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Spectral action and neutrino mass

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Abstract – We propose the extension of the spectral action principle to fermions and show that the neutrino mass terms then appear naturally as next-order corrections.

Introduction. – The problem whether neutrino is a massive or a massless particle, which seemed to be one of the long-standing puzzles of particle physics, seems to be solved by now. Convincing strong evidence of neutrino oscillations, coming from various experiments ([1], see [2] for a review of further results), gave the necessary experimental proof. Still, the subject is far from being closed for theoretical and experimental physics as there remain open, at least, three major questions: whether all neutrinos are indeed massive, whether they are Majorana fermions, and, probably one of the most fascinating —still, if massive, why their masses are so small.

There exist, of course, various possible theoretical mechanisms, which could be, in principle, verified experimentally (see, for instance, [3] for a review). Here, we would like to address the issue from the point of view of geometry. In fact, none of the fundamental symmetry principles seemed to enforce the neutrino to be massless. Only recently (compared to the time-scale of neutrino investigations), the early application of methods of noncommutative geometry suggested that massless neutrinos agree with the geometry of the Standard Model described as a finite spectral triple or, in other words, a zero-dimensional manifold, satisfying Poincaré duality in K-theory [4–6].

The fact that the observed neutrinos are massive indicated that the model was not very realistic and has led to some modified propositions, which admit massive neutrinos and see-saw mechanism. This, however, required some adjustments to the notion of geometry, in particular, the geometry of the finite space was required to have a homological dimension different from 0 (mod 8) [7,8].

There are probably many possible ways to adapt the model and its axioms to the physical reality, or to interpret the physical data within the existing framework with slight changes of the particle content of the model [9].

Here, we shall discuss a possible mechanism, which relates the earlier explanations [10] with the current formalism of spectral geometry. This will have the advantage of providing not only the explanation for the mass but also for its scale, while leaving the original geometry untouched.

Standard Model and geometry. – Within the framework of noncommutative geometry, the standard Dirac operator has a natural extension by a finite-dimensional matrix, which has a natural interpretation of the mass matrix and mixing angles —we shall not discuss here in detail the restrictions set on this matrix by the spectral triple axioms (like the order-one condition). Let us briefly remind that for a manifold, the Dirac operator satisfies certain commutation rules with the chirality γ and the charge conjugation C operators, which (we consider only even dimensions) are fixed. The difference between Minkowski and Euclidean approach is in the mutual relations between the charge conjugation and the chiral projections: in $1+3$ dimensions charge conjugation necessarily changes chirality, which is not the case in 0 or 4 dimensions.

The new notion of the Dirac operator allows for possible finite components of D, so that it becomes

$$D=\gamma^\mu\partial_\mu+\omega_{\mathrm{s}}+D_{\mathrm{F}}, \qquad (1)$$

where ω_{s} is the spin connection and D_{F} is a linear operator on the full Hilbert space. The existence of this finite matrix component leads to the introduction of additional gauge fields that could be naturally identified with the Higgs doublet.

However, what is crucial for us is the assumption that the action function is purely *spectral*, that is, for the

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bosonic part it is taken as a cutoff trace of the Dirac operator [11]:

$$S_{\mathrm{b}} = \mathrm{Tr}\, f(D^2) \tag{2}$$

with suitable choice of the cutoff function f.

For the fermions, one usually takes the expectation value of D

$$S_{\mathrm{f}} = \langle \Psi | D \Psi \rangle. \tag{3}$$

The bosonic spectral action in its asymptotic expansion gives rise to all gravity-dependent terms, which include the cosmological constant and the Einstein-Hilbert action as well as all terms of the Yang-Mills-Higgs model, with the symmetry-breaking Higgs potential. This model is still open for possible extensions, like models including leptoquarks [12] or extended symmetries [13].

The mass terms for fermions arise in (3) through the nonzero expectation value of the Higgs from the minimal couplings between the Higgs and the leptons, due to the symmetry-breaking potential term for the Higgs doublet, which arises naturally in (2) [6,14].

Note that the spectral action works only in the Euclidean setup, hence in order to pass to the Minkowski setup, one needs first to derive all local terms and then Wick rotate them.

We shall ignore this issue and also we shall not assume any particular noncommutative geometry model, thus not touching the fermion-doubling problem [15,16] and the need of reducing the product of the Hilbert spaces to a certain physical subspace (as in [8]). Instead, we take a minimalist approach and consider only the spectral action of the already given part of the Dirac operator from the physical Standard Model.

Fermionic mass terms and neutrinos. – Majorana particles are fermions, which are invariant under the charge conjugation operation. From the mathematical point of view, in the physically relevant Minkowski setup they are real representations of the real Clifford algebra Cliff $(1,3)$. It is allowed that Standard Model neutrinos are Majorana, with a left-handed neutrino and its right-handed antipartner (in the Minkowski setup). In principle, there are no obstacles that such particles might be massive, and one introduces a Majorana mass term of the form:

$$m_{\mathrm{l}} \overline{\Psi_L^c} \Psi_L + \mathrm{h.c.}, \tag{4}$$

where Ψ_L and Ψ_L^c are chiral components of the Majorana field Ψ.

It is important to realize that such terms are not excluded by the Standard Model, however, to preserve the gauge invariance, they would require the interactions of the type [10]:

$$\mathcal{L}_m = \kappa \left(\overline{e_L^c} H^+ - \overline{\nu^c} H^0 \right) \left(H^+ e_L - H^0 \nu \right), \tag{5}$$

where e_L, ν is the doublet of left-handed leptons, H^+ and H^0 are the Higgs field components, and κ is a coefficient (or a matrix, if we take into account flavors). Such term, though acceptable from the classical point of view, introduces, in the context of quantum field theory, a nonrenormalizable interaction [17]. It is also clear that it cannot appear in the standard noncommutative formulation of the fermionic action (3), since the coupling between the generalized Dirac operator and the fermions is linear in D. The speculation that such terms arise from quantum gravity corrections [10] gives the neutrino mass of the range of $10^{-5}\,\mathrm{eV}$, which is much less than the experimental estimations.

In the formulation of noncommutative geometry, an important role is played by a finite spectral triple, that is, a geometry over a finite algebra. We shall not assume any particular model here, taking the resulting generalized Dirac operator and the Higgs potential for granted.

The extended spectral action principle. – In the model-building principle of noncommutative geometry, the main role is played by an algebra, which corresponds to the functions on the space-time, its representation on the Hilbert space, and the Dirac operator, which encodes all the information about differentiation, metric as well all internal degrees of freedom. Dirac operator could be modified by its *internal fluctuations*, thus leading to a family of operators.

The physics is set by the action (2) with a suitable cutoff function f. The asymptotic expansion of the bosonic action leads to leading terms providing the volume (cosmological constant), the traditional Einstein-Hilbert action of pure gravity as well as the Yang-Mills gauge functionals for the internal fluctuations of the Dirac operator, which are identified as gauge fields. Clearly, one has to make a Wick rotation to Minkowski geometry to consider the physical fields, nevertheless the consistency with the standard physical picture is striking.

As it has been already observed by Chamseddine [18], there is a huge difference between the bosonic and fermionic (3) parts of the action in Noncommutative Geometry, especially in the formulation of the spectral action principle. At first, the bosonic (and gravitational) part of the action depends solely on the eigenvalues (or, more precisely, on the eigenvalue asymptotic) of the generalized Dirac operator. This is not the case for fermion fields, where the action principle is the expectation value of the Dirac operator in the state set by Ψ. Thus all the eigenvalues of the Dirac do intervene. The generalization proposed by [18] and tested on a simple example led to the interpretation of additional terms as arising from the supersymmetric theory.

Although Chamseddine's efforts were concentrated on the couplings between gauge fields and fermions, there is still a place in such models for couplings between discrete gauge field strength and fermions, in particular, for coupling between two fermions and a term quadratic in the Higgs. The spectral approach to the action in noncommutative geometry offers a feasible theoretical mechanism for the appearance of such terms.

First of all, one might consider a *total* action of the type:

$$S = \mathrm{Tr}\, f\,\big((D+P_\Psi)^2\big), \tag{6}$$

where f is some cutoff function and P_Ψ is a projection on the field Ψ.

To give an example of what are the consequences, consider the terms in the asymptotic expansion that shall arise. Using standard results of Gilkey [19] for the asymptotic expansion, we obtain that the relevant terms involving fermions read

$$\begin{aligned} S_\Psi \;=\; & \Lambda^2 \int_M \mathrm{Tr}\,(DP_\Psi + P_\Psi D) + \frac{1}{360}\left(\int_M -60R\mathrm{Tr}\, DP_\Psi \right.\\ & \left. +180\mathrm{Tr}\,(DP_\Psi + P_\Psi D)^2 + 60\mathrm{Tr}\,\Delta(DP_\Psi)\right). \end{aligned} \tag{7}$$

The leading term in the expansion is nothing else than the standard fermion action:

$$\int_M \langle \Psi, D\Psi\rangle,$$

which gives the minimal couplings between fermion and gauge fields, as well as coupling between fermions and the Higgs and then fermion masses.

The terms of the next order contain some higher-derivative components, coupling of fermions to the scalar curvature as well as a nonlinear coupling of the fermions to the Dirac operator. Let us analyze the latter in more detail. Simple calculation yields that the next-order terms that do not involve derivatives are

$$\mathrm{Tr}\,(DP_\Psi + P_\Psi D)^2 = 3(\langle\Psi|D\Psi\rangle)^2 + \langle\Psi|D^2\Psi\rangle. \tag{8}$$

From our point of view, the interesting terms are these, which include the square of the Dirac operator.

The Ansatz. – Let us postulate a simple-minded solution to the question "how to obtain quadratic terms" (5) using an extension of the spectral action principle. Of course, since the minimal Standard Model alone does not provide the answer, we must find a way so that no linear term, giving *bare* neutrino mass appears but there will be a quadratic term, originating in a nonzero contribution (8).

First, observe that the expression

$$\big(H^{\mathrm{T}}\sigma L\big), \tag{9}$$

where L is a lepton doublet, H is the Higgs doublet, and σ is a σ^2 Pauli matrix is itself gauge invariant. Recall that under $U(1)\times SU(2)$ gauge transformations, the lepton doublet and the Higgs field transform as follows:

$$L \mapsto hL\bar{z}, \qquad H \mapsto hHz,$$

with $h\in SU(2)$, $z\in U(1)$. Hence, the term (9) remains invariant due to the fact that σ intertwines the fundamental representation of the quaternions with its conjugate:

$$\sigma h \sigma^{-1} = h^*. \tag{10}$$

For this reason, any spinor field N, which is totally noninteracting shall give a gauge-invariant term of the form

$$\langle N|H^{\mathrm{T}}\sigma L\rangle. \tag{11}$$

Clearly, adding the field N to the family of all fermion fields is the solution, which is comparable to the addition of sterile neutrinos.

However, we propose a way so that the effective Standard Model action (as we see it) ignores those particles. Our *Ansatz* is for the spectral action including fermions and the form of the cutoff function f that excludes a subspace of the Hilbert space.

We assume that the full Hilbert space includes some sterile, noninteracting (*i.e.*, one which is not in the representation of the discrete algebra) fermion N. Furthermore, instead of taking f to be scalar-valued, let us assume that in addition to the cutoff in the eigenvalues of D, the function projects on a subspace of the Hilbert space, which consists of all fermions, which are in the representation of the discrete algebra. One can explain this restriction saying that the spectral action presents an effective action up to some fixed energy scale and the cutoff is implemented also by the restriction to some subspace of the Hilbert space.

If we look now at the spectral action:

$$S_{\mathrm{eff}} = \mathrm{Tr}_{\mathrm{ph}}\, f_\Lambda((D+P_\Psi)^2), \tag{12}$$

we see that neither the bosonic part nor the standard fermionic part (*i.e.*, linear in D) shall change with respect to the well-known action of the Standard Model. The difference shall appear, however, at the level of correction terms, that is, second-order with respect to the leading term. There, we shall see contributions from D^2, which are exactly of the form

$$\sigma H^* H^{\mathrm{T}} \sigma.$$

Even though the details of the effective action depend on the particular form of f, it is not important for our considerations. The modified action provides just corrections to the original action (3) of order $o(\frac{1}{\Lambda^2})$.

If we take the value of the cutoff parameter [11] $\Lambda = 10^{15}$ GeV and estimate the resulting neutrino mass (taking the coefficient in the term to be of order 1), we obtain the values of order 10^{-2} eV, which agrees with the current experimental data. In fact, it was pointed out that such order of neutrino mass suggests in turn [17] that the scale of 10^{15} GeV (which is well below the Planck scale) is the one at which one can expect to change the effective model.

Conclusions. – We have shown that a minimal adjustment of the contents of the Standard Model together with the extension of the spectral action principle to fermions might provide an explanation of the small neutrino mass. Clearly, the correction terms have no measurable influence on the masses of other particles as they are many orders of magnitude smaller. In the case of originally massless neutrino this correction shall be, however, a leading term. Therefore, even within the simplest description of the Standard Model in noncommutative geometry, it is possible to generate a neutrino mass *via* correction from the spectral action. We presented argument for one family, but its generalization to many generations is straightforward. Of course, the terms are not renormalizable. However, we might treat the model as an effective one and the neutrino mass term as effective at a given energy scale. Incidentally, the extra gravity terms that appear in the spectral action principle as next-order corrections to the Einstein-Hilbert action lead to similar problems.

If we take the noncommutative geometry description and the spectral action as an approximation of the real geometry, then the cutoff parameter Λ of the bosonic action has a natural physical meaning of the scale of possible fluctuations of the geometry. It is conceivable that the physics observed at today's energy scales is the restriction of some more general model. The spectral action appears to be well-suited for this purpose.

The proposition shown in this paper indicates a way of introducing the neutrino mass in a purely dynamical way. Although the term (5) that induces this mass at the nonzero Higgs expectation value is not new, the method of obtaining it is set into the noncommutative geometry. Since the part of the Hilbert space, which is cut off through the spectral action, corresponds to some hypothetical particles of the sterile neutrino type, one might view it as a form of see-saw mechanism [20]. Although no direct mass of the extra particles is needed, the cutoff mechanism allows for the appearance of correction terms, which are significant only in the case of originally massless neutrinos. Note that contrary to the see-saw mechanism, we do not need to justify the nondynamical and large mass terms for the sterile neutrinos, as in our case the role of the "see-saw weight" is played by the cutoff parameter. We believe that the possibility appears to be feasible and requires more research on the action principle for fermions and quantum theory of fields in this noncommutative setup. Finally, let us mention that since the presented model keeps the postulate that there are only left neutrino currents, the experimental results (in particular, from neutrinoless double β decay) might distinguish whether the "discrete" noncommutative geometry description of the Standard Model is right or wrong. We believe that the geometric approach, that we advocate here, might also shed new light on the issues of physics beyond the Standard Model.

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REFERENCES

[1] Super-Kamiokande Collaboration (Kajita T.), *Nucl. Phys. Proc. Suppl.*, **100** (2001) 107.

[2] Alberico W. M. and Bilenky S. M., *Phys. Part. Nucl.*, **35** (2004) 297; *Fiz. Elem. Chastits At. Yadra*, **35** (2004) 545.

[3] Akhmedov E. K., *Nucl. Phys. Proc. Suppl.*, **100** (2001) 295.

[4] Chamseddine A., Connes A. and Marcolli M., *Adv. Theor. Math. Phys.*, **11** (2007) 991.

[5] Connes A. and Marcolli M., *Noncommutative Geometry, Quantum Fields and Motives, AMS Colloq. Publ.*, Vol. **55** (AMS, Hindustan Book Agency) 2008.

[6] Connes A., *J. Math. Phys.*, **36** (1995) 6194.

[7] Barrett J. W., *J. Math. Phys.*, **48** (2007) 012303.

[8] Connes A., *JHEP*, **0611** (2006) 081.

[9] Schelp R., *Int. J. Mod. Phys. B*, **14** (2000) 2477.

[10] Weinberg S., *Phys. Rev. Lett.*, **43** (1979) 1566.

[11] Chamseddine A. and Connes A., *Commun. Math. Phys.*, **186** (1997) 731.

[12] Paschke M., Scheck F. and Sitarz A., *Phys. Rev. D*, **59** (1999) 035003.

[13] Stephan Ch., *Almost-Commutative Geometries Beyond the Standard Model II: New Colours*, arXiv:0706.0595.

[14] Iochum B., Kastler D. and Schucker T., *J. Math. Phys.*, **38** (1997) 4929.

[15] Gracia-Bondia J. M., Iochum B. and Schucker T., *Phys. Lett. B*, **416** (1998) 123.

[16] Lizzi F., Mangano G., Miele G. and Sparano G., *Phys. Rev. D*, **55** (1997) 6357.

[17] Wilczek F., *Nucl. Phys. Proc. Suppl.*, **77** (1999) 511.

[18] Chamseddine A., *Phys. Lett. B*, **436** (1998) 84.

[19] Gilkey P., *Invariance Theory, The Heat Equation and the Atiyah-Singer Index Theorem* (Publish or Perish, Dilmington) 1984.

[20] Jureit J. H., Krajewski T., Schucker T. and Stephan Ch., *Phys. Lett. B*, **654** (2007) 127.