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# Nonequilibrium mesoscopic superconductors in a fluctuational regime

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**Abstract** – We develop a non-equilibrium Ginzburg-Landau-type theory of the far-from-equilibrium dynamics of superconductors in a fluctuational regime and apply our approach to quantitative description of a superconductor island in a stationary nonequilibrium state. We derive the effective temperature of the nonequilibrium state and find fluctuational contributions to the magnetic susceptibility showing that it becomes a singular function of  $\sqrt{V - V_c}$ , where  $V$  is the external drive and  $V_c$  is its “critical” value at which the nonequilibrium phase transition takes place.

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Time-dependent Ginzburg-Landau equation (TDGL) successfully describes *weakly* non-equilibrium dynamics of the systems experiencing a second-order phase transition, including the ferromagnetic, superconducting, and the superfluid transitions, to name a few [1–10], in the vicinity of the critical point. A marked progress has been recently achieved [9] in formulation Keldysh technique-based approach aimed at extension of TDGL onto strongly non-equilibrium situation, see, *e.g.* the review ref. [11]. Yet constructing a theory of far from the equilibrium dynamics for the GL systems remains a major challenge of the nonequilibrium statistical physics.

A nonequilibrium extension of the Ginsburg-Landau theory (NGL) requires, in principle, the details of the underlying microscopic physics. The latter enter the theory through the quasiparticle density of states and relaxation rates that appear as parameters in corresponding kinetic equations (KE) for quasiparticle distribution functions which, in their turn, depend on the order parameter  $\Delta$  [3,4]. However, as we show below, one can construct a phenomenological non-equilibrium theory in a critical region without invoking the details of the microscopic behavior of a low-symmetry phase, making use of the symmetry considerations in a spirit of ref. [12].

In this letter we develop a theory of the *far-from-equilibrium* fluctuation effects in superconductors generalizing

a standard description of superconducting fluctuations [9,13] to the far-from-equilibrium state and derive a closed GL-like equations set, describing quantitatively far-from-equilibrium dynamics of fluctuations. In particular, we calculate the nonequilibrium fluctuation contribution to magnetic susceptibility and find an analytical expression for the effective temperature,  $T_{\text{eff}} = T \cosh^2(V/4T)$ , depending on the bias,  $V$ . As an illustration of the proposed general technique, we discuss the dynamics of a superconducting granule connected to reservoir via disordered normal wires.

In the framework of a phenomenological theory of a second order phase transition, the properties of the low-temperature phase near the transition are determined by the free-energy functional written as an expansion in the order parameter, see, *e.g.* refs. [12,14]. In an equilibrium, the density matrix space  $\mathbb{M}$  is parameterized by the temperature and the transition occurs at  $T = T_c$ ; the corresponding dimensionless parameter of the Landau expansion is  $|\tau_{\text{GL}}^{(\text{eq})} T_c|^{-1} \sim |T - T_c|/T_c \ll 1$  (we will use the units where  $\hbar = k_B = e = 1$  throughout the paper), where  $\tau_{\text{GL}}^{(\text{eq})}$  is the Ginzburg-Landau relaxation time in equilibrium. In the general nonequilibrium case the transition extends over some *surface* in  $\mathbb{M}$  (equilibrium density matrices form a zero measure subspace of  $\mathbb{M}$ ).

We consider a system where the excitations and their kinetics are well defined. Then the nonequilibrium GL theory has a form similar to that of the equilibrium one,

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provided the distribution functions of the excitations are stationary. Following the general recipe [11] for treating an out of the equilibrium system, one is to use the Keldysh partition function, instead of the conventional partition function for the equilibrium case, and, accordingly, the Keldysh action for the order parameter replaces the GL free energy in equilibrium. The parameters of the nonequilibrium GL-expansion are functionals of the excitation distribution functions. The closeness of the system to the (non-equilibrium) phase transition surface is determined by the dimensionless parameter  $|\tau_{\text{GL}}T_{\text{eff}}|^{-1} \ll 1$ , where  $T_{\text{eff}}$  is the nonequilibrium energy scale replacing  $T$  in an out-of-equilibrium state and  $\tau_{\text{GL}}$  is a general, nonequilibrium, GL relaxation time replacing  $\tau_{\text{GL}}^{(\text{eq})}$ . This inequality is the necessary condition for applicability of our approach. Note that the existence of  $T_{\text{eff}}$  does not imply the local equilibrium form of the excitation distributions.

The coefficients of the nonequilibrium GL functional behave differently as compared to those of the equilibrium. In particular, the coefficient at the fourth-order term in the order parameter,  $\Delta$ , can even change the sign at large driving forces. This signals the onset of an instability of the NGL equations solution and means that the NGL functional should be expanded to higher orders with respect to  $\Delta$ . In the context of superconductivity, the equilibrium GL-expansion has the usual form [14]:

$$\Omega[\Delta] = \nu a \int_0^\beta d\tau d\mathbf{r} \left\{ \Delta^* L^{-1} \Delta + \frac{b}{2T} |\Delta|^4 \right\}, \quad (1)$$

where  $L^{-1} = i\partial_\tau + (\tau_{\text{GL}}^{(\text{eq})})^{-1} - D\partial_{\mathbf{r}}^2$ ,  $(\tau_{\text{GL}}^{(\text{eq})})^{-1} = (T - T_c)\alpha$ ,  $Z_\Delta = \int D\Delta D\Delta^* \exp\{-\Omega[\Delta, \Delta^*]\}$  is the partition function, and  $\nu$  is the density of states (DoS) at the Fermi shell. In a disordered metal model  $a = \pi/8T$ ,  $\alpha = 8/\pi$ ,  $D$  is the diffusion coefficient and  $b = 7\zeta(3)/\pi^3$  [1].

The order parameter in the Keldysh space has two components, corresponding to the lower and upper branches of the Keldysh time-contour. To simplify the structure of the action, it is convenient to use the rotated basis and introduce ‘‘classical’’ ( $\Delta_1$ ) and ‘‘quantum’’ ( $\Delta_2$ ) components of the order parameter (half-sum and half-difference of the order parameter values at the lower and upper branches of the Keldysh time-contour) [9,15]. Thus  $\vec{\Delta} = (\Delta_1, \Delta_2)^\tau$ . The Keldysh analog of the partition function  $Z_\Delta$  is

$$\mathcal{Z}_\Delta = \int D\vec{\Delta} D\vec{\Delta}^* \exp\{iS_\Delta[\vec{\Delta}, \vec{\Delta}^*]\}. \quad (2)$$

The average order parameter should be identified with  $\langle \Delta_1 \rangle$ , while  $\langle \Delta_2 \rangle = 0$ , where averaging is based on  $\mathcal{Z}_\Delta$ :

$$\langle \dots \rangle = \int D\vec{\Delta} D\vec{\Delta}^* [\dots] \exp\{iS_\Delta[\vec{\Delta}, \vec{\Delta}^*]\}. \quad (3)$$

Here we used the fact that in the absence of the quantum components of external source-fields,  $\mathcal{Z}_\Delta \equiv 1$ , this is a manifestation of the causality principle. The saddle point

equation

$$\frac{\delta S_\Delta[\vec{\Delta}, \vec{\Delta}^*]}{\delta \Delta_2^*} = 0, \quad (4)$$

at the manifold  $\Delta_2^* = \Delta_2 = 0$  produces the nonequilibrium generalization of the GL equations.

Near the phase transition we can expand  $iS_\Delta[\vec{\Delta}, \vec{\Delta}^*]$  over  $\Delta$ . This expansion over quantum components of the order parameter should be performed up to the first order, see eq. (4). Carrying out the microscopic calculation based on the Keldysh functional representation of the BCS theory [16] in a form of the nonlinear  $\sigma$ -model we find that the shape of  $S_\Delta[\vec{\Delta}, \vec{\Delta}^*]$  above and below [the restrictions are discussed] the phase transition can be written in a form similar to that of 1:

$$S_\Delta = \nu a \text{Tr} \left\{ \vec{\Delta}^\dagger \left[ \hat{L}^{-1} - \frac{b}{T_{\text{eff}}} |\Delta_1|^2 \hat{\sigma}_0 \right] \vec{\Delta} \right\}, \quad (5)$$

where  $\text{Tr}$  means the trace with respect to times and the integration over coordinates, while  $\hat{\sigma}_0$  is the identity matrix in the Keldysh space. We show below that eq. (5) can be derived as well from the symmetry considerations similar to those of ref. [12].

The microscopic Keldysh nonlinear  $\sigma$ -model calculation expresses GL coefficients  $a$  and  $b$  through the gauge invariant electron and hole distribution functions [4],  $\tilde{F}_e$  and  $\tilde{F}_h$  (or  $\tilde{F}_L = (\tilde{F}_e + \tilde{F}_h)/2$  and  $\tilde{F}_T = (\tilde{F}_e - \tilde{F}_h)/2$ ), where in equilibrium,  $\tilde{F}_{e(h)} = \tanh(\epsilon/2T)$ :

$$a = \frac{\pi}{4} \lim_{\epsilon \rightarrow 0} \partial_\epsilon \tilde{F}_L(\epsilon) \equiv \frac{\pi}{8T_{\text{eff}}}, \quad (6)$$

$$b = 4T_{\text{eff}}^2 \int_0^\infty t \tilde{Y}(t) dt, \quad (7)$$

where  $2\pi\tilde{Y}(t) = \int_0^\infty d\epsilon \exp[-iEt] \tilde{F}_L(\epsilon)/(E + i0)$ . One can see straightforwardly that in the equilibrium limit eqs. (6), (7) reproduce conventional values  $a = \pi/8T$  and  $b = 7\zeta(3)/\pi^3$ , respectively. Formulas (6) and (7) for the coefficients of the far-from-equilibrium GL functional constitute the main result of our work.

A structure of  $\hat{L}$  in a Keldysh space is determined by the causality principle[11] and by the comparison with the standard form of GL expression given by eq. (1):

$$\hat{L}^{-1} = \begin{pmatrix} 0 & L_A^{-1} \\ L_R^{-1} & [L^{(-1)}]_K \end{pmatrix}, \quad (8)$$

where  $R(A)$  and  $K$  subscripts denote the retarded (advanced) and Keldysh propagators, respectively. The form of the  $R(A)$ -component can be found from the correspondence between the imaginary and real time representations that can be transformed into each other by the Euclidean rotation:

$$L_{R(A)}^{-1} = \pm(\partial_t - 2i\varphi) - \tau_{\text{GL}}^{-1} - D(\nabla_{\mathbf{r}} - 2i\mathbf{A})^2, \quad (9)$$

where  $(\varphi, \mathbf{A})$  is the electromagnetic potential<sup>1</sup>.

<sup>1</sup>The quasiparticle imbalance effect [4], causes a change in the electric potential,  $\varphi = \varphi_0 + \delta\varphi$ ,  $\varphi_0 = \varphi(\Delta = 0)$ . Generally,  $\delta\varphi$ , depends on  $\langle \Delta_1 \rangle$  and  $\tilde{F}_T$  ( $\delta\varphi \sim \int_\epsilon N_\epsilon \tilde{F}_T(\epsilon)$ , where  $N$  is BCS DoS). The same applies to  $j_s$ . As we restrict ourselves to superconducting systems with small gradients of  $\varphi$  ( $j_s$ ) and strong nonequilibrium ( $|V| \gg \langle \Delta_1 \rangle$ ), fig. 1, so  $\delta\varphi \sim o(\langle \Delta_1 \rangle / T_{\text{eff}})$  and the NGL coefficients should be calculated using  $\tilde{F}_{e(h)}|_{\delta\varphi=0}$  ( $\delta\varphi$  is important, *e.g.*, for the proximity coupled S-islands [17]).

The Keldysh component of the fluctuation propagator is expressed via the additional temperature scale  $T^*$ :

$$[L^{(-1)}]_K \equiv 4iT^*, \quad (10)$$

where  $T^* = [1 - \tilde{F}_e(0)\tilde{F}_h(0)]T_{\text{eff}}$  is determined from the microscopic calculation and  $\tau_{\text{GL}}^{-1}$  is determined by (provided the time reversal symmetry is not broken)

$$\tau_{\text{GL}}^{-1} = T_{\text{eff}} \int_{\epsilon} K^{(L)}(\epsilon) [\tilde{F}_L(\epsilon) - \tilde{F}_L^{(0)}(\epsilon)], \quad (11)$$

where  $K^{(L)}(\epsilon) = \alpha/2\epsilon$ . Here  $\tilde{F}_L^{(0)}(\epsilon)$  is the distribution function at the phase transition surface.

The current can be found analogously to ref. [9] by adding to the action the quantum source in a form of the vector potential,  $\mathbf{A}_2$ , in addition to the (classical) external fields potentials  $(\varphi, \mathbf{A})$  discussed above, and then varying the term proportional to  $\mathbf{A}_2$  in  $S_{\Delta}$ . Thus, this yields the supercurrent as

$$\mathbf{j}_s = \pi e \nu D |\Delta_1|^2 \partial_{\mathbf{r}} \arg(\Delta_1) / 2T_{\text{eff}}. \quad (12)$$

The conditions that electromagnetic fields and quasi-particle distribution functions are stationary and weakly depend on coordinates on the scale of the Cooper pair size,  $\xi = \sqrt{D\tau_{\text{GL}}}$ , ensure the applicability of the NGL functional. We verify that our highly NGL equations reduce to the standard *weakly*-nonequilibrium form in the case where the deviations from the equilibrium are small. Indeed, if  $\tilde{F}$  differs slightly from  $\tanh(\epsilon/2T)$ , one has to use the equilibrium values for  $a$  and  $b$  and keep nonequilibrium  $\tau_{\text{GL}}$  (since the nonequilibrium corrections are of the next order if the deviation from the equilibrium is small). This means that our theory is restricted to the neighborhood of the intersection of the phase transition surface with the equilibrium density matrix subspace, see fig. 1. Rewriting eq. (11) as  $1/(\tau_{\text{GL}}T_{\text{eff}}) = \alpha(T - T_c) + \alpha \int_{\epsilon} [\tilde{F}_L(\epsilon) - \tanh(\epsilon/2T)]/2\epsilon$ , we recover the weakly-nonequilibrium version of the GL theory [2,6].

Most of the quantities related to superconducting fluctuations are the singular functions of  $\tau_{\text{GL}}^{(\text{eq})}$  in equilibrium [13]. When we move out of the equilibrium, the fluctuation corrections are parameterized by the nonequilibrium  $\tau_{\text{GL}}$  and depend on  $T_{\text{eff}}$ . Recently zero-dimensional superconducting fluctuations and fluctuating diamagnetism in the lead nanoparticles were experimentally investigated, see, *e.g.*, ref. [18]. Motivated by the experiments we find, as an example, the nonequilibrium,  $V > V_c$ , fluctuation contribution to the magnetic moment of a small (of size  $L \ll \xi$ ) superconductor:

$$M = -2T_{\text{eff}}\tau_{\text{GL}}HDL^2\eta, \quad (13)$$

where  $H$  is the magnetic field and  $\eta = 1/10$  for a spherical island. Nonequilibrium fluctuation corrections to other quantities, *e.g.*, related to the diffusion propagator contributions to  $\mathcal{Z}$  (and the Langevin noise corrections with the correlator proportional  $T^*$ ), will be presented in [17].

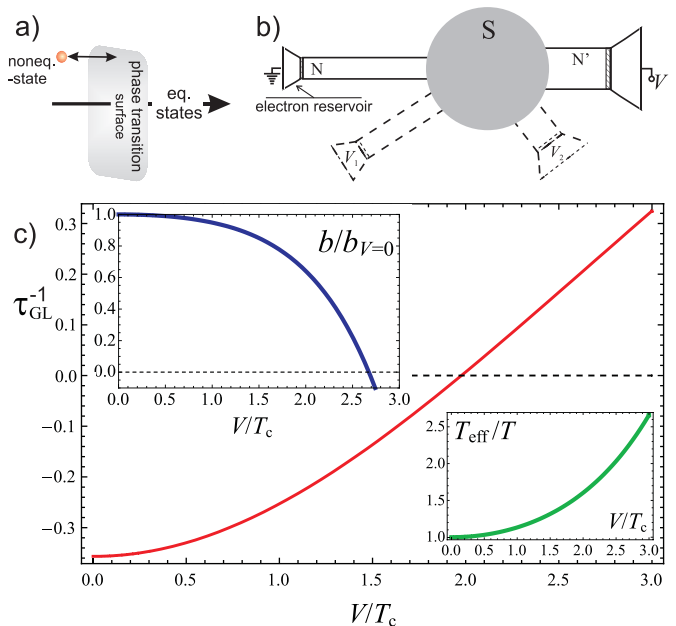


Fig. 1: (Color online) a) Density matrix space. b) An exemplary system: a superconducting granule weakly connected to the reservoirs. c) The GL nonequilibrium relaxation time as a function of the applied voltage  $V$  at the reservoir temperature  $T = 0.7T_c$ . (The units of  $\tau_{\text{GL}}^{-1}$  are chosen to match  $\ln(T/T_c)$  in equilibrium.) The lower inset illustrates that  $T_{\text{eff}}$  differs from  $V$ . The upper inset shows  $b(V)$  and illustrates the difference between the nonequilibrium and equilibrium behaviors; in the latter case  $b$  is the temperature-independent constant.

Now we sketch a general procedure for calculating fluctuation-related quantities. In an out-of-equilibrium state, one uses the Keldysh real-time partition function instead of the conventional thermodynamic partition function [9]:

$$\mathcal{Z} = \int D[\mathcal{A}, \Psi, \bar{\Psi}, \Delta, \Delta^*] \exp\{iS\}, \quad (14)$$

where the Grassman fields,  $\Psi, \bar{\Psi}$ , describe the fermion (superconductivity-related) degrees of freedom (on Keldysh contour),  $\mathcal{A} = (\varphi, \mathbf{A})$  and  $S[\mathcal{A}, \Psi, \bar{\Psi}, \Delta, \Delta^*]$  is the microscopic Keldysh action of the system. The Ginzburg-Landau expansion of the effective thermodynamical potential is an example of the so-called low-energy field theory, *i.e.* a theory, where the order parameter fields change negligibly on the microscopic scales, *e.g.*, the lattice constant. The low-energy theory in Keldysh formalism appears after integrating out the high-energy part of the fields. The resulting low-energy effective action consists of three parts. The first one,  $S_{\Delta}$ , describes quantum dynamics of the  $\Delta$ -field, the second part generates the kinetic equations for the excitations. It looks schematically like  $\text{Tr}[z \circ (\text{KE})]$ , where the dynamic variable  $z$  is closely related to the anti-Keldysh component of the  $Q$ -matrix in the nonlinear  $\sigma$ -model formalism. Integrating over  $z$  we obtain the functional  $\delta$ -function ensuring that

the distribution function obeys the kinetic equations. The third part  $S_A$  describes electromagnetic fields. The variation of the effective action over  $S_A$  produces Maxwell equations.

The fluctuations of the  $\Delta$ -field enter the collision integrals of the kinetic equations and the collisionless terms (the fluctuation renormalizations of the KE coefficients), while  $\langle \Delta_1 \rangle$  enter the (nonlinear) kinetic equations as external fields [4,17]. The  $\Delta$ -fluctuations in KE contribute to the fluctuation corrections to the kinetic coefficients [17]. The  $\langle \Delta_1 \rangle$ -terms in KE are important while  $\tilde{F}_\epsilon$  differs essentially from  $\tanh(\epsilon/2T)$  only at small energies,  $\epsilon \sim \langle \Delta_1 \rangle$ . But if quasiparticles are excited in the wide energy range above the gap,  $\langle \Delta_1 \rangle \lesssim |\epsilon| \lesssim T_{\text{eff}}$ , then  $\langle \Delta_1 \rangle$ -terms in KE induce small,  $\sim o(\langle \Delta_1 \rangle/T_{\text{eff}}) \ll 1$ , perturbation of  $\tilde{F}_\epsilon$  and subleading ( $\sim \langle \Delta_1 \rangle/T_{\text{eff}} \ll 1$ ) terms in  $S_\Delta$  compared to the terms given in eq. (5).

The phenomenological Landau theory predicts  $(\tau_{\text{GL}}^{\text{(eq)}})^{-1}$  to depend linearly on  $|T - T_c|$  and  $|\tau_{\text{GL}}^{\text{(eq)}}|T_c \gg 1$ . In the nonequilibrium state the role of  $(T - T_c)$  is taken by some functional of the electron and hole distribution functions, which characterizes the effective “distance” from the phase transition. We expect that  $\tau_{\text{GL}}^{-1}$  is a linear functional of  $\delta\tilde{F}_{e(h)} = \tilde{F}_{L(T)} - \tilde{F}_{L(T)}^{(0)}$ , otherwise the contribution proportional to  $(T - T_c)^3$  to  $\tau_{\text{GL}}^{-1}$  would have appeared in equilibrium. We write thus

$$\tau_{\text{GL}}^{-1} = T_{\text{eff}} \int_\epsilon \{K_\epsilon^{(L)} \delta\tilde{F}_L(\epsilon) + K_\epsilon^{(T)} \delta\tilde{F}_T(\epsilon)\}, \quad (15)$$

where the kernels  $K$  are some functions of the energy. In equilibrium  $\tilde{F}_T = 0$  and  $\tilde{F}_L = \tanh(\epsilon/2T)$ . Then we reproduce the equilibrium value of  $\tau_{\text{GL}}^{-1} = \alpha(T - T_c)$  choosing  $K^{(L)}(\epsilon) = \alpha/2\epsilon$ . We consider the system invariant under the time reversal symmetry. So we should choose  $K^{(T)} = 0$  because otherwise this term would give the unnatural contribution to  $\tau_{\text{GL}}^{-1}$  changing its sign when, *e.g.*, we reverse the direction of all currents in the system.

The important question is how this formalism describes the phase transition interface in the density-matrix space, see fig. 1. The parameter  $\tau_{\text{GL}}^{-1}$  should not depend upon the choice of  $F_L^{(0)}$ , thus  $F_L^{(0)}$  and  $F_L^{(0')}$  belonging to the interface should satisfy the relation

$$\int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{2\epsilon} [\tilde{F}_L^{(0')}(\epsilon) - \tilde{F}_L^{(0)}(\epsilon)] = 0, \quad (16)$$

where  $\omega_D$  is the Debye energy.

The microscopic derivation of the GL action shows that eq. (16) can be interpreted as the integral representation of the electron-phonon interaction constant,  $\lambda$ , giving the BCS-superconductivity:

$$\int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{2\epsilon} \tilde{F}_L^{(0)}(\epsilon) = \int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{2\epsilon} \tanh(\epsilon/2T_c) = \frac{1}{\nu\lambda}, \quad (17)$$

where  $T_c = 2\gamma\omega_D/\pi e^{-1/\nu\lambda}$  and  $\gamma = e^C$ , with  $C = 0.577\dots$  being the Euler constant.

In the Fourier space  $L_{R(A)}^{-1} = \pm i\omega + \tau_{\text{GL}}^{-1} + Dq^2$ . The Keldysh component  $L_K^{(-1)}$  in equilibrium should satisfy the relation following from the fluctuation-dissipation theorem (FDT) [9]:

$$[L^{(-1)}]_K = B_\omega(L_R^{-1} - L_A^{-1})_\omega, \quad (18)$$

where  $B_\omega = \coth\omega/2T$  is the equilibrium distribution function of the complex  $\Delta$ -field. The similar relation holds for the out of the equilibrium state where the gradients of  $B$  with respect the “the center of mass” Wigner transformation variables are irrelevant, which is the case we consider. The main (infrared) frequency scale of the Landau theory is  $\tau_{\text{GL}}^{-1}$  and  $Dq^2 \sim \omega \sim \tau_{\text{GL}}^{-1}$ . In equilibrium  $\tau_{\text{GL}}^{-1} \sim (T - T_c) \ll T_c$  and, therefore,  $2i\omega B_\omega \rightarrow 4iT_c$ . In the out-of-equilibrium state we should choose  $\tau_{\text{GL}}^{-1}$  smaller than any relevant energy scale of  $B_\omega$ . Then we can also replace  $2\omega B_\omega$  by

$$\lim_{\omega \rightarrow 0} 2\omega B_\omega \equiv T^* \quad (19)$$

and consider  $T^*$  as the second effective temperature. So

$$[L^{(-1)}]_K \equiv 4iT^*. \quad (20)$$

A recipe for constructing the stationary nonequilibrium distribution function is sketched in fig. 1b. The superconductor is connected to the electron reservoirs through the wires with the normal resistances  $R$  and  $R'$  (*e.g.*, the disordered quasi-1D normal metal wires). The reservoir biases and temperatures are, in general, different. If the wire resistance satisfies the relation,  $R \gg 1/L\sigma_N$  ( $L$  and  $\sigma_N$  are the S-diameter and the normal conductivity of the S-material) then the current coming from the wires spreads over the island that ensures the weakness of  $\tilde{F}_{e(h)}$  ( $\varphi$ ) gradients. The supercurrent  $j_s$  is much smaller than the critical current if  $\xi T_{\text{eff}}/(L\langle \Delta_1 \rangle) \gg L\sigma_N R$  (the normal current - supercurrent conversion at the NS interface is considered in ref. [17] using the generalized boundary conditions [19], which take into account the proximity effect). We neglect the Coulomb blockade effects considering the transmission probability (per channel) between the island and the wires to be close to unity. Normal wire - superconductor island structures used in the experiments [20] satisfy most of these conditions. Solving the normal state KE,  $D\partial_{\mathbf{r}}^2 F_{e(h)} = 0$ , we find that on the island,  $F_{e(h)} = \sum_n p_n \tanh[(E - V_n)/2T_n]$ ,  $\sum_n p_n = 1$ , where  $V_n$  is the voltage at the terminal  $n$  and  $p_n$  is determined by the resistivity of the wires (the exemplary system size is smaller than the quasiparticle inelastic length scale). For the two-terminal case,  $p_2 = R/R_\Sigma$ , where  $R_\Sigma = R + R'$ . A similar structure of  $p_n$  holds for a multiterminal case.

To illustrate the developed approach, we consider an exemplary system with  $p_2 = p_1 = 1/2$ . Then

$$\tilde{F}_{e(h)} = \frac{1}{2} \left[ \tanh \frac{E \pm V/2}{2T} + \tanh \frac{E \mp V/2}{2T} \right], \quad (21)$$

where  $\delta\varphi = 0$  (see footnote <sup>1</sup>),  $T^* = T_{\text{eff}}$ , and

$$T_{\text{eff}} = T \cosh^2(V/4T), \quad (22)$$

$$\tau_{\text{GL}}^{-1} = \alpha T_{\text{eff}} \left\{ \text{Re} H_{-\frac{1}{2}+iu} + 2 \ln 2 + \ln(T/T_c) \right\}, \quad (23)$$

$$b = -\frac{T_{\text{eff}}}{2\pi^3 T} \text{Re} \Psi^{(2)}(1/2 + iu). \quad (24)$$

Here  $u = V/4\pi T$ ,  $H$  is the Harmonic number and  $\Psi$  is the Digamma function. Importantly,  $b$  is very sensitive to the degree of nonequilibrium, see fig. 1: it changes sign at  $u \approx 0.3$  remaining negative at larger  $u$  that signals of the possible instability (in agreement with [10]) and requires keeping  $\sim \Delta^6$ -terms in the action [17]. While  $u \lesssim 1$ ,  $\tau_{\text{GL}}^{-1} \approx [\theta u^2 + \ln(T/T_c)] \alpha T_{\text{eff}}$  with  $\theta = 7\zeta[3] \approx 8.4$ . In fig. 1,  $\langle \Delta \rangle \sim \sqrt{V_c - V}$ , with  $V_c \approx 4\pi T \sqrt{\ln(T_c/T)}/\theta$ . Taking, *e.g.*,  $T = 1.3T_c$  in the reservoirs and  $V = 2T_c$  we get:  $T_{\text{eff}}/T \sim 0.87$ ,  $\tau_{\text{GL}}/\tau_{\text{GL}}^{(\text{eq})} \sim 0.69$  and then the fluctuation susceptibilities, see eq. (13):  $\chi/\chi^{(\text{eq})} \sim 0.6$ .

To conclude, we have constructed the nonequilibrium GL theory on the symmetry grounds under the condition that the kinetics of the high-symmetric phase is established. The coefficients of the nonequilibrium GL functional, which are the constants in an equilibrium, become strongly dependent on the external drive in a nonequilibrium state. In particular, the coefficient at the fourth-order (in the order parameter) term can change its sign at large driving forces; this would signal the onset of the instability which requires the higher-order expansion. The energy parameter  $T_{\text{eff}}$  replacing the equilibrium temperature  $T$ , is now a nonlinear function of the bias voltage. We have demonstrated that the fluctuation corrections to observable quantities, *e.g.*, to the magnetic susceptibility, in a superconducting island get strongly renormalized and become the singular functions of  $\sqrt{V - V_c}$  when out of equilibrium rather than being functions of  $\sqrt{T - T_c}$  in equilibrium. Accordingly, the order parameter vanishes like  $\sqrt{V_c - V}$  in the out-of-equilibrium state replacing its  $\sqrt{T_c - T}$ -dependence of the equilibrium state.

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