Completely integrable N-body quantum systems in three dimensions

J. Phys. A: Math. Gen. **14** (1981) 2581–2594. Printed in Great Britain

Completely integrable N-body quantum systems in three dimensions

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Received 2 March 1981

Abstract. A family of completely integrable three-dimensional N-body quantum systems is introduced and completely solved by the dynamical algebras $O(3N+1, 2)$ and their representations. In a particular realisation of these algebras the particles interact by N-body 'Coulomb-type' potentials. A complete set of commuting integrals of motions, their spectra, the energy levels for both discrete and continuous spectra, and their degeneracy have been explicitly determined. Relativistic generalisations and applications are briefly discussed.

1. Introduction

There are a number of exactly soluble many-body problems in one space-dimension, i.e. N particles moving in a straight line and interacting with some model potentials (see Lax 1968, Calogero 1969, 1971, 1974, Toda 1970, Moser 1965, Olshanetsky and Perelomov 1976, 1977, Airault *et al* 1977, Kostant 1980). In three dimensions the only known exactly soluble N-body problem is the problem of N-coupled harmonic oscillators, and, for $N = 2$ only, the two-body Coulomb problem.

In this paper we present a class of exactly soluble N-body problems in three space dimensions with non-trivial N-particle interaction potentials which, under certain conditions, can also approximate the realistic many-body problems of atomic and nuclear physics. However, beyond this, the main virtue of exactly soluble model systems is to provide us with a complete list of global quantum numbers, besides the usual energy and angular momentum quantum numbers, needed to specify a many-body quantum system. And it is interesting that we obtain for our models conserved 'good' quantum numbers which can be taken to be angular momenta of subsystems. The way in which various choices of a complete set of commuting operators (CSCO) appear here will also be relevant to more realistic problems.

Beside the purely theoretical interest, the exactly soluble model systems presented here constitute a new starting point of perturbation theory to realistic systems. It is perhaps the other extreme to the usual starting point of perturbation theory, namely the N non-interacting particles. In the latter case we introduce the configuration mixing to account for particle interactions. In our case the starting point is a more rigidly coupled system so that we have to introduce a kind of configuration *decoupling*. These problems will be discussed in the second part of these series of papers.

0305-4470/81/102581+14$01.50

Because our approach is algebraic and group theoretical, we begin in § 2 with a general discussion of quantum dynamics based on dynamical groups, and then treat the details of the soluble models in successive section.

2. The dynamical group approach to integrable systems

Our approach generalises the dynamical group methods used in recent years for systems like the H atom or the oscillator. The group theory of these relatively simple systems has been reviewed in several places (Wybourne 1974, Barut 1972, Barut and Rączka 1977, Wulfman 1971). Originally, the dynamical group properties of quantum systems have been obtained from the known quantum mechanical solutions of these systems. Once the general features are realised, one can turn the argument around. Quantisation can be directly based on group representations ('dynamical group quantisation' (Barut 1977)). More generally stated: quantum systems can be defined from the representations of abstract Lie groups (Barut 1980a). We shall now show that 'every Lie algebra and its representation determines in principle a class of completely integrable dynamical systems, once the Hamiltonian and other integrals of the motion have been physically identified'.

This approach has certain advantages:

(i) it provides a coordinate independent treatment,

(ii) all integrals of motion, including the Hamiltonian, are on the same footing (we can define the concept of symmetry *with respect to* any integral of the motion, not just symmetry of the Hamiltonian),

(iii) it provides a unified treatment of classical and quantum physics,

(iv) it provides a method of passage to relativistic dynamics, once the Lorentz group within the dynamical group is physically identified. (The passage to field theory is also possible in principle by the limit $N \to \infty$.) The general postulates for a relativistic treatment of composite systems have been given elsewhere (Barut 1980b).

Let L be a Lie algebra with elements $\{X_i\}$, $i = 1, \ldots, n$, and $\mathscr{E}(L)$ the enveloping algebra of L, (essentially polynomicals in X_i). A complete set of commuting operators (CSCO) can be chosen in L, or in $\mathscr{E}(L)$, or having elements both in L and $\mathscr{E}(L)$. Any such CSCO can in principle be identified with the integrals of motion J_m of a dynamical system. A representation $\pi(L)$ of L, (which automatically extends to a representation of $\mathscr{E}(L)$ on the same Hilbert space) provides us with the space of states which can then be labelled by the eigenvalues of CSCO. Actually, in general, we have to go to the enveloping field of L in which besides the polynomial functions of L we may have operators like $\sqrt{\boldsymbol{q}^2}$ or $1/\sqrt{\boldsymbol{q}^2}$, etc. But this does not change the general procedure.

The more difficult question is whether the representation $\pi(L)$ can be realised canonically in some underlying phase space $\Gamma(p, q)$, if we wish to make an identification with a mechanical N-body problem, for example. On the other hand, in quantum physics, the underlying phase space is not always known. The primary observed quantities are the spectral lines, i.e. the spectra of observables, and their intensities. We probe an unknown quantum system (atom, nucleus, hadron) by external agents (photons, electrons, neutrinos) and try to infer from the response (i.e. spectra) a possible internal dynamics, and a phase space, including spin and other internal degrees of freedom. Thus in this sense the group theory approach may be more direct and at the same time more general than the simple canonical quantum mechanics which has been abstracted from macroscopic classical physics.

In the models treated here we have both dynamical groups and canonical realisations, but the former allow other, more general, realisations or systems and can be generalised to relativistic dynamics.

3. The exactly soluble model systems

We consider the system of N interacting particles characterised by the Hamiltonian

$$H = \frac{1}{2m}\sum_{i=1}^{N} \boldsymbol{p}_i^2 + U(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N). \tag{3.1}$$

Here we present two different types of model systems which are completely integrable. One is the *oscillator-type N-body problem* and the other is the *Coulomb-type N-body problem*. The oscillator-type problems are characterised by the potential

$$U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \tfrac{1}{2}k \sum_{i=1}^{N} \boldsymbol{r}_i^2 + V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \tag{3.2a}$$

and the Coulomb-type problems are characterised by

$$U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = -\zeta\left(\sum_{i=1}^{N} \boldsymbol{r}_i^2\right)^{-1/2} + V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \tag{3.2b}$$

where k and ζ are constants. In order for these problems to have exact analytical solutions, the additional term in the potential, V, must satisfy the equation

$$\sum_{i=1}^{N} \boldsymbol{r}_i \cdot \nabla_i V + 2V = 0. \tag{3.3}$$

Namely, the potential V is a homogeneous function of degree -2. This is true for both the oscillator-type and the Coulomb-type problems. The restriction (3.3) is discussed further in § 4.

The energy eigenvalues and eigenstates of the oscillator-type problems were investigated by Gambardella (1975), using the technique of SU(1, 1) spectrum-generating algebra. He did not discuss, however, the degeneracy structure of the energy eigenspaces. In this paper, we shall consider mainly the Coulomb-type problems which provide also interesting applications to atomic physics (see, for example, White and Stillinger 1970). Mathematically similar problems were discussed before in the form of the so-called 'H atom in n dimensions' (e.g. Bander and Itzykson 1966, Herrick and Sinanoğlu 1973, Rasmussen and Salamo 1979). Here we introduce a new physical interpretation, namely in the form of an N-body problem in three dimensions. We think this is the more proper interpretation because the Coulomb potential in n dimensions is not simply $1/r$ but $1/r^{n-2}$. Using the dynamical group approach, we will study both the energy spectrum and its degeneracy structure thus providing a complete classification of state vectors.

We shall first restrict ourselves to the discussion of the *system of N distinguishable spinless particles*. For the application to atomic physics, however, we must also consider the system of N identical spin-$\frac{1}{2}$ particles, and the strong restrictions imposed by the Pauli principle have to be taken into account. This problem will be discussed in a subsequent paper.

We shall see that both the discrete and continuous spectra can be treated in a unified manner by our formalism. Furthermore, there is a definite group theoretical procedure to extend the theory to the relativistic dynamics.

4. Spectrum-generating algebra O(2, 1) and the energy eigenvalues of the system

We first determine the energy spectrum of an exactly soluble Coulomb-type N-body problem using the techniques of the O(2, 1) spectrum-generating algebra. For simplicity of notation we relabel the position variables and momentum operators as follows:

$$(x_1, y_1, z_1; x_2, y_2, z_2; \ldots; x_N, y_N, z_N)$$
$$\equiv (X_1, X_2, X_3; X_4, X_5, X_6; \ldots; X_{3N-2}, X_{3N-1}, X_{3N}),$$
$$(p_{x_1}, p_{y_1}, p_{z_1}; p_{x_2}, p_{y_2}, p_{z_2}; \ldots; p_{x_N}, p_{y_N}, p_{z_N})$$
$$\equiv (P_1, P_2, P_3; P_4, P_5, P_6; \ldots; P_{3N-2}, P_{3N-1}, P_{3N}). \tag{4.1}$$

We also define

$$r^2 \equiv X_k X_k \qquad p^2 \equiv P_k P_k \qquad r = \sqrt{r^2} \qquad p = \sqrt{p^2} \tag{4.2}$$

where the summation convention is implied by the repeated subindices, e.g. $r^2 = \Sigma_{k=1}^{3N} (X_k)^2$. The first result is that the operators

$$T_1 = \tfrac{1}{2}(rp^2 - r) + rV \qquad T_2 = X_k P_k - \tfrac{1}{2}\mathrm{i}(3N-1) \qquad T_3 = \tfrac{1}{2}(rp^2 + r) + rV \tag{4.3}$$

satisfy the commutation relations of the Lie algebra O(2, 1)

$$[T_1, T_2] = -\mathrm{i}T_3 \qquad [T_2, T_3] = \mathrm{i}T_1 \qquad [T_3, T_1] = \mathrm{i}T_2 \tag{4.4}$$

provided the potential V satisfies

$$[T_2, V] = \mathrm{i}2V. \tag{4.5}$$

In (4.4) the generators are so chosen that the 'compact' generator T_3 has a discrete spectrum, while T_1 and T_2 have continuous spectra. The condition (4.5) is equivalent to

$$X_k \frac{\partial V}{\partial X_k} + 2V = 0 \tag{4.6}$$

namely, the potential V is a homogeneous function of degree -2.

Secondly, the Casimir operator of the O(2, 1) Lie algebra,

$$Q^2_{\mathrm{O}(2,1)} = T_3^2 - T_1^2 - T_2^2 \tag{4.7}$$

is evaluated to be

$$Q^2_{\mathrm{O}(2,1)} = \Lambda^2 + 2r^2 V + \tfrac{3}{4}(3N-1)(N-1) \tag{4.8}$$

where

$$\Lambda^2 = \tfrac{1}{2} \sum_{i,k=1}^{3N} (X_i P_k - X_k P_i)^2. \tag{4.9}$$

In the case

$$V = d/r^2 \qquad d = \text{constant} \tag{4.10}$$

equation (4.8) becomes

$$Q^2_{O(2,1)} = \Lambda^2 + 2d + \tfrac{3}{4}(3N-1)(N-1). \tag{4.11}$$

It is well known that the operator Λ^2 provides the eigenvalue equation (Erdelyi *et al* 1953, Grynberg and Koba 1964)

$$\Lambda^2|\lambda\rangle = \lambda(\lambda + 3N - 2)|\lambda\rangle. \tag{4.12}$$

We now consider the following Hamiltonian equation for the stationary states of our system,

$$H\Psi = \left[\frac{1}{2m}\left(\sum_{i=1}^{N} \boldsymbol{p}_i^2\right) - \zeta\left(\sum_{i=1}^{N} \boldsymbol{r}_i^2\right)^{-1/2} + \kappa\left(\sum_{i=1}^{N} \boldsymbol{r}_i^2\right)^{-1}\right]\Psi = E\Psi \tag{4.13}$$

where κ is a constant.

We introduce the associated equation

$$\Theta\Psi \equiv [r(H-E)]\Psi = 0. \tag{4.14}$$

With (4.3) and (4.13), the operator Θ can be expressed as a linear combination of the O(2, 1) generators as

$$\Theta = \left(\frac{1}{2m} - E\right)T_3 + \left(\frac{1}{2m} + E\right)T_1 + \zeta. \tag{4.15}$$

In equation (4.15), we have replaced κ by d/m. We now diagonalise the operator Θ as follows. Defining

$$\tilde{\Psi} \equiv e^{-i\theta T_2}\Psi \tag{4.16}$$

$$\tilde{\Theta} \equiv e^{-i\theta T_2}\Theta\, e^{i\theta T_2} \tag{4.17}$$

we obtain the transformed equation

$$\tilde{\Theta}\tilde{\Psi} = 0. \tag{4.18}$$

The transformed operator $\tilde{\Theta}$ is calculated to be

$$\tilde{\Theta} = \left(\frac{1}{2m} - E\right)[(\cosh\theta)T_3 + (\sinh\theta)T_1] + \left(\frac{1}{2m} + E\right)[(\sinh\theta)T_3 + (\cosh\theta)T_1] + \zeta. \tag{4.19}$$

4.1. Discrete spectrum

Here we diagonalise T_3 which has a discrete spectrum. If we choose

$$\tanh\theta = \frac{E + 1/2m}{E - 1/2m} \tag{4.20}$$

we obtain from (4.18) and (4.19) the simple equation

$$[(-2E/m)^{1/2}T_3 - \zeta]\tilde{\Psi} = 0. \tag{4.21}$$

In the D^+ representation of the O(2, 1) algebra (the discussion can be found in Wybourne (1974), Barut (1972), Barut and Rączka (1977)), we can write the joint

eigenvalue equations of $Q^2_{O(2,1)}$ and T_3 as

$$Q^2_{O(2,1)}|\phi, s\rangle = \phi(\phi+1)|\phi, s\rangle \tag{4.22}$$

$$T_3|\phi, s\rangle = (-\phi+s)|\phi, s\rangle \tag{4.23}$$

where s is a non-negative integer and ϕ is a real negative number.

Combining equations (4.11), (4.12) and (4.22), we obtain

$$\phi(\phi+1) = \lambda(\lambda+3N-2)+2d+\tfrac{3}{4}(3N-1)(N-1). \tag{4.24}$$

Solving this equation for ϕ, we can write the eigenvalue of T_3 as

$$\begin{aligned} n &= -\phi+s \\ &= s+\tfrac{1}{2}+[\lambda(\lambda+3N-2)+2d+\tfrac{1}{4}(3N-2)^2]^{1/2} \\ s &= 0, 1, 2, 3, \ldots . \end{aligned} \tag{4.25}$$

Comparing equation (4.21) with (4.23), we obtain

$$\zeta(-m/2E)^{1/2} = n.$$

Thus we have obtained the energy spectrum

$$E_n = -\zeta^2 m/2n^2. \tag{4.26}$$

In the special case, when

$$d = 0 \tag{4.27}$$

the principal quantum number n has a very simple form

$$n = s+\lambda+\tfrac{1}{2}(3N-1) \tag{4.28}$$

and for $N = 1$ and $d = 0$, equation (4.26) reduces to the Balmer formula of the hydrogen atom.

4.2. Continuous spectrum

In equation (4.19) we can also diagonalise the generator T_1 instead of T_3. In this case we choose the tilting angle θ such that the coefficient of T_3 vanishes, namely,

$$\tanh\theta = \frac{E-1/2m}{E+1/2m} \tag{4.29}$$

Whence $\sinh\theta = (E-1/2m)/(2E/m)^{1/2}$, $\cosh\theta = (E+1/2m)/(2E/m)^{1/2}$ and equation (4.19) becomes

$$[(2E/m)^{1/2}T_1+\zeta]\tilde{\Psi} = 0. \tag{4.30}$$

Denoting the real continuous spectrum of T_1 by k, $T_1|k\rangle = k|k\rangle$, we obtain

$$E_k = m\zeta^2/2k^2 \qquad -\infty < k < +\infty. \tag{4.31}$$

5. Dynamical algebra O(3N + 1, 2) and a classification of degenerate states

The spectrum-generating algebra (4.3) does not solve the complete degeneracy of levels of the system, because we have not yet studied the complete set of commuting operators

(CSCO). Since we are restricting our discussion to the system of N distinguishable spinless particles, we take the angular momentum of each particle as a constant of the motion rather than the particle exchange operators P_{ij} (which mix the states of different angular momenta of particles).

In order to exhibit the CSCO we extend the O(2, 1) algebra to a larger algebra $O(3N+1, 2)$ for the special case

$$V = 0. \tag{5.1}$$

We find that the operators L_{AB} in equation (5.4) satisfy the commutation relations of the Lie algebra $O(3N+1, 2)$

$$[L_{AB}, L_{CD}] = -\mathrm{i}(g_{AC}L_{BD} + g_{BD}L_{AC} - g_{AD}L_{BC} - g_{BC}L_{AD}) \tag{5.2}$$

with

$$g_{MM} = (\underset{1}{-}, \underset{2}{-}, \ldots, \underset{3N}{-}, \underset{3N+1}{-}, \underset{3N+2}{+}, \underset{3N+3}{+}) \tag{5.3}$$

$$g_{MN} = 0 \qquad M \neq N.$$

A basis of this Lie algebra is given by the operators

$$\begin{aligned}
&\Lambda \to L_{ij} = X_iP_j - X_jP_i \\
&A_j \equiv L_{j,3N+1} = \tfrac{1}{2}X_jp^2 - P_jX_kP_k + \mathrm{i}\tfrac{3}{2}(N-1)P_j - \tfrac{1}{2}X_j \\
&M_j \equiv L_{j,3N+2} = \tfrac{1}{2}X_jp^2 - P_jX_kP_k + \mathrm{i}\tfrac{3}{2}(N-1)P_j + \tfrac{1}{2}X_j \\
&\Gamma_j \equiv L_{j,3N+3} = rP_j \\
&T_2 \equiv L_{3N+1,3N+2} = X_kP_k - \tfrac{1}{2}\mathrm{i}(3N-1) \\
&T_3 \equiv L_{3N+2,3N+3} = \tfrac{1}{2}(rp^2 + r) \\
&T_1 \equiv L_{3N+1,3N+3} = \tfrac{1}{2}(rp^2 - r) \qquad i, j, k = 1, 2, \ldots, 3N.
\end{aligned} \tag{5.4}$$

Some of the second-order Casimir operators of the $O(3N+1, 2)$ and its subalgebras are

$$\begin{aligned}
Q^2_{O(2,1)} &= T_3^2 - T_1^2 - T_2^2 = \Lambda^2 + \tfrac{3}{4}(3N-1)(N-1) \\
&= Q^2_{O(3N)} + \tfrac{3}{4}(3N-1)(N-1) && (5.5a)
\end{aligned}$$

$$Q^2_{O(3N)} = \Lambda^2 \tag{5.5b}$$

$$Q^2_{O(3N+1)} = \Lambda^2 + A_kA_k = T_3^2 - \tfrac{1}{4}(3N-1)^2 \tag{5.5c}$$

$$\begin{aligned}
Q^2_{O(3N+1,1)} &= \Lambda^2 + A_kA_k - M_kM_k - T_2^2 \\
&= -\tfrac{1}{4}(3N-1)(3N+1) && (5.5d)
\end{aligned}$$

$$\begin{aligned}
Q^2_{O(3N+1,2)} &= \Lambda^2 + A_kA_k - M_kM_k - \Gamma_k\Gamma_k + T_3^2 - T_1^2 - T_2^2 \\
&= -\tfrac{3}{4}(3N-1)(N+1). && (5.5e)
\end{aligned}$$

We shall now show that the set of Casimir operators of the following subgroup chain (5.6) gives a complete set of commuting operators of our system.

$$\begin{array}{ccc}
O(3N+1, 2) \supset O(\overset{\lambda}{3N}) \times O(\overset{\lambda}{2}, 1) & & \\
\cup & & \\
O(2) & & \\
{\scriptstyle n} & &
\end{array} \tag{5.6a}$$

$$\begin{array}{l} O(3N+1,2) \supset O(3\overset{n}{N}+1)\times \overset{n}{O}(2) \\ \qquad\qquad\qquad \cup \\ \qquad\qquad\qquad \overset{\lambda}{O}(3N) \\ \overset{\lambda}{O}(3N) \supset O(3\overset{\lambda_{N-1}}{N}-3)\times^{N} O(3)^{(l_N,m_N)} \\ \qquad\qquad \cup \\ \qquad O(3\overset{\lambda_{N-2}}{N}-6)\times^{N-1} O(3)^{(l_{n-1},m_{N-1})} \\ \qquad\qquad \cup \\ \qquad\qquad \vdots \\ \qquad\qquad \cup \\ \qquad O(6)^{\lambda_2}\times^{3} O(3)^{(l_3,m_3)} \\ \qquad\qquad \cup \\ \qquad {}^{1}O(3)^{(l_1,m_1)}\times^{2} O(3)^{(l_2,m_2)}. \end{array} \tag{5.6b,c}$$

Here we also indicated the quantum numbers due to these subgroups.

In equation (5.6c), Casimir operators of $O(3N)$, $O(3N-3)$, $O(3N-6), \ldots, O(6)$, $O(3)$ are

$$Q^2_{O(3j)} \equiv \Lambda_j^2 \equiv \tfrac{1}{2}\sum_{i,k=1}^{3j} (X_i P_k - X_k P_i)^2 \tag{5.7}$$

where $j = 1, 2, \ldots, N$ and $\Lambda^2 = \Lambda_N^2$, $\boldsymbol{L}_1^2 = \Lambda_1^2$.

In terms of harmonic polynomials (Erdelyi *et al* 1953, Grynberg and Koba 1964), one can construct an orthonormal set of simultaneous eigenfunctions of the Casimir operators which label all the subgroups in equation (5.6c). We denote these polynomials by

$$H_\lambda(\gamma, l, m \,|\, \xi_1, \xi_2, \ldots, \xi_{3N}) \tag{5.8}$$

with $\gamma = \{\gamma_2, \gamma_3, \ldots, \gamma_N\}$, $l = \{l_1, l_2, \ldots, l_N\}$, $m = \{m_1, m_2, \ldots, m_N\}$ and $\xi_i = X_i/r$. These quantum numbers must satisfy

$$\lambda_j = \lambda_{j-1} + l_j + 2\gamma_j \qquad \gamma_j = 0, 1, 2, \ldots \tag{5.9a}$$

and

$$-l_i \leqslant m_i \leqslant l_i \tag{5.9b}$$

with $\lambda_1 = l_1$, $\lambda_N = \lambda$. The corresponding eigenvalue equations are

$$\begin{aligned} &\Lambda_j^2 H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) = \lambda_j(\lambda_j + 3j - 2) H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) \\ &\boldsymbol{L}_i^2 H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) = l_i(l_i+1) H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) \\ &(L_z)_i H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) = m_i H_\lambda(\gamma, l, m \,|\, \xi_1, \ldots, \xi_{3N}) \end{aligned} \tag{5.10}$$

where

$$\boldsymbol{L}_i^2 = Q^2_{i_{O(3)}}. \tag{5.11}$$

In equation (5.8) we regarded γ_j as a quantum number and λ_j as simply a convenient

name for the sum in equation (5.9*a*). The polynomials (5.8) are homogeneous of degree λ and satisfy the Laplace equation

$$\sum_{i=1}^{3N} \frac{\partial^2}{\partial X_i^2}[r^\lambda H_\lambda(\gamma, l, m\,|\xi_1, \ldots, \xi_{3N})] = 0. \tag{5.12}$$

For each λ, there are

$$d_\lambda(N) = (2\lambda + 3N - 2)\frac{(\lambda + 3N - 3)!}{\lambda!(3N-2)!} \tag{5.13}$$

linearly independent polynomials. Enumeration of different sets of quantum numbers allowed by equation (5.9) leads to equation (5.13).

In the subgroup chain (5.6), we notice that $Q^2_{O(3N)}$ and $Q^2_{O(2,1)}$ are labelled by the same quantum number λ, and $Q^2_{O(3N+1)}$ and T_3 are labelled by the same quantum number n. We can take the basis to be the simultaneous eigenvectors of the Casimir operators of the subgroups shown in equation (5.6). Thus the state vectors are denoted by

$$|\tilde{n}\gamma lm\rangle \equiv |\tilde{n};\ \gamma_N, \gamma_{N-1}, \ldots, \gamma_2;\ l_N, l_{N-1}, \ldots, l_1;\ m_N, m_{N-1}, \ldots, m_1\rangle. \tag{5.14}$$

They satisfy the eigenvalue equations

$$T_3|\tilde{n}\gamma lm\rangle = n|\tilde{n}\gamma lm\rangle \tag{5.15a}$$

$$\Lambda_j^2|\tilde{n}\gamma lm\rangle = \lambda_j(\lambda_j + 3j - 2)|\tilde{n}\gamma lm\rangle \tag{5.15b}$$

$$\boldsymbol{L}_i^2|\tilde{n}\gamma lm\rangle = l_i(l_i+1)|\tilde{n}\gamma lm\rangle \tag{5.15c}$$

$$(L_z)_i|\tilde{n}\gamma lm\rangle = m_i|\tilde{n}\gamma lm\rangle. \tag{5.15d}$$

The vectors $|\tilde{n}\gamma lm\rangle$ are basis states of the representation of the dynamical group $O(3N+1, 2)$. The Schrödinger states $|n\gamma lm\rangle$ are written by virtue of equation (4.16) as

$$|n\gamma lm\rangle = e^{i\theta T_2}|\tilde{n}\gamma lm\rangle. \tag{5.16}$$

We can of course diagonalise other sets of operators, but the diagonalisation shown above is convenient for many applications because the angular momenta of individual particles are retained. Because of equations (5.5*a*) and (5.5*c*), we can write the weight diagram given in figure 1. Each point in the diagram represents a subspace of state

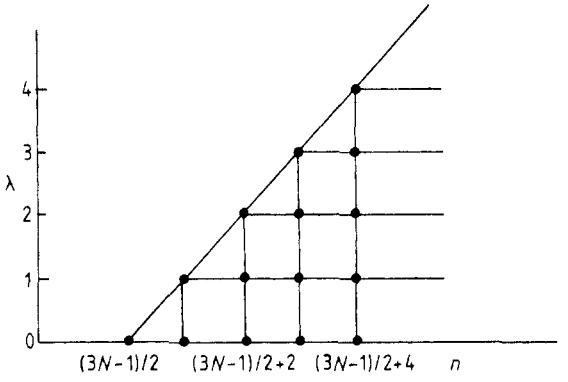


Figure 1. The weight diagram of the dynamical algebra $O(3N+1, 2)$.

vectors which are further classified by the quantum numbers $\gamma_{N-1}, \gamma_{N-2}, \ldots, \gamma_2$; l_N, $l_{N-1}, \ldots, l_1$; $m_N, m_{N-1}, \ldots, m_1$. By O(2, 1) raising and lowering operators, a state vector can be transformed into another which has the same λ but different n. These O(2, 1) transformations are represented by a horizontal line in figure 1. States on each vertical line give a basis set of the degeneracy group O($3N+1$). In the next section, we determine the dimensions of the representations of the degeneracy group.

6. Dimensions and representations of the degeneracy group O($3N+1$)

In this section we calculate dimensions of degenerate levels, from which we determine the representations of the degeneracy group O($3N+1$). The dimension of a degenerate level can be obtained by counting the number of states having the same principal quantum number n. Comparing the result with the dimension of a given representation of O($3N+1$), we can identify the representation of our degeneracy group. We can use Weyl's formula (Wybourne 1974, ch 14) for a given representation to calculate its dimension.

The dimension of O($3N+1$) labelled by n is calculated by summing the dimensions of allowed representations of its subgroup O($3N$) labelled by λ;

$$D_n(N) = \sum_{\lambda=0}^{n-\frac{1}{2}(3N-1)} d_\lambda(N) \tag{6.1}$$

where $D_n(N)$ and $d_\lambda(N)$ are dimensions of the O($3N+1$) and the O($3N$) representations, respectively. The expression for $d_\lambda(N)$ is given in equation (5.13). We obtain

$$D_n(1) = n^2 \qquad D_n(2) = \tfrac{1}{60} n (n^2 - \tfrac{9}{4})(n^2 - \tfrac{1}{4}) \ldots . \tag{6.2}$$

Some numerical values are given in table 1.

Table 1. Values of $D_n(N)$, $(k_1 = n - \frac{1}{2}(3N-1))$.

	$N=1$	$N=2$	$N=3$	$N=4$
k_1	O(4) D_2	O(7) B_3	O(10) D_5	O(13) B_6
0	1	1	1	1
1	4	7	10	13
2	9	27	54	90
3	16	77	210	442
4	25	182	660	1729

On the other hand, we can use Weyl's formula to calculate the dimensions of the representations of a compact group (Wybourne 1974, ch 14). We consider the special representations of O($3N+1$) denoted by

$$\begin{aligned} &[k_1, \underbrace{0, 0, 0, \ldots}_{\{\frac{1}{2}(3N+1)-1\} \text{ times}}] && \text{for } N \text{ odd} \\ &[k_1, \underbrace{0, 0, 0, \ldots}_{(\frac{1}{2}N-1) \text{ times}}] && \text{for } N \text{ even.} \end{aligned} \tag{6.3}$$

Here we used Cartan–Weyl labelling scheme (Wybourne 1974, ch 12) for $O(3N+1)$. For the representations of (6.3), Weyl's formula leads to expressions (Wybourne 1974, ch 14)

$$\begin{aligned} D_{[k_1,0,0,0,\ldots]} &= \prod_{i=2}^{\frac{1}{2}(3N+1)} \left(\frac{w_1^2-g_i^2}{g_1^2-g_i^2}\right) && N \text{ odd} \\ D_{[k_1,0,0,0,\ldots]} &= \left(\frac{v_1}{f_1}\right) \prod_{i=2}^{(3N/2)} \left(\frac{v_1^2-f_i^2}{f_1^2-f_i^2}\right) && N \text{ even} \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} g_i &= \tfrac{1}{2}(3N+1)-i & w_1 &= k_1+g_1 \\ f_i &= \tfrac{3}{2}N-i+\tfrac{1}{2} & v_1 &= k_1+f_1. \end{aligned} \tag{6.5}$$

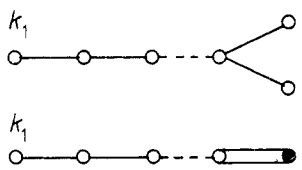
These equations give exactly the same results as those in table 1;

$$\begin{aligned} D_n(N) &= D_{[n-\frac{1}{2}(3N-1),\underbrace{0,0,0,\ldots}_{\{\frac{1}{2}(3N+1)-1\} \text{ times}}]} && N \text{ odd} \\ D_n(N) &= D_{[n-\frac{1}{2}(3N-1),\underbrace{0,0,0,\ldots}_{\{\frac{3}{2}N-1\} \text{ times}}]} && N \text{ even.} \end{aligned} \tag{6.6}$$

Hence the representations of the degeneracy group $O(3N+1)$ can be denoted by

$$\begin{aligned} &[n-\tfrac{1}{2}(3N-1), \overbrace{0,0,0,\ldots}^{\{\frac{1}{2}(3N+1)-1\} \text{ times}}] && N \text{ odd} \\ &[n-\tfrac{1}{2}(3N-1), \overbrace{0,0,0,\ldots}^{\{\frac{3}{2}N-1\} \text{ times}}] && N \text{ even.} \end{aligned} \tag{6.7}$$

In another notation using Dynkin diagrams (Wybourne 1974, ch 12), these are



for N odd

for N even. (6.8)

7. More about the dynamical algebra for $N = 2$

In equation (5.4), we presented a realisation of the $O(3N+1, 2)$ algebra. There is another realisation of the $O(3N+1, 2)$ given by Kyriakopoulos (1969). It is interesting to compare these two. Here we restrict our discussion to the case of $N = 2$, the $O(7, 2)$ algebra. Kyriakopoulos realisation of the $O(7, 2)$ is given by operators

$$\begin{aligned} L_{\alpha\beta} &= -\mathrm{i}(a_\alpha^\dagger a_\beta - a_\beta^\dagger a_\alpha) \\ L_{\alpha 8} &= \tfrac{1}{2}\mathrm{i}(a_\rho^\dagger a_\rho^\dagger a_\alpha - 2a_\alpha^\dagger a_\rho^\dagger a_\rho - 5a_\alpha^\dagger + a_\alpha) \\ L_{\alpha 9} &= -\tfrac{1}{2}(a_\rho^\dagger a_\rho^\dagger a_\alpha - 2a_\alpha^\dagger a_\rho^\dagger a_\rho - 5a_\alpha^\dagger - a_\alpha) \\ L_{89} &= a_\rho^\dagger a_\rho + \tfrac{5}{2} \qquad \alpha, \beta, \rho = 1, 2, \ldots, 7 \end{aligned} \tag{7.1}$$

In the above expressions summation is assumed over repeated indices. The Boson creation and annihilation operators $a_\rho^\dagger$, a_ρ are used and they satisfy the commutation relations

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta} \qquad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0. \tag{7.2}$$

We now define a new set of creation and annihilation operators which also satisfy

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta} \qquad [b_\alpha, b_\beta] = [b_\alpha^\dagger, b_\beta^\dagger] = 0 \tag{7.3a}$$

by

$$b_1^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + \mathrm{i}a_2^\dagger) \qquad b_2^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger - \mathrm{i}a_2^\dagger) \qquad b_3^\dagger = a_3^\dagger$$

$$b_4^\dagger = \frac{1}{\sqrt{2}}(a_4^\dagger + \mathrm{i}a_5^\dagger) \qquad b_5^\dagger = \frac{1}{\sqrt{2}}(a_4^\dagger - \mathrm{i}a_5^\dagger) \qquad b_6^\dagger = a_6^\dagger$$

$$b_7^\dagger = a_7^\dagger. \tag{7.3b}$$

In order to construct representations, we choose basis states of the form

$$|n_1 n_2 n_3; n_4 n_5 n_6; n_7\rangle = (n_1! n_2! n_3! n_4! n_5! n_6! n_7!)^{-1/2} b_1^{\dagger n_1} b_2^{\dagger n_2} b_3^{\dagger n_3} b_4^{\dagger n_4} b_5^{\dagger n_5} b_6^{\dagger n_6} b_7^{\dagger n_7} |0\rangle \tag{7.4}$$

where the operators $b_\rho^\dagger$ operate on the vacuum state $|0\rangle$.

We try to express the state vectors $|\tilde{n}\gamma lm\rangle$ in equation (5.14) in terms of basis states $|n_1 n_2 n_3; n_4 n_5 n_6; n_7\rangle$. Namely, we construct the simultaneous eigenvectors of the operators L_{89}, $Q^2_{O(6)}$, $L^2_{(1)}$, $L^2_{(2)}$, L_{12}, L_{45}, where

$$Q^2_{O(6)} = \sum_{i=1}^{5}\sum_{j=2}^{6} L_{ij}^2 \qquad L^2_{(1)} = L_{12}^2 + L_{13}^2 + L_{23}^2 \qquad L^2_{(2)} = L_{45}^2 + L_{46}^2 + L_{56}^2 \tag{7.5}$$

and they satisfy eigenvalue equations

$$\begin{aligned}
&L_{89}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = n|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \\
&Q^2_{O(6)}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = \lambda(\lambda+4)|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \\
&L^2_{(1)}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = l_1(l_1+1)|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \\
&L^2_{(2)}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = l_2(l_2+1)|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \\
&L_{12}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = m_1|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \\
&L_{45}|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle = m_2|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle.
\end{aligned} \tag{7.6}$$

In the above equations,

$$\lambda = l_1 + l_2 + 2\gamma_2 \qquad \gamma_2 = 0, 1, 2, \ldots \tag{7.7}$$

and

$$|\tilde{n}, \gamma_2, l_1 m_1, l_2 m_2\rangle \equiv |\tilde{n}; \gamma_2; l_2, l_1; m_2, m_1\rangle. \tag{7.8}$$

Operating L_{89}, $Q^2_{O(6)}$, $L^2_{(1)}$, $L^2_{(2)}$, L_{12}, L_{45} and

$$Q^2_{O(7)} = \sum_{i=1}^{6}\sum_{\alpha=2}^{7} L_{i\alpha}^2 \qquad Q^2_{O(2,1)} = L_{89}^2 - L_{79}^2 - L_{78}^2 \tag{7.9}$$

on the basis vectors $|n_1n_2n_3; n_4n_5n_6; n_7\rangle$, we find the following results. For $n=\frac{5}{2}$,

$$|\tilde{n}=\tfrac{5}{2}, \gamma_2=0, S(1), S(2)\rangle = |000; 000; 0\rangle. \tag{7.10a}$$

For $n=\frac{7}{2}$,

$$\begin{aligned}
&|\tilde{n}=\tfrac{7}{2}, \gamma_2=0, p_1(1), s(2)\rangle = |100; 000; 0\rangle\\
&|\tfrac{7}{2}, 0, p_{-1}(1), s(2)\rangle = |010; 000; 0\rangle\\
&|\tfrac{7}{2}, 0, p_0(1), s(2)\rangle = |001; 000; 0\rangle\\
&|\tfrac{7}{2}, 0, s(1), p_1(2)\rangle = |000; 100; 0\rangle\\
&|\tfrac{7}{2}, 0, s(1), p_{-1}(2)\rangle = |000; 010; 0\rangle\\
&|\tfrac{7}{2}, 0, s(1), p_0(2)\rangle = |000; 001; 0\rangle\\
&|\tfrac{7}{2}, 0, s(1), s(2)\rangle = |000; 000; 1\rangle.
\end{aligned} \tag{7.10b}$$

In these equations, $p_{-1}(2)$ means $l_2=1$, $m_2=-1$, etc. In general, $|\tilde{n}, \gamma_2, l_1m_1, l_2m_2\rangle$ is expressed by a linear combination of basis vectors $|n_1n_2n_3; n_4n_5n_6; n_7\rangle$, for example

$$|\tfrac{9}{2}, 0, s(1), d_0(2)\rangle = (1/\sqrt{3})|000; 110; 0\rangle - (2/3)^{1/2}|000; 002; 0\rangle$$

$$\begin{aligned}
&|\tfrac{9}{2}, 2, s(1), s(2)\rangle\\
&\quad = (1/\sqrt{3})\{|110; 000; 0\rangle - |000; 110; 0\rangle\}\\
&\qquad + (1/\sqrt{6})\{|002; 000; 0\rangle - |000; 002; 0\rangle\}
\end{aligned} \tag{7.11}$$

For a realisation of the dynamical algebra given by equation (5.4), we identically have the relations

$$Q^2_{O(7)} = L^2_{89} - \tfrac{25}{4} \qquad Q^2_{O(6)} = Q^2_{O(2,1)} - \tfrac{15}{4}. \tag{7.12}$$

However, for the Kyriakopoulos realisation, equations in (7.12) do not hold identically. They hold as operator identities for state vectors in equations (7.10), (7.11).

The realisation of O(7, 2) discussed in § 5 and the Kyriakopoulos realisation with basis vectors $|n_1n_2n_3; n_4n_5n_6; n_7\rangle$, both correspond to the special representation designated by

$$[n-\tfrac{5}{2}, 0, 0, 0, \ldots]. \tag{7.13}$$

8. Conclusions

We have completely solved a family of three-dimensional N-body problems at the quantum level. The corresponding classical problem is also soluble by using Poisson brackets (Barut 1980a). The results have been given for a particular choice of the complete set of commuting integrals of the motion, but our group theoretical procedure suggests how to choose many other CSCO's appropriate for other applications.

As these systems represent to our knowledge, the first non-trivial three-dimensional integrable N-body problems, one can learn quite a bit about the general properties of integrable N-body quantum systems in general. Moreover, they can be taken to be the 'unperturbed' problem for a new perturbation theory for more realistic N-body problems in atomic and nuclear physics. These and other generalisations will be reported in subsequent work.

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