-valued.

The phase space is thus a copy of $\mathbb{R}^2$ with symplectic form

$$\omega = B\, dx^1 \wedge dx^2 \,. \tag{6.30}$$

To quantize, we can take x^2 to be a multiplication operator and

$$x^1 = -i\frac{1}{B}\frac{\partial}{\partial x^2} = 2\pi i\Theta\frac{\partial}{\partial x^2} \,. \tag{6.31}$$

Hence, the algebra $\mathcal{A}$ of functions on the noncommutative $\mathbf{T}^2$ is generated by

$$U_1 = \exp(ix^1) = \exp\left(-2\pi\Theta\tfrac{\partial}{\partial x^2}\right) U_2 = \exp(ix^2), \tag{6.32}$$

again obeying

$$U_1 U_2 = e^{-2\pi i\Theta} U_2 U_1 \,. \tag{6.33}$$

The commutant of the U_i is generated by

$$\begin{aligned} \widetilde{U}_1 &= \exp\left(\frac{ix_1}{\Theta}\right) = \exp\left(-2\pi\frac{\partial}{\partial x^2}\right) \\ \widetilde{U}_2 &= \exp\left(\frac{ix_2}{\Theta}\right) . \end{aligned} \tag{6.34}$$

We note that unlike U_i which are invariant under $x_i \to x_i + 2\pi$, $\widetilde{U}_i$ are not invariant under this shift. Yet, they are valid operators on our Hilbert space because x_i are points in $\mathbb{R}^2$, rather than in $\mathbf{T}^2$. These operators obey

$$\widetilde{U}_1\widetilde{U}_2 = e^{-2\pi i\widetilde{\Theta}}\widetilde{U}_2\widetilde{U}_1 \,, \tag{6.35}$$

where

$$\widetilde{\Theta} = \frac{1}{\Theta} \,. \tag{6.36}$$

The formulas are again standard, and the interpretation is as follows [4]. With twobrane boundary conditions, the vertex operators at $\sigma = \pi$ generate, as we saw above, a noncommutative torus with $\Theta' = -\Theta$. The T-duality transformation that converts twobranes to zerobranes acts on Θ' by $\Theta' \to -1/\Theta'$, giving us a noncommutative torus $\mathcal{A}'$ with noncommutativity parameter $\widetilde{\Theta} = -1/\Theta' = 1/\Theta$.

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In physical terms, the algebra $\mathcal{A}'$ is the algebra of ground state 0-0 strings acting on the 2-0 open strings at $\sigma = \pi$. The relevant 0-0 strings are open strings that wind on a geodesic around the torus, starting and ending at the origin. That these open strings in the small volume limit generate the algebra of a noncommutative torus was the starting point [4, 12, 13] in applications of noncommutative Yang-Mills theory to string theory. From the point of view of the present paper, the statement can be justified by computing the OPE's of the 0-0 open strings, which are equivalent by T-duality to the OPE's of 2-2 tachyon vertex operators. We studied the 2-2 OPE's in section 2.1.

6.3.3 $(1, m)$ boundary conditions

Now we consider a more general case: an open string that at $\sigma = 0$ has the same twobrane boundary conditions as before while at $\sigma = \pi$ it terminates on a twobrane that carries m units of zerobrane charge. The zerobrane charge can be incorporated by placing on the twobrane in question a magnetic field of constant curvature $m/2\pi$. The boundary terms in the action become now

$$L_B = -\frac{i}{2}\int dt B_{ij} x^i \frac{dx^j}{dt} + \frac{i}{2}\int dt \left(B_{ij} + \frac{m}{2\pi}\epsilon_{ij}\right) \widetilde{x}^i \frac{d\widetilde{x}^j}{dt} . \tag{6.37}$$

(Here ϵ_{ij} is an antisymmetric tensor with $\epsilon_{12} = 1$.) The symplectic structure is

$$\omega = B dx^1 \wedge dx^2 - \left(B + \frac{m}{2\pi}\right) d\widetilde{x}^1 \wedge d\widetilde{x}^2 . \tag{6.38}$$

While the algebra acting at $\sigma = 0$ is a noncommutative torus with $\Theta = -1/2\pi B$, the algebra acting at $\sigma = \pi$ is now a noncommutative torus $\mathcal{A}'$ with

$$\widetilde{\Theta} = \frac{1}{2\pi B + m} . \tag{6.39}$$

No essentially new computation is required here; the algebra at $\sigma = \pi$ is determined in the usual way in terms of the boundary conditions at $\sigma = \pi$. (At $m = 0$, we had in section 6.3.1 above $\widetilde{\Theta} = -\Theta = 1/2\pi B$; replacing B by $B + m/2\pi$ gives (6.39).) We can write this as

$$\widetilde{\Theta} = \frac{\Theta'}{1 + m\Theta'} , \tag{6.40}$$

where $\Theta' = -\Theta$ is the noncommutativity parameter we found in section 6.3.1 above for the algebra at $\sigma = \pi$. $\widetilde{\Theta}$ is obtained from Θ' by the T-duality transformation that maps $(1, 0)$ boundary conditions (twobrane charge 1 and zerobrane charge 0) to $(1, m)$. In fact, in the zero area limit, the modular parameter $\tau = 2\pi B + i(\text{Area})$ reduces to $\tau = 2\pi B$. The modular transformation that maps $(1, 0)$ to $(1, m)$ is

$$2\pi B \to 2\pi B + m , \tag{6.41}$$

which in terms of $\Theta' = 1/2\pi B$ is

$$\Theta' \to \frac{\Theta'}{1 + m\Theta'}. \tag{6.42}$$

Of course, we can also construct explicitly the $\mathcal{A} \times \mathcal{A}'$ bimodule by quantizing the open strings. (The details are a bit lengthy and might be omitted on first reading.) For this, we set

$$\begin{aligned} x^i &= y^i + \lambda s^i \\ \widetilde{x}^i &= y^i + (1+\lambda) s^i \,, \end{aligned} \tag{6.43}$$

$y^i \in \mathbf{T}^2$ and s^i real-valued. If

$$\frac{m}{2\pi}\lambda^2 + (2\lambda + 1)\left(B + \frac{m}{2\pi}\right) = 0 \,, \tag{6.44}$$

then the symplectic form in these variables has no $ds^1 \wedge ds^2$ term, and reads

$$\omega = -\left(\frac{m}{2\pi}\lambda + (B + \frac{m}{2\pi})\right)\left(ds^1 \wedge dy^2 - ds^2 \wedge dy^1\right) - \frac{m}{2\pi} dy^1 \wedge dy^2 \,. \tag{6.45}$$

A further rescaling

$$s^i = -w t^i \,, \tag{6.46}$$

with

$$w = \frac{1}{\frac{m}{2\pi}\lambda + \left(B + \frac{m}{2\pi}\right)} \tag{6.47}$$

gives

$$\omega = dt^1 \wedge dy^2 - dt^2 \wedge dy^1 - \frac{m}{2\pi} dy^1 \wedge dy^2 \,. \tag{6.48}$$

Because there is no $dt^1 \wedge dt^2$ term, the y^i commute and can be represented by multiplication operators. The remaining commutators can be represented by taking $t^1 = -iD/Dy^2$, $t^2 = iD/Dy^1$, where

$$\left[\frac{D}{Dy^1}, \frac{D}{Dy^2}\right] = i\frac{m}{2\pi}. \tag{6.49}$$

Hence, the Hilbert space is the space of sections of a line bundle over $\mathbf{T}^2$ with first Chern class m. The algebra $\mathcal{A}$ that acts at $\sigma = 0$ is generated by

$$\begin{aligned} U_1 &= \exp(ix^1) = \exp\left(iy^1 - w\lambda\frac{D}{Dy^2}\right) \\ U_2 &= \exp(ix^2) = \exp\left(iy^2 + w\lambda\frac{D}{Dy^1}\right) , \end{aligned} \tag{6.50}$$

and using the above formulas, one can verify that $U_1 U_2 = \exp(-2\pi i \Theta) U_2 U_1$, with as usual $\Theta = -1/2\pi B$. One can similarly describe the algebra $\mathcal{A}'$ that acts at $\sigma = \pi$.

We can pick a gauge in which

$$\begin{aligned}\frac{D}{Dt^1} &= \frac{\partial}{\partial t^1}\\ \frac{D}{Dt^2} &= \frac{\partial}{\partial t^2} + i\frac{m}{2\pi}t^1\,,\end{aligned} \tag{6.51}$$

with wave functions obeying $\psi(t^1, t^2+2\pi) = \psi(t^1, t^2)$, $\psi(t^1+2\pi, t^2) = e^{-imt^2}\psi(t^1, t^2)$. In this gauge, we expand $\psi(t^1, t^2) = \sum_{k\in\mathbb{Z}} f_k(t^1)e^{ikt^2}$, where $f_k(t^1+2\pi) = f_{k+m}(t^1)$. The f_k's can thus be grouped together into m functions $f_0(t^1), \ldots, f_{m-1}(t^1)$ of a *real* variable t^1. This leads to the description of the module used in [4].

6.3.4 (n, m) boundary conditions

The general case of this type is to consider an open string that terminates at $\sigma = 0$ on a twobrane, and at $\sigma = \pi$ on a cluster of n twobranes with zerobrane charge m. We call this a cluster of charges (n, m). For simplicity, we suppose that m and n are relatively prime.

Such a cluster can be described as a system of n twobranes that bear a $\mathrm{U}(n)$ gauge bundle E with a connection of constant curvature $m/2\pi n$. The center of $\mathrm{U}(n)$ is $\mathrm{U}(1)$, and the curvature of E lies in this $\mathrm{U}(1)$. Nonetheless, E cannot be obtained by tensoring a $\mathrm{U}(1)$ bundle with a trivial $\mathrm{U}(n)$ bundle, because the first Chern class of the $\mathrm{U}(1)$ bundle would have to be m/n, not an integer. This leads to some complications in the direct description [19] of the module derived from E.

However, E, and the open string Hilbert space that comes with it, is naturally described by an "orbifolding" procedure. We let $\widehat{\mathbf{T}}^2$ be obtained from $\mathbf{T}^2$ by an n^2-fold cover, obtained by taking an n-fold cover in each direction. (So $\widehat{\mathbf{T}}^2$ is described with the same coordinates x^1, x^2 as $\mathbf{T}^2$, but they have period $2\pi n$.) When pulled back to $\widehat{\mathbf{T}}^2$, E has a central curvature with mn Dirac flux units. In particular, on $\widehat{\mathbf{T}}^2$, we can write $E = \mathcal{L} \otimes V$, where $\mathcal{L}$ is a $\mathrm{U}(1)$ line bundle with first Chern class mn, and V is a trivial $\mathrm{U}(n)$ bundle, with trivial connection.

To get from $\widehat{\mathbf{T}}^2$ to $\mathbf{T}^2$, we must divide by the symmetries $T_i : x^i \to x^i + 2\pi$, $i = 1, 2$. These symmetries act on the bundle $\mathcal{L}$, but in their action on $\mathcal{L}$ they do not commute. If they commuted in acting on $\mathcal{L}$, then after dividing by the group generated by the T_i, $\mathcal{L}$ would descend to a line bundle over the original $\mathbf{T}^2$ of first Chern class m/n, a contradiction as this is not an integer. Rather than T_1 and T_2 commuting in their action on $\mathcal{L}$, we have

$$T_1T_2 = T_2T_1e^{-2\pi im/n}. \tag{6.52}$$

To get translation operators that do commute, we let W_1 and W_2 be elements of $\mathrm{U}(n)$, regarded as constant gauge transformations of V, that obey

$$W_1W_2 = W_2W_1e^{2\pi im/n}. \tag{6.53}$$

Then the operators

$$\mathcal{T}_1 = T_1 W_1,\ \mathcal{T}_2 = T_2 W_2 \tag{6.54}$$

do commute. By imposing invariance under the $\mathcal{T}_i$, the bundle $\mathcal{L}\otimes V$ on $\widehat{\mathbf{T}}^2$ descends to the desired U(n) bundle E over $\mathbf{T}^2$.

Now, let us construct the algebras that act at the two ends of the string. At $\sigma = 0$, the boundary coupling is just

$$-\frac{i}{2}\int d\tau B\left(x^1\frac{dx^2}{d\tau} - x^2\frac{dx^1}{d\tau}\right). \tag{6.55}$$

There are no $n \times n$ matrices to consider, so after descending to $\mathbf{T}^2$, the algebra $\mathcal{A}$ of tachyon operators at $\sigma = 0$ is generated simply by $U_i = \exp(ix^i)$, with the usual algebra $U_1U_2 = e^{-2\pi i\Theta}U_2U_1$, with $\Theta = -1/2\pi B$.

Life is more interesting at $\sigma = \pi$. Before orbifolding, with the target space understood as $\widehat{\mathbf{T}}^2$ so that the x^i have periods $2\pi n$, the boundary couplings are

$$\frac{i}{2}\int d\tau\left(B + \frac{m}{2\pi n}\right)\left(\widetilde{x}^1\frac{d\widetilde{x}^2}{d\tau} - \widetilde{x}^2\frac{d\widetilde{x}^1}{d\tau}\right). \tag{6.56}$$

We have included the central curvature of $\mathcal{L}$. The algebra of functions of the $\widetilde{x}^i$ at $\sigma = \pi$ is generated, before orbifolding, by $Y_i = \exp(i\widetilde{x}^i/n)$ with

$$Y_1Y_2 = Y_2Y_1\exp\left(-\frac{2\pi i}{n^2}\left(2\pi B + \frac{m}{n}\right)\right). \tag{6.57}$$

Here we have shifted $B \to B + m/2\pi n$ in the usual formula (which at $\sigma = \pi$ has a phase $\exp(2\pi i\Theta) = \exp(-i/B)$), and also taken account of the fact that the exponent of Y_i is n times smaller than usual. Since we assume that m and n are relatively prime, the algebra of $n \times n$ matrices is generated by W_1 and W_2. Hence, the Y_i and the W_i together generate the algebra of operators that can act on the open string ground states at $\sigma = \pi$. However, for the orbifolding, we want to consider the subalgebra of operators that commute with the $\mathcal{T}_i$. It is generated by

$$\begin{aligned}\widetilde{U}_1 &= Y_1W_2^b\,,\\ \widetilde{U}_2 &= Y_2W_1^{-b}\,,\end{aligned} \tag{6.58}$$

where b is an integer such that mb is congruent to -1 mod n, or in other words there exists an integer a with

$$1 = an - mb\,. \tag{6.59}$$

Equivalently,

$$P = \begin{pmatrix} n & m \\ b & a \end{pmatrix} \tag{6.60}$$

is an element of SL$(2,\mathbb{Z})$. The $\widetilde{U}_i$ obey $\widetilde{U}_1\widetilde{U}_2 = \exp(-2\pi i\widetilde{\Theta})\widetilde{U}_2\widetilde{U}_1$, with

$$\widetilde{\Theta} = \frac{1}{n^2}\frac{1}{2\pi B + (m/n)} - \frac{mb^2}{n}. \tag{6.61}$$

Using (6.59) and $\Theta' = -\Theta = 1/2\pi B$, this is

$$\widetilde{\Theta} = \frac{a\Theta' + b}{m\Theta' + n} \quad \text{modulo } \mathbb{Z}\,. \tag{6.62}$$

This shows that the algebra $\mathcal{A}'$ that acts at $\sigma = \pi$ is the algebra of a noncommutative torus, with $\widetilde{\Theta}$ obtained from Θ' by a T-duality transformation that maps a brane cluster of charges $(1,0)$ to a brane cluster of charges (n,m).

The $\mathcal{A}\times\mathcal{A}'$ bimodule can be described explicitly by quantizing the open strings. In fact, before orbifolding, it arises from open strings on $\widehat{\mathbf{T}}^2$ that end at $\sigma = \pi$ on a cluster with brane charges $(1,nm)$. The module with this boundary condition was described in section 6.3.3 above, and the general case follows by dividing by $\mathbb{Z}_n \times \mathbb{Z}_n$. We will omit details.

A few loose ends. We conclude this subsection by clearing up a few loose ends.

The modules we have constructed are all called projective modules mathematically. (An $\mathcal{A}$ module M is called projective if there is another $\mathcal{A}$ module N such that $M \oplus N$ is equivalent to a direct sum of free modules.) We have constructed all of our modules (except the module of 2-0 strings) starting with a complex line bundle or vector bundle over an ordinary commutative $\mathbf{T}^2$, which determines the boundary conditions at the $\sigma = \pi$ end. Given a complex vector bundle E over $\mathbf{T}^2$, there is a complex bundle F such that $E \oplus F$ is trivial. This is the starting point of complex K-theory; it means that the space of sections of E is a projective module for the ring of functions on the commutative $\mathbf{T}^2$. From this it follows that the modules over the noncommutative $\mathbf{T}^2$ obtained by quantizing open strings are likewise projective. ¿From this point of view, the 2-0 module needs special treatment, because it is not determined by a vector bundle at the $\sigma = \pi$ end. However, the 2-0 module is determined by a boundary condition at $\sigma = \pi$ that is associated with an element of $K(\mathbf{T}^2)$, so it still leads to a projective module [17].

To an $\mathcal{A}$ module $M(E)$ determined by E (which is a bundle over $\mathbf{T}^2$ or in the exceptional case a K-theory element of $\mathbf{T}^2$), we can associate the Chern classes of E in $H^*(\mathbf{T}^2,\mathbb{Z})$. This natural topological invariant corresponds to $\mu(M)$ in the language of [17]. It is related to but differs from the Θ-dependent "Chern character in noncommutative geometry," which we will not try to elucidate in a physical language.

6.4 Theoretical issues

In this subsection, we will make contact with mathematical approaches [4, 17]–[21] to T-duality of noncommutative Yang-Mills theory on a torus via Morita equivalence of algebras, and then we will discuss how this language can be extended, to some extent, to open string field theory.

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Let α be any boundary condition for open strings on $\mathbf{T}^p$, in the zero slope limit with fixed open string parameters. The α-α ground states form an algebra $\mathcal{A}_\alpha$. For any other boundary condition γ that can be introduced in the same closed string theory, let $M_{\alpha,\gamma}$ denote the open string tachyon states with α boundary conditions on the left and γ boundary conditions on the right. It is an $\mathcal{A}_\alpha \times \mathcal{A}_\gamma$ bimodule.

$\mathcal{A}_\gamma$ is the commutant of $\mathcal{A}_\alpha$ in this module, and vice-versa. This means that the set of operators on $M_{\alpha,\gamma}$ that commute with $\mathcal{A}_\alpha$ is precisely $\mathcal{A}_\gamma$, and vice-versa. That $\mathcal{A}_\alpha$ and $\mathcal{A}_\gamma$ commute is clear from the fact that they act at opposite ends of the open strings. That they commute only with each other follows from the fact that, as was clear in the construction of the modules, together they generate the full algebra of observables in the string ground states. As usual in quantum mechanics, this full algebra of observables acts irreducibly on $M_{\alpha,\gamma}$.

There is an interesting analogy between the present open string discussion and rational conformal field theory. Consider, for example, the WZW model in two dimensions on a closed Riemann surface, with target space a Lie group G. As long as one considers the full closed string theory, this model has ordinary $G\times G$ symmetry. Quantum groups arise if one tries to factorize the model into separate left and right-moving sectors. Likewise, for open strings, the full algebra of operators acting on the string ground states, namely $\mathcal{A}_\alpha \times \mathcal{A}_\gamma$, acts irreducibly on the quantum mechanical (ground state) Hilbert space $M_{\alpha,\gamma}$, which as we have seen in the introduction to this section can be naturally realized in terms of ordinary functions (or sections of ordinary bundles) over an ordinary torus. A quantum torus arises if one attempts to focus attention on just one end of the open string, and to interpret just $\mathcal{A}_\alpha$ (or $\mathcal{A}_\gamma$) geometrically.

Morita equivalence. For any two boundary conditions α and β, there is a natural correspondence between $\mathcal{A}_\alpha$ modules that are obtained by quantizing open strings, and the analogous $\mathcal{A}_\beta$ modules. Indeed, there is one of each for every boundary condition γ; thus, the correspondence is $M_{\alpha,\gamma} \leftrightarrow M_{\beta,\gamma}$. This natural one-to-one correspondence between (projective) modules for two rings is called mathematically a Morita equivalence. In fact, the α-β and β-α open strings give bimodules $M_{\alpha,\beta}$ and $M_{\beta,\alpha}$ for $A_\alpha \times A_\beta$ and $A_\beta \times A_\alpha$. Using $M_{\alpha,\beta}$ one can define a map from left $\mathcal{A}_\beta$ modules to left $\mathcal{A}_\alpha$ modules by

$$N \to M_{\alpha,\beta} \otimes_{\mathcal{A}_\beta} N\,, \tag{6.63}$$

for every left $\mathcal{A}_\beta$ module N, with the inverse being $L \to M_{\beta,\alpha} \otimes_{\mathcal{A}_\alpha} L$ for a left $\mathcal{A}_\alpha$ module L. In physical terms, if N is of the form $M_{\beta,\gamma}$ for some γ, then the map from $M_{\alpha,\beta} \times N = M_{\alpha,\beta} \times M_{\beta,\gamma}$ to $M_{\alpha,\gamma}$ (which coincides with $M_{\alpha,\beta} \otimes_{\mathcal{A}_\beta} M_{\beta,\gamma}$) is just the natural string vertex combining an α-β open string and a β-γ open string to make an α-γ open string. In other words, an α-β state with vertex operator $\mathcal{O}$ and a β-γ state with vertex operator $\mathcal{O}'$ is mapped to an α-γ state with vertex

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operator given by the product $\lim_{\tau\to\tau'}\mathcal{O}(\tau)\mathcal{O}'(\tau')$. This gives a well-defined map from $M_{\alpha,\beta}\times M_{\beta,\gamma}$ to $M_{\alpha,\beta}$ because the dimensions of the operators vanish, and it can be interpreted as a map from $M_{\alpha,\beta}\otimes_{\mathcal{A}_\beta} M_{\beta,\gamma}$ to $M_{\alpha,\gamma}$ because of associativity of the operator product expansion, which states that $(\mathcal{O}\mathcal{O}'')\mathcal{O}'=\mathcal{O}(\mathcal{O}''\mathcal{O}')$ for any β-β vertex operator $\mathcal{O}''$.

Relation to T-duality. Now, we will make a few remarks on the relation between Morita equivalence and T-duality, as exploited in [4, 17]–[21]. First we need to understand the mathematical notion of "a connection on a module M over a noncommutative algebra $\mathcal{A}_\alpha$." If $\mathcal{A}_\alpha$ is the algebra of a noncommutative torus, generated by $U_i=\exp(ix^i)$, with $U_1U_2=e^{-2\pi i\Theta}U_2U_1$, then according to the standard mathematical definition used in the above-cited papers, such a connection is supposed to be given by operators

$$D_i=\frac{\partial}{\partial x^i}-iA_i\,, \tag{6.64}$$

where one requires that the A_i commute with the U_j. If M is a module $M_{\alpha,\gamma}$, constructed from α-γ open strings (for some γ), then this definition means that the A_i are elements of $\mathcal{A}_\gamma$. Physically, to describe strings propagating with such a connection, we must perturb the open string worldsheet action by adding a boundary perturbation at the γ end, without modifying the action at the α end.[14]

Now we can see Morita equivalence of noncommutative gauge fields (a more precise notion than Morita equivalence of modules), in the sense defined and exploited in [17]. This asserts that gauge theory over $\mathcal{A}_\alpha$ in the module $M_{\alpha,\gamma}$ is equivalent to gauge theory over $\mathcal{A}_\beta$ in the module $M_{\beta,\gamma}$. In fact, to do gauge theory over $\mathcal{A}_\alpha$ in the module $M_{\alpha,\gamma}$, we add a boundary perturbation at the γ end; likewise, to do gauge theory over $\mathcal{A}_\beta$ in the module $M_{\beta,\gamma}$, we add a boundary perturbation at the γ end. By using in the two cases the *same* boundary perturbation at the γ end, we get the equivalence between gauge theory over $\mathcal{A}_\alpha$ and gauge theory over $\mathcal{A}_\beta$ that is claimed mathematically. If $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are algebras of functions on noncommutative tori, then these must be T-dual tori (as proved in [4, 17, 19, 21] and in section 6.3). Conversely, any pair of T-dual tori can be boundary conditions in the same closed string field theory. So this gives the T-duality of noncommutative Yang-Mills theory on different tori as described mathematically.

Let w be a T-duality transformation and suppose that $\beta=w^{-1}\alpha$. A special case of what was just said is that gauge theory over $\mathcal{A}_\alpha$ with the module $M_{\alpha,\gamma}$ is equivalent to gauge theory over $\mathcal{A}_{w^{-1}\alpha}$ with the module $M_{w^{-1}\alpha,\gamma}$. Acting with T-duality on the boundary conditions at both ends of the string is certainly a symmetry of the full string theory (even before taking a zero slope limit). This operation transforms $\mathcal{A}_{w^{-1}\alpha}$

[14]This mathematical definition differs from a more naive physical notion, adopted in most of this paper, which is that if the gauge fields are elements of $\mathcal{A}_\gamma$, then we say we are working on the torus whose algebra of functions is $\mathcal{A}_\gamma$.

back to $\mathcal{A}_\alpha$ while transforming $M_{w^{-1}\alpha,\gamma}$ to $M_{\alpha,w\gamma}$. So it follows that gauge theory over $\mathcal{A}_\alpha$ with module $M_{\alpha,\gamma}$ is equivalent to gauge theory over $\mathcal{A}_\alpha$ with module $M_{\alpha,w\gamma}$. Thus we can, if we wish, consider the T-duality to leave fixed the torus $\mathcal{A}_\alpha$ and act only on the commutant $\mathcal{A}_\gamma$. This alternative formulation of how the T-duality acts is more in line with the naive notion of "gauge theory on a noncommutative torus" mentioned in the last footnote.

Regardless of which approach one takes, the key simplification that causes the $\widehat{F}^2$ action to be invariant under T-duality, while the conventional F^2 Yang-Mills action is not, is that in the zero slope limit, one can consider independent T-duality transformations at the two ends of the open string, in this way defining a transformation that leaves the torus fixed and acts only on the quantum numbers of the gauge bundle.

Relation to open string field theory. In all of this discussion, we have considered a $*$ product constructed just from the string ground states, in the limit $\alpha' \to 0$. It is natural to ask whether one can define a more general $*$ product that incorporates all of the string states. The only apparent way to do this is to use the $*$ product defined by gluing open strings in the noncommutative geometry approach to open string field theory [39]. This $*$ product can be introduced for open strings defined with any boundary condition in any closed string conformal field theory. Consider the case of oriented bosonic open string theory. In flat $\mathbb{R}^{26}$, with $B = 0$, taking free (Neumann) boundary conditions for the open strings, one gets a $*$ product (considered in [39]) whose $\alpha' \to 0$ limit is the ordinary commutative multiplication of functions on $\mathbb{R}^{26}$. What if we repeat the same exercise with a constant nonzero B-field? Using the relation of the $*$ product of open strings to the operator product algebra, in this situation the string field theory $*$ product reduces in the zero slope limit to the $*$ product of noncommutative Yang-Mills theory on $\mathbb{R}^{26}$. (If instead of Neumann boundary conditions for open strings, one takes p-brane boundary conditions for some $p < 25$, one gets instead noncommutative Yang-Mills on the p-brane worldvolume.) Thus, noncommutative Yang-Mills theory can be regarded as a low energy limit of string field theory. This gives an interesting illustration of the open string field theory philosophy, though it remains that the open string field theory does not seem particularly useful for computation (being superseded by the $\widehat{F}^2$ action).

The discussion of Morita equivalence of algebras makes sense in the full generality of open string field theory. Keeping fixed the closed string conformal field theory, let α be any possible boundary condition for the open strings. Then the $*$ product of the α-α open strings (keeping all of the excited open string states, and without any zero slope limit) gives an algebra $\mathcal{A}_\alpha$. For any other boundary condition β, the α-β open strings give an $\mathcal{A}_\alpha \times \mathcal{A}_\beta$ bimodule $M_{\alpha,\beta}$ that by the same construction as above establishes a Morita equivalence between $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$. This Morita equivalence gives a natural equivalence between the K-theory of $\mathcal{A}_\alpha$ and that of $\mathcal{A}_\beta$. Presumably, in

Type II superstring theory (in an arbitrary closed string background) the D-brane charge takes values in this K-theory — generalizing the relation of D-brane charge to ordinary complex K-theory in the long distance limit [66, 67].

7. Relation to M-theory in DLCQ

Motivated by the spectrum of BPS states, Connes, Douglas and Schwarz [4] proposed that M-theory compactified on a null circle (DLCQ)

$$x^- = \frac{1}{2}(x^0 - x^1) \sim x^- + 2\pi R \tag{7.1}$$

with nonzero C_{-ij} leads to noncommutative geometry. This suggestion was further explored in [19]. Here, we will examine this problem in more detail from our perspective.

The compactification on a null circle needs a careful definition. Here we will define it as a limit of a compactification on a space-like circle in the (x^0, x^1) plane which is almost null [68, 69]. Its invariant radius ϵR will eventually be taken to zero. In order to study this system we follow the following steps:

1. We boost the system to bring the space-like circle to be along x^1.
2. We scale the energy such that energies of order one before the boost remain of order one after the boost.
3. We scale the transverse directions and momenta in order to let them affect the energy as before the boost.
4. We interpret the system with the small circle as type IIA string theory.
5. If the original system is also compactified on a spatial torus, we perform T-duality on all its sides.

After this sequence of steps we find a system which typically has a smooth limit as ϵ is taken to zero. This limit was first discussed in [70] and later in the context of this definition of the Matrix Model [5] in [71, 69]. We will now follow these steps for the case of compactification on a null circle with constant nonzero $C_{\pm ij}$.

We consider M-theory compactified on the null circle (7.1). We assume that the metric is $g_{+-} = 1$, $g_{\pm i} = 0$, but the metric in the transverse directions g_{ij} can be arbitrary. If the transverse space includes a torus, we identify the transverse coordinates as $x^i \sim x^i + 2\pi r^i$.

It is important that we keep all the parameters of the M-theory compactification fixed. We hold the Planck scale M_p, the metric g and the identification parameters r^i fixed of order one as we let $\epsilon \to 0$.

Since x^- is compact in the DLCQ, the time coordinate is x^+. Therefore, the hamiltonian, which is the generator of time translation, is $P_+ = P^-$. The parameter R in (7.1) can be changed by a longitudinal boost. Therefore, the dependence of various quantities on R is easily determined using longitudinal boost invariance. In particular, the DLCQ hamiltonian is of the general form

$$P^- = P_+ = RM_p^2 F\left(x^i M_p, \frac{RC_{-ij}}{M_p^2}, \frac{C_{+ij}}{RM_p^4}\right), \tag{7.2}$$

for some function F (we suppress the transverse momenta $P \sim 1/x$). The dependence on the Planck scale M_p was determined on dimensional grounds. The system is invariant under translation along the null circle. The corresponding conserved charge, the longitudinal momentum, is

$$P_- = P^+ = \frac{N}{R}. \tag{7.3}$$

The Hilbert space is split into sectors of fixed N. Since R can be changed by a boost, the way to describe the decompactification of the null circle is to consider the limit $N \to \infty$, $R \to \infty$ with fixed P_-.

As we said above, we view this system as the $\epsilon \to 0$ limit of a compactification on a space-like circle of invariant radius ϵR. Since we plan to take ϵ to zero, we will expand various expressions below in powers of ϵ keeping only the terms which will be of significance. The hamiltonian is also changed to $P_+ + \epsilon P_-$, so that it does not generate translations along the space-like circle.

We now perform step 1 above and boost the system such that the circle is along the x^1 direction. We denote the various quantities after the boost with the subscript ϵ. The generator of time translation, $P_{\epsilon,0}$ receives a large additive contribution from $P_{\epsilon,-} = N/\epsilon R$, which we are not interested in. Therefore, we consider the new hamiltonian

$$H = P_{\epsilon,0} - \frac{N}{\epsilon R} = \epsilon RM_p^2 F\left(x^i M_p, \frac{\epsilon RC_{\epsilon,-ij}}{M_p^2}, \frac{C_{\epsilon,+ij}}{\epsilon RM_p^4}\right). \tag{7.4}$$

$C_{\epsilon,\pm ij}$ are related to $C_{\pm ij}$ by a boost

$$C_{\epsilon,-ij} = \frac{1}{\epsilon}C_{-ij}, \qquad C_{\epsilon,+ij} = \epsilon C_{+ij}. \tag{7.5}$$

One of the consequences of the large boost is that energies which were originally of order one are now very small. This is clear from (7.4) where there is an explicit factor of ϵ in front of the hamiltonian. In order to focus on these low energies, we perform step 2 above and scale the energy by $1/\epsilon$. We do that by replacing the system with Planck scale M_p with a similar system with Planck scale

$$\widetilde{M}_p^2 = \frac{1}{\epsilon}M_p^2. \tag{7.6}$$

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Then, the new hamiltonian

$$\widetilde{H} = \epsilon R \widetilde{M}_p^2 F = R M_p^2 F \tag{7.7}$$

is of order one.

In order to keep the dependence on x and C as it was before the scaling of M_p, we should follow step 3 above and also scale them such that the arguments of F are unchanged:

$$\widetilde{H} = R M_p^2 F\left(\widetilde{x}^i \widetilde{M}_p, \frac{\epsilon R \widetilde{C}_{\epsilon,-ij}}{\widetilde{M}_p^2}, \frac{\widetilde{C}_{\epsilon,+ij}}{\epsilon R \widetilde{M}_p^4}\right). \tag{7.8}$$

That is

$$\widetilde{x}^i = x^i \frac{M_p}{\widetilde{M}_p} = x^i \epsilon^{1/2}\,, \quad \widetilde{C}_{\epsilon,-ij} = C_{\epsilon,-ij}\frac{\widetilde{M}_p^2}{M_p^2} = C_{-ij}\frac{1}{\epsilon^2}\,, \quad \widetilde{C}_{\epsilon,+ij} = C_{\epsilon,+ij}\frac{\widetilde{M}_p^4}{M_p^4} = C_{+ij}\frac{1}{\epsilon}. \tag{7.9}$$

We now move to step 4 above and interpret this system, M-theory on a spatial circle of radius ϵR, as type IIA string theory. The parameter N is now interpreted as the number of D0-branes. The string theory parameters are

$$\widetilde{\alpha}' \sim \frac{1}{\widetilde{M}_p^3 R\epsilon} = \frac{1}{M_p^3 R}\epsilon^{1/2}, \qquad \widetilde{g}_s \sim (\widetilde{M}_p R\epsilon)^{3/2} = (M_p^2 R^2 \epsilon)^{3/4}, \tag{7.10}$$

and there is a nonzero B-field

$$\widetilde{B}_{\epsilon,ij} = \frac{RC_{-ij}}{\epsilon} \tag{7.11}$$

(the contribution of C_{+ij} to $\widetilde{B}$ is negligible for small ϵ). We note as a consistency check of our various changes of variables that just as the periods of C around a three cycle including the null circle are of order one, so are the periods of $\widetilde{B}$ around a two cycle $\widetilde{\Sigma}^{(2)}$

$$\int_{\widetilde{\Sigma}^{(2)}} \widetilde{B} \sim 1\,. \tag{7.12}$$

We now move to step 5 above. If the target space includes a torus T^p, $\widetilde{x}^i$ is identified with $\widetilde{x}^i + 2\pi\widetilde{r}^i$, where $\widetilde{r}^i = r^i\epsilon^{1/2}$. The metric on the torus was not changed by our various changes of variables and remained as it was in the original M-theory problem of order one, $g_{ij} \sim 1$. Therefore, the volume of the torus $\widetilde{V} = (\det g)^{1/2}\prod_i 2\pi\widetilde{r}^i \sim \epsilon^{\frac{p}{2}}$.

Let the rank of $\widetilde{B}$ be $r \leq p$ (r is even). For simplicity we assume that $\widetilde{B}_{ij}$ is nonzero only for $i,j = 1,\ldots,r$. We now perform T-duality on this torus converting the D0-branes to Dp-branes. After the transformation, the new coordinates $\widehat{x}^i$ are identified as

$$\widehat{x}^i \sim \widehat{x}^i + 2\pi\widehat{r}^i \tag{7.13}$$

with

$$\widehat{r}^i = \frac{\widetilde{\alpha}'}{\widetilde{r}^i} \sim \frac{1}{M_p^3 R r^i}. \tag{7.14}$$

Hence, the periods of $\widehat{x}^i$ are of order one.

The new metric and B-field are

$$\widehat{g}_{ij} = \left(\frac{1}{g + 2\pi\widetilde{\alpha}'\widetilde{B}}\right)_{Sij} \sim \begin{cases} \epsilon & \text{for } i,j = 1,\dots,r \\ 1 & \text{otherwise} \end{cases}$$

$$\widehat{B}_{ij} = \frac{1}{2\pi\widetilde{\alpha}'}\left(\frac{1}{g + 2\pi\widetilde{\alpha}'\widetilde{B}}\right)_{Aij} = \begin{cases} \left(\frac{1}{(2\pi\widetilde{\alpha}')^2\widetilde{B}}\right)_{ij} \sim 1 & \text{for } i,j = 1,\dots,r \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the new volume of the torus is $\widehat{V} = (\det\widehat{g})^{1/2}\prod_i 2\pi\widehat{r}^i \sim \epsilon^{\frac{r}{2}}$, and the new string coupling is

$$\widehat{g}_s = \widetilde{g}_s \frac{\widehat{V}^{1/2}}{\widetilde{V}^{1/2}} \sim \epsilon^{\frac{3-p+r}{4}}. \tag{7.15}$$

This is exactly the limit studied above in (2.14) and (2.47), which leads to quantum Yang-Mills with finite coupling constant on a noncommutative space with finite volume. More explicitly, we can determine the metric G, the noncommutativity parameter θ and the Yang-Mills coupling using (2.15) and (2.46), and the expressions for $\widehat{g}$, $\widehat{B}$ and $\widehat{g}_s$ in (7.15) and (7.15):

$$G_{ij} = g^{ij}, \qquad \theta^{ij} = \frac{C_{-ij}}{RM_p^6}, \qquad g_{YM}^2 \sim \frac{M_p^6 R^3}{(M_p^3 R)^p V}, \tag{7.16}$$

where $V = (\det g)^{1/2}\prod_i 2\pi r^i$ is the volume of the original torus. We would like to make a few comments:

1. The reason the indices in the left hand side and the right hand side of these equations do not seem to match is because the torus we end up with is dual to the original torus of the underlying M theory.
2. It is important that all these quantities are independent of ϵ and hence the limit $\epsilon \to 0$ leads to a well defined theory.
3. The expressions for G and g_{YM} are as in the case without the C field [6, 71, 69].
4. Even though in the zero slope limit of string theory $\theta = 1/B$, our various changes of variables lead to $\theta \sim C$ indicating that the behavior of the theory is smooth for C near zero.
5. The DLCQ limit $\epsilon \to 0$ is clearly smooth in C in the theory of the D0-branes. The T-duality transformation to the Dp-branes leads to the metric $\widehat{g}$ and string coupling $\widehat{g}_s$, which do not depend smoothly on C. However, the noncommutative Yang-Mills parameters (7.16) depend smoothly on C.

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6. The discussion of T-duality in [4, 17]–[21] is in terms of the D0-branes and the torus with coordinates $\widetilde{x}$ and metric $\widetilde{g}$. The discussion in section 6 is in terms of the torus with coordinates $\widehat{x}$ and metric $\widehat{g}$. Hence, the expressions for the T-duality transformations on G, Θ and g_{YM} differ by conjugation by the T-duality transformation which relates the two tori $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; i.e. they are related by $a \leftrightarrow d$ and $b \leftrightarrow c$.

7. For $p \geq 4$ the parameters in (7.16) including the Yang-Mills coupling are finite, but the theory is not likely to make sense, as it is not renormalizable [54]–[61]. For $C = 0$ ($r = 0$) this follows from $\widehat{g}_s \to \infty$, which forces the use of strongly coupled string theory leading to the $(2,0)$ theory for $p = 4$ [72, 73] and the little string theory for $p = 5$ [73, 34]. This suggests that also for nonzero C there exist new consistent theories for which noncommutative Yang-Mills theory is the low energy approximation.

8. Noncommutative version of the six dimensional $(2,0)$ theory

So far we have been studying open strings ending on D-branes in the presence of a B-field yielding a generalization of the ordinary gauge theory on D-branes. It is natural to ask whether the discussion can be extended to M5-branes in M-theory in the presence of a C field, or perhaps even to NS5-branes in string theory with background RR fields. This could lead to a generalization of the $(2,0)$ six dimensional field theory and the enigmatic little string theory [34].

There are several reasons to expect that such generalizations exist. First, reference [37] proposed a DLCQ description of a deformation of the $(2,0)$ field theory in terms of quantum mechanics on the deformed moduli space of instantons. This was motivated as a regularization of the $(2,0)$ theory in which the small instanton singularities are absent. A similar two-dimensional field theory in that target space could lead to a deformation of the little string theory. The second reason for the existence of these theories is the fact that the 4+1 and 5+1 dimensional noncommutative Yang-Mills theories are not renormalizable [54]–[61]. Yet, they arise, as in section 7, in the DLCQ description of M-theory on $\mathbf{T}^4$ and $\mathbf{T}^5$. It is natural to ask then whether they can be embedded in consistent theories without gravity. In the commutative case the supersymmetric Yang-Mills theory in five dimensions can be embedded in the six-dimensional $(2,0)$ theory and the six-dimensional gauge theory can be embedded in the little string theory. Therefore, we are led to look for generalizations of these theories in which the noncommutative gauge theories can be embedded.

We want to study M5-branes in a background C-field in an appropriate limit. In $\mathbb{R}^{11}$ a constant C-field can be gauged away. But in the presence of M5-branes it leads to a constant H-field at infinity. This is similar to the fact that a constant B-field can be gauged away but it leads to a constant background F in the presence of D-branes. There is, however, a crucial difference between these cases. The constant background F on D-branes is arbitrary. Here, however the background H satisfies an algebraic constraint (for example, when H is small it is selfdual), and therefore only half of its degrees of freedom can be specified.

We start by analyzing the constraints on H on a single M5-brane in $\mathbb{R}^6$. There is no completely satisfactory covariant lagrangean describing the dynamical twoform B on the M5-brane. We can use either a non-covariant lagrangean [74] or covariant equations of motion [75]. Here we will follow the latter approach.

The background H field transforms as $\mathbf{10}\oplus\mathbf{10'}$ under the six-dimensional Lorentz group. It can be written as $H = H^+ + H^-$, where H^+ is selfdual ($\mathbf{10}$ of the Lorentz group), and H^- is anti-selfdual ($\mathbf{10'}$ of the Lorentz group). Out of H^+ we can form

$$\begin{aligned} K_i^j &= H^+_{ikl} H^{+jkl} \,, \\ U_{ijk} &= K_i^l H^+_{ljk} \,, \\ D &= K_i^j K_j^i \,. \end{aligned} \tag{8.1}$$

which transform as $\mathbf{20'}$ (K_{ij} is a traceless symmetric tensor), $\mathbf{10'}$ (U_{ijk} is an anti-selfdual threeform) and $\mathbf{1}$ (D is a scalar). For slowly varying H the equation of motion express H^- in terms of H^+

$$H^- = \frac{1}{f(D)} U \,, \tag{8.2}$$

where U is the anti-selfdual form in (8.1) and $f(D)$ is a function of the scalar D in (8.1), which is given implicitly by

$$D = 6 f(D)^2 \left(1 - \frac{2\sqrt{2} M_p^3}{f(D)^{1/2}} \right) . \tag{8.3}$$

In (8.3), we take the solution which continuously interpolates between $f(0) = 8M_p^6$ and $\lim_{D\to\infty} f(D) = \sqrt{D/6}$.

We are looking for constant H solutions. For simplicity, we start with the metric $g_{ij} = \eta_{ij} = (-1,1,1,1,1,1)$ (we will later scale it). Up to a Lorentz transformation, the nonzero components of the generic[15] H satisfying (8.2) are

$$H_{012} = h_0 \,, \qquad H_{345} = h \,, \tag{8.4}$$

[15] By taking a limit as $h, h_0 \to 0$ while boosting, one can get a nongeneric constant H solution of (8.2) which in a suitable coordinate system takes the form $H_{012} = -H_{125} = H_{034} = -H_{345} = h$ with any constant h. For this solution $K_0^0 = -K_5^5 = K_0^5 = -K_5^0 = -4h^2$, $U = D = 0$. It is selfdual both in the subspace spanned by x^0, x^5, and in the subspace spanned by x^1, x^2, x^3, x^4. Therefore, it is possible that this solution is relevant to the discussion in [37], which was based on such a selfdual tensor.

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with a relation between h_0 and h. These nonzero values of H break the SO(5, 1) Lorentz group to SO(2, 1) × SO(3). Using $\epsilon^{012345} = 1$ we find

$$
\begin{aligned}
H^+_{012} &= -H^+_{345} = -\frac{1}{2}(h - h_0)\,,\\
K^0_0 &= K^1_1 = K^2_2 = -K^3_3 = -K^4_4 = -K^5_5 = -\frac{1}{2}(h-h_0)^2\,,\\
U_{012} &= U_{345} = \frac{1}{4}(h-h_0)^3\,, \qquad D = \frac{3}{2}(h-h_0)^4\,.
\end{aligned}
\tag{8.5}
$$

Then equation (8.2) determines the relation between h_0 and h

$$
h_0^2 - h^2 + M_p^{-6} h_0^2 h^2 = 0\,. \tag{8.6}
$$

For a given h, we have to take the solution

$$
h_0 = -\frac{h}{(1 + M_p^{-6} h^2)^{1/2}} \tag{8.7}
$$

(for an anti-M5-brane we have the opposite sign in (8.7)). For small h we have $h_0 \approx -h$ and $H^- \approx 0$, i.e. H is selfdual. For large h we have $h_0 \approx -M_p^3$ and H is dominated by its spatial components $H_{345} = h$.

We are looking for a limit in which the theory in the bulk of spacetime decouples. It should be such that after compactification on a circle, the low energy theory will be the five-dimensional $\widehat{F}^2$ theory we found in the zero slope limit. In order to motivate the appropriate limit, we examine a compactification of the theory on a spacelike $\S^1$ of invariant radius r to five dimensions. The resulting five-dimensional theory includes a gauge field with $F_{ij} = \oint dx^k H_{ijk}$, whose dynamics is controlled for slowly varying fields by the DBI action $\mathcal{L}_{DBI} \sim \sqrt{-\det(g + \frac{1}{2\pi r M_p^3} F)}$. The magnetic dual of F is the threeform $\widetilde{F} = {}^*(\partial\mathcal{L}_{DBI}/\partial F)$, where ${}^*(\;)$ denotes a dual in five dimensions. The six-dimensional equation (8.2) states that $\widetilde{F} = H$ with indices in the five noncompact dimensions.

Let us examine various different compactifications:

1. The compactification is along a circle in the subspace spanned by $x^{3,4,5}$. It can be taken, without loss of generality, to be along x^5. Here the background H we study leads to magnetic field of rank two $F_{34} = 2\pi r h$ and a three-form $\widetilde{F}_{012} = h_0$. The limit $h \to \infty$ leads to $\widetilde{F}$ of order one.

2. The compactification is along a circle in the subspace spanned by $x^{0,1,2}$. It can be taken without loss of generality to be along x^1. Here the background H we study leads to electric field $F_{02} = 2\pi r h_0$ and a three-form $\widetilde{F}_{345} = h$. In the limit $h \to \infty$ the electric field reaches its critical value. It is satisfying that the large magnetic field and the critical electric field limits of the five-dimensional

theory are the same limit in six dimensions. Also, the six-dimensional equation of motion (8.2) guarantees that the five-dimensional electric field cannot exceed its maximal value.

3. The compactification is along a circle, which can be brought using a boost along $x^{1,2}$ to the subspace spanned by $x^{1,2,3,4,5}$. It can be taken to be along x^1 and x^5. This leads to a background magnetic field of rank two $F_{34} \sim h$ and an electric field $F_{02} \sim h_0$. There is also a background three form.

4. The generic compactification along a spacelike circle which is not of the form 3 above, can be taken, using the $SO(2,1) \times SO(3)$ freedom, to be in the $x^{0,5}$ plane,
$$(x^0, x^5) \sim (x^0, x^5) + 2\pi r \left(-a, \sqrt{1+a^2}\right) \tag{8.8}$$
for some constant a. Or in terms of the boosted and rescaled coordinates $\widetilde{x}^0 = \sqrt{1+a^2}x^0 + ax^5$, $\widetilde{x}^5 = \frac{1}{r}(ax^0 + \sqrt{1+a^2}x^5)$, it is
$$(\widetilde{x}^0, \widetilde{x}^5) \sim (\widetilde{x}^0, \widetilde{x}^5) + (0, 2\pi) . \tag{8.9}$$
We will use these new coordinates and will drop the tilde over $x^{0,5}$. Then, the metric is $g_{ij} = (-1,1,1,1,1,r^2)$ and the background H field is $H_{012} = h_0\sqrt{1+a^2}$, $H_{034} = -ha$, $H_{125} = -h_0 ar$, $H_{345} = hr\sqrt{1+a^2}$. The five-dimensional theory has background magnetic field of rank four $F_{12} = -2\pi r h_0 a$, $F_{34} = 2\pi r h\sqrt{1+a^2}$, there is no background electric field, and there is a background three-form $\widetilde{F}_{034} = -ah$, $\widetilde{F}_{012} = \sqrt{1+a^2}h_0$.

We are interested in the zero slope limit in which $\frac{1}{2\pi r M_p^3}F \gg g$ and F is held fixed with rank four. The nonlinearity of the equation of motion (8.2) imposes restrictions on possible scalings. Since we want $g_{IJ} = \epsilon\delta_{IJ}$ for $I, J = 1,2,3,4$, but keep $g_{00} = -1$, $g_{55} = r^2$, we should scale $M_p \sim \epsilon^{-1/2}$ and change H to $H_{012} = h_0\sqrt{1+a^2}\epsilon^{-1/2}$, $H_{034} = -ha\epsilon^{-1/2}$, $H_{125} = -h_0 ar\epsilon^{-1/2}$, $H_{345} = hr\sqrt{1+a^2}\epsilon^{-1/2}$. Our desired scaling $M_s^2 = 2\pi r M_p^3 \sim \epsilon^{-1/2}$ and $g_s \sim (rM_p)^{3/2} \sim \epsilon^{3/4}$ is reproduced with $g_{55} = r^2 \sim \epsilon^2$. In order to keep F_{IJ} fixed we should also have $a = \epsilon^{-1/2}$. Then, the background threeform field $\widetilde{F}$ is of order $1/\epsilon$.

We conclude that we analyze the theory with
$$\begin{gathered} g_{IJ} = \epsilon\delta_{IJ}\,, \qquad g_{55} = r^2 \sim \epsilon^2\,, \qquad g_{00} = -1\,, \qquad M_p \sim \epsilon^{-1/2}\,, \\ H_{012} \approx h_0\epsilon^{-1}\,, \qquad H_{034} \approx -h\epsilon^{-1}\,, \qquad H_{125} \approx -h_0 r\epsilon^{-1}\,, \qquad H_{345} \approx hr\epsilon^{-1} \end{gathered} \tag{8.10}$$
compactified on $x^5 \sim x^5 + 2\pi$. In terms of the null coordinates $x^\pm = \frac{1}{2}(x^0 \pm rx^5)$, the dominant components of H are $H_{-12} \approx 2h_0\epsilon^{-1}$, $H_{-34} \approx -2h\epsilon^{-1}$, while H_{+IJ} are of order one.

We would like to make a few comments:

1. $H_{-IJ} \sim \epsilon^{-1}$, but $\oint dx^- H_{-IJ} \sim 1$ is a finite, generic rank four twoform.

2. $H_{+IJ} \ll H_{-IJ}$ in the limit. This fact has a simple reason. In the the zero slope limit with $F_{0I} = 0$ the lagrangean $\sqrt{-\det(g + \frac{1}{2\pi r M_p^3} F)} \sim \mathrm{Pf}F$. Therefore, $\widetilde{F}_{0IJ} \sim \epsilon^{0IJKL} F_{KL}$. Hence $H_{0IJ} \approx -r^{-1} H_{5IJ}$, and H is dominated by its H_{-IJ} components.

3. The limit of the five-dimensional theory is the same as a DLCQ compactification of the six-dimensional theory. This fact is consistent with the way this theory was derived in section 7 by starting with M-theory in DLCQ on $\mathbf{T}^4$. In the last step we performed T-duality to convert the theory to a five dimensional field theory. But because of the T-duality symmetry of the noncommutative gauge theories, we must have the same limit as before the T-duality transformation.

This discussion motivates us to examine the theory in $\mathbb{R}^6$ with

$$\begin{gathered} g_{IJ} = \epsilon\delta_{IJ}\,, \qquad g_{55} = \epsilon^2\,, \qquad g_{00} = -1\,, \qquad M_p = A\epsilon^{-1/2}\,, \\ H_{012} = h_0\epsilon^{-1}\sqrt{1+\epsilon}\,, \qquad H_{034} = -h\epsilon^{-1}\,, \qquad H_{125} = -h_0\,, \qquad H_{345} = h\sqrt{1+\epsilon} \\ h_0^2 - h^2 + A^{-6} h_0^2 h^2 = 0\,, \qquad \text{for } \epsilon \to 0\,, \quad h, h_0, A, x^i = \text{fixed} \end{gathered} \tag{8.11}$$

(the subleading terms in H_{012} and H_{345} are needed in order to satisfy (8.2)). We note that in the limit H_{012} and H_{034} diverge like ϵ^{-1}. In terms of null coordinates $x^\pm = \frac{1}{2}(x^0 \pm \epsilon x^5)$,

$$H_{-12} \approx 2h_0\epsilon^{-1}\,, \qquad H_{-34} \approx -2h\epsilon^{-1}\,, \qquad H_{+12} \approx \frac{1}{2}h_0\,, \qquad H_{+34} \approx \frac{1}{2}h\,, \tag{8.12}$$

i.e. H_{-IJ} are taken to infinity and H_{+IJ} are of order one.

It is not clear to us whether the limit (8.11) of M-theory with M5-branes exists. What we showed is that this limit satisfies the equation of motion on the M5-branes (8.2), and that after compactification on $\S^1$ it leads to D4-branes in the zero slope limit, i.e. to five dimensional noncommutative Yang-Mills with the $\frac{1}{g_{YM}^2}\widehat{F}^2$ lagrangean with finite g_{YM}. Therefore, if this theory can be embedded in a consistent theory within M-theory, it must come from the limit (8.11).

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