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Analysis of the periodicity of voltage versus applied flux curves of planar and three-dimensional SQUIDs in the presence of coupling inhomogeneity

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Abstract
Coupling inhomogeneities in the Josephson junctions of planar 0-SQUIDs and π-SQUIDs and of three-dimensional SQUIDs are considered. Starting from the properties of planar SQUIDs, it is shown that slight non-homogeneous couplings of the twelve junctions in three-dimensional superconducting interferometers do not affect the periodicity properties of voltage versus applied flux curves. The results are obtained by applying a rigorous and general analytic approach to planar and three-dimensional superconducting devices.

1. Introduction

Superconducting quantum interference devices (SQUIDs) are widely used instruments in experimental physics [1–3] and their dynamical properties are often explained in the limit of null values of their characteristic parameter $\beta = \frac{LI_J}{\Phi_0}$, where L is the inductance of a single branch of the device, I_J is the average value of the maximum Josephson currents of the two junctions and Φ_0 is the elementary flux quantum. In this limiting case, indeed, and for equal values of the maximum Josephson currents and of the resistive parameters of the two junctions in the device in the absence of noise, the physical properties of these superconducting systems are studied by means of an effective single-junction model:

$$\frac{d\varphi_A}{d\tau} + \cos(\pi\psi_{ex})\sin\varphi_A = \frac{i_B}{2}, \tag{1}$$

where $\varphi_A = \frac{\varphi_1+\varphi_2}{2}$ is the average superconducting phase difference across the two junctions, φ_1 and φ_2 being the gauge-invariant superconducting phase differences across the first and second junctions, respectively. In equation (1) ψ_{ex} is the applied flux Φ_{ex} divided by the elementary flux quantum Φ_0, i_B is the bias current normalized to the maximum Josephson current of both junctions and τ is a normalized time variable which we shall define in the following section. From the differential equation (1) it can be inferred that the critical current of the device is periodic, with respect to ψ_{ex}, with period $\Delta\psi_{ex} = 1$; the same is true for the voltage versus applied flux curves. Strictly speaking, however, the model represented by equation (1) is only valid for $\beta = 0$ and for homogeneous and symmetric systems. A planar dc SQUID is represented, schematically, in figure 1(a).

Recently, three-dimensional (3D) SQUIDs [4, 5] have been conceived as ultra-sensitive vectorial magnetic field detectors. A schematic representation of a 3D SQUID is given in figure 1(b), where it appears that this system is equivalent to an elementary cubic network of Josephson junctions. When small structural deformations of these circuital models are considered, it can be shown that the periodicity of voltage versus applied flux curves is still preserved, while the time period suffers slight variations with respect to the symmetrical case, as seen in a first-order perturbation analysis by De Luca and Romeo [6]. The analytic approach developed by the authors, indeed, indicates that the field directions for which the voltage versus applied fluxes show periodicity can be represented on a unitary sphere. These representations have the same properties as those pertaining to the homogeneous network, which models the behaviour of a hypothetical vectorial magnetic field sensor. In this way, also considering future applications of this system, it could be useful to determine its electrodynamic response under coupling inhomogeneity of the Josephson junctions. Starting from a similar analysis for 0-SQUIDs and π-SQUIDs, therefore, we develop a rigorous analysis by which it is shown that the periodicity properties of the voltage versus applied flux curves is preserved in the presence of small coupling inhomogeneity of the Josephson junctions.

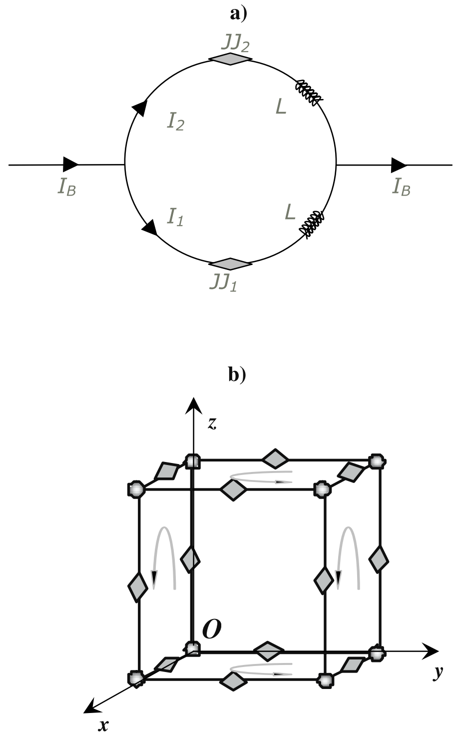

Figure 1. (a) Schematic representation of a planar dc SQUID. Notice that only one loop current can be present. (b) Pictorial representation of a 3D SQUID consisting of six current loops (one for each cubic face) and twelve Josephson junctions. In this system mutual inductance coefficients between loops are to be considered.

The present work is organized as follows. In the following section we write the unified equations for 0-SQUIDs and π-SQUIDs in the presence of coupling inhomogeneity adopting a matrix approach; the dynamical equations for 3D SQUIDs having twelve Josephson junctions with small differences in the Josephson current are also determined. In section 3 the periodicity of the voltage versus applied flux curves is studied for all these systems. Conclusions are drawn in section 4.

2. Dynamical equations in matrix form

2.1. DC SQUIDs

We start by describing the dynamics of the gauge-invariant superconducting phase differences, φ_1 and φ_2, across the two junctions in the dc SQUID represented in figure 1(a) by means of the resistively shunted junction (RSJ) model [1, 2]. Let us assume that the resistive parameters and the maximum Josephson currents of the junctions can be written as follows:

$$R_1 = (1+\delta)R, \qquad R_2 = (1-\delta)R, \qquad (2a)$$

$$I_{J1} = (1+\varepsilon)I_J, \qquad I_{J2} = (1-\varepsilon)I_J, \qquad (2b)$$

where δ and ε describe the relative deviations of the model parameters from the corresponding average values R and I_J. In this way we may set $\tau = \frac{2\pi R I_J}{\Phi_0} t$, where t is the ordinary time variable. The flux Φ threading the superconducting loop is given by

$$\Phi = \Phi_{ex} + L(I_1 - I_2), \qquad (3)$$

where I_1 and I_2 are the two branch currents. Moreover, by imposing the quantization condition for the SQUID loop, we write

$$\frac{2\pi}{\Phi_0}\Phi + \varphi_1 - \varphi_2 = \pi N, \qquad (4)$$

where N is an even integer for 0-SQUIDs and an odd integer for π-SQUIDs [6]. In this way, by means of the RSJ model [1], the dynamical equations for the variables φ_1 and φ_2 can be written as follows:

$$\frac{1}{1+\delta}\frac{d\varphi_1}{d\tau} + (1+\varepsilon)\sin\varphi_1 + \frac{\varphi_1 - \varphi_2}{4\pi\beta} = \frac{1}{2}\left[i_B - \frac{\psi_{ex} - N/2}{\beta}\right], \qquad (5a)$$

$$\frac{1}{1-\delta}\frac{d\varphi_2}{d\tau} + (1-\varepsilon)\sin\varphi_2 - \frac{\varphi_1 - \varphi_2}{4\pi\beta} = \frac{1}{2}\left[i_B + \frac{\psi_{ex} - N/2}{\beta}\right] \qquad (5b)$$

where the following normalized quantities have been introduced:

$$i_1 = \frac{I_1}{I_J}, \qquad i_2 = \frac{I_2}{I_J}, \qquad i_B = \frac{I_B}{I_J}, \qquad \psi_{ex} = \frac{\Phi_{ex}}{\Phi_0}. \qquad (6)$$

By now noticing that $\frac{1}{1\pm\delta} \approx 1 \mp \delta$, to first order in the perturbation parameters, an approximation to which we hereby abide, and by introducing the following matrices:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}; \qquad \mathbf{\Delta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (7)$$

we can symbolically write equations (5a) and (5b) as follows:

$$(\mathbf{1} - \delta\mathbf{\Delta})\frac{d\varphi}{d\tau} + (\mathbf{1} + \varepsilon\mathbf{\Delta})\sin\varphi + \frac{1}{2\pi\beta}\mathbf{A}\varphi = f, \qquad (8)$$

where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \qquad \sin\varphi = \begin{pmatrix} \sin\varphi_1 \\ \sin\varphi_2 \end{pmatrix},$$

and

$$f = \frac{1}{2}\begin{pmatrix} i_B - \frac{\psi_{ex} - N/2}{\beta} \\ i_B + \frac{\psi_{ex} - N/2}{\beta} \end{pmatrix}.$$

Equation (8) can be further simplified by multiplying both sides of the equation by $(\mathbf{1} - \varepsilon\mathbf{\Delta})$ and by noticing that, to first order in the inhomogeneity parameters, we have: $(\mathbf{1} - \varepsilon\mathbf{\Delta})(\mathbf{1} - \delta\mathbf{\Delta}) \approx [\mathbf{1} - (\varepsilon + \delta)\mathbf{\Delta}]$ and $(\mathbf{1} - \varepsilon\mathbf{\Delta})(\mathbf{1} + \varepsilon\mathbf{\Delta}) \approx \mathbf{1}$. We also notice that, when the electrodes of the two junctions are made with the same material, the parameters ε and δ are such that

$$R_1 I_{J1} = R_2 I_{J2} \Rightarrow (1+\varepsilon)(1+\delta)RI_J = (1-\varepsilon)(1-\delta)RI_J, \qquad (9)$$

given that the product $R_k I_{Jk}$ depends only on the electrode's material and on the temperature [2]. In this way, $\varepsilon + \delta = 0$ and equation (8) can be rewritten, after having multiplied both sides by $(\mathbf{1} + \varepsilon\mathbf{\Delta})$, as

$$\frac{d\varphi}{d\tau} + \sin\varphi + \frac{1}{2\pi\beta}\hat{\mathbf{A}}\varphi = \hat{f}, \qquad (10)$$

where $\hat{\mathbf{A}} = (\mathbf{1} - \varepsilon\boldsymbol{\Delta})\mathbf{A}$ and $\hat{f} = (\mathbf{1} - \varepsilon\boldsymbol{\Delta})f$. We have thus obtained a dynamical equation for the non-homogeneous system which is formally identical to the one written in the homogeneous case and, depending on the integer N, written both for a 0-SQUID or a π-SQUID. Notice, however, that the definition of the matrix $\mathbf{A}$ in equation (7) is such as to allow a direct comparison with results in [6], rather than [7].

2.2. 3D SQUIDs

We now turn to analyse the case of a 3D SQUID consisting of twelve Josephson junctions having maximum Josephson current $I_{Jk} = (1 + \varepsilon_k)I_J$ and resistive parameters $R_k = (1 + \delta_k)R$, $k = 1, 2, \ldots, 12$, located at the midpoints of the twelve superconducting edges of a cube, as represented in figure 1(b). The dynamical equations according to the resistively shunted junction (RSJ) model are as follows:

$$\frac{1}{1+\delta_k}\frac{\mathrm{d}\varphi_k}{\mathrm{d}\tau} + (1 + \varepsilon_k)\sin\varphi_k = i_k \tag{11}$$

where the superconducting phase differences φ_k, $k = 1, 2, \ldots, 12$ may be indexed by means of the position vector and the direction in which they lie, as done in [8]. However, here we shall utilize these equations only in their matrix form, so that it is not important how the junctions are indexed. Indeed, by defining the twelve-component vector φ of components φ_k, we may write equation (11) as follows:

$$(\mathbf{1} - \boldsymbol{\Delta})\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + (\mathbf{1} + \boldsymbol{\Lambda})\sin\varphi + \frac{1}{2\pi\beta}\mathbf{A}\varphi = \mathbf{f} \tag{12}$$

where we set $\boldsymbol{\Delta}_{ii} = \delta_i$, $\boldsymbol{\Delta}_{ij} = 0$ for $i \neq j$, $\boldsymbol{\Lambda}_{ii} = \varepsilon_i$, $\boldsymbol{\Lambda}_{ij} = 0$ for $i \neq j$, and the twelve-component current vector equal to $\mathbf{i} = \mathbf{f} - \frac{1}{2\pi\beta}\mathbf{A}\varphi$. The exact form of the matrix $\mathbf{A}$ and of the vector $\mathbf{f}$ has been calculated in [9]. Here we only need to notice that, generalizing equations (2*a*) and (2*b*) and the property in equation (9), we can set: $\sum_{i=1}^{12}\delta_i = 0$, $\sum_{i=1}^{12}\varepsilon_i = 0$ and $\boldsymbol{\Delta} + \boldsymbol{\Lambda} = 0$. As done in the planar SQUIDs case, by multiplying both members of equation (12) by $(\mathbf{1} - \boldsymbol{\Lambda})$, we obtain, to first order in the parameters δ_i and ε_i, $i = 1, 2, \ldots, 12$, the following dynamical equation:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + \sin\varphi + \frac{1}{2\pi\beta}(\mathbf{1} - \boldsymbol{\Lambda})\mathbf{A}\varphi = (\mathbf{1} - \boldsymbol{\Lambda})\mathbf{f}, \tag{13}$$

where again $\beta = \frac{LI_J}{\Phi_0}$, L being in this case the self-inductance of a single branch of the cubic network. It is easy to notice that this equation is formally similar to equation (10).

3. Voltage versus applied flux curves

In the present section we analyse the periodicity of voltage versus applied flux curves both for planar SQUIDs and for 3D SQUIDs.

3.1. Planar SQUIDs

By starting with planar SQUIDs, we first write equation (10) for an increase:

$$\Delta\hat{f} = (\mathbf{1} - \varepsilon\Delta)\Delta f = (\mathbf{1} - \varepsilon\Delta)\frac{\Delta\psi_{\mathrm{ex}}}{2\beta}\begin{pmatrix}1\\-1\end{pmatrix}$$

of the forcing term, where we keep the bias current fixed. We may define the voltage across the junction, normalized to RI_J, by setting $v(\tau) = \frac{\mathrm{d}\varphi_A}{\mathrm{d}\tau}$, where $\varphi_A = \frac{\varphi_1+\varphi_2}{2}$. In this way, we can define the vector $V(\tau)$, whose two components are exactly equal to $v(\tau)$, as follows:

$$V(\tau) = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \mathbf{C}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \tag{14}$$

with the obvious definition of the matrix $\mathbf{C}$. The voltage can also be obtained by equation (10), by writing

$$V(\tau) = \mathbf{C}\left(\hat{f} - \sin\varphi - \frac{1}{2\pi\beta}\hat{\mathbf{A}}\varphi\right). \tag{15}$$

Assume that, by increasing the value of the applied magnetic flux by $\Delta\psi_{\mathrm{ex}}$, we get an increase of the superconducting phases given by $\varphi^{(d)} - \varphi = 2\pi K$, where $K = \binom{k_1}{k_2}$ is a vector with two integer components. At the same, assume the function $V(\tau)$ remains unchanged, so that the quantity $\Delta\psi_{\mathrm{ex}}$ is the periodicity of the time-averaged value $\langle V\rangle$ of the voltage, when expressed as a function of the externally applied flux ψ_{ex}. In this way, we need to have

$$\hat{f}^* - \sin\varphi - \frac{1}{2\pi\beta}\hat{\mathbf{A}}\varphi = \hat{f}^{**} - \sin\varphi^{(d)} - \frac{1}{2\pi\beta}\hat{\mathbf{A}}\varphi^{(d)} \tag{16}$$

where

$$\hat{f}_c^* = \frac{1}{2}(\mathbf{1} - \varepsilon\Delta)\begin{pmatrix}i_B - \frac{\psi_{\mathrm{ex}} - N/2}{\beta}\\ i_B + \frac{\psi_{\mathrm{ex}} - N/2}{\beta}\end{pmatrix}$$

and

$$\hat{f}_c^{**} = \frac{1}{2}(\mathbf{1} - \varepsilon\Delta)\begin{pmatrix}i_B - \frac{\psi_{\mathrm{ex}} + \Delta\psi_{\mathrm{ex}} - N/2}{\beta}\\ i_B + \frac{\psi_{\mathrm{ex}} + \Delta\psi_{\mathrm{ex}} - N/2}{\beta}\end{pmatrix}.$$

Since now $\sin\varphi = \sin\varphi^{(d)}$, equation (13) can be written as follows:

$$\Delta\psi_{\mathrm{ex}}\begin{pmatrix}1\\-1\end{pmatrix} = 2\mathbf{A}K. \tag{17}$$

From equation (17) we have $\Delta\psi_{\mathrm{ex}} = k_2 - k_1$, which implies that the period $\Delta\psi_{\mathrm{ex}}$ is an integer, not necessarily equal to one.

In figures 2(a) and (b) we show the difference between the time-averaged voltage curves obtained numerically for a 0-SQUID and a π-SQUID with $\beta = 0.05$, respectively, and those obtained in the case these systems are not homogeneous ($\varepsilon = -\delta = 0.1$). In these figures the full curves give the homogeneous SQUID behaviour, while the dotted curves are representative of the response of SQUIDs in the presence of inhomogeneous parameters. We notice that, apart from a translation in the ψ_{ex} axis of $\frac{1}{2}$, the curves are perfectly identical. This is clearly a consequence of the fact that the π-SQUID behaviour is obtained by giving a shift of $\psi_{\mathrm{ex}} = \frac{1}{2}$ in the equations of the motion of a conventional dc SQUID.

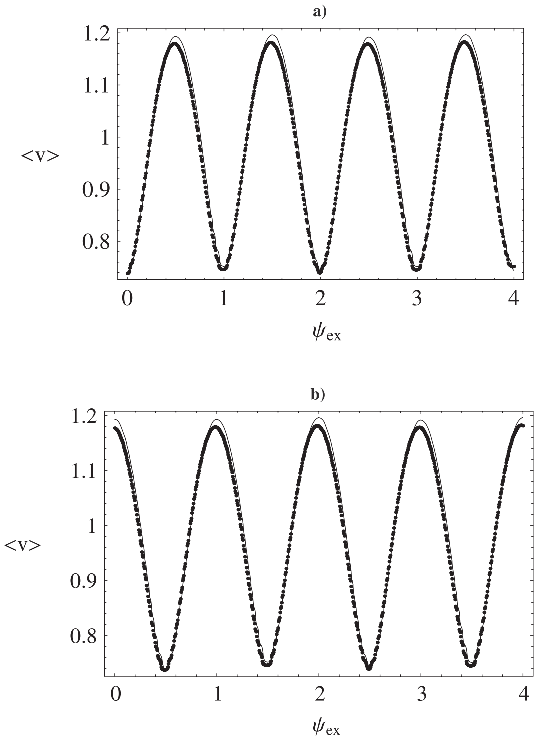

Figure 2. (a) Numerically evaluated normalized voltage versus normalized applied flux for a 0-SQUID for $\beta = 0.05$ and $i_B = 2.5$ and for the homogeneous case (full line) and the inhomogeneous case with $\varepsilon = -\delta = 0.1$ (dotted line). (b) Numerically evaluated normalized voltage versus normalized applied flux for a π-SQUID for $\beta = 0.05$ and $i_B = 2.5$ and for the homogeneous case (full line) and the inhomogeneous case with $\varepsilon = -\delta = 0.1$ (dotted line).

Similar to what can be done in the symmetric case, the asymmetric dc SQUID behaviour can be treated analytically for $\beta = 0$. Let us write down the effective single-junction model in this case:

$$\frac{d\varphi_A}{d\tau} + \cos(\pi\psi_{ex})\sin\varphi_A - \varepsilon\sin(\pi\psi_{ex})\cos\varphi_A = \frac{i_B}{2}. \quad (18)$$

Of course, the model represented by equation (18) reduces to the usual $\beta = 0$ model for symmetric two-junction interferometers given in equation (1). We notice that it is possible, for $\frac{i_B}{2} > \sqrt{\cos^2(\pi\psi_{ex}) + \varepsilon^2\sin^2(\pi\psi_{ex})}$, to integrate the differential equation (18) by separation of variables, so that

$$\tau = \frac{2}{\sqrt{S}}\tan^{-1}\left[\frac{\left(\frac{i_B}{2} - \varepsilon\sin(\pi\psi_{ex})\right)\tan\left(\frac{\varphi_A}{2}\right) - \cos(\pi\psi_{ex})}{\sqrt{S}}\right], \quad (19)$$

where $S = \frac{i_B^2}{4} - \cos^2(\pi\psi_{ex}) - \varepsilon^2\sin^2(\pi\psi_{ex}) > 0$. By inverting the relation found above, one finally has

$$\varphi_A(\tau) = 2\tan^{-1}\left[\frac{\cos(\pi\psi_{ex}) + \sqrt{S}\tan\left(\frac{\sqrt{S}\tau}{2}\right)}{\left(\frac{i_B}{2} - \varepsilon\sin(\pi\psi_{ex})\right)}\right] + 2k\pi, \quad (20)$$

where k is an integer. The pseudo-period

$$T = \frac{2\pi}{\sqrt{S}} \approx \frac{2\pi}{\sqrt{\frac{i_B^2}{4} - \cos^2(\pi\psi_{ex}) - \varepsilon^2\sin^2(\pi\psi_{ex})}}$$

of the average superconducting phase in equation (20) is related to the time-dependent voltage by the simple expression $\langle v\rangle = \frac{2\pi}{T}$ [1]. Therefore, the average voltage is

$$\langle v\rangle = \sqrt{\frac{i_B^2}{4} - \cos^2(\pi\psi_{ex}) - \varepsilon^2\sin^2(\pi\psi_{ex})}. \quad (21)$$

Notice that the quantity $i_{max} = 2\sqrt{\cos^2(\pi\psi_{ex}) + \varepsilon^2\sin^2(\pi\psi_{ex})}$ represents the normalized critical current of the device. The above equation reduces to the well-known expression for the homogeneous and symmetric dc SQUID with $\beta = 0$ to first order in ε. This is thus a direct analytic example that the voltage versus applied is left qualitatively unaltered by inhomogeneity in the junction parameters. Furthermore, in the above example we notice that $\Delta\psi_{ex} = 1$ and that, to first order in ε, the system is seen to behave as if inhomogeneity were absent.

For arbitrary values of β, however, a simple single-junction model is, in general, not appropriate to describe the system's response. In this way, closed analytic expressions cannot be obtained and the analysis of the periodicity of the $\langle v\rangle$ versus ψ_{ex} curves needs to be done by numerical methods. One possible way of tackling the problem is the phase plane analysis for the time-averaged voltage versus normalized applied flux. In this type of analysis, a periodicity in the normalized external flux can be detected by looking at the interval of ψ_{ex} in which the curve closes. One can thus report the numerical derivative $d\langle v\rangle/d\psi_{ex}$ of the time-averaged normalized voltage $\langle v\rangle$ with respect to ψ_{ex} as a function of $\langle v\rangle$. However, the numerical evaluation of $\langle v\rangle$ is generally done by averaging a set of numerical data over a rather long interval of time. This type of evaluation carries along an indeterminacy which may affect the phase plane analysis, giving not conclusive results on a possible period doubling, not excluded by the theoretical analysis done in this section.

In order to illustrate this technique, let us consider the $d\langle v\rangle/d\psi_{ex}$ versus $\langle v\rangle$ curves for a symmetric dc with $\beta = 0$ and $\varepsilon = 0$, and with $\beta = 0$ and $\varepsilon = 0.25$, by retaining the second-order term in equation (21) in the second case. To see the effect on the $\langle v\rangle$ versus ψ_{ex} curves given by junction parameters inhomogeneity, these curves are shown in figure 3(a) for $i_B = 2.5$. One notices that the periodicity $\Delta\psi_{ex} = 1$ is left unaltered by the junction parameters' inhomogeneity, so that the corresponding $d\langle v\rangle/d\psi_{ex}$ versus $\langle v\rangle$ curves shown in figure 3(b) both close after exactly one turn (the ψ_{ex} variable goes though a variation $\Delta\psi_{ex} = 1$). A period doubling effect would have implied a closing of the curves after a variation $\Delta\psi_{ex} = 2$ of the ψ_{ex} variable, which would have appeared as two complete turns in the $d\langle v\rangle/d\psi_{ex}$ versus $\langle v\rangle$ graphs.

Finally, we can notice that in the $\beta = 0$ case we represent the response of the system by a first-order dynamical model, while for $\beta \neq 0$ we need to adopt a second-order model. This dimensionality crossover might induce possible period doubling effects ($\Delta\psi_{ex} = 2$) in the $\langle v\rangle$ versus ψ_{ex} curves. However, due to indeterminacy in the numerical value of $\langle v\rangle$, proof of the existence of period doubling effects becomes a very delicate task which cannot be easily accomplished without a very high control of the numerical accuracy of the $\langle v\rangle$ versus ψ_{ex} curves.

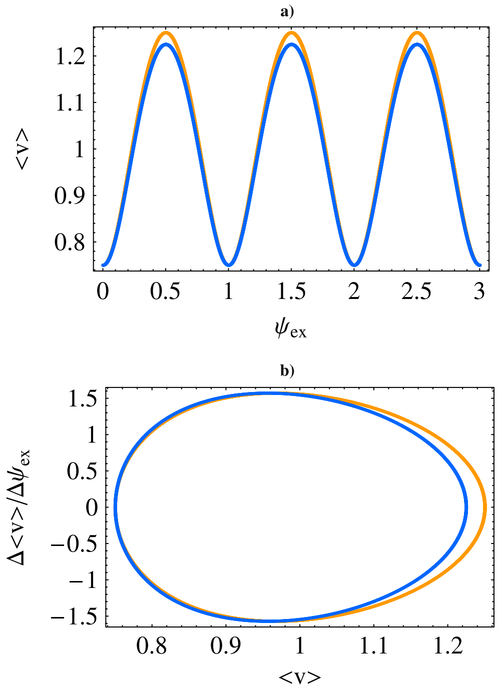

Figure 3. (a) Normalized voltage versus normalized applied flux for a 0-SQUID for $\beta = 0$ and $i_B = 2.5$ and for the homogeneous case (orange curve) and the inhomogeneous case with $\varepsilon = -\delta = 0.25$ (blue curve). (b) Phase space representation of the normalized voltage of a homogeneous 0-SQUID (a) and of an inhomogeneous 0-SQUID for $\beta = 0$, $i_B = 2.5$. On the ordinate the numerical derivative $\mathrm{d}\langle v\rangle/\mathrm{d}\psi_{ex}$ of the time-averaged voltage $\langle v\rangle$ with respect to ψ_{ex} is represented for the homogeneous case (orange curve) and for the inhomogeneous case with $\varepsilon = -\delta = 0.25$ (blue curve).

3.2. 3D SQUIDs

By now recalling that the dynamical equations of 3D SQUIDs have the same form as equation (10), we might infer that the periodicity of the voltage versus applied flux curves is left unaltered by small perturbation of the resistive and coupling parameters, as will be done by briefly recalling the results obtained in [6] and [9]. In order to give the condition for the existence of periodicity of the time-averaged voltage across the branches of an elementary cubic network, for a given direction in space, we start noticing that the voltage vector is given by $v(\tau) = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$, so that the time-averaged voltage is simply $\langle v(\tau)\rangle = \langle\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\rangle$ and a relation similar to equation (17) is valid also for homogeneous 3D SQUIDs [9], namely

$$\Delta\mathbf{f} = -\frac{1}{\beta}\mathbf{A}K, \tag{22}$$

where $\Delta\mathbf{f}$ is the variation in the forcing term, $\mathbf{A}$ is a matrix containing information on the properties of the system, comprising the mutual inductance between loops, and K is a twelve-dimensional vector whose components are integers. In the case of inhomogeneous 3D SQUIDs, the same matrix $(\mathbf{1} - \mathbf{\Lambda})$ multiplies $\Delta\mathbf{f}$ and $\mathbf{A}$ in equation (13), so that the relation corresponding to equation (22) is obtained by multiplying both sides of equation (22) by $(\mathbf{1} - \mathbf{\Lambda})$. The resulting periodicity $\Delta\psi_{ex}$, as was found in the case of planar SQUIDs, is left unaltered and is given by

$$\Delta\psi_{ex} = \sqrt{i^2 + j^2 + k^2}, \tag{23}$$

where the integer indices i, j, k are related to the allowed directions in space of the applied field $\hat{\mathbf{H}}$ given by

$$\hat{\mathbf{H}} = \frac{(i, j, k)}{\sqrt{i^2 + j^2 + k^2}}. \tag{24}$$

It has already been noted [10] that number theory is useful in determining the periodicities in the voltage versus applied flux curves of 3D SQUID. Indeed, so-called *equatorial gaps* appear on the unitary sphere, where field directions in equation (24) are represented. These striped regions, characterized by the absence of periodic behaviour of the voltage versus applied flux curves, can be understood by means of the Gauss condition, by which periodicities of the type $\Delta\psi^*_{ex} = 2^a\sqrt{8b + 7}$, where a and b are nonnegative integers, cannot be observed. In [10] it is also argued, however, that these regions on the unitary sphere are of null measure, so that the probability related to non-periodic voltage versus applied flux curves is effectively zero.

4. Conclusions

The question of the period of voltage versus applied flux curves of planar dc SQUIDs (0-SQUIDs and π-SQUIDs) and 3D SQUIDs for arbitrary values of the parameter β in the presence of small perturbations, ε and δ, in the maximum Josephson currents and resistive parameters of the junctions, respectively, is addressed in the present work.

By a general and rigorous analytical approach it is shown that, to first order in the perturbation parameters ε and δ, the periodicity of the voltage versus applied flux curves of the planar devices is an integer multiple m of Φ_0, or $\Delta\Phi_{ex} = m\Phi_0$, as in the unperturbed case. It is also noted that for $\beta = 0$ the periodicity of voltage versus applied flux curves is $\Delta\Phi_{ex} = \Phi_0$ and that a detailed numerical analysis is needed to show possible period doubling effects in the case $\beta \neq 0$.

For 3D SQUIDs, on the other hand, a similar analysis confirms that the periodicities of $\langle v\rangle$ versus ψ_{ex} curves are left unaltered with respect to the homogeneous system consisting of perfectly identical junctions, when a small perturbation in the maximum Josephson currents and resistive parameters is allowed. These results confirm the possibility of utilizing 3D SQUIDs as ultra-sensitive vectorial magnetic field sensors even in the presence of slight non-homogeneous couplings of the twelve junctions in this three-dimensional superconducting interferometer.

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